Polynomials (Problem Set)

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(*) might require calculus knowledge. (*) are past ISI problems

1. Find all positive integers a, b such that each of the equations

$$x^2 - ax + b = 0 \quad \text{and} \quad x^2 - bx + a = 0$$

has distinct positive integral roots.

2. $f(x)$ is a degree 4 polynomial satisfying $f(n) = \frac{1}{n}$ for $n = 1, 2, 3, 4, 5$. If $f(0) = \frac{a}{b}$, (where a and b are co-prime positive integers), then what is $a + b$?

3. Find the number of real solutions of the equation:

$$(x-1)(x-3)(x-4)\ldots(x-2025) = (x-2)(x-4)(x-6)\ldots(x-2024)$$

4. Let a, b be the roots of the equation

$$x^2 - 10cx - 11d = 0$$

and those of

$$x^2 - 10ax - 11b = 0$$

are c, d. Then what is $a + b + c + d$? $\quad (a \neq b \neq c \neq d)$

5. Let $x_1, x_2, \ldots, x_n$ be complex numbers satisfying the equations

$$x_1 + x_2 + \cdots + x_n = n$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = n$$
$$x_1^3 + x_2^3 + \cdots + x_n^3 = n$$
$$\vdots$$
$$x_1^n + x_2^n + \cdots + x_n^n = n$$

Then, prove that $x_i = 1 \quad \forall i = 1, 2, \ldots, n$.

6. (*) Let $P(x), Q(x)$ be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is s. If

$$P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3),$$

then prove that

- $P(x) - Q(x) = (x-1)^a r(x)$ for some integer $a \geq 1$ and a polynomial $r(x)$ with $r(1) \neq 0$.
- $s^2 = 3^{a-1}$, where a is as given in the previous.

7. $x_1^2 + px_1 + q = x_2, x_2^2 + px_2 + q = x_3, x_3^2 + px_3 + q = x_1$. Let p, q be real numbers with $\alpha < \beta$ be the roots of the equation $x^2 + (p-1)x + q = 0$. What is the maximum number of solutions of the system of the equations above where $x_1, x_2, x_3 \in [\alpha, \beta]$ is?

8. Consider all the numbers of the form $1 \pm \sqrt{2} \pm \sqrt{3} \pm \cdots \pm \sqrt{2024}$. Prove that when all of the numbers are multiplied, the product will belong to $\mathbb{Z}$.

9. $w, x, y, z \in \mathbb{N}, w^2 + x^2 + y^2 + z^2 = wxy + xyz + wxz + wyz$. Prove that there exists a solution (w_0, x_0, y_0, z_0) such that each of them is greater than 2025^{2025}. [Hint: Note that $(1, 1, 1, 1)$ is a solution. Now, suppose (w^*, x^*, y^*, z^*) is a general solution. WLOG x^* be the minimum of them. So, fixing w^*, y^*, z^*, get another x. Then can you proceed similarly?]

10. $a, b, c \in \mathbb{R}$ such that $(a + c)(a + b + c) < 0$. Prove that $\left(\frac{b-c}{2}\right)^2 \geq a(a + b + c)$.

11. Let $P(x)$ be a polynomial such that $(x + 1)P(x - 1) = (x - 1)P(x)$ for all $x \in \mathbb{R}$. Determine the maximum possible degree of $P(x)$.

12. Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0, q \in \mathbb{Z}$ has no integer root.

13. (*)(*) Consider the equation $x^5 + x = 10$. Show that
    (a) The equation has only one real root.
    (b) The root lies between 1 and 2.
    (c) This root must be irrational.

14. $f(x)$ i sa cubic polynomial $x^3 + ax^2 + bx + c$ such that $f(x) = 0$ has three distinct integral roots and $f(g(x)) = 0$ doesn't have any real roots, where $g(x) = x^2 + 2x - 5$. Then find minimum value of $a + b + c$.

15. (*) $P(x) \in \mathbb{Z}[x], P(1) = 7$, and $P(n)$ is prime for all $n \in \mathbb{N}$. Find $P(2025)$.

16. $f, g \in \mathbb{R}[x]$. Also, we have $f(x^2 + x + 1) = f(x).g(x)$ for all $x \in \mathbb{R}$. Prove that $deg(g)$ must be even.

17. Suppose a, b, c are distinct integers. $P(x) \in \mathbb{Z}[x]$. Also, we have $P(a) = P(b) = P(c) = -1$. Find all integer roots of P.

18. $0 < a \leq b \leq c \in \mathbb{R}$. Also, $a \leq x \leq y \leq z \leq c$, where $x, y, z \in \mathbb{R}$. It is given that $a+b+c = x+y+z$, and $xyz = abc$. Prove that $a = x, b = y, c = z$.

19. $a_1, a_2, \cdots, a_{2025}$ are distinct reals. $P(x) \in \mathbb{R}[x]$. Degree of P is 2024. Given that $P(1) = 2026$, and $|P(a_i) - P(a_j)| = |a_i - a_j|$ for all $i, j \in \{1, \cdots, 2025\}$. Find all such polynomials P.

20. $P(x) \in \mathbb{Z}[x]$. a, b, c distinct integers such that $P(a) = b, P(b) = c, P(c) = a$. Prove that such polynomials don't exist.

21. $a \in \mathbb{Z}, P(x) \in \mathbb{Z}[x]$ such that $P(P(P(P(a)))) = a$. Prove that $P(P(a)) = a$.

22. f is a polynomial such that $f(x) \in \mathbb{Z}[x]$ such that $f(0), f(1)$ both are odd. Find all integer roots of f.

23. f is a polynomial such that $f(x) \in \mathbb{R}[x]$. $f(1) = 3$ and $f(x + 1) = f(x) + 3x^2 + 3x + 1$. Find $f(\frac{1}{2})$.

24. f is a polynomial such that $f(x) \in \mathbb{R}[x]$. Degree of f is n. Also, $f(i) = \frac{i}{i+1}$ for $i = 0, 1, \cdots, n$. Find $f(n + 1)$.

25. $P(x) = x^n + x^{n-1} + x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1x + a_0$ for $n \geq 3$. Can all roots of P be real?

26. $f(x) = x^n - nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1x + a_0$. It is given that all roots of f are real. Find f.

27. If the sum of the real roots x to each of the equations

$$2^{2x} - 2^{x+1} + 1 - \frac{1}{k^2} = 0$$

for $k = 2, 3, \ldots, 2023$ is N, what is 2^N?

28. Let x, y, z be nonzero numbers, not necessarily real, such that

$$(x-y)^2+(y-z)^2+(z-x)^2=24yz$$

and

$$\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=3.$$

Compute $\frac{x^2}{yz}$.

29. Suppose that $p(x), q(x)$ are monic polynomials with nonnegative integer coefficients such that

$$\frac{1}{5x} \geq \frac{1}{q(x)}-\frac{1}{p(x)} \geq \frac{1}{3x^2}$$

for all integers $x \geq 2$. Compute the minimum possible value of $p(1) \cdot q(1)$.

30. Let the roots of

$$x^{2022}-7x^{2021}+8x^2+4x+2$$

be $r_1, r_2, \ldots, r_{2022}$, the roots of

$$x^{2022}-8x^{2021}+27x^2+9x+3$$

be $s_1, s_2, \ldots, s_{2022}$, and the roots of

$$x^{2022}-9x^{2021}+64x^2+16x+4$$

be $t_1, t_2, \ldots, t_{2022}$. Compute the value of

$$\sum_{1 \leq i,j \leq 2022} r_i s_j + \sum_{1 \leq i,j \leq 2022} s_i t_j + \sum_{1 \leq i,j \leq 2022} t_i r_j.$$

31. Let $f^1(x)=x^3-3x$. Let $f^n(x)=f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of

$$\frac{f^{2022}(x)}{x}.$$

If

$$\sum_{r \in \mathcal{R}} \frac{1}{r^2}=\frac{a^b-c}{d}$$

for positive integers a, b, c, d, where b is as large as possible and c and d are relatively prime, find $a+b+c+d$.

32. Let x, y, z be positive real numbers with $1<x<y<z$ such that

$$\log_x y+\log_y z+\log_z x=8, \quad \text{and}$$

$$\log_x z+\log_z y+\log_y x=\frac{25}{2}.$$

The value of $\log_y z$ can then be written as $\frac{p+\sqrt{q}}{r}$ for positive integers p, q, and r such that q is not divisible by the square of any prime. Compute $p+q+r$.

33. Find the sum of all possible values of a such that there exists a non-zero complex number z such that the four roots, labeled r_1 through r_4, of the polynomial

$$x^4-6ax^3+(8a^2+5a)x^2-12a^2x+4a^2$$

satisfy $|\Re(r_i)|=|r_i-z|$ for $1 \leq i \leq 4$. Note, for a complex number x, $\Re(x)$ denotes the real component of x.

34. Suppose that the polynomial $x^2 + ax + b$ has the property such that if s is a root, then $s^2 - 6$ is a root. What is the largest possible value of $a + b$?

35. Suppose $f(x)$ is a monic quadratic polynomial such that there exists an increasing arithmetic sequence $z_1 < z_2 < z_3 < z_4$ where $|f(z_1)| = |f(z_2)| = |f(z_3)| = |f(z_4)| = 2020$. Compute the absolute difference of the two roots of $f(z)$.

36. (*)(*) Let a, b, c be three real numbers which are roots of a cubic polynomial, and satisfy $a+b+c = 6$ and $ab + bc + ca = 9$. Suppose $a < b < c$. Show that

$$0 < a < 1 < b < 3 < c < 4.$$

37. (*) Let $a_0, a_1, \ldots, a_{19} \in \mathbb{R}$ and

$$P(x) = x^{20} + \sum_{i=0}^{19} a_i x^i, x \in \mathbb{R}.$$

If $P(x) = P(-x)$ for all $x \in \mathbb{R}$, and

$$P(k) = k^2,$$

for $k = 0, 1, 2, \ldots, 9$. Find $P(x)$.

38. (*) Let c be a fixed real number. Show that a root of the equation

$$x(x + 1)(x + 2) \cdots (x + 2009) = c$$

can have multiplicity at most 2. Determine the number of values of c for which the equation has a root of multiplicity 2.

39. (*) Let $P(X)$ be a polynomial with integer coefficients of degree $d > 0$.

    (a) If α and β are two integers such that $P(\alpha) = 1$ and $P(\beta) = -1$, then prove that $|\beta - \alpha|$ divides 2.

    (b) Prove that the number of distinct integer roots of $P^2(x) - 1$ is atmost $d + 2$.

40. (*) We are given $a, b, c \in \mathbb{R}$ and a polynomial $f(x) = x^3 + ax^2 + bx + c$ such that all roots (real or complex) of $f(x)$ have same absolute value. Show that $a = 0$ iff $b = 0$.

41. (*) Let $f(x)$ be a polynomial with integer co-efficients. Assume that 3 divides the value $f(n)$ for each integer n. Prove that when $f(x)$ is divided by $x^3 - x$, the remainder is of the form $3r(x)$ where $r(x)$ is a polynomial with integer coefficients.

42. (*) Let $f(x) = ax^2 + bx + c$ where a, b, c are real numbers. Suppose $f(-1), f(0), f(1) \in [-1, 1]$. Prove that $|f(x)| \le \frac{3}{2}$ for all $x \in [-1, 1]$.

43. (*) Suppose that $P(x)$ is a polynomial with real coefficients, such that for some positive real numbers c and d, and for all natural numbers n, we have $c|n|^3 \le |P(n)| \le d|n|^3$.

    Prove that $P(x)$ has a real zero.

44. (*) If a polynomial P with integer coefficients has three distinct integer zeroes , then show that $P(n) \ne 1$ for any integer n.

45. (*) Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $P(X) = X$ has no real solution. Prove that $P(P(X)) = X$ has no real solution.

46. (*) Let a, b, c be nonzero real numbers such that $a + b + c \ne 0$. Assume that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c}$$

Show that for any odd integer k,

$$\frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k} = \frac{1}{a^k + b^k + c^k}.$$

47. (*) Let k, n and r be positive integers.

(a) Let $Q(x) = x^k + a_1 x^{k+1} + \cdots + a_n x^{k+n}$ be a polynomial with real coefficients. Show that the function $\frac{Q(x)}{x^k}$ is strictly positive for all real x satisfying

$$0 < |x| < \frac{1}{1 + \sum_{i=1}^{n} |a_i|}$$

(b) Let $P(x) = b_0 + b_1 x + \cdots + b_r x^r$ be a non zero polynomial with real coefficients. Let m be the smallest number such that $b_m \neq 0$. Prove that the graph of $y = P(x)$ cuts the x-axis at the origin (i.e., P changes signs at $x = 0$) if and only if m is an odd integer.

48. (*) Consider the polynomial $ax^3 + bx^2 + cx + d$ where a, b, c, d are integers such that ad is odd and bc is even.Prove that not all of its roots are rational.

49. (*) If $P(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}$ be a polynomial with real coefficients and $a_1^2 < a_2$ then prove that not all roots of $P(x)$ are real.

50. (*) Let $p(x) = x^7 + x^6 + b_5 x^5 + \cdots + b_0$ and $q(x) = x^5 + c_4 x^4 + \cdots + c_0$. If $p(i) = q(i)$ for $i = 1, 2, 3, \cdots, 6$. Show that there exists a negative integer r such that $p(r) = q(r)$.