Basics of Cauchy Functional Equation

2.6.23

1) $f(x+y) = f(x) + f(y)\ \forall x, y \in \mathbb{R}$. This equation is called Cauchy functional equation. f is continuous on $\mathbb{R}$. Then prove that:

A) f is an odd function.

B) If m is a natural number, $f(mx) = mf(x)$

C) If n is a natural number, $f\left(\frac{x}{n}\right) = \frac{1}{n} f(x)$

D) If r is a rational number, $f(rx) = rf(x)$

E) So, we are done with all the naturals and rationals. Now we will prove it for the irrationals. For that, take any irrational point $y \in \mathbb{R}$. Take a sequence of rationals $\{r_n\}$ such that $r_n \to y$. Hence prove that $f(yx) = yf(x)\ \forall y \in \mathbb{R}$.

F) Hence, the solution of the Cauchy equation is $f(x) = ax$ for some real valued a and for all x.

2) Now from question 1, if continuity was not given, how much could you conclude for the function f?

3) Now, in the question 1, it was given that f is continuous on the whole of $\mathbb{R}$, but now let say, in spite of whole $\mathbb{R}$, f is continuous only at a point $x_0 \in \mathbb{R}$. Prove that under the same assumption that f is linear, the same conclusion of 1 hold, i.e., $f(x) = ax$ for some a in $\mathbb{R}$ and for all x.

4) Now, say f is not continuous, but f is given monotonically increasing and linear, then the same conclusion of 1 hold, i.e., $f(x) = ax$ for some a in $\mathbb{R}$ and for all x.

5) Now in the question, if it was given that $f(x + y) = f(x) + f(y) + b$, for some $b \in \mathbb{R}$, and all the rest assumptions are same, then what you could conclude?

6) Now, we are done with the different types of Cauchy equation. Now say, $g(x + y) = g(x)g(y)$ and g is given to be continuous on the whole of $\mathbb{R}$, then what you could conclude about g?