Module I

Partial Differential Equations

Syllabus

Origin of partial differential equations, linear and non-linear partial equations of first order, lagrange's equations, Charpit's method, Cauchy's method of characteristics, solution of linear partial differential equation of higher order with constant coefficients, equations reducible to linear partial differential equations with constant coefficients.

Contents

Partial Differential Equations

Applications

- Partial differential equations are used to mathematically formulate the solution of physical and other problems involving function of several variables, such as the propagation of heat or sound, fluid flow, Waves, elasticity, electrostatic, electrodynamics etc.
- Fluid mechanics, heat and mass transfer, are all modelled by partial differential equations and all have plenty of real life applications.
  For example,
    - Fluid mechanics is used to understand how the circulatory system works, how to get rockets and planes to fly, and even some extent how the weather behaves.
    - Heat and mass transfer is used to understand how drug delivery devices work, how kidney dialysis works, and how to control heat for temperature-sensitive things.

Partial differential equations in Stokes flow-

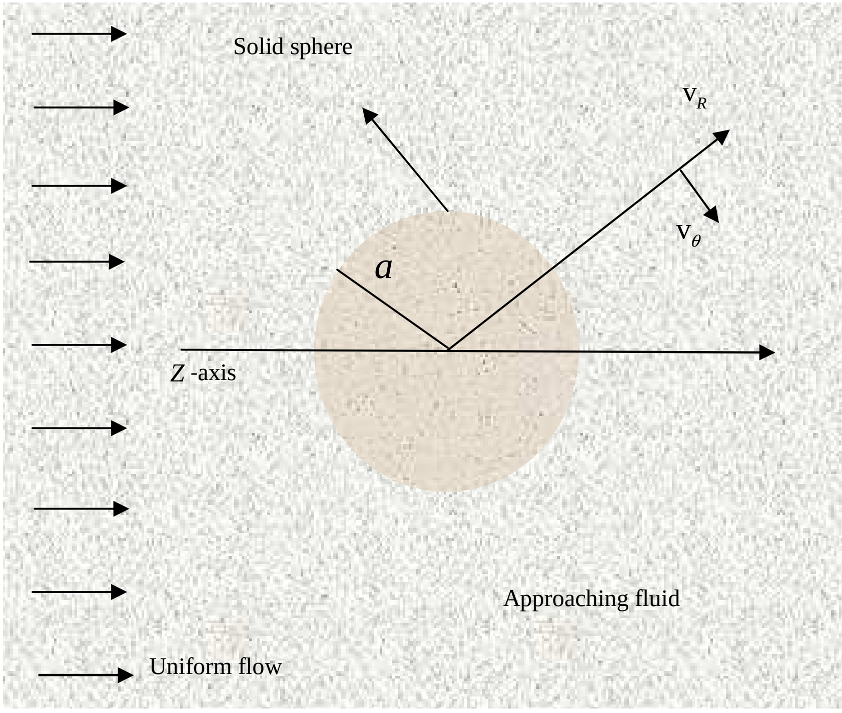

Figure 1: Newtonian fluid around a solid sphere

The governing equation for the creeping viscous flow of an incompressible fluid are

$P = \mu \quad ^2\,\mathbf{v}$(1)

and

$.\mathbf{v} = 0$ (2)

where, $= \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{\partial}{\partial \phi}$,

vis the velocity and $P$is the pressure. Introducing the stream function $\psi(r,\theta)$, which is related to velocity in spherical coordinate system (r,θ,ϕ) by

$\mathrm{v}_r(r,\theta)\hat{e}_r + \mathrm{v}_\theta(r,\theta)\hat{e}_\theta = - \quad \times \left(\frac{\psi}{r \sin\theta}\hat{e}_\theta\right)$ (3)

Using equation (1) and (3), we get fourth order partial differential equation

$E^4\psi = 0$ (4)

where$E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\zeta^2)}{r^2}\frac{\partial^2}{\partial \zeta^2}, \zeta = \cos\theta$

Applying the method of separation of variable we get the general solution of equation (4) and after ignoring all terms which are not applied in presented model (Figure 1), we get the stream function solution for the flow field for the Figure 1, which is in the form of partial differential equation.

$$\psi = (cr^{-1} + fr^4)G_2(\zeta)$$

where$c$ and f are constant and $G_2(\zeta)$ is a Gigenbaur function.

1.1 Introduction of Partial Differential Equation

Differential Equations

An equation involving derivatives or differential coefficient is called differential equation.

Or

An equation involving derivatives or differentials of one or more dependent variables w.r.t. one or more independent variables is called a differential equations

Example

$$\frac{dy}{dx} + y = 0,$$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 2y = 0$$

Ordinary Differential Equation

A differential equation involving derivative w.r.t. a single independent variable is called an ordinary differential equation

Example

$$\frac{d^4x}{dt^4} + y\frac{d^2t}{dx^2} + \left(\frac{dy}{dx}\right)^5 = e^t$$

Partial Differential Equation

A differential equation involving partial derivatives with respect to more than one independent variables is called partial differential equation.

Example

$$\frac{\partial^2 \mathrm{v}}{\partial t^2} = k\left(\frac{\partial^3 \mathrm{v}}{\partial x^3}\right)^2,$$

$$\frac{\partial^2 \mathrm{u}}{\partial x^2} + \frac{\partial^2 \mathrm{u}}{\partial y^2} + \frac{\partial^2 \mathrm{u}}{\partial z^2} = 0$$

1.2Order of a Partial Differential Equation

Order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

Example

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$ {first order}

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial y}\right) \text{ \{third order\}}$$

$z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$ {first order}

$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$ {second order}

1.3Degree of a Partial Differential Equation

The degree of a partial differential equation is a degree of highest order derivative occurs in it. When it has been made free from radical sign and fraction power.

For example,

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$ {first degree}

$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial y}\right)$ {first degree}

$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$ {second degree}

$y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right)$ {second degree}

1.4Linear and Non-linear Partial differential Equation

A partial differential equation is said to be linear if the dependent variable and its partial derivative occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation.

For example

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$ {linear}

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz${linear}

$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ {non-linear}

$z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x$ {non-linear}

$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}${non-linear}

Notation:- In partial derivative we use the following notations:

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

We usually assume x and y as independent variable and z to be as dependent variable.

Example 1: Find the order and degree of the following PDE:

(1) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2\left(\frac{\partial z}{\partial x}\right)$, Order 3, Degree 1

(2) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial x}\right)^2 = 1$, Order 2, Degree 1

(3) $x\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z\frac{\partial z}{\partial y}$, Order 1, Degree 2

(4) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{-1/2}$

$$\Rightarrow \left(\frac{\partial^2 z}{\partial x^2}\right)^2 = \left(1 + \frac{\partial z}{\partial y}\right)^{-1}$$

$$\Rightarrow \left(1+\frac{\partial z}{\partial y}\right)\left(\frac{\partial^2 z}{\partial x^2}\right)^2 = 1, \qquad \text{Order 2, Degree 2}$$

(5) $\left(1+\frac{\partial z}{\partial x}\right)^{1/2} = \left(\frac{\partial^2 z}{\partial y^2}\right)^{1/3}$

$$\Rightarrow \left(1+\frac{\partial z}{\partial x}\right)^3 = \left(\frac{\partial^2 z}{\partial y^2}\right)^2 \qquad \text{Order 2, Degree 2}$$

1.5Classification of first order partial differential equations

(1) Linear equation

Example $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$

(2) Semi linear equation

Example $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = \frac{x^2z^2}{y^2}$

(3) Quasi-linear equation

Example $x^2zp + y^2zq = xy$

(4) Non-linear equation

Example $p^2 + q^2 = 1,$ and $pq = 1$

1.6Formation of a Partial Differential Equation-

(1) By elimination of arbitrary constants

a) When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives more than one partial differential equations.

Example 1:

Let

$z = ax + y \qquad (1)$

Differentiating (1) partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a \quad (2)$

Differentiating (1) partially w.r.t. y, we get

$\frac{\partial z}{\partial y} = 1 \qquad (3)$

Eliminating $'a'$ from (1) and (2)

$z = \frac{\partial z}{\partial x}x + y \quad (4)$

Eq. (3) does not contain arbitrary constant, so (3) is also a partial differential equation under the consideration. We get two partial differential equations (3) and (4).

b) When number of arbitrary constants are equal to the number of independent variables, then the elimination of arbitrary constants shall give a unique P.D.E. of order one.

c) When number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants lead to a P.D.E. of order usually greater than one.

Example 2: Form partial differential equation from the following equations by eliminating the arbitrary constants.

(a) $z = ax + by + a^2 + b^2$ (b) $z = (x+a)(y+b)$

(c) $z = (x-a)^2 + (y-b)^2$ (d) $az + b = a^2x + y$

Solution (a) Given $z = ax + by + a^2 + b^2 \quad (1)$

Differentiating (1) partially w.r.t. $x$and y, we get

$\frac{\partial z}{\partial x} = aand\frac{\partial z}{\partial y} = b$

Substituting these values of a and b in (1), we get

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Which is the required partial differential equation.

Solution(b) Given $z = (x + a)(y + b)$ (1)

Differentiating z partially w.r.t. x

$$\frac{\partial z}{\partial x} = p = (y + b)$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = (x + a)$$

Form (1), we get

$$z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

Or

$$z = pq$$

Solution(c) Given $z = (x - a)^2 + (y - b)^2$ (1)

Differentiating (1) partially w.r.t. $x$and y, we get

$$\frac{\partial z}{\partial x} = 2(x - a)$$

$$\frac{\partial z}{\partial y} = 2(y - b)$$

$$4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Solution(d) Given $az + b = a^2x + y$ (1)

Differentiating (1) partially w.r.t. x, we get

$a\frac{\partial z}{\partial x} = a^2 \Rightarrow p = a(2)$

Differentiating (1) partially w.r.t. y, we get

$a\frac{\partial z}{\partial y} = 1 \Rightarrow q = \frac{1}{a}$ (3)

Multiplying (2) and (3)

$$pq = 1$$

(2) By the elimination of arbitrary function

Example 3: Form the partial differential equation by eliminating the arbitrary functions from the following

(a)$z = f(x^2 - y^2)$**(b)**$z = e^y f(x + y)$

(c)$z = \phi(x).\psi(y)$**(d)**$z = f(x + it) + g(x - it)$

Solution(a):$z = f(x^2 - y^2)$ (1)

Differentiating (1) partially w.r.t. $x$and y, we get

$\frac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$ (2)

$\frac{\partial z}{\partial y} = (-2y)f'(x^2 - y^2)$ (3)

Dividing (2) by (3), we obtain

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{x}{y}$$

$$\frac{p}{q} = -\frac{x}{y}$$

$$\Longrightarrow py + qx = 0$$

Solution (b): Given $z = e^y f(x + y)$ (1)

Differentiating (1) partially w.r.t. $x$and y, we get

$\frac{\partial z}{\partial x} = e^y f'(x + y)$ (2)

$\frac{\partial z}{\partial y} = e^y f(x + y) + e^y f'(x + y)$ (3)

Using (1), (2) and (3), we get

$$\frac{\partial z}{\partial y} = z + \frac{\partial z}{\partial x}$$

Solution(c) Given $z = \phi(x).\psi(y)$ (1)

Differentiating (1) partially w.r.t. x

$\frac{\partial z}{\partial x} = p = \phi'(x).\psi(y)$ (2)

$\frac{\partial z}{\partial y} = q = \phi(x).\psi'(y)$ (3)

Differentiating (2) partially w.r.t. y

$\frac{\partial^2 z}{\partial x \partial x} = s = \phi'(x).\psi'(y)$(4)

Multiplying (2) and (3), we get

$$pq = \phi(x)\psi(y)\phi'(x)\psi'(y)$$

$$pq - zs = 0$$

Solution(d) Given$z = f(x + it) + g(x - it)$

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial z}{\partial t} = if'(x + it) - ig'(x - it)$$

$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it)$(2)

$$\frac{\partial^2 z}{\partial t^2} = i^2 f(x+it) + i^2 g'(x-it)$$

$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g'(x-it)$(3)

Using eq. (2) and (3), we get

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

Practice questions

`1. Form partial differential equations from the following equations by eliminating the arbitrary constants:

(a) $z = ax + by + ab$. **Ans.** $z = px + qy + pq$
(b) $z = ax + a^2y^2 + b$. **Ans.** $q = 2p^2y$
(c) $z = (x^2 + a)(y^2 + b)$. **Ans.** $pq = 4xyz$
(d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **Ans.** $\boldsymbol{xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0}$

2.Form partial differential equations from the following equations by eliminating the arbitrary functions:

(a)$z = f(x^2 + y^2)$**Ans.** $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

(b)$x + y + z = f(x^2 + y^2 + z^2)$.**Ans.** $\boldsymbol{(y - z)p + (z - x)q = x - y}$

(c) $\boldsymbol{z = f(xy/z)}$. **Ans.** $\boldsymbol{px - qy = x - y}$

1.7Linear Partial Differential Equations of order One

A quasi-linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are the functions of x, y, z. Such a partial differential equation is known as Lagrange equation

For example $xyp + yzq = zx$

Lagrange's method of Solving$\boldsymbol{Pp + Qq = R}$, when P, Q and R are the function of $\boldsymbol{x,\ y,\ z}$

Theorem. The general Solution of the Lagrange equation

$Pp + Qq = R$ (1)

is

$\phi(\mathrm{u}, \mathrm{v}) = 0$ (2)

whereϕ is an arbitrary function and

$\mathrm{u}(x, y, z) = \mathrm{c}_1and\mathrm{v}(x,\ y,\ z) = \mathrm{c}_2$ are two independent Solution of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here, c_1 and c_2 are arbitrary constant.

Proof. Differentiating (2) partially w.r.t. x and y

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial \phi}{\partial \mathrm{v}}\frac{\partial \mathrm{v}}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial \phi}{\partial \mathrm{v}}\frac{\partial \mathrm{v}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \mathrm{v}}{\partial \mathrm{v}}\frac{\partial \mathrm{v}}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial \mathrm{v}}\left(\frac{\partial \mathrm{v}}{\partial x}+\frac{\partial \mathrm{v}}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial \phi}{\partial \mathrm{v}}\left(\frac{\partial \mathrm{v}}{\partial x}+\frac{\partial \mathrm{v}}{\partial z}p\right)=0$ (4)

Similarly

$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial \phi}{\partial \mathrm{v}}\left(\frac{\partial \mathrm{v}}{\partial y}+\frac{\partial \mathrm{v}}{\partial z}q\right)=0$ (5)

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial \mathrm{v}}$ from (4) & (5), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial \mathrm{v}}{\partial x}+p\frac{\partial \mathrm{v}}{\partial z} \\ \frac{\partial \mathrm{v}}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial \mathrm{v}}{\partial y}+q\frac{\partial \mathrm{v}}{\partial z} \end{vmatrix}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial \mathrm{v}}{\partial y}+q\frac{\partial \mathrm{v}}{\partial z}\right)-\left(\frac{\partial \mathrm{v}}{\partial x}+p\frac{\partial \mathrm{v}}{\partial z}\right)\left(\frac{\partial \mathrm{u}}{\partial y}+q\frac{\partial u}{\partial z}\right)=0$$

$$\Rightarrow \left[\frac{\partial u}{\partial z}\frac{\partial \mathrm{v}}{\partial y}-\frac{\partial \mathrm{v}}{\partial z}\frac{\partial \mathrm{u}}{\partial y}\right]p-\left[\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial z}-\frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial z}\right]q+\left[\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}\frac{\partial \mathrm{u}}{\partial y}\right]=0$$

$\Rightarrow \left[\frac{\partial \mathrm{v}}{\partial z}\frac{\partial \mathrm{u}}{\partial y}-\frac{\partial u}{\partial z}\frac{\partial \mathrm{v}}{\partial y}\right]p-\left[\frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial z}\right]q=\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}\frac{\partial \mathrm{u}}{\partial y}$(6)

which is same as eq. (1), with

$P=\frac{\partial \mathrm{v}}{\partial z}\frac{\partial \mathrm{u}}{\partial y}-\frac{\partial u}{\partial z}\frac{\partial \mathrm{v}}{\partial y}$, $\quad Q=\frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial z}$ and $R=\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}\frac{\partial \mathrm{u}}{\partial y}$

$\therefore \phi(\mathrm{u},\mathrm{v})=0$is the Solution of Lagrange's equation. To determine u and vfrom$P$, Q and R, suppose $u=a$ and $\mathrm{v}=\mathrm{b}$, where a and b constant so that

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du=0$$

$$\frac{\partial \mathrm{v}}{\partial x}dx+\frac{\partial \mathrm{v}}{\partial y}dy+\frac{\partial \mathrm{v}}{\partial z}dz=d\mathrm{v}=0$$

By cross multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial \mathrm{v}}{\partial z}-\frac{\partial \mathrm{v}}{\partial y}\frac{\partial u}{\partial z}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial \mathrm{v}}{\partial x}-\frac{\partial \mathrm{v}}{\partial z}\frac{\partial u}{\partial x}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial y}}$$

Or

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Thus, we have Solution of partial differential equation $Pp+Qq=R$

is$\phi(\mathrm{u},\mathrm{v})=0$ or $\quad \mathrm{v}=f(\mathrm{u})$

Working Rule

1. The standard form of Lagrange's equation
   $Pp+Qq=R$(1)
2. Find auxiliary equation for eq. (1)

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ (2)

3. Solve auxiliary equation by

(i) Grouping method

(ii) Method of multipliers

(iii) Combination of (i) & (ii)

4. Suppose u $= a$ and v $= b$ are two Solution of eq. (1) which obtained by (2), where u and v are function of x, y, z and a and b are constant.
5. Complete Solution of (1) is $f(\mathrm{u}, \mathrm{v}) = 0$ or u $= \phi(\mathrm{v})$ or v $= \psi(\mathrm{u})$or $f(a, b)$.

Example 1: Solve the following partial differential equations

(i)$yzp - xzq = xy$**(ii)**$p\ \tan x + q\ \tan y = \tan z$

Solution (i). The given differential equation is

$yzp - xzq = xy$(1)

Comparing eq. (1)with $Pp + Qq = R$, we get

$P = yz, Q = -xz$and$R = xy$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\Rightarrow \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$ (2)

Taking first and second fractions of eq. (2)

$$\frac{dx}{yz} = \frac{dy}{-xz} \Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$\Rightarrow xdx + ydy = 0$ (3)

Integrating (3), we get

$\Rightarrow x^2 + y^2 = c_1$(4)

Next, taking second and third fractions

$$\frac{dy}{-xz} = \frac{dz}{xy} \Rightarrow \frac{dy}{-z} = \frac{dz}{y}$$

Integrating, we get

$$\Rightarrow y^2 + z^2 = c_2$$

Therefore, the general Solution is

$$\phi(x^2 + y^2, y^2 + z^2) = 0$$

(ii) Given differential equation

$p\ \tan x + q\ \tan y = \tan z$(1)

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$(2)

Taking first two fraction of (2)

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\Rightarrow \cot x dx - \cot y dy$$

On integrating

$\log \sin x - \log \sin y = \log c_1or\frac{\sin x}{\sin y} = c_1$(3)

Taking last two fractions of eq. (2)

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\Rightarrow \cot y dy = \cot z dz$$

$\log \sin y - \log \sin z = \log c_2or\frac{\sin y}{\sin z} = c_2$ (4)

Therefore, the general solution is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

Example 2: Solve the following partial differential equations

(i)$p + 3q = 5z + \tan y - 3x$

(ii) $py + qx = xyz^2(x^2 - y^2)$

Solution. (i) Given$p + 3q = 5z + \tan y - 3x$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan y-3x}$ (2)

Taking first two fractions of (2)

$$\frac{dx}{1} = \frac{dy}{3}$$

$y - 3x = c_1$ (3)

Taking first and last fraction of (2)

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

On integrating, we get

$$x - \frac{1}{5}\log(5z + \tan c_1) = \frac{1}{5}c_2$$

Or

$5x - \log[5z + \tan(y - 3x)] = c_2$,usingeq. (3) (4)

From eq. (3) and (4) the required general Solution

$$\phi(y - 3x, 5x - \log[5z + \tan(y - 3x)] = c_2) = 0$$

Solution. (ii) Given

$py + qx = xyz^2(x^2 - y^2)$ (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$ (2)

Taking the first two fraction of (2), we get

$$\frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow xdx - ydy = 0$$

Integrating

$x^2 - y^2 = c_1$ (3)

Taking last two fraction of (2)

$$\frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$$

$$\Rightarrow \frac{dy}{x} = \frac{dz}{xyz^2c_1}, \qquad \text{using (3)}$$

$$\Rightarrow yc_1dy = \frac{dz}{z^2}$$

On integrating

$$c_1y^2 + \frac{2}{z} = c_2$$

$$\Rightarrow y^2(x^2 - y^2) + \frac{2}{z} = c_2, \qquad \text{using (3)}$$

$$\phi\left(x^2 - y^2, y^2(x^2 - y^2) + \frac{2}{z}\right) = 0$$

Example 3: Solve

(i)$(mz - ny)p + (nx - lz)q = ly - mx$

(ii) $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

Solution.(i) Given partial differential equation

$$(mz - ny)p + (nx - lz)q = ly - mx$$

The Lagrange's auxiliary equations for the given equation are

$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$ (1)

Choosing x, y, z as multipliers, each fraction of (1) is equal to

$$\frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore\ xdx + ydy + zdz = 0$$

Integrating,$x^2 + y^2 + z^2 = c_1$(2)

Again, choosing $l, m, n,$ as multipliers, then each fraction of (1) is equal to

$$\frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

$$\Longrightarrow ldx + mdy + ndz = 0$$

on integrating

$lx + my + nz = c_2$ (3)

From (2) and (3)

$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0,\ \phi$ being and arbitrary function.

Solution (ii) Given partial differential equation

$$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$$

The Lagrange's auxiliary equations for the given equation are

$\frac{\boldsymbol{dx}}{x(y^2+z)} = \frac{\boldsymbol{dy}}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$ (1)

Choosing$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$$

On integrating, we get $\log x + \log y + \log z = \log c_1$

$\Rightarrow xyz = c_1$ (2)

Now, choosing as $x, y, -1$ as multipliers, each fraction of (1)

$$\frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow xdx + ydy - dz$$

$\Rightarrow x^2 + y^2 - 2z = c_2$ (3)

∴ From eq. (2) and (3) the general Solution is

$$\phi(xyz, x^2 + y^2 - 2z) = 0$$

Practice questions

Solve the following partial differential equations

1. $\frac{y^2z}{x}p + xzq = y^2$**Ans.**$\phi(x^3 - y^3,\ x^2 - z^2) = 0$
2. $x^2p + y^2q = z^2$**Ans.**$\phi\left(\frac{1}{x} - \frac{1}{y},\ \frac{1}{y} + \frac{1}{z}\right) = 0$
3. $pz - qz = z^2 + (x + y)^2$**Ans.**$\phi(x + y, \log(x^2 + y^2 + z^2 + 2xy) - 2x) = 0$
4. $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$**Ans.**$\phi(x^2 + y^2 + z^2,\ x/yz) = 0$
5.

Hint: Use multiplier as $x, y, z$and $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$

6. $(z - y)p + (x - z)q = y - x$**Ans.**$\phi(x^2 + y^2 + z^2, x + y + z) = 0$
7. $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \left(\frac{x-y}{xy}\right),$ **Ans.**$\phi(xyz, x + y + z) = 0$

8. $(y^2+z^2)p - xyq = -zx$, **Ans.** $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$

1.8 Non-linear Partial Differential Equations of Order One

Charpit's method: General method of Solving partial differential equations of order one but of any degree.

Auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Working Rule

1. Transfer all terms of the given equation to the L.H.S. and denote entire expression by f.
2. Write down charpit's auxiliary equation
3. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$ and put these in charpit's auxiliary.
4. Select two proper fraction to find at least one of p and q and then find other
5. Put these value of p and q in $dz = pdx + qdy$, which on integration gives the complete Solution of given equation.

Example 1: Find a complete integral of $z = px + qy + p^2 + q^2$.

Solution. Let $f(x, y, y, p, q) = z - px - qy - p^2 - q^2$ (1)

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

From eq. (1),

$$\frac{\partial f}{\partial x} = -p, \frac{\partial f}{\partial y} = -q, \frac{\partial f}{\partial z} = 1, \frac{\partial f}{\partial p} = -x - 2p, \frac{\partial f}{\partial q} = -y - 2q \quad (3)$$

Using eq. (2) and (3)

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \quad (4)$$

Taking first fraction of eq. (4), we get

$dp = 0 \Rightarrow p = a$ (5)

Taking second fraction of eq. (5),

$$dq = 0 \Rightarrow q = b$$

Putting value of p and q in $dz = pdx + qdy$

$$\Rightarrow z = adx + bdy$$

$$\Rightarrow z = ax + by + c$$

Or

$$\Rightarrow z = ax + by + a^2 + b^2$$

Example 2: Solve $(p^2 + q^2)y = qz$.

Solution. Let $f \equiv (p^2 + q^2)y - qz = 0$ (1)

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = p^2 + q^2, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial p} = 2py, \quad \frac{\partial f}{\partial q} = 2qy - z \quad (3)$$

From eq. (2) and (3)

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{2qy+z} \quad (4)$$

Taking first two fraction

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$\Rightarrow p^2 + q^2 = a^2$ (5)

Now, from eq. (5) and (1), $a^2 y = qz$

$q = \frac{a^2 y}{z}$ (6)

Again from (5) eq. $p^2 = a^2 - q^2 \Rightarrow p = \sqrt{a^2 - \left(\frac{a^4y^2}{z^2}\right)} = \frac{q}{z}\sqrt{z^2 - a^2y^2}$

Now putting these values of p and q in $dz = pdx + qdy$

$$\Rightarrow dz = pdx + qdy$$

$$\Rightarrow dz = \frac{q}{z}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dy$$

$$\Rightarrow \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = adx$$

On integrating, $\left(z^2 - a^2y^2\right)^{1/2} = ax + b$

$\Rightarrow z^2 - a^2y^2 = (ax + b)^2$, where a and b are arbitrary constant

Example 3: Find a complete integral of $pqy + pq + qy = yz$.

Solution. Given

$f(x, y, z, p, q) \equiv pqy + pq + qy - yz = 0$ (1)

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

$\Rightarrow \frac{dp}{0} = \frac{dq}{(px+q)+qy} = \frac{dz}{-p(xy+q)-q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$, (from eq. (1) and (2)) (3)

From first fraction, we have $dp = 0 \Rightarrow p = a$

Putting $p = a$ in eq. (1), we get $q = \frac{y(z-ax)}{(a+y)}$.

Putting these values of p and q in the equation $dz = pdx + qdy$, we get

$$dz = adx + \frac{y(z-ax)}{(a+y)}dy$$

$$\Rightarrow \frac{dz - adx}{z - ax} = \frac{y}{(a+y)}dy$$

$$\Rightarrow \frac{dz - adx}{z - ax} = \left(1 - \frac{a}{a+y}\right)dy$$

On integrating

$$(z - ax)(y + a)^a = be^y$$

Example 4: Find a complete integral of $p^2x + q^2y = z$

Solution. Given equation is $f(x, y, z, p, q) = p^2x + q^2y - z = 0$ (1)

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

$\Rightarrow \frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}$, (from eq. (1) and (2)) (3)

Now, each fraction of eq. (2)

$$= \frac{2pxdp + p^2dx}{2px(-p+p^2) + p^2(-2px)} = \frac{2qydq + q^2dy}{2qy(-q+q^2) + q^2(-2qy)}$$

$$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

$$\Rightarrow \log(p^2x) = \log(q^2y) + \log a$$

$\Rightarrow p^2x = q^2ya$ (4)

From (1) and (4)

$$q^2ya + q^2y = z$$

$\Rightarrow q = \{z/(1+a)y\}^{1/2}$ (5)

Using (4) and (5), we have

$$p = \left\{\frac{za}{(1+a)x}\right\}^{1/2}$$

Putting the value of p and q in $dz = pdx + qdy$,

We get $dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$,

$$\Rightarrow (1+a)^{1/2}z^{-1/2}dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy$$

On integrating,

$(1+a)^{1/2}\sqrt{z}=\sqrt{a}\sqrt{x}+\sqrt{y}+b$,where$a,b$ are arbitrary constants.

Example 5: Find the complete integral of $2(z+px+qy)=yp^2$

Solution. Given equation is $f(x,y,z,p,q)=2(z+px+qy)-yp^2=0$ (1)

Charpit's auxiliary equation

$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$ (2)

$\Rightarrow \frac{dp}{4p}=\frac{dq}{2q-p^2+2q}=\frac{dz}{-p(2x-2yp)-2yq}=\frac{dx}{-(2x-2yp)}=\frac{dy}{-2y}$, (from eq. (1) and (2)) (3)

$$\frac{dp}{4p}=\frac{dy}{-2y}$$

On integrating, we get $py^2=a$ (4)

From eq. (4) and (1), we have

$$q=-\frac{z}{y}-\frac{ax}{y^3}+\frac{a^2}{2y^4}$$

Putting the value of p and $q$in $dz=pdx+qdy$, we get

$$dz=\frac{a}{y^2}dx+\left(-\frac{z}{y}-\frac{ax}{y^3}+\frac{a^2}{2y^4}\right)dy$$

$$\Rightarrow (ydz+zdy)-a\left(\frac{ydx-xdy}{y^2}\right)-\frac{a^2}{2y^3}dy=0$$

Or

$$d(yz)-ad\left(\frac{x}{y}\right)-\frac{a^2}{2}y^{-3}dy=0$$

On integrating

$$yz-a\frac{x}{y}+\frac{a^2}{4y^2}=b$$

Where a and b are arbitrary constants.

Example 6 Find the complete integral of $zpq=p+q$

Solution Let $f(x,y,z,p,q)=zpq-p+q=0$ (1)

Charpit's auxiliary equations are

$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$ (2)

From eq. (1), $f_x=0, f_y=0,\ f_z=pq, f_p=zq-1, f_q=zp-1$ (3)

Using eq. (2) and (3),

$\Rightarrow \frac{dp}{p^2q}=\frac{dq}{pq^2}=\frac{dz}{-p(zq-1)-q(zp-1)}=\frac{dx}{1-zq}=\frac{dy}{-zp}$, (from eq. (1) and (2)) (4)

Taking first two fraction

$\frac{dp}{p^2q} = \frac{dq}{pq^2} \Rightarrow p = qa$ (5)

From eq. (1) and (5), we get $p = (1+a)/z$ and $q = (1+a)/az$

$$\therefore dz = pdx + qdy = [(1+a)/z]dx + [(1+a)/az]dy$$

On integrating $z^2 = 2(1+a)[x + (1/a)y] + b$, where a and b are arbitrary constants.

Example 7: Find the compete integral of $p^2 - y^2q = y^2 - x^2$

Solution. Given $f(x,y,z,p,q) = p^2 - y^2q - y^2 + x^2 = 0$ (1)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad (2)$$

$$\frac{dp}{2x} = \frac{dq}{-2qy-2y} = \frac{dz}{-p(2p)+q^2} = \frac{dx}{-2p} = \frac{dy}{y^2} \quad (3)$$

Taking first and fourth fraction

$$\frac{dp}{2x} = \frac{dx}{-2p} \Rightarrow pdp + xdx = 0$$

Integrating $p^2 + x^2 = a^2$ (4)

Using eq. (1) and (4), we get $p = (a^2 - x^2)^{1/2}$, $q = a^2y^{-2} - 1$

$$\therefore dz = pdx + qdy = (a^2 - x^2)^{1/2} + (a^2y^{-2} - 1)dy$$

On integrating $z = (x/2)(a^2 - x^2)^{1/2} + (a^2/2)\sin^{-1}\left(\frac{x}{a}\right) - \left(\frac{a^2}{y}\right) - y + b$

Practice questions

Using Charpit's method, find a complete integral of the following equations:

1. $z = px + qy + pq$. **Ans.** $z = ax + by + ab$
2. $(p+q)(px+qy) - 1 = 0$. **Ans.** $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b$
3. $pq = px + qy$. **Ans.** $z = (1/2a)(ax+y)^2 + b$
4. $2zx - px^2 - 2qxy + pq$. **Ans.** $z = ay + b(x^2 - a)$
5. $p = (qy+z)^2$. **Ans.** $yz = ax + 2\sqrt{(ay)} + b$

1.9 Some important forms

(I) Standard form I: $\boldsymbol{f(p,q) = 0}$.

Equation of the form $f(p,q) = 0$ (i.e. equation containing p and q only).

Working Rule

Given equations is $f(p,q) = 0$ (1)

The complete solution of eq. (1) is $z = ax + by + c$ (2)

where a and b are connected by the relation $f(a,b) = 0$ (3)

From eq. (3), we can find the relation $b = \phi(a)$

$\therefore$ The complete integral (2) becomes

$$z = ax + \phi(a)y + c$$

Example 1: Solve $p^2 - q^2 = 25$.

Solution: The given equation is of the standard form I i.e. $f(p, q) = 0$

$\therefore$ Its complete integral is $z = ax + by + c$

where$a$ and b are connected by the relation $f(a, b) = 0$

$\therefore$ We have $a^2 - b^2 = 25 \Rightarrow b = \sqrt{a^2 - 25}$

$\therefore$ The complete integral is $z = ax + \sqrt{a^2 - 25}y + c$.

Example 2:$x^2p^2 + y^2q^2 = z^2$.

Solution: Given eq. is$x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial x}\right)^2 = z^2$ (1)

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial x}\right)^2 = 1$$

Put $X = \log x$, $Y = \log y$,$Z = \log z$

$$\Longrightarrow dX = \frac{1}{x}dx, dY = \frac{1}{y}dy, dZ = \frac{1}{z}dz$$

$\therefore$ eq. (1) becomes

$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

or

$P^2 + Q^2 = 1$,where $P = \frac{\partial Z}{\partial X}, Q = \frac{\partial Z}{\partial Y}$. (2)

Clearly eq. (2) is standard form I i.e. $f(P, Q) = 0$.

$\therefore$ Its complete integral is

$Z = aX + bY + c_1$ (3)

$$\text{where} a^2 + b^2 = 1 \Longrightarrow b\sqrt{1 - a^2}.$$

$\therefore$ Eq. (3) becomes:

$$Z = aX + \left(\sqrt{1 - a^2}\right)Y + c_1$$

or

$$\log z = a \log x + \sqrt{(1 - a^2)} \log y + c_1$$

Now putting $a = \cos\alpha$ and $c_1 = \log c$, we get

$$\Longrightarrow \log z = \cos\alpha \log x + \sin\alpha \log y + \log c$$

$$\Longrightarrow z = cx^{\cos\alpha}y^{\sin\alpha}.$$

Standard form II: $f(p, q, z) = 0$

Equation involving only $p, q$and z. i.e. equation of the form $f(p, q, z) = 0$

Working Rule

Step I: Given equation is of the form: $f(p,q,z) = 0$. (1)

Step II: Putting $p = \frac{dz}{dX}$ and $q = a\frac{dz}{dX}$ in (1), where $z = f(X)$ and $X = x + ay$.

Step III: Solve the resulting ordinary differential equation between x and z.Then result so obtained is the complete integral of (1)

Example I: Solve $z = p^2 + q^2$

Solution: Given $z = p^2 + q^2$ (1)

Eq. (1) is of standard form II i. e. $f(z,p,q) = 0$

Putting $z = f(X)$, where $X = x + ay$,

$p = \frac{dz}{dX}$and$q = a\frac{dz}{dX}$ in (1), we get

$$z = \left(\frac{dz}{dX}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2$$

$$\Rightarrow z = (1 + a^2)\left(\frac{dz}{dX}\right)^2$$

$$\Rightarrow \frac{dz}{dX} = \frac{\sqrt{z}}{\sqrt{1 + a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{dX}{\sqrt{1 + a^2}}$$

On integrating, we get

$$2\sqrt{z} = \frac{1}{\sqrt{1 + a^2}}X + c_1$$

$$\Rightarrow 2\sqrt{z} = \frac{1}{\sqrt{1 + a^2}}(X + b)$$

$$\Rightarrow 2\sqrt{z} = \frac{x + ay + b}{\sqrt{1 + a^2}}$$

$$\Rightarrow (x + ay + b)^2 = 4z(1 + a^2).$$

Example 2: find the complete solution of $z^2(p^2z^2 + q^2) = 1$.

Solution: Given $z^2(p^2z^2 + q^2) = 1$

This is of the form $f(z,p,q) = 0$. i.e. standard form II.

Putting $z = f(x + ay) = f(X)$, so that $p = \frac{dz}{dX}$ and $q = a\frac{dz}{dX}$ in (1), we get

$$z^2\left[\left(\frac{dz}{dX}\right)^2 z^2 + a^2\left(\frac{dz}{dX}\right)^2\right] = 1 \Rightarrow z^2(z^2 + a^2)\left(\frac{dz}{dX}\right)^2 = 1$$

$$\Rightarrow z\sqrt{(z^2 + a^2)}dz = dX$$

On integrating, $\frac{1}{3}(z^2 + a^2)^{3/2} = X + b$

$\Rightarrow 9(x + ay + b)^2 = (z^2 + a^2)^3.$

Standard form III: $\boldsymbol{f_1(x,p) = f_2(y,q)}$.

Working Rule
Step 1: Given $f_1(x,p) = f_2(y,q)$

Putting $f_1(x,p) = a$ and $f_2(y,q) = a$

Step 2: Solving eq. (2) for p and q, we get

$p = \phi_1(x,a)$and$q = \phi_2(y,a)$

Step 3: Complete integral is

$$dz = pdx + qdy$$

$$\therefore dz = \phi_1(x,a)dx + \phi_2(y,a)$$

$$\Rightarrow z = \int [\phi_1(x,a)dx + \phi_2(y,a)] + b.$$

Example 1: Solve $q = px + q^2$.

Solution: Given $px = q - q^2$

Which is of the standard form III. i.e. $f_1(x,p) = f_2(y,q)$.

Here $f_1(x,p) = px$ and $f_2(y,q) = q - q^2$

Let $f_1(x,p) = a$ and $f_2(y,q) = a$

$\Rightarrow px = a$and$q - q^2 = a$

$\Rightarrow p = \frac{a}{x}andq^2 - q + a = 0 \Rightarrow q = \frac{1\pm\sqrt{1-4a}}{2}$

We know that complete integral is

$$dz = pdx + qdy$$

$$= \frac{a}{x}dx + \frac{1 \pm \sqrt{1-4a}}{2}dy$$

On integrating, we get

$$z = a\log x + \frac{1 \pm \sqrt{1-4a}}{2}y + b.$$

Example 2: Solve $z^2(p^2 + q^2) = x^2 + y^2$

Solution: Given $z^2(p^2 + q^2) = x^2 + y^2$

Or $\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$ (1)

Putting $zdz = dZ \Rightarrow Z = \frac{z^2}{2}$

$\therefore$ Eq. (1) becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$$

$\Longrightarrow P^2 + Q^2 = x^2 + y^2$, where $P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial y}$

$$\Longrightarrow P^2 - x^2 = -Q^2 + y^2$$

Which is in standard form III, i.e. $f_1(x, P) = f_2(y, Q)$.

Let $P^2 - x^2 = a$ and $y^2 - Q^2 = a$

$$\Longrightarrow P = \sqrt{(a + x^2)}, Q = \sqrt{(y^2 - a)}$$

∴ The complete integral is $dZ = Pdx + Qdy$,

$$\Longrightarrow dZ = \sqrt{(a + x^2)}dx + \sqrt{(y^2 - a)}dy$$

On integrating, we have

$$Z = \frac{x}{2}\sqrt{(a + x^2)} + \frac{a}{2}\log\left\{x + \sqrt{(a + x^2)}\right\} + \frac{y}{2}\sqrt{(y^2 - a)} - \frac{a}{2}\log\left\{y + \sqrt{y^2 - a}\right\} + b$$

$$\Longrightarrow z^2 = x\sqrt{(a + x^2)} + a\ \log\left\{x + \sqrt{(a + x^2)}\right\} + y\sqrt{(y^2 - a)} - a\log\{y + \sqrt{(y^2 - a)}\} + b.$$

Standard form IV (Clairaut'sForm): $\boldsymbol{z = px + qy + f(p, q)}$.

Working Rule

Step I: Given equation is of the form $z = px + qy + f(p, q)$. (1)

Step 2: Put $p = a$ and $q = b$, we get complete solution or complete integral

$z = ax + by + f(a, b)$ (2)

Step 3: Differentiating (2) partially w.r.t. a and b, we get

$0 = x + \frac{\partial f}{\partial a}$ (3)

$0 = y + \frac{\partial f}{\partial b}$ (4)

The singular solution is obtained by eliminating a and b from (2), (3) and (4).

Example 1: Find the complete solution of $z = px + qy + c\sqrt{1 + p^2 + q^2}$.

Solution: Given $z = px + qy + c\sqrt{1 + p^2 + q^2}$, (1)

Which is of standard form IV i.e. Clairaut's form i.e. $z = px + qy + f(p, q)$.

Putting $p = a$ and $q = b$ in (1)

The complete integral is

$z = ax + by + c\sqrt{1 + a^2 + b^2}$.

Example 2: Find the complete and singular solution of $z = px + qy + \log pq$.

Solution: Given $z = px + qy + \log pq$ (1)

which is of Clairaut's form i.e. $z = px + qy + f(p, q)$.

Putting $p = a$ and $q = b$ in (1), we get the compete integral is

$z = ax + by + \log ab$.(2)

Differentiating (2) partially w.r.t. a and b, we get

$0 = x + \frac{1}{a}$ and $0 = y + \frac{1}{b}$

$\Rightarrow a = -\frac{1}{x}$ and $b = -\frac{1}{y}$

Putting these values of a and b in eq. (2), we get the singular solution

$$z = -2 - \log(xy).$$

Practice Questions

Solve the following partial differential equations:

1. $p^2 + p = q^2$. **Ans.** $z = ax + \sqrt{a^2 + a}y + c$
2. $pq = p + q$. **Ans.** $z = ax + \frac{a}{a-1}y + c$
3. $z^2(p^2 + q^2 + 1) = a^2$. **Ans.** $(1 + b^2)(a^2 - z^2) = (x + by + c)^2$
4. $z^2(p^2x^2 + q^2) = 1$. **Ans.** $\sqrt{1 + a^2}z^2 = \pm 2(\log x + ay) + c$
5. $yp = 2yx + \log q$. **Ans.** $z = x^2 + ax + \frac{1}{a}e^{ay} + b$
6. $\sqrt{p} + \sqrt{q} = x + y$. **Ans.** $z = \frac{1}{3}(x + a)^3 + \frac{1}{3}(y - a)^3 + b$
7. $z = px + qy - 2\sqrt{pq}$ **Ans.** $z = ax + by - 2\sqrt{ab}$
8. $z = px + qy + \sin(p + q)$ **Ans.** $= ax + by + \sin(a + b)$

1.10 Partial differential equation of higher order

Homogenous linear partial differential equations with constant coefficients-

The general form of such equation of such equation is

$$A_0 \frac{\partial^n y}{\partial x^n} + A_1 \frac{\partial^n y}{\partial x^{n-1}\partial y} + \cdots \ldots \ldots \ldots . + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial y^{n-1}} + \cdots \ldots + P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} + Rz = f(x, y)$$

where $A_0, A_1, \ldots \ldots A_n, B_0 \ldots \ldots , P, Q, R$ are constants.

In the above PDE all the partial derivatives are of the same order is called homogenous; otherwise it is called non-homogenous.

For examples:

(1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$ is homogenous linear partial differential equation.

(2) $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} - 3z = 2\cos(x + 2y)$ is non-homogenous linear partial differential equation.

Solution of homogenous linear partial differential equation with constant coefficients-

Rules for finding complementary function (C. F.):

To find the auxiliary equation we put $D = m$ and $D' = 1$ in $f(D, D') = 0$

Case I: The roots $m_1, m_2, \ldots \ldots \ldots m_n$ are distinct.

Then $C.F. = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \cdots \ldots \ldots \phi_n(y + m_nx)$

where $\phi_1, \phi_2, \ldots \ldots \phi_n$ are arbitrary functions.

Case II: The auxiliary equation has repeated roots. Suppose $m_1 = m_2 = \cdots = m_n = m$

Then

C.F.=$\phi_1(y+mx)+x\phi_2(y+mx)+\cdots\ldots\ldots x^{r-1}\phi_r(y+mx)$.

Example 1: Find the complimentary function of $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=0$.

Solution: The given equation is

$(D^2-DD'-6D'^2)z=0$,where$D=\frac{\partial}{\partial x}$ and $D'=\frac{\partial}{\partial y}$

The auxiliary equation is

$$m^2-m-6=0$$

$$\Rightarrow (m-3)(m+2)=0 \Rightarrow m=3,-2$$

$$\therefore C.F.=\phi_1(y+3x)+\phi_2(y-2x)$$

Example 2: Solve $(D+2D')(D-3D')^2z=0$.

Solution: Given $(D+2D')(D-3D')^2z=0$ (1)

The A. E. of eq. (1) is

$$(m+2)(m-3)^2=0$$

$$\Rightarrow m=-2,3,3$$

$$\therefore C.F.=\phi_1(y-2x)+\phi_2(y+3x)+x\phi_3(y+3x)$$

Rules for finding particular integral (P.I.)

P.I. of the equation $F(D,\ D')z=\phi(x,y)$ is given by $\frac{1}{F(D,\ D')}\phi(x,y)$.

Case I: When $\phi(x,y)$ is of the form $\phi(ax+by)$

Formula (i) when $F(a,b)\neq 0$ then

$P.I.=\frac{1}{F(a,b)}\int\int\int \ldots\ldots\int \phi(u)dudu\ldots\ldots du$, where $F(a,b)\neq 0$

where, $u=ax+by$ and D,D' are replaced by a and b respectively.

Formula (ii) when $F(a,b)=0$, then

$P.I.=x\frac{1}{\frac{\partial}{\partial D}F(D,D')}\phi(ax+by)$, where$F'(D,D')\neq 0$

If $F'(D,D')=0$ then

$$P.I.=x^2\frac{1}{\frac{\partial}{\partial D}F'(D,D')}\phi(ax+by),$$

Example.1: Solve the linear partial differential equation

$$\frac{\partial^3 u}{\partial x^3}-3\frac{\partial^3 u}{\partial x^2\partial y}+4\frac{\partial^3 u}{\partial y^3}=e^{x+2y}$$

Solution: The given equation is

$$(D^3 - 3D^2D' + 4D'^3)u = e^{x+2y}$$

where$D = \frac{\partial}{\partial x}$ $and\ D' = \frac{\partial}{\partial y}$

Auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$\Rightarrow m = 2, 2, -1$$

$\Rightarrow C.F. = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x)$ (1)

$$\text{P.I.} = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$= \frac{1}{1^3 - 3(1)^2(2) + 4(2)^3} \int \int \int e^u du\, du\, du$, where$x + 2y = u$

$$= \frac{1}{27} e^{x+2y}$$

∴ Complete solution is $C.F. + P.I. = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x) + \frac{1}{27} e^{x+2y}$.

Example 2: Solve $\frac{\partial^z z}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x - 2y)$.

Solution: Given equation is $\left(D^2 - 3DD' + 2D'^2\right)z = e^{2x+3y} + \sin(x - 2y)$ (1)

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$\Rightarrow m = 1, 2$$

$$\therefore C.F. = f_1(y + x) + f_2(y + 2x)$$

$$P.I. = \frac{1}{\left(D^2 - 3DD' + 2D'^2\right)} e^{2x+3y} + \frac{1}{\left(D^2 - 3DD' + 2D'^2\right)} \sin(x - 2y)$$

$$= P_1 + P_2$$

where, $P_1 = \frac{1}{\left(D^2 - 3DD' + 2D'^2\right)} e^{2x+3y}$

$= \frac{1}{2^2 - 3(2)(3) + 2(3)^2} \int \int e^u du\, du$,where$u = 2x + 3y$

$= \frac{1}{4} e^u = \frac{1}{4} e^{2x+3y}$ (2)

$$P_2 = \frac{1}{\left(D^2 - 3DD' + 2D'^2\right)} \sin(x - 2y)$$

$= \frac{1}{(1^2 - 3(1)(-2) + 2(-2)^2)} \int \int \sin v\, dv\, dv$,where$v = (x - 2y)$

$= -\frac{1}{15} \sin(x - 2y)$ (3)

∴ The complete solution is $C.F. + P.I.$

$$= f_1(y + x) + f_2(y + 2x) + \frac{1}{4} e^{2x+3y} - \frac{1}{15} \sin(x - 2y).$$

Case II: When $\phi(x, y)$ is of the form $x^m y^n$.

The particular integral (P.I.) is evaluated by expanding the symbolic function $1/f(D, D')$ in an infinite series of ascending powers of D or D'.

Note: If $n < m, 1/f(D, D')$ should be expanded in power of D'/D whereas if $m < n, 1/f(D, D')$ should be expanded in power of D/D'.

Example 1: Solve $\left(D^2 - 2DD' + D'^2\right)z = 12xy$.

Solution: Given $\left(D^2 - 2DD' + D'^2\right)z = 12xy$

The auxiliary equation is $m^2 - 2m + 1 = 0 \Longrightarrow m = 1, 1$

$$\therefore C.F. = f_1(y + x) + xf_2(y + x).$$

Now, $P.I. = \frac{1}{D^2 - 2DD' + D'^2} 12xy = \frac{1}{(D - D')^2} 12xy$

$$= \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2}.12xy$$

$= \frac{1}{D^2}(1 + \frac{2D'}{D} + \cdots).12xy$ {by Binomial theorem}

$$= \frac{1}{D^2}\left(12xy + \frac{2}{D}.12x\right).12xy = \frac{1}{D^2}(12xy) + \frac{1}{D^3}(24x)$$

$$= \left[\frac{1}{D^2}x\right]12y + \left[\frac{1}{D^3}x\right]24$$

$$= 2x^3y + x^4.$$

Hence the general solution $z = C.F. + P.I.$

$\Rightarrow z = f_1(y + x) + xf_2(y + x) + 2x^3y + x^4.$

Example 2: Solve $\frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}$.

Solution: Given $(D^2D' - 2DD'^2 + D'^3)z = \frac{1}{x^2}$.

or

$(D^2 - 2DD' + D'^2)D'z = \frac{1}{x^2}$. (1)

$C.F. = f_1(x) + f_2(y + x) + xf_3(y + x)$ (2)

Now, $P.I. = \frac{1}{(D - D')^2 D'}\frac{1}{x^2} = \frac{1}{(D - D')^2 D'}\frac{1}{x^2}$

$$\frac{1}{(D - D')^2}\left(\frac{y}{x^2}\right) = \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2}\frac{y}{x^2}$$

$$\frac{1}{D^2}\left\{1 + \frac{2D'}{D} + \cdots\right\}\frac{y}{x^2} = \frac{1}{D^2}\left\{\frac{y}{x^2} + \frac{2}{D}\left(\frac{1}{x^2}\right)\right\}$$

$$= y.\frac{1}{D^2}\left(\frac{1}{x^2}\right) + \frac{2}{D^3}\left(\frac{1}{x^2}\right) = y.\frac{1}{D}\left(-\frac{1}{x}\right) + 2\frac{1}{D^2}\left(-\frac{1}{x}\right)$$

$$-y\log x-\frac{2}{D}(\log x)$$

$$-y\log x-2(x\log x-x)$$

$$=-y\log x-2x\log x+2x$$

Hence the complete solution is $z=C.F.+P.I.$

Or $z=f_1(x)+f_2(y+x)+xf_3(y+x)-y\log x-2x\log x+2x$

General method for finding P.I. of $F(D,D')=\phi(x,y)$

$$P.I.=\frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')}\phi(x,y)$$

$$\frac{1}{(D-mD')}\phi(x,y)=\int\phi(x,c-mx)dx$$

Example 1: Solve $(D^2-DD'-2D'^2)z=(y-1)\,e^x$.

Solution: Given $(D^2-DD'-2D'^2)z=(y-1)\,e^x$

Its auxiliary equation

$$m^2-m-2=0\Rightarrow(m-2)(m+1)=0$$

$$\Rightarrow m=2,-1$$

$$\therefore C.F.=f_1(y-x)+f_2(y+2x)$$

$$P.I.=\frac{1}{D^2-DD'-2D'^2}(y-1)e^x$$

$$=\frac{1}{(D-2D')(D+D')}(y-1)e^x$$

$$=\frac{1}{(D+D')}\left[\frac{1}{(D-2D')}(y-1)e^x\right]$$

$$=\frac{1}{(D+D')}\int(c-2x-1)e^xdx,\qquad \{y=c-2x$$

$$=\frac{1}{(D+D')}[(c-2x-1)e^x+2e^x]$$

$$=\frac{1}{(D+D')}[(y-1)e^x+2e^x]$$

Or $P.I.=\frac{1}{(D+D')}(y+1)e^x=\int(c+x+1)e^xdx$

$$=(c+x+1)e^x-e^x$$

$$=(y+1)e^x-e^x$$

Or $P.I.=ye^x$

$\therefore$The complete solution is $z=C.F.+P.I.$

$$\Rightarrow z=f_1(y-x)+f_2(y+2x)+ye^x$$

Example 2: Solve $r+s-6t=y\cos x$.

Solution: Given eq. $(D^2+DD'-6D'^2)z = y\cos x$

Its auxiliary eq. $m^2+m-6=0 \Rightarrow m = 2,-3.$

$$\therefore C.F = f_1(y+2x)+f_2(y-3x)$$

$$P.I. = \frac{1}{D^2+DD'-6D'^2} y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')} y\cos x$$

$= \frac{1}{(D-2D')}\int(3x+c)\cos x dx$,where$c = y-3x$

$$= \frac{1}{(D-2D')}\left[(3x+c)\sin x - \int 3\sin x dx\right]$$

$= \frac{1}{(D-2D')}[y\sin x + 3\cos x]$, as$c = y-3x$

$= \int[(c'-2x)\sin x + 3\cos x]dx$,where$c' = y+2x$

$$= (c'-2x)(-\cos x) - \int(-2)(-\cos x)dx + 3\sin x$$

$= y(-\cos x) - 2\sin x + 3\sin x$,as$c' = y+2x$

$$= \sin x - y\mathit{sos}x$$

Hence the general solution is $z = f_1(y+2x)+f_2(y-3x)+\sin x - y\mathit{sos}x$.

Practice Questions

Solve the following partial differential equations:

1. $\frac{\partial^2 z}{\partial x^2} - 7\frac{\partial^2 z}{\partial x\partial y} + 12\frac{\partial^2 z}{\partial y^2} = e^{x-y}$. **Ans.**$z = f_1(y+3x)+f_2(y+4x)+\frac{1}{20}e^{x-y}$
2. $(D^2+7DD'+12D'^2)z = \sinh x$.**Ans.**$z = f_1(y-x)+f_2(y-2x)+\frac{1}{4}e^{2x-3y}$
3. $(D^3-4D^2D'+5DD'^2-2D'^3)z = e^{y+2x}+(y+x)^{1/2}$.

   Ans. $z = f_1(y+x)+xf_2(y+x)+f_3(y+2x)+xe^{y+x}-\frac{x^2}{3}(y+x)^{3/2}$

4. $[D^2+(a+b)DD'+abD'^2]z = xy$. **Ans.**$z = f_1(y-ax)+f_2(y-bx)+\frac{1}{6}x^3y-\left(\frac{a+b}{24}\right)x^4$
5. $\left[D^2-2DD'+D'^2\right]z = e^{x+2y}+x^3$.**Ans.**$z = f_1(y+x)+xf_2(y+x)+e^{x+2y}+\frac{x^5}{20}$
6. $(D^3-D'^3)z = x^3y^3$.**Ans.**$z = f_1(y+x)+f_2(y+\omega x)+f_3(y+\omega^2x)+\frac{x^6y^3}{120}+\frac{x^9}{10080}$.
7. $(D^2+5DD'+6D'^2)z = \frac{1}{y-2x}$.**Ans.**$z = f_1(y-x)+f_2(y+2x)+x\log(y-2x)$.
8. $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$**Ans.**$z = f_1(y-x)+xf_2(y-x)+x\sin y$.