Miscellaneous Problem Set

1) Let $n \geq 1$ be an integer. Then

$a)$Prove that $X^n + Y^n + Z^n$ can be written as a polynomial with integers coefficients in the variables $\alpha = X + Y + Z,\ \beta = XY + YZ + ZX$ and $\gamma = XYZ$.

$b)$ Let $G_n = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC)$, where x, y, z, A, B, C are real numbers such that $A + B + C$ is an Integral multiple of π. Using (a) or otherwise show that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integers n.

2) The Fibonacci sequence is defined by $a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n$. Find number of n for which

$$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots \frac{a_n}{2^n} > 2$$

3) Suppose that $x_1, x_2, \ldots, x_n$ are nonnegative real numbers for which $x_1 + x_2 + x_3 + \ldots + x_n < \frac{1}{2}$. Prove that

$$\left(1 - x_1\right)\left(1 - x_2\right)\ldots\left(1 - x_n\right) > \frac{1}{2}$$

4) If z_1, z_2, z_3 are non-zero complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Then prove that z_1, z_2, z_3 lie on a circle passing through the origin.

5) Let $a = \sqrt[2023]{2023}$ which is greater between 2023 and $a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$, where a appears 2023 times.

6) Prove that the sum of entries of the table situated in different rows and different columns is not less than 1.

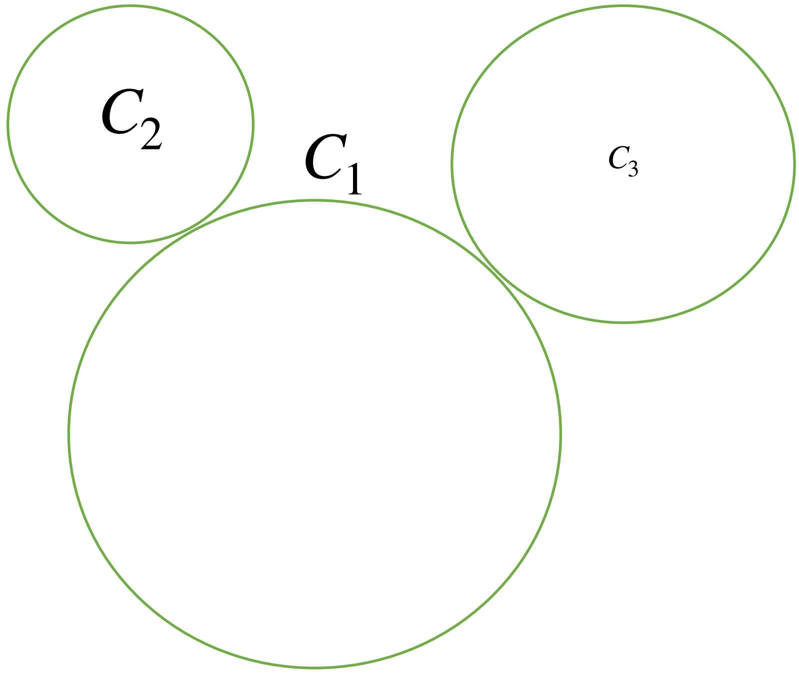
$$\begin{array}{cccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \cdots\cdots\cdots & \frac{1}{n} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots\cdots\cdots & \frac{1}{n+1} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots\cdots\cdots & \frac{1}{n+2} \\
\vdots & & & & & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots\cdots\cdots & \frac{1}{2n-1}
\end{array}$$

7) How many distinct integers are in the sequence

$$\left\lfloor \frac{1^2}{2023} \right\rfloor, \left\lfloor \frac{2^2}{2023} \right\rfloor, \left\lfloor \frac{3^2}{2023} \right\rfloor, \cdots\cdots, \left\lfloor \frac{2023^2}{2023} \right\rfloor.$$

8)Consider three circles, C_1, C_2 and C_3. Say, the center of C_1 is O. The center of C_2 is A and C_3 is C. Radius of $C_2 \neq$ Radius of C_3. C_2 *and* C_3 touches circle C_1 at B *and* D respectively. Now AC and BD extends to meet at P. BP meets C_3 at E. Prove that

$$\frac{PE}{PB} = \frac{PC}{PA}.$$


C_2
C_1
C_3