Calculus (Problem Set)

Srijan Chattopadhyay

July 8, 2025

1. $X_0, X_1, \alpha \in (0,1)$. $X_{n+1} = \alpha X_n + (1-\alpha)X_{n-1}\ \forall n \geq 1$. Prove that, $\{X_n\}$ converges and find the limit.

2. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ and f is monotonically increasing. Then, prove that $\exists a$ such that $f(x) = ax\ \forall x \in \mathbb{R}$.

3. **Algorithm to Compute $\sqrt{\alpha}, \alpha > 0$:** Start with any $x_1 > \sqrt{\alpha}$. Define $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$. Show that $\{x_n\}$ is monotonically decreasing and hence find the limit of $\{x_n\}$.

4. Prove that for any prime $p \geq 3$, the number $\binom{2p-1}{p-1} - 1$ is divisible by p^2.

5. For sets A, B, let $f : A \to B$ and $g : B \to A$ be functions such that $f(g(x)) = x$ for each x. Prove that :
   - f need not be one one
   - f must be onto
   - g must be one one
   - g need not to be onto.

6. Let $x_n \to x$ and $y_n \to y$. Then, prove that $\frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \to xy$.

7. Let $0 < a \leq x_1 \leq x_2 \leq b$. Define $x_n = \sqrt{x_{n-1}x_{n-2}}$ for $n \geq 3$. Show that $a \leq x_n \leq b$ and $|x_{n+1} - x_n| \leq \frac{b}{a+b}|x_n - x_{n-1}|$ for $n \geq 2$. Prove $\{x_n\}$ is convergent.

8. Let $0 < y_1 < x_1$. Define $x_{n+1} = \frac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{x_ny_n}$, for $n \in \mathbb{N}$. Prove that:
   - $\{y_n\}$ is increasing and bounded above while $\{x_n\}$ is decreasing and bounded below.
   - $0 < x_{n+1} - y_{n+1} < 2^{-n}(x_1 - y_1)$ for $n \in \mathbb{N}$.
   - Prove that x_n and y_n converge to the same limit.

9. **Euler's Constant:** Let
$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n t^{-1} dt$$
   - Show that γ_n is a decreasing sequence.
   - Show that $0 < \gamma_n \leq 1$ for all n.
   - $lim\gamma_n$ exists and is denoted by γ. The real real number γ is called the Euler's constant.

10. Evaluate $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n.3^m + m.3^n)}$

11. Find the value of $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty}{}_{i \neq j \neq k} \frac{1}{3^i 3^j 3^k}$, where the sum is taken over i, j, k such that no two of them can be equal.

12. Let $S_n, n = 1, 2, 3, \cdots$ be the sum of infinite geometric series, whose first term is n and the common ratio is $\frac{1}{n+1}$. Evaluate
$$lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1}{S_1^2 + S_2^2 + \cdots + S_n^2}$$

13. $t_n = \frac{n^5+n^3}{n^4+n^2+1}$, $S_r = \sum_{n=1}^{r} t_r$. Find an explicit expression of S_r.

14. Find the following limits (if exists):

    - $\underset{x\to 0}{lim} \frac{(x-\sin x)}{x^3}$
    - $\underset{x\to 0}{lim} \frac{(e^x-1-x)}{x^2}$
    - $\underset{x\to 0}{lim} (\frac{a^x+b^x+c^x}{3})^{1/x}, a, b, c > 0$
    - $\underset{x\to 0}{lim} ((x+a)(x+b)(x+c))^{1/3} - x$

15. $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}$ for some $\lambda > 0$. Prove that f is continuous (This is called **Lipschitz continuous**).

16. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq \lambda (x - y)^2$ for all $x, y \in \mathbb{R}$ for some $\lambda > 0$.

17. $f(x) = x \sin(1/x)$ if $x \neq 0$ and $= 0$ if $x = 0$. Prove that x is continuous.

18. Can you find minimum value of r such that if $f(x) = x^r \sin(1/x)$ if $x \neq 0$ and $= 0$ if $x = 0$, then f is at least once differentiable?

19.
$$f(x) = \left\{ \begin{array}{ll} 0, & \text{for } x \in Q \\ 1, & \text{for } x \in Q^c \end{array} \right\}$$

    Q is the set of all rational numbers. Find all continuity points of f (This function is called Dirichlet Function).

20. Can there exist any continuous $f : \mathbb{R} \to \mathbb{R}$ which is rational on irrational points and irrational on rational points?

21. $f : \mathbb{R} \to \mathbb{R}$ and $\exists \alpha \in (0, 1)$ such that $|f(x) - f(y)| \leq \alpha |x - y|$. Prove that f has a unique fixed point. (A point x is called a fixed point of f is $f(x) = x$).

22. $f, g : [0, 1] \to [0, 1]$ are continuous functions such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$. Show that $\exists c \in [0, 1]$ such that $f(c) = g(c)$.

23. Suppose that a function f is continuous on the interval $[a, b]$ and differentiable on (a, b). If the graph of f is not a line segment, prove that there exists a point $c \in (a, b)$ such that $|f'(c)| > |\frac{f(b)-f(a)}{b-a}|$

24. Let f be a twice differentiable function on the open interval $(-1, 1)$ such that $f(0) = 1$. Suppose f also satisfies $f(x) \geq 0, f'(x) \leq 0$ and $f''(x) \leq f(x)$, for all $x \geq 0$. Show that $f'(0) \geq -\sqrt{2}$.

25. Find the limit

$$\lim_{x\to 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}$$

26. Calculate the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}$$

27. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection of the positive integers. Prove that at least one of the following limits is true:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n + f(n)} = \infty; \qquad \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{f(n)} \right) = \infty.$$

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x) > 0$ for all real numbers x. Show that $f(8) > 2024 f(0)$.

29. Evaluate
$$\frac{1/2}{1+\sqrt{2}}+\frac{1/4}{1+\sqrt[4]{2}}+\frac{1/8}{1+\sqrt[8]{2}}+\frac{1/16}{1+\sqrt[16]{2}}+\cdots$$

30. Prove that for any function $f:\mathbb{Q}\to\mathbb{Z}$, there exist $a,b,c\in\mathbb{Q}$ such that $a<b<c$, $f(b)\geq f(a)$ and $f(b)\geq f(c)$.

31. Define the sequence $x_1,x_2,\ldots$ by the initial terms $x_1=2, x_2=4$, and the recurrence relation
$$x_{n+2}=3x_{n+1}-2x_n+\frac{2^n}{x_n}\quad \text{for}\quad n\geq 1.$$
Prove that $\lim_{n\to\infty}\frac{x_n}{2^n}$ exists and satisfies
$$\frac{1+\sqrt{3}}{2}\leq\lim_{n\to\infty}\frac{x_n}{2^n}\leq\frac{3}{2}.$$

32. Let $x_1=2021$, $x_n^2-2(x_n+1)x_{n+1}+2021=0$ $(n\geq 1)$. Prove that the sequence x_n converges. Find the limit $\lim_{n\to\infty}x_n$

33. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function. Prove that
$$\left|f(1)-\int_0^1 f(x)dx\right|\leq\frac{1}{2}\max_{x\in[0,1]}|f'(x)|.$$

34. Let $0<a<1$. Find all functions $f:\mathbb{R}\to\mathbb{R}$ continuous at $x=0$ such that $f(x)+f(ax)=x,\ \forall x\in\mathbb{R}$.

35. How many ordered pairs of real numbers (a,b) satisfy equality $\lim_{x\to 0}\frac{\sin^2 x}{e^{ax}-2bx-1}=\frac{1}{2}$?

36. For a given integer $n\geq 1$, let $f:[0,1]\to\mathbb{R}$ be a non-decreasing function. Prove that
$$\int_0^1 f(x)dx\leq(n+1)\int_0^1 x^n f(x)dx.$$

37. For $n=1,2,\ldots$ let
$$S_n=\log\left(\sqrt[n^2]{1^1\cdot 2^2\cdot\ldots\cdot n^n}\right)-\log(\sqrt{n}),$$
where log denotes the natural logarithm. Find $\lim_{n\to\infty}S_n$.

38. Let V be the set of all continuous functions $f\colon[0,1]\to\mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0)=0$ and $f(1)=1$. Determine all $\alpha\in\mathbb{R}$ such that for every $f\in V$, there exists some $\xi\in(0,1)$ such that
$$f(\xi)+\alpha=f'(\xi)$$

39. Let $f:\mathbb{R}\to\mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

40. Calculate the exact value of the series $\sum_{n=2}^{\infty}\log(n^3+1)-\log(n^3-1)$ and provide justification.

41. Find the limit
$$\lim_{n\to\infty}\left(\frac{(1+\frac{1}{n})^n}{e}\right)^n.$$

42. A real-valued function f defined in (a,b) is said to be convex if
$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$
whenever $a<x<b$, $a<y<b$, $0<\lambda<1$.

- Prove that every convex function is continuous.

- Prove that every increasing convex function of a convex function is convex.
- If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

- Let $f : (a, b) \to \mathbb{R}$ such that f'' exists on (a, b). Then f is convex iff $f''(x) > 0 \; \forall x \in (a, b)$.

43. $f : [0, 1] \to \mathbb{R}$ differentiable on $(0, 1)$, such that $\exists a, b \in (0, 1)$ such that $\int_0^a f(x)dx = 0$ and $\int_b^1 f(x)dx = 0$. Also, $|f'(x)| \leq M \; \forall x \in (0, 1)$. Prove that $|\int_0^1 f(x)dx| \leq \frac{(1-a+b)M}{4}$.
    - First prove that, $|\int_0^1 f(x)dx| \leq \frac{M(1-a)}{2}$ [Hint: Use MVT and a substitution to use $\int_0^a f(x)dx = 0$, Think how you can transform the interval $(0, 1)$ to $(0, a)$].
    - Then, prove that $|\int_0^1 f(x)dx| \leq \frac{Mb}{2}$ [Hint: Again use MVT and a substitution to use $\int_b^1 f(x)dx = 0$, Think how you can transform the interval $(0, 1)$ to $(b, 1)$]

44. Show that there doesn't exist any increasing differentiable function $f : \mathbb{R} \to \mathbb{R}^+$ such that $f(f(x)) = f'(x)$

45. Determine all continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy $\int_0^1 f(x)(1 - f(x))dx = \frac{1}{12}$

46. Let $f(x) = \int_0^1 |t - x|tdt \; \forall x \in \mathbb{R}$. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$.

47. $f(x) = \int_x^{x+1} \sin(u^2)du$, find $\underset{x\to\infty}{lim} f(x)$, if it exists.

48. The sequence $\{q_n(x)\}$ of polynomials is defined by

$$q_1(x) = 1 + x, \quad q_2(x) = 1 + 2x$$

and for $m \geq 1$ by

$$q_{2m+1}(x) = q_{2m}(x) + (m + 1)xq_{2m-1}(x),$$
$$q_{2m+2}(x) = q_{2m+1}(x) + (m + 1)xq_{2m}(x).$$

Let x_n be the largest real solution of $q_n(x) = 0$. Prove that

(a) the sequence $\{x_n\}$ is increasing.

(b) $x_{2m+2} > -\frac{1}{m+1}$ for $m \geq 1$.

(c) the sequence $\{x_n\}$ converges to 0.

49. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x - 2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right-angled triangle.

50. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and

$$f(0) = 0, \quad f(1) = 1.$$

(a) Show that there exists x_1 in $(0, 1)$ such that

$$\frac{1}{f'(x_1)} = 1.$$

(b) Show that there exist distinct x_1, x_2 in $(0, 1)$ such that

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2.$$

(c) Show that for a positive integer n, there exist distinct $x_1, x_2, \ldots, x_n$ in $(0, 1)$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

51. Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{-3/2}}$. Is $\lim_{n\to\infty} a_n < \infty$?

52. Let $P(x) = x^{100}+20x^{99}+198x^{98}+a_{97}x^{97}+\ldots+a_1x+1$ be a polynomial where the a_i $(1 \leq i \leq 97)$ are real numbers. Prove that the equation $P(x) = 0$ has at least one nonreal root.

53. Find $\lim_{x\to\infty} \left((2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x\right)$

54. Find $\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$.

55. A sequence (a_n) is defined by $a_0 = -1, a_1 = 0$, and $a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$ for all positive integers n. Find a_{100}.

56. Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies

$$\left(\int_{-1}^{1} e^x f(x)dx\right)^2 \geq \left(\int_{-1}^{1} f(x)dx\right)\left(\int_{-1}^{1} e^{2x} f(x)dx\right).$$

Prove that there exists a point $c \in (-1, 1)$ such that $f(c) = 0$. [You can assume CS inequality for integrals.]

57. Let f and g be two continuous, distinct functions from $[0, 1] \to (0, +\infty)$ such that
$\int_0^1 f(x)dx = \int_0^1 g(x)dx$
Let
$y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)}dx$, for $n \geq 0$, natural.
Prove that (y_n) is an increasing and divergent sequence.

58. Consider a function $f : [0, 1] \to [0, 1]$ satisfying the following property $|f(x) - f(y)| < |x - y|$ for all $x, y \in X, x \neq y$. Show that f has a fixed point. Is the fixed point unique? [Hint : Define $d(x) = |x - f(x)|$. Suppose $\inf_x d(x) \geq \epsilon > 0$. Assume $\exists x_0$ such that the infimum is attained. (Can you prove this? You can skip if you can't!). Then, use the property of f to arrive at a contradiction.]

59. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Find continuity points of f.

60. Show that the polynomial equation with real coefficients $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_3x^3+x^2+x+1 = 0$ cannot have all real roots.

61. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-to-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

62. Consider $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + \frac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

63. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that

$$f''(x) \leq 0 \text{ for all } x \text{ and } f'(x_0) < 0 \text{ for some } x_0.$$

(b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that

$$f^{(k)}(x) \leq 0 \text{ for all } x \text{ and } f^{(k-1)}(x_0) < 0 \text{ for some } x_0.$$

Here, $f^{(k)}$ denotes the k^{th} derivative of f.

64. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Can you put appropriate conditions on f, so that f restricted to (a, b), $-\infty < a < b < \infty$ can't attain it's maximum and minimum inside (a, b)?

65. Let $a, b \in \mathbb{Z}$ and $b > 0$. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that

(a) f is continuous if and only if $a > 0$.

(b) $f'(0)$ exists if and only if $a > 1$.

(c) f' is bounded if and only if $a \geq 1 + b$.

(d) f' is continuous if and only if $a > 1 + b$.

66. Let $f : (0, 1] \to \mathbb{R}$ be a differentiable function with f' bounded on $(0, 1]$. Define

$$a_n = f\left(\frac{1}{n}\right), \quad n \geq 1.$$

Show that $\{a_n\}$ is a convergent sequence.

67. Let f be a thrice differentiable function on $(0, 1)$ such that $f(x) \geq 0$ for all $x \in (0, 1)$. If $f(x) = 0$ for at least two values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$.

68. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that $f(x + 1) - f(x)$ converges to 0 as $x \to \infty$. Then show that

$$\frac{f(x)}{x} \to 0 \quad \text{as } x \to \infty.$$

69. Let f be continuous, non-negative, and assume that

$$\int_0^\infty f(x)\, dx < \infty.$$

Then show that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$

70. Let $f : [0, 1] \to \mathbb{R}$ be a real-valued continuous function which is differentiable on (0, 1) and satisfies $f(0) = 0$. Suppose that there exists a constant $c \in (0, 1)$ such that

$$|f'(x)| \leq c|f(x)| \text{ for every } x \in (0, 1).$$

Show that $f(x) = 0$ for all $x \in [0, 1]$. etermine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy

$$f(x) = (x + 1)f(x^2),$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

71. Suppose $F : \mathbb{R} \to \{0, 1\}$ is a function, i.e. F only takes two values 0 and 1. Suppose F is non decreasing, right continuous, and $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$. Prove that there exists some $x \in \mathbb{R}$ such that $F(y) = 1$ for all real $y \geq x$, and $F(y) = 0$ for all real $y < x$.

72. Consider a container of the shape obtained by revolving a segment (From $y = 0$ to $y = 5$) of the curve $x = e^{\frac{y}{5}}$ around the y-axis. The container is initially empty. Water is poured at a constant rate of $\pi\,\text{cm}^3$ into the container. Let $h(t)$ be the height of water inside the container at time t. Find $h(t)$ as a function of t.

73. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable and satisfy

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$. For $n \in \mathbb{N}$, define

$$\mu_n = \int_0^\infty x^n f(x) dx.$$

a. Express μ_n in terms of μ_0. b. Prove that the sequence $\frac{3^n \mu_n}{n!}$ always converges, and the the limit is 0 only if μ_0.

74. Suppose $f : \mathbb{R} \to [0, \infty)$. For $\epsilon > 0$, define$f_\epsilon : \mathbb{R} \to [0, \infty)$ by

$$f_\epsilon(x) = n\epsilon \ \text{ when } \ n\epsilon \leq f(x) < (n+1)\epsilon$$

Prove that $f_\epsilon(x) \to f(x)$ for all $x \in \mathbb{R}$ as $\epsilon \to 0$.

75. Suppose $f_1, f_2, \cdots, f_n$ are convex function from $\mathbb{R}$ to $\mathbb{R}$. Prove that $M(x) = \max\{f_1(x), \cdots, f_n(x)\}$ is also convex. [A function is called convex if for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$]. Can you say anything about the minimum function? 9 + 1

76. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

77. Suppose $f : \mathbb{R} \to [0, \infty)$. Define the following functions for $n \in \mathbb{N}$

$$s_n(x) = \begin{cases} n & \text{if } \ f(x) \geq n \\ 2^{-n}i & \text{if } \ 2^{-n}i \leq f(x) \leq 2^{-n}(i+1), i = 0, 1, 2, \cdots, n.2^n - 1 \end{cases}$$

Prove that $s_1 \leq s_2 \leq s_3 \leq \cdots$. Also, prove that $\forall x$, $s_n(x) \to f(x)$ as $n \to \infty$.

78. Define $\{x_n\}_{n \in \mathbb{N}}$ as $x_0 = 1$, and, $x_{n+1} = \ln(e^{x_n} - x_n)$. Prove that $\sum_{n=0}^{\infty} x_n$ converges, hence find the value.

79. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $0 \leq f'(x) \leq 1$ and $f(a) = 0$. Show that

$$3 \left(\int_a^b f(x)^2 \, dx \right)^3 \geq \int_a^b f(x)^8 \, dx.$$

80. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let

$$J = \left\{ \frac{f(b) - f(a)}{b - a} : a, b \in I, a < b \right\}$$

Show that

a) J is an interval.

b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements.

81. Let f be a continuous function on $[0, 1]$ such that for every $x \in [0, 1]$, we have $\int\limits_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\int\limits_0^1 f(t)^2 dt \geq \frac{1}{3}$.

82. Let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined as $x_n = \sin(2\pi n! e)$ for all $n \in \mathbb{N}$. Compute $\lim_{n \to \infty} x_n$.

83. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at 0. Define another function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x)\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose that g is also differentiable at 0. Prove that

$$g'(0) = f'(0) = f(0) = g(0) = 0.$$

84. Determine all the pairs of positive real numbers (a, b) with $a < b$ such that the following series

$$\sum_{k=1}^{\infty} \int_a^b \{x\}^k dx = \int_a^b \{x\} dx + \int_a^b \{x\}^2 dx + \int_a^b \{x\}^3 dx + \cdots$$

is convergent and determine its value in function of a and b. $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. Assume you can swap the integral and summation.

85. Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy

$$f(x) = (x + 1)f(x^2),$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

86. Let $f : [0, 1] \to (0, \infty)$ be a continuous function satisfying $\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0, 1)$ such that $\int_0^c f(t)dt = c - \frac{1}{2}$.

87. Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

88. Let $f : (a, b) \to \mathbb{R}$ is continuously differentiable, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

89. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

90. Consider the sequence $(a_n)_{n\geqslant 1}$ defined by $a_1 = 1/2$ and $2n \cdot a_{n+1} = (n + 1)a_n$. Determine the general formula for a_n. Let $b_n = a_1 + a_2 + \cdots + a_n$. Prove that $\{b_n\} - \{b_{n+1}\} \neq \{b_{n+1}\} - \{b_{n+2}\}$.

91. If $f : [0, \infty] \to \mathbb{N}$ is a right continuous function having left limits. Show that $\forall t \in [0, \infty)$, the set $\{y : f(x) = y, x \in [0, t]\}$ has a finite number of points, i.e., the range for each closed and bounded interval starting from 0 has a finite number of points. [Hint: You can use the fact that each bounded sequence has a convergent subsequence.]