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Fine Properties of Functions with Applications in Brownian Motion and Partial Differential Equations

Final Report

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Abstract

We have covered several topics regarding fine properties of functions and its various applications. At first, there are discussions about **Hausdorff Measure** (Section 1). Then we discuss the Area and Coarea Formulae, and the **General Change of Variables Formula** for two subcases (Section 2). We study an important space of functions named **Sobolev Space** which is relevant in applications like solutions to Partial Differential Equations (Section 3,8). Then we move on to **Fine Properties of Functions** such as approximation by more regular functions (Section 4), traces and extensions (Section 5) and differentiability (Section 6). Then we discuss about **Sobolev Inequalities**, which are useful in proving certain estimates in Partial Differential Equations and Sobolev embeddings (Section 8). We discuss a notion of capturing "bad" sets (where singularity of a function arises) called **Capacity** and observe its Hausdorff dimensional properties (Section 9). Then we deal with **Quasicontinuity** of (precise representative of) Sobolev functions with the knowledge of capacity (Section 9). Then we see several applications of these theories, such as in **Brownian Motion** (Section 10) and **Weak Solutions of Partial Differential Equations** (Section 11). Due to insufficiency of space, we include various topics we have covered, in many cases without mentioning the proofs here (even though they were studied scrupulously during the project) and instead refer to the resources.

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Contents

1 Hausdorff Measure **4**

1.1 Introduction . . . 4
1.2 Properties of Hausdorff Measure . . . 4
1.3 Isodiametric Inequality . . . 4
1.4 Hausdorff and Lebesgue Measure . . . 4
1.5 Densities . . . 5
1.6 Hausdorff Measure and Lipschitz Mapping . . . 5
1.7 Example: The Set where a Summable Function is Large . . . 5

2 Area and Coarea Formulae **5**

2.1 Area Formula . . . 6
2.2 Coarea Formula . . . 7
2.3 General Change of Variables Formula . . . 7
2.4 Applications of Change of Variables . . . 8

3 Sobolev and BV Functions **9**

3.1 Sobolev Function: Definitions and elementary properties . . . 9
3.2 $W^{1,\infty}$ functions and Lipschitz Functions . . . 9
3.3 Bounded Variation Functions . . . 9
3.4 Structure Theorem for BV_{loc} Functions . . . 10
3.5 General Sobolev Spaces . . . 10

4 Approximations of Sobolev Functions by Smooth Functions **10**

4.1 Local Approximation . . . 10
4.2 Global Approximation . . . 12

5 Traces and Extensions **12**

5.1 Extension of Lipschitz Functions . . . 12
5.2 Trace of Sobolev Functions . . . 12
5.3 Extension of Sobolev Functions . . . 12
5.4 Whitney's Extension Theorem . . . 12

6 Differentiability **13**

6.1 Rademacher's Theorem . . . 13
6.2 Differentiability of Sobolev Functions on a.e. Line . . . 13
6.3 L^p differentiability a.e. for BV_{loc} Functions . . . 13
6.4 Differentiability a.e. for $W^{1,p}$ . . . 13
6.5 Aleksandrov's Theorem . . . 14

7 Sobolev Inequalities **14**

7.1 Gagliardo-Sobolev-Nirenberg Inequality . . . 14
7.2 Poincaré Inequality . . . 15
7.3 Morrey's Inequality . . . 15

8 More on the Sobolev Spaces **16**

8.1 The Space $H^{k,p}$ and $H_0^{k,p}$ . . . 16
8.2 Embedding Theorems . . . 17

9 Capacity **18**
9.1 Definition and Elementary Properties . . . 18
9.2 Capacity and Hausdorff Dimension . . . 18
9.3 Quasicontinuity: Precise Representatives of Sobolev Functions . . . 18

10 Application: Brownian Motion **19**
10.1 Strategies to Determine Hausdorff Dimension . . . 19
10.2 Energy Criterion and Frostman's Theorem . . . 19
10.3 Hausdorff Dimension of Brownian Paths . . . 20
10.4 An open problem: Percolation Dimension . . . 21

11 Application: Regularity of Weak Solutions of PDE **22**
11.1 Examples . . . 22
11.2 General Strategy to Solve: Weak Solutions . . . 22
11.3 Variational Problem . . . 23
11.4 Regularity of Solutions . . . 23

1 Hausdorff Measure

1.1 Introduction

Hausdorff measures are introduced to capture even small subsets of $\mathbb{R}^n$, which has dimension s (may be fractional) and the s-dimensional Hausdorff measure is denoted as $\mathcal{H}^s$. For example, Cantor Set, which has zero measure on $\mathbb{R}$ (i.e., $\mathcal{L}^1$ measure zero), has Hausdorff dimension $\log 2/\log 3$ (For general treatment, see [5], p. 14). The formal definitions are given below,

Let $A \subset \mathbb{R}^n,\ 0 \le s < \infty,\ 0 < \delta \le \infty$. Then

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s | A \subset \bigcup_{j=1}^{\infty} C_j, \operatorname{diam} C_j \le \delta\right\}, \tag{1}$$

where

$$\alpha(s) \coloneqq \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}, \tag{2}$$

$$\mathcal{H}^s(A) \coloneqq \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A). \tag{3}$$

Hausdorff dimension of a set is defined as

$$\dim(A) \coloneqq \inf\{0 \le s < \infty | \mathcal{H}^s(A) = 0\}. \tag{4}$$

1.2 Properties of Hausdorff Measure

Here are some interesting properties of Hausdorff measures.

(i) $\mathcal{H}^0$ is counting measure,

(ii) $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^n$ for all $s > n$,

(iii) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0, A \subset \mathbb{R}^n$,

(iv) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for affine isometry $L : \mathbb{R}^n \to \mathbb{R}^m, A \subset \mathbb{R}^n$,

(v) Let $A \subset \mathbb{R}^n$ and $0 \le s < t < \infty$. Then $\mathcal{H}^s(A) < \infty \implies \mathcal{H}^t(A) = 0$ and $\mathcal{H}^t(A) > 0 \implies \mathcal{H}^s(A) = \infty$ (Hence, the definition of Hausdorff dimension makes sense).

1.3 Isodiametric Inequality

The following inequality gives bound of Lebesgue measure of a set in terms of its diameter.

Theorem 1. $\mathcal{L}^n(A) \le \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$.

Interesting thing is, the set A need not be in a ball of diameter $\operatorname{diam} A$. The proof uses a technique called **Steiner Symmetrization**.

1.4 Hausdorff and Lebesgue Measure

When working on usual Euclidean Space $\mathbb{R}^n$ with equipped measure $\mathcal{L}^n$, $\mathcal{H}^n$ also gives the same measure, which makes Hausdorff measure a great generalization of Lebesgue measure.

Theorem 2. $\mathcal{H}^n = \mathcal{L}^n$ *on* $\mathbb{R}^n$.

1.5 Densities

A well known density result for Lebesgue measure is,

$$\lim_{r\to 0}\frac{\mathcal{L}^n(B(x,r)\cap E)}{\alpha(n)r^n}=\begin{cases}1 & \text{for } \mathcal{L}^n \text{ a.e. } x\in E\\ 0 & \text{for } \mathcal{L}^n \text{ a.e. } x\in \mathbb{R}^n-E\end{cases}. \tag{5}$$

For Hausdorff measure, the following holds.

Theorem 3. *Assume $E\subset\mathbb{R}^n$, E is $\mathcal{H}^s$-measurable and $\mathcal{H}^s<\infty$. Then,*

$$\lim_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s}=0 \tag{6}$$

for $\mathcal{H}^s$ a.e. $x\in\mathbb{R}^n-E$, and

$$\limsup_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s}\in[2^{-s},1] \tag{7}$$

for $\mathcal{H}^s$ a.e. $x\in E$.

1.6 Hausdorff Measure and Lipschitz Mapping

For **Lipschitz maps**, we have the following results

Theorem 4. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be Lipschitz, $A\subset\mathbb{R}^n$, $0\le s<\infty$. Define $G(f;A)=\{(x,f(x))|x\in A\}\subset\mathbb{R}^n\times\mathbb{R}^m$ Then,*

$$\mathcal{H}^s(f(A))\le(\operatorname{Lip} f)^s\mathcal{H}^s(A), \tag{8}$$

$$\dim(G(f;A))=n. \tag{9}$$

1.7 Example: The Set where a Summable Function is Large

If a function is locally summable, then the Hausdorff dimension of the set where the function is "locally large" can be estimated.

Theorem 5. *Let $f\in L^1_{loc}(\mathbb{R}^n)$. suppose $0\le s<n$ and define*

$$\Lambda_s:=\left\{x\in\mathbb{R}^n\,\middle|\,\limsup_{r\to 0}\frac{1}{r^s}\int_{B(x,r)}|f|dy>0\right\}. \tag{10}$$

Then

$$\mathcal{H}^s(\Lambda_s)=0. \tag{11}$$

2 Area and Coarea Formulae

Area and Coarea formula yields the formula for general change of variable where the function may not be a bijection or domain and range are of different dimension.

Due to Theorem 18, the Jacobian Jf for a Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}^m$ is defined as $\sqrt{\det(Df^*\circ Df)}$ if $n\le m$, and $\sqrt{\det(Df\circ Df^*)}$ if $n\ge m$, wherever Df exists (it exists a.e.).

2.1 Area Formula

Theorem 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \le m$. Then for each $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$,*

$$\int_A Jf dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) d\mathcal{H}^n(y). \tag{12}$$

Outline of Proof. Due to Theorem 18, we can assume that $Df(x)$ and $Jf(x)$ exist for all $x \in A$ and also suppose $\mathcal{L}^n(A) < \infty$.

Case 1: $A \subset \{Jf > 0\}$. Then we apply the following Lemma,

Lemma 7. Fix $t > 1$. we can choose Borel sets $\{E_k\}$ such that

(i) $B = \cup_{k=1}^{\infty} E_k$,

(ii) $f|_{E_k}$ is one-one ($k \in \mathbb{N}$), and

(iii) for each $k \in \mathbb{N}$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathrm{Lip}((f|_{Ek}) \circ T_k^{-1}) \le t, \quad \mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \le t, \quad t^{-n}|\det T_k| \le Jf|_{E_k} \le t^n |\det T_k|. \tag{13}$$

Set $F_j^i = E_j \cap Q_i \cap A$, where $Q_i \in \mathcal{B}_k := \bigg\{ Q \mid Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i = \frac{c_i}{k}, b_i = \frac{c_{i+1}}{k}, c_i \in \mathbb{Z}, i \in \mathbb{N} \bigg\}$. Now since $g_k := \sum_{i,j=1}^{\infty} \chi_{f(F_j^i)} \uparrow \mathcal{H}^0(A \cap f^{-1}\{y\})$ as $k \uparrow \infty$, by Monotone Convergence Theorem,

$$\lim_{k\to\infty} \sum_{i,j=1}^{\infty} \mathcal{H}^n(f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) d\mathcal{H}^n. \tag{14}$$

By Lemma 7,

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(f|_{E_j} \circ T_j^{-1} \circ T_j) \le t^n \mathcal{L}^n(T_j(F_j^i)) = t^n |\det T_j| \mathcal{L}^n(F_j^i), \tag{15}$$

$$t^{-n}|\det T_j| \mathcal{L}^n(F_j^i) = \mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j \circ (f|_{E_j})^{-1} \circ f(F_j^i)) \le t^n \mathcal{H}^n(f(F_j^i)). \tag{16}$$

Thus by Equation 15, 16 and Theorem 7

$$t^{-2n}\mathcal{H}^n(f(F_j^i)) \le t^{-n}|\det T_j| \mathcal{L}^n(F_j^i) \le \int_{F_j^i} Jf dx \le t^n |\det T_j| \mathcal{L}^n(F_j^i) \le t^{2n} \mathcal{H}^n(f(F_j^i)). \tag{17}$$

Now sum on i and j, apply Equation 14 by taking $k \to$ and finally $t \to 1^+$ to obtain Equation 12 for Case 1.

Case 2: $A \subset \{Jf = 0\}$. Factor $f = p \circ g$, where $g : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$, $g(x) := (f(x), \epsilon x)$ $\forall x \in \mathbb{R}^n$ and $p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $p(y, z) = y$ $\forall y \in \mathbb{R}^m, z \in \mathbb{R}^n$. So $Dg(x) = \begin{pmatrix} Df(x) \\ \epsilon I \end{pmatrix}_{(n+m)\times n}$. By Binet-Cauchy formula

$$\begin{aligned} 0 < \epsilon^{2n} = (\det(\epsilon I))^2 \le Jg(x)^2 = Jf(x)^2 + \{&\text{sum of squares of} \\ &\text{terms each involving at least one } \epsilon\} \le C\epsilon^2 \ \forall x \in A. \end{aligned} \tag{18}$$

Since p is a projection, using Theorem 4 and Case 1

$$\mathcal{H}^n(f(A)) \leq \mathcal{H}^n(g(A)) \leq \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}\{y,z\}) d\mathcal{H}^n(y,z) = \int_A Jg(x)dx \leq \epsilon C \mathcal{L}^n(A). \tag{19}$$

Now let $\epsilon \to 0$ to conclude $\mathcal{H}^n(f(A)) = 0$ and thus, since $\operatorname{spt} \mathcal{H}^0(A \cap f^{-1}\{y\}) \subset f(A)$, $\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}\{y\}) d\mathcal{H}^n = 0$. Then

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}\{y\}) d\mathcal{H}^n = 0 = \int_A Jf dx. \tag{20}$$

In the general case, apply Case 1 and Case 2 by writing $A = (A \cap \{Jf > 0\}) \sqcup (A \cap \{Jf = 0\})$. □

Note, it is important to show that integrands are measurable with respect to integrating measure. We have omitted such consideration in the proof, but these hold and can be proved.

2.2 Coarea Formula

Theorem 8. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for each $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$,*

$$\int_A Jf dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) dy. \tag{21}$$

Remark: The proof of Coarea formula uses analogous techniques and lemmae to Area formula, but is more cumbersome and technical, so we leave the proof here. For reference, see [4], Section 3.4 .

2.3 General Change of Variables Formula

We also obtain **General Change of Variables** formulae for these two cases.

Theorem 9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-summable function, then*

$$n \leq m \implies \int_A g(x) Jf(x) dx = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}\{y\}} g(x) \right] d\mathcal{H}^n(y). \tag{22}$$

$$n \geq m \implies \int_A g(x) Jf(x) dx = \int_{\mathbb{R}^m} \left[\int_{f^{-1}\{y\}} g \; d\mathcal{H}^{n-m} \right] dy. \tag{23}$$

Proof. Consider Equation 22. Since $g = g^+ - g^-$, we assume without loss of generality that $g \geq 0$. Then by partition of unity, we can write g as

$$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i} \tag{24}$$

for appropriate $\mathcal{L}^n$-measurable sets $\{A_i\}_{i\in\mathbb{N}}$. Then the Monotone Convergence Theorem and Area formula implies

$$\begin{aligned}
\int_{\mathbb{R}^n} gJf dx &= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^n} \chi_{A_i} Jf dx \\
&= \sum_{i=1}^{\infty} \frac{1}{i} \int_{A_i} Jf dx \\
&= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^0(A_i \cap f^{-1}\{y\}) d\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \sum_{x\in f^{-1}\{y\}} \chi_{A_i}(x) d\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^m} \sum_{x\in f^{-1}\{y\}} \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}(x) d\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^m} \left[\sum_{x\in f^{-1}\{y\}} g(x) \right] d\mathcal{H}^n(y).
\end{aligned}$$

This proves Equation 22. Equation 23 can be proved similarly using Coarea formula $\square$

2.4 Applications of Change of Variables

1. **Length of a curve** ($n = 1, m \geq 1$): Assume $f : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and one-one. Define the curve $C = f([a, b]) \subset \mathbb{R}^m$. Then "length" of C is,

$$\mathcal{H}^1(C) = \int_a^b |\boldsymbol{\nabla}_t f| dt. \tag{25}$$

2. **Surface Area of a graph** ($n \geq 1, m = n + 1$): Assume $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Define for an open set $U \subset \mathbb{R}^n$, $G = \{(x, g(x)) | x \in U\} \subset \mathbb{R}^{n+1}$. Then "surface area" of G is,

$$\mathcal{H}^n(G) = \int_U (1 + |Dg|^2)^{\frac{1}{2}} dx. \tag{26}$$

3. **Submanifolds**: Let $M \subset \mathbb{R}^m$ be a Lipschitz, n-dimensional embedded submanifold. Suppose $U \subset \mathbb{R}^n$ and $f : U \to M$ is a chart for M. Let $A \subset f(U)$, Borel. Then define, $g_{ij} = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}$, $(1 \leq i, j \leq n)$, $g = \det((g_{ij}))$. Then the "volume" of A in M is

$$\mathcal{H}^n(A) = \int_B g^{\frac{1}{2}} dx. \tag{27}$$

4. **Polar Coordinates**: Let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-summable. Then

$$\int_{\mathbb{R}^n} g dx = \int_0^{\infty} \left(\int_{\partial B(0,r)} g d\mathcal{H}^{n-1} \right) dr. \tag{28}$$

5. **Level Sets**: Assume $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Then

$$\int_{\mathbb{R}^n} |\nabla f| dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\}) dt. \tag{29}$$

3 Sobolev and BV Functions

3.1 Sobolev Function: Definitions and elementary properties

Assume $U \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(U)$, $1 \leq i \leq n$. Then $g_i \in L^1_{loc}$ is called the **Weak Partial Derivative** of f with respect to x_i in U if

$$\int_U f \frac{\partial \varphi}{\partial x_i} dx = -\int_U g_i \varphi dx \tag{30}$$

for all $\varphi \in C^1_c(U)$.
Denote $g_i = \frac{\partial f}{\partial x_i}$, $\nabla f = (\frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_n})$ provided they exist.

A function belongs to **Sobolev space** $W^{1,p}(U)$ if $f \in L^p(U)$ and all weak partial derivatives $\frac{\partial f}{\partial x_i}$ exist and belong to $L^p(U)$ $(i = 1, \cdots, n)$. The function f belongs to $W^{1,p}_{\text{loc}}(U)$ if $f \in W^{1,p}_{\text{loc}}(V)$ for all $V \subset\subset U$. f is a **Sobolev function** if $f \in W^{1,p}_{\text{loc}}(U)$ for some $1 \leq p \leq \infty$.

Note, if f is a Sobolev function, then the "Integration by Parts" formula holds,

$$\int_U f \frac{\partial \varphi}{\partial x_i} dx = -\int_U \frac{\partial f}{\partial x_i} \varphi dx \tag{31}$$

for all $\varphi \in C^1_c(U)$.

If $f \in W^{1,p}(U)$ then define

$$||f||_{W^{1,p}(U)} = \left(\int_U |f|^p + |\nabla f|^p dx \right)^{\frac{1}{p}} \quad , p \in [1, \infty), \tag{32}$$

$$||f||_{W^{1,\infty}(U)} = \operatorname*{ess\,sup}_U (|f| + |\nabla f|). \tag{33}$$

3.2 $W^{1,\infty}$ functions and Lipschitz Functions

Theorem 10. *Let $f : U \in \mathbb{R}$. Then f is locally Lipschitz in U if and only if $f \in W^{1,\infty}_{loc}(U)$.*

3.3 Bounded Variation Functions

A function $f \in L^1(U)$ has **bounded variation** in $U \subset \mathbb{R}^n$ if

$$\sup \left\{ \int_U f \ \nabla \cdot \varphi dx \middle| \varphi \in C^1_c(U; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty. \tag{34}$$

A function $f \in L^1_{loc}(U)$ has **locally bounded variation** in U if for each open set $V \subset\subset U$,

$$\sup \left\{ \int_V f \; \boldsymbol{\nabla} \cdot \varphi dx \middle| \varphi \in C^1_c(V;\mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty. \tag{35}$$

Denote the corresponding spaces of such functions as $BV(U)$ and $BV_{loc}(U)$.

3.4 Structure Theorem for BV_{loc} Functions

Theorem 11. *Let $f \in BV_{loc}(U)$. Then there exists a Radon measure μ on U and a μ-measurable function $\sigma : U \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ μ *a.e. and*

(ii) $\int_U f \; \boldsymbol{\nabla} \cdot \varphi dx = -\int_U \varphi \cdot \sigma d\mu$

for all $\varphi \in C^1_c(U;\mathbb{R}^n)$.

3.5 General Sobolev Spaces

So far, we have only discussed about $W^{1,p}$ spaces, which is equivalent to having weak first partial derivative. Now we generalize the notion.

Let $u : \Omega \to \mathbb{R}$ be integrable, $\boldsymbol{\alpha} := (\alpha_1, \cdots, \alpha_d), |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$ and

$$D_{\boldsymbol{\alpha}}\varphi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \quad \text{for } \varphi \in C^{|\boldsymbol{\alpha}|}(\Omega). \tag{36}$$

An integrable function $v : \Omega \to \mathbb{R}$ is called an $\boldsymbol{\alpha}$-th weak derivative of u, in symbols $v = D_\alpha u$, if

$$\int_\Omega \varphi v dx = (-1)^\alpha \int_\Omega u D_\alpha \varphi dx \text{ for all } \varphi \in C_0^{|\boldsymbol{\alpha}|}(\Omega). \tag{37}$$

Define Sobolev Space $W^{k,p}$ $(k \in \mathbb{N}, 1 \leq p < \infty)$ as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D_{\boldsymbol{\alpha}} u \text{ exists and is contained in } L^p(\Omega) \text{ for all } |\boldsymbol{\alpha}| \leq k\} \tag{38}$$

and

$$||u||_{W^{k,p}} := \left(\sum_{|\boldsymbol{\alpha}| \leq k} \int_\Omega |D_{\boldsymbol{\alpha}} u|^p \right)^{\frac{1}{p}}. \tag{39}$$

4 Approximations of Sobolev Functions by Smooth Functions

4.1 Local Approximation

Theorem 12. *Assume $f \in W^{1,p}(U)$ for some $1 \leq p < \infty$, U open subset of $\mathbb{R}^n$. Then there exists a sequence $\{f_k\}_{k=1}^\infty \subset W^{1,p}(U) \cap C^\infty(U)$ such that*

$$f_k \to f \text{ in } W^{1,p}(U).$$

Proof. Fix $\epsilon > 0$ and define

$$\begin{cases} U_k := \{x \in U | \operatorname{dist}(x, \partial U) > k^{-1}\} \cap U(0,k) & k \in \mathbb{N} \\ U_0 = \emptyset. \end{cases} \tag{40}$$

Set

$$V_k := U_{k+1} - \bar{U}_{k-1}, \quad k \in \mathbb{N}, \tag{41}$$

and let $\{\zeta_k\}_{k\in\mathbb{N}}$ be a sequence of smooth functions such that

$$\begin{cases} \zeta_k \in C_c^\infty(V_k), & 0 \le \zeta_k \le 1, k \in \mathbb{N} \\ \sum_{k=1}^{\infty} \zeta_k = 1 & \text{on } U. \end{cases} \tag{42}$$

For each $k \in \mathbb{N}$, $f\zeta_k \in W^{1,p}(U)$, with $\operatorname{spt}(f\zeta_k) \subset V_k$; hence there exists $\epsilon_k > 0$ such that

$$\begin{cases} \operatorname{spt}(\eta_{\epsilon_k} * (f\zeta_k)) \subset V_k \\ \left(\int_U |\eta_{\epsilon_k} * (f\zeta_k) - f\zeta_k|^p dx \right)^{\frac{1}{p}} < \frac{\epsilon}{2^k} \\ \left(\int_U |\eta_{\epsilon_k} * (D(f\zeta_k)) - D(f\zeta_k)|^p dx \right)^{\frac{1}{p}} < \frac{\epsilon}{2^k} \end{cases} \tag{43}$$

where the mollifier η_ϵ is defined as following, Define the C^∞-function $\eta : \mathbb{R}^n \to \mathbb{R}$

$$\eta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}} & if |x| < 1 \\ 0 & if |x| \ge 1, \end{cases} \tag{44}$$

the constant c is such that

$$\int_{\mathbb{R}^n} \eta(x) dx = 1. \tag{45}$$

Now define,

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right) \quad (\epsilon > 0, x \in \mathbb{R}^n). \tag{46}$$

Define

$$f_\epsilon := \sum_{k=1}^{\infty} \eta_{\epsilon_k} * (f\zeta_k) \tag{47}$$

in some neighbourhood of each point $x \in U$, there are only finitely many nonzero terms in this sum. Hence $f_\epsilon \in C_c^\infty(U)$.

Since $f = \sum_{k=1}^{\infty} f\zeta_k$, Equation 43 implies

$$||f_\epsilon - f||_{L^p(U)} \le \sum_{k=1}^{\infty} \left(\int_U |\eta_{\epsilon_k} * (f\zeta_k) - f\zeta_k|^p dx \right)^{\frac{1}{p}} < \epsilon \tag{48}$$

and

$$||Df_\epsilon - Df||_{L^p(U)} \le \sum_{k=1}^{\infty} \left(\int_U |\eta_{\epsilon_k} * (D(f\zeta_k)) - D(f\zeta_k)|^p dx \right)^{\frac{1}{p}} < \epsilon. \tag{49}$$

Consequently $f_\epsilon \in W^{1,p}(U)$ and

$$f_\epsilon \to f \text{ in } W^{1,p}(U) \text{ as } \epsilon \to 0. \tag{50}$$

□

4.2 Global Approximation

Theorem 13. *Assume U is bounded, ∂U is Lipschitz, i.e., there exist $r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon rotating and re-labelling the coordinate axes if necessary - we have $U \cap Q(x,r) = \{y | \gamma(y_1, \cdots, y_{n-1}) < y_n\} \cap (x,r)$ where $Q(x,r) = \{y || y_i - x_i| < r, i = 1, \cdots, n\}$. Then for $f \in W^{1,p}$ for some $1 \le p < \infty$, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset W^{1,p}(U) \cap C^{\infty}(\bar{U})$ such that*

$$f_k \to f \text{ in } W^{1,p}(U).$$

5 Traces and Extensions

5.1 Extension of Lipschitz Functions

Theorem 14. *Assume $A \in \mathbb{R}^n$, and let $f : A \to \mathbb{R}^n$ be Lipschitz. There exists a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that*

(i) $\bar{f} = f$,

(ii) $\mathrm{Lip}(\bar{f}) \le \sqrt{m}\,\mathrm{Lip}(f)$.

Remark: *Kirszbraun's Theorem* asserts than there exists such extension with $\mathrm{Lip}(\bar{f}) = Lip(f)$, see [7], Section 2.10.43.

5.2 Trace of Sobolev Functions

Theorem 15. *Assume U bounded, ∂U is Lipschitz. $1 \le p < \infty$.*

(i) There exists a bounded operator $T : W^{1,p}(U) \to L^p(\partial U; \mathcal{H}^{n-1})$ such that $Tf = f$ for all $f \in W^{1,p}(U) \cap C(\bar{U})$.

(ii) Furthermore, for all $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f \in W^{1,p}(U)$,

$$\int_U f \nabla \cdot \varphi dx = -\int_U \nabla f \cdot \varphi dx + \int_{\partial U} (\varphi \cdot \nu) Tf \; d\mathcal{H}^{n-1}, \tag{51}$$

ν denoting the unit outer normal to ∂U.

5.3 Extension of Sobolev Functions

Theorem 16. *Assume U is bounded, ∂U Lipschitz, $1 \le p < \infty$. Let $U \subset\subset V$. There exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that $Ef = f$ on U and $spt(Ef) \subset V$ for all $f \in W^{1,p}(U)$.*

5.4 Whitney's Extension Theorem

Theorem 17. *Let $C \subset \mathbb{R}^n, f : C \to \mathbb{R}$ continuous, $d : C \to \mathbb{R}^n$ continuous, and for each compact set $K \subset C$,*

$$\lim_{\delta \to 0} \sup \left\{ \left| \frac{|f(y) - f(x) - d(x) \cdot (y-x)|}{|x-y|} \right| \middle| 0 < |x-y| \le \delta, \; x, y \in K \right\} = 0. \tag{52}$$

Then there exists a function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\bar{f}$ *is* C^1,

(ii) $\bar{f} = f,\ D\bar{f} = d$ *on* C.

6 Differentiability

6.1 Rademacher's Theorem

Theorem 18 (Rademacher's Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$ a.e.*

6.2 Differentiability of Sobolev Functions on a.e. Line

Theorem 19. *(i) If $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, then for each $k = 1, \cdots, n$, the functions*

$$f^*_k(x', t) = f^*(\cdots, x_{k-1}, t, x_{k+1}, \cdots)$$

are absolutely continuous in t on compact subsets of $\mathbb{R}$, for $\mathcal{L}^{n-1}$ a.e. point $x' = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$. Also $(f^*_k)' \in L^p_{loc}(\mathbb{R}^n)$.

(ii) Conversely, suppose $f \in L^p_{loc}(\mathbb{R}^n)$ and $f = g$ $\mathcal{L}^n$ a.e., where for each $k = 1, \cdots, n$, the functions

$$g_k(x', t) = g(\cdots, x_{k-1}, t, x_{k+1}, \cdots)$$

are absolutely continuous in t on compact subsets of $\mathbb{R}$ for $\mathcal{L}^{n-1}$ a.e. point $x' = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$, and $g'_k \in L^p_{loc}(\mathbb{R}^n)$. Then $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

6.3 L^p differentiability a.e. for BV_{loc} Functions

Theorem 20. *Assume $f \in BV_{\mathrm{loc}}(\mathbb{R}^n)$. Then for $\mathcal{L}^n$ a e. $x \in \mathbb{R}^n$,*

$$\left(\fint_{B(x,r)} |f(y) - f(x) - \nabla f(x) \cdot (x - y)|^{1^*} \right)^{\frac{1}{1^*}} = o(r) \quad \text{as } r \to 0. \tag{53}$$

6.4 Differentiability a.e. for $W^{1,p}$

Theorem 21. *Assume $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < n$. Then for $\mathcal{L}^n$ a e. $x \in \mathbb{R}^n$*

$$\left(\fint_{B(x,r)} |f(y) - f(x) - \nabla f(x) \cdot (x - y)|^{p^*} \right)^{\frac{1}{p^*}} = o(r) \quad \text{as } r \to 0 \tag{54}$$

Theorem 22. *Let $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $n < p \le \infty$. Then f is differentiable $\mathcal{L}^n$ a.e. and its derivative equals its weak derivative $\mathcal{L}^n$ a.e.*

6.5 Aleksandrov's Theorem

Theorem 23. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then f is* ***twice differentiable $\mathcal{L}^n$ a.e.****, more precisely, for $\mathcal{L}^n$ a.e. x (can construct and define D^2 in a specific way such that),*

$$\left|f(x+h) - f(x) - Df(x)\cdot h - \frac{1}{2}h^T \cdot D^2 f(x)\cdot h\right| = o(|h|^2) \text{ as } h \to 0. \tag{55}$$

7 Sobolev Inequalities

7.1 Gagliardo-Sobolev-Nirenberg Inequality

Let for $1 \le p < n$, Sobolev conjugate $p^* = \frac{np}{n-p}$. Note $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Theorem 24. *Assume $1 \le p < n$. There exists a constant C_1, depending only on n and p, such that*

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} dx\right)^{\frac{1}{p^*}} \le C_1 \left(\int |\boldsymbol{\nabla} f|^p dx\right)^{\frac{1}{p}} \tag{56}$$

for all $f \in W^{1,p}(\mathbb{R}^n)$.

Proof. By Theorem 4.1, it is enough to assume that $f \in C_c^1(\mathbb{R}^n)$. Then for $1 \le i \le n$,

$$f(x_1, \cdots, x_i, \cdots, x_n) = \int_{-\infty}^{t_i} \frac{\partial f}{\partial x_i} dt_i, \tag{57}$$

and so

$$|f(x)| \le \int_{-\infty}^{\infty} |\boldsymbol{\nabla} f(x_1, \cdots, t_i, \cdots, x_n)| dt_i. \tag{58}$$

Thus

$$|f(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f(x_1, \cdots, t_i, \cdots, x_n)| dt_i\right)^{\frac{1}{n-1}}. \tag{59}$$

Integrate with respect to x_1,

$$\begin{aligned}\int_{-\infty}^{\infty} |f|^{1^*} dx_1 &= \left(\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f| dt_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f| dt_i\right)^{\frac{1}{n-1}} dx_1 \\ &\le \left(\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f| dt_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f| dx_1 dt_i\right)^{\frac{1}{n-1}}.\end{aligned}$$

Now iteratively integrate with respect to x_2 and so on to eventually obtain

$$\int_{\mathbb{R}^n} |f|^{1^*} dx \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\boldsymbol{\nabla} f| dx_1 \cdots dt_i \cdots dx_n\right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |\boldsymbol{\nabla} f| dx\right)^{\frac{n}{n-1}}. \tag{60}$$

So,

$$\left(\int_{\mathbb{R}^n} |f|^{1^*} dx\right)^{\frac{1}{1^*}} \le \int_{\mathbb{R}^n} |\boldsymbol{\nabla} f| dx, \tag{61}$$

so it proves for $p = 1$. Now if $1 < p < n$, set $g = |f|^{\gamma}$ so that $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = p^*$. So applying Equation 61 to g we find

$$\left(\int_{\mathbb{R}^n} |f|^{\gamma n} n - 1 dx\right)^{\frac{n-1}{n}} \leq \gamma \int_{\mathbb{R}^n} |f|^{\gamma-1} |\nabla f| dx \leq \gamma \left(\int_{\mathbb{R}^n} |f|^{\frac{(\gamma-1)p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}. \quad (62)$$

So we obtain Equation 56 from definition of γ and Equation 62. □

7.2 Poincaré Inequality

Denote $(f)_{x,r} = ⨍_{B(x,r)} f dy = (\alpha(n) r^n)^{-1} \int_{B(x,r) f dy}$, $U(x,r)$ and $B(x,r)$ closed and open ball respectively.

Theorem 25. *Assume $1 \leq p < n$. There exists a constant C_2, depending only on n and p, such that*

$$\left(⨍_{B(x,r)} |f - (f)_{x,r}|^{p^*} dy\right)^{\frac{1}{p^*}} \leq C_2 r \left(⨍_{B(x,r)} |\nabla f|^p dx\right)^{\frac{1}{p}} \quad (63)$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in W^{1,p}(U(x,r))$.

We refer to [4], Section 4.5.2 for a proof of this theorem. But a lemma is needed to proof Poincaré Inequality as well as Morrey's Estimate

Lemma 26. For each $1 \leq p < \infty$, there exists a constant C, depending only on n and p, such that

$$\int_{B(x,r)} |f(y) - f(z)|^p dy \leq C r^{n+p-1} \int_{B(x,r)} |\nabla f(y)|^p |y - z|^{1-n} dy \quad (64)$$

for all $B(x,r) \subset \mathbb{R}^n, f \in C^1(B(x,r))$ and $z \in B(x,r)$.

For a proof, refer [4], Section 4.5.2.

7.3 Morrey's Inequality

Theorem 27. *For each $n < p < \infty$, there exists a constant C_3, depending only on n and p, such that*

$$|f(y) - f(z)| \leq C_3 r \left(⨍_{B(x,r)} |\nabla f|^p dw\right)^{\frac{1}{p}} \quad (65)$$

for all $B(x,r) \in \mathbb{R}^n$, $f \in W^{1,p}(U(x,r))$ and $\mathcal{L}^n$ a.e. $y, z \in U(x,r)$
In particular, if $f \in W^{1,p}(\mathbb{R}^n)$, then the limit

$$\lim_{r \to 0} (f)_{x,r} = f^*(x) \quad (66)$$

exists for all $x \in \mathbb{R}^n$ and f^* is Hölder continuous with exponent $1 - \frac{n}{p}$.

Proof. Assume f is C^1 and using Lemma 26 with $p = 1$, and Hölder Inequality,

$$\begin{aligned}
|f(y) - f(z)| &\leq \fint_{B(x,r)} |f(y) - f(w)| + |f(w) - f(z)| dw \\
&\leq C \int_{B(x,r)} |\nabla f(w)| (|y - w|^{1-n} + |z - w|^{1-n}) dw \\
&\leq C \left(\int_{B(x,r)} (|y - w|^{1-n} + |z - w|^{1-n})^{\frac{p}{p-1}} dw \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla f|^p dw \right)^{\frac{1}{p}} \\
&\leq C r^{(n-(n-1)\frac{p}{p-1})\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla f|^p dw \right)^{\frac{1}{p}} \\
&= C r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla f|^p dw \right)^{\frac{1}{p}}.
\end{aligned}$$

By Theorem 4.1, we can approximate $f \in W^{1,p}(U(x,r))$ and the same estimate holds for $\mathcal{L}^n$ a.e. $y, z \in U(x,r)$. This proves Equation 65.

Now suppose $f \in W^{1,p}(\mathbb{R}^n)$. then for $\mathcal{L}^n$ a.e., x, y, apply the estimate of 65 with $r = |x - y|$ to obtain

$$|f(y) - f(x)| \leq C|x - y|^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla f|^p dw \right)^{\frac{1}{p}} \leq C ||\nabla f||_{L^p(\mathbb{R}^n)} |x - y|^{1-\frac{n}{p}}. \tag{67}$$

Thus f is equal $\mathcal{L}^n$ a.e. to a Hölder continuous function $\bar{f}$. So $f^* = \bar{f}$ everywhere in $\mathbb{R}^n$. □

8 More on the Sobolev Spaces

8.1 The Space $H^{k,p}$ and $H_0^{k,p}$

Let $1 \leq p < \infty, \Omega \subset \mathbb{R}^d$. The spaces $H^{k,p}$ and $H_0^{k,p}$ are defined to be the closures of C^∞ and C_0^∞ respectively with respect to the norm $|| \cdot ||_{W^{k,p}(\Omega)}$.

Due to the result 4.1, we have the following theorem.

Theorem 28. $W^{1,p}(\Omega) = H^{1,p}(\Omega)$.

In fact, it holds in general that

Theorem 29. $W^{k,p}(\Omega) = H^{k,p}(\Omega)$. *The space* $W^{k,p}(\Omega)$ *is complete with respect to* $|| \cdot ||_{W^{k,p}(\Omega)}$*, i.e., it is a Banach space.*

In fact, $W^{k,p}$ is a separable space, if $p < \infty$. Note that, $W^{k,\infty}$ is a Banach Space too.

The space $W^{m,2}$ is interesting since it is a **Hilbert Space**, equipped with the inner product $\langle u, v \rangle = \sum_{|\boldsymbol{\alpha}| \leq k} \int_\Omega D_{\boldsymbol{\alpha}} u D_{\boldsymbol{\alpha}} v$. In fact, $W^{1,2}$ appears so commonly in analysis of partial differential equations that it is denoted as H^1 (See Section 11 for further discussion).

8.2 Embedding Theorems

Now we mention the important **Sobolev Embedding Theorems**, which eludes the interplay between different functional spaces.

Theorem 30. $H_0^{1,p} \subset \begin{cases} L^{\frac{dp}{d-p}}(\Omega) & \textit{for } p < d \\ C^0(\bar{\Omega}) & \textit{for } p > d \end{cases}.$

Note that the $p < d$ case is a direct consequence of Gagliardo-Sobolev-Nirenberg Inequality (Theorem 7.1)

Inductively, the following can be concluded further.

Corollary 31. $H_0^{k,p}(\Omega) \subset \begin{cases} L^{\frac{dp}{d-kp}}(\Omega) & \text{for } kp < d \\ C^m(\bar{\Omega}) & \text{for } 0 \leq m < k - \frac{d}{p} \end{cases}.$

Corollary 32. If $u \in H_0^{k,p}(\Omega)$ for some p and all $k \in \mathbb{N}$, then $u \in C^\infty(\Omega)$.

The following conclusions use Inequalities in Section 7 and Extension results in Section 5

Theorem 33. *Let Ω be an open set, $\subset \mathbb{R}^n$. Then*

(i) $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, *if* $p < n$,

(ii) $W^{1,p}(\Omega) \subset L^q(\Omega)\ \forall q \in [p, \infty)$, *if* $p = n$,

(iii) $W^{1,p}(\Omega) \subset L^\infty(\Omega)$, *if* $p > n$,

(iv) $W^{1,p}(\Omega) \subset C(\bar{\Omega})$, *if* $p > n$.

Outline of Proof. At first, it is convenient to prove for $\mathbb{R}^n$ and then use Theorem 5.3 to extend $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$. *(i)* is direct consequence of Gagliardo-Sobolev-Nirenberg Inequality (Theorem 7.1) .*(ii)* follows from an application of Young's Inequality and intermediate step of proving Gagliardo-Sobolev-Nirenberg Inequality (Theorem 7.1). *(iii)* follows from Morrey' estimate (Theorem 7.3) and Hölder's Inequality. □

In general, by induction method, the following can be proved.

Theorem 34. *Let Ω be an open set, $\subset \mathbb{R}^n$. Then*

(i) $W^{m,p}(\Omega) \subset L^q(\Omega), \frac{1}{q} = \frac{1}{p} - \frac{m}{n}$, *if* $\frac{1}{p} - \frac{m}{n} > 0$,

(ii) $W^{m,p}(\Omega) \subset L^q(\Omega)\ \forall q \in [p, \infty)$, *if* $\frac{1}{p} - \frac{m}{n} = 0$,

(iii) $W^{m,p}(\Omega) \subset L^\infty(\Omega)$, *if* $\frac{1}{p} - \frac{m}{n} < 0$,

(iv) $W^{m,p}(\Omega) \subset C^k(\bar{\Omega})\ \forall 0 \leq k \leq m - \frac{n}{p}$, *if* $\frac{1}{p} - \frac{m}{n} < 0$.

More generally,

Theorem 35 (Sobolev Embedding Theorem)**.** *Let Ω be an open set, $\subset \mathbb{R}^n$. If $k < l, p < n$ and $1 \leq p < q < \infty$ such that $\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$, then*

$$W^{k,p}(\Omega) \subset W^{l,q}(\Omega). \tag{68}$$

9 Capacity

9.1 Definition and Elementary Properties

Theorem 36. *Define K^p, a space of a function as following*

$$K^p = \{f : \mathbb{R}^n \to \mathbb{R} | f \geq 0, f \in L^{p^*}(\mathbb{R}^n), \nabla f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}. \tag{69}$$

*If $A \subset \mathbb{R}^n$, define p-**capacity** of A, $\operatorname{Cap}_p(A)$ as*

$$\operatorname{Cap}_p(A) = \inf\left\{\int_{\mathbb{R}^n} |\nabla f|^p dx | f \in K^p, A \subset \{f \geq 1\}^o\right\}. \tag{70}$$

Note that, Cap_p is a (outer) measure on $\mathbb{R}^n$. Here are some properties of Capacity. (Let $A \subset \mathbb{R}^n$)

(i) $\operatorname{Cap}_p(A) = \inf\{\operatorname{Cap}_p(U) | U \text{ open}, A \subset U\}$,

(ii) $\operatorname{Cap}_p(\lambda A) = \lambda^{n-p} \operatorname{Cap}_p(A)$,

(iii) $\operatorname{Cap}_p(L(A)) = \operatorname{Cap}_p(A)$ for affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$,

(iv) $\operatorname{Cap}_p(B(x,r)) = r^{n-p} \operatorname{Cap}_p(B(0,1))$,

(v) $\operatorname{Cap}_p(A) \leq C\mathcal{H}^{n-p}(A)$ (The constant C depends only on n and p),

(vi) $\mathcal{L}^n(A) \leq C \operatorname{Cap}_p(A)^{\frac{n}{n-p}}$ (The constant C depends only on n and p).

9.2 Capacity and Hausdorff Dimension

Theorem 37. *(i) If $\mathcal{H}^{n-p}(A) < \infty$, then $\operatorname{Cap}_p(A) = 0$ $(1 < p < n)$.*

(ii) Assume $A \subset \mathbb{R}^n$ and $1 \leq p < \infty$. If $\operatorname{Cap}_p(A) = 0$ then $\mathcal{H}^s(A) = 0$ for all $s > n - p$.

9.3 Quasicontinuity: Precise Representatives of Sobolev Functions

Theorem 38. *Assume $f \in K^p$ and $\epsilon > 0$. Let*

$$A = \{x \in \mathbb{R}^n | (f)_{x,r} > \epsilon \text{ for some } r > 0\}. \tag{71}$$

Then

$$\operatorname{Cap}_p(A) \leq \frac{C}{\epsilon^p} \int_{\mathbb{R}^n} |\nabla f|^p dx, \tag{72}$$

where C depends only on n and p .

A function f is p-quasicontinuous if for each $\epsilon > 0$, there exists an open set V such that $Cap_p(V) \leq \epsilon$ and $f\big|_{\mathbb{R}^n - V}$ is continuous

Theorem 39. *Suppose $f \in W^{1,p}(\mathbb{R}^n), 1 \leq p < n$.*

(i) There is a Borel set $E \subset \mathbb{R}^n$ such that $\operatorname{Cap}_p(E) = 0$ and $f^*(x)$ exists,

(ii) $\lim_{r\to 0} ⨍_{B(x,r)} |f - f^*(x)|^{p^*} dy = 0$ for each $x \in \mathbb{R}^n - E$,

(iii) The precise representative f^* is p-quasicontinuous.

10 Application: Brownian Motion

10.1 Strategies to Determine Hausdorff Dimension

We will determine the Hausdorff measure by obtaining upper and lower bound. For **upper bound**, we need to exhibit α such that $\mathcal{H}^\alpha(E) = 0$ which implies $\dim E \leq \alpha$ (Section 1.2). This is usually the simpler part as we just need to exhibit suitable $\delta = \delta_n$-covers where $\delta \to 0$ and corresponding expression $\sum_{j\geq 1}(\operatorname{diam} E_j^n)^\alpha$ becomes small as $n \to \infty$.

For **lower bound**, we need to exhibit a β such that $\mathcal{H}^\beta(E) > 0$ which implies $\dim E \geq \beta$ (Section 1.2). It is usually more difficult and we need to use clever covering strategies or capacitary methods.

For upper bound of $f(E)$ where f is Hölder continuous with exponent $\gamma \in (0, 1]$, it follows without difficulty that

$$\dim f(E) \leq \gamma^{-1} \dim E. \tag{73}$$

10.2 Energy Criterion and Frostman's Theorem

Here are results involving capacitary methods, that we will use to determine several Hausdorff dimensional properties of Brownian Motion.

Theorem 40 (Energy Criterion, Frostman 1935)**.** *Let μ be a probability measure on E. Then*

$$\iint_{E\times E} \frac{\mu(dx)\mu(dy)}{|x-y|^\alpha} < \infty \ \Rightarrow \mathcal{H}^\alpha(E) > 0 \ \Rightarrow \dim E \geq \alpha. \tag{74}$$

Proof. Since the double integral is finite, there exists a Borel set B with $\mu(B) > 0$ such that $\int_E \frac{\mu(dy)}{|x-y|^\alpha}$ for all $x \in B$. Let $(B_j)_{j\geq 1}$ be any cover of B by Borel sets. For all $x, y \in B_j$, $(\operatorname{diam} B_j)^\alpha$. So $\mu(B_j) \leq (\operatorname{diam} B_j)^\alpha \int_{B_j} \frac{\mu(dy)}{|x-y|^\alpha}$ for all $x \in B_j \cap B$. Thus $0 < M^{-1}\mu(B) \leq M^{-1}\sum_{j\geq 1}\mu(B_j) \leq \sum_{j\geq 1}(\operatorname{diam} B_j)^\alpha$. So $\mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha(B) \geq M^{-1}\mu(B) > 0$. □

The next theorem of Frostman requires a lemma in order to prove, which involves exquisite graph theoretic argument.

Lemma 41 (Frostman's Lemma)**.** If $K \subset \mathbb{R}^d$ is a closed set such that $\mathcal{H}^\alpha(K) > 0$, then there exists a nonzero Borel probability measure μ supported on K such that $\mu(D) < C|D|^\alpha$ for all Borel sets D, where C is a constant not depending on D and $|D|$ denotes the diameter of D.

Proof. Assume $K \subset [0, 1]^d$ without loss of generality. We create a tree with a root that we associate with the cube and each vertex in the tree 2^d edges emanating from it, each vertex corresponding to one of the 2^d sub-cubes of half the side length of the original cube. We then remove the edges ending in vertices associated with non-K-intersecting subcubes. Assign bound of flow through each edges as following: at the level n, define $b(e) = 2^{-n\alpha}$. Since for each $x \in K$, there is an infinite path emanating from the root, all of whose vertices are associated with cubes that contain x and thus intersect K, any cutset must contain one of the edges emanating from these vertices. So the corresponding cubes with the given endpoints of the edges in any cutset must cover K. Since $\mathcal{H}^\alpha(K) > 0$, there is $\delta > 0$ such that

$$\inf\left\{\sum_j |A_j|^\alpha : K \in \cup_j A_j, |A_j| \leq \delta\right\} > 0.$$

So by **MaxFlow-MinCut Theorem** (Ford and Fulkerson (1956), for proof, see [11], Theorem 4.3.11), there exists a flow f **of positive strength** such that meets the bound b. Now define $\tilde{\mu}(\{\text{all paths through } e\}) = f(e)$. This collection $\mathcal{C}$ of sets of the form all paths through e forms a semi-algebra. Also, due to preservation of flow, $\tilde{\mu}$ is countably additive. Thus (by [3], Theorem A.1.3), we can extend $\tilde{\mu}$ to a measure μ on $\sigma(\mathcal{C})$. By construction, μ is supported on K.

Suppose D is a Borel subset of $\mathbb{R}^d$. Let n be such integer tha $2^{-n} < |D \cap [0,1]^d| \le 2^{-(n-1)}$. Then $D \cap [0,1]^d$ can be covered with 3^d of the cubes in the construction having the side length 2^{-n}. So $\mu(D) \le 3^d 2^{-n\alpha} \le 3^d |D|^\alpha$, so we have a fintie measure $]u$ satisfying the condition of Lemma. Renormalize the constant to make it a probability measure and our proof is done. $\square$

Now we are ready to prove Frostman's Theorem, which is converse of Energy Criterion.

Theorem 42 (Frostman's Theorem)**.** *If $K \subset \mathbb{R}^d$ is closed and $\dim K > \alpha$, then there exists a Borel probability measure μ on K such that*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha} < \infty. \tag{75}$$

Proof. Since $\dim K > \alpha$, there exists a $\beta > \alpha$ such that $\mathcal{H}^\beta(K) > 0$. By Frostman's Lemma (Lemma 41), there exists a non-zero Borel probability measure μ on K and C such that $\mu(D) \le C|D|^\alpha$. Restrict μ to support with diameter less than 1 if necessary. Fix $x \in K$ and let $S_k(x) = \{y : 2^{-k} < |x-y| \le 2^{1-k}\}$. Now,

$$\int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} = \sum_{k=1}^{\infty} \int_{S_k(x)} \frac{d\mu(y)}{|x-y|^\alpha} \le \sum_{k=1}^{\alpha} \mu(S_k(x)) 2^{k\alpha} \le C \sum_{k=1}^{\infty} |2^{2-k}|^\beta 2^{k\alpha} = C' \sum_{k=1}^{\infty} 2^{k(\alpha-\beta)}.$$

So,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha} < C' \sum_{k=1}^{\infty} 2^{k(\alpha-\beta)} < \infty.$$

$\square$

Note: There exist another type of capacity, that can be defined as following. The α-capacity of a set K is

$$\mathrm{Cap}_\alpha(K) = \left(\inf_\mu \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha} \right)^{-1}. \tag{76}$$

10.3 Hausdorff Dimension of Brownian Paths

Denote

$$B(E, \omega) \coloneqq \{B_t(\omega) : t \in E\},$$
$$GrB(E, \omega) \coloneqq \{(t, B_t(\omega)) : t \in E\}.$$

Theorem 43 (Taylor 1953; McKean 1955)**.** *Let $(B_t)_{t \ge 0}$ be a BM^d and $F \in [0,1]$ a closed set. Then*

$$\dim B(F) = \min\{d, 2\dim F\} = \begin{cases} 2\dim F, & d \ge 2 \\ \min\{1, 2\dim F\}, & d = 1 \end{cases} \quad a.s. \tag{77}$$

Proof. The upper bound follows from the Hölder continuity with exponent $\gamma \in (0, \frac{1}{2})$, so by Equation 73, $\dim B(E) \le \gamma^{-1} \dim E$ a.s. for any Borel set $E \subset [0,1]$ and then take $\gamma \to \frac{1}{2}$.

For the lower bound, let $F \in [0,1]$ be a closed set and set $\alpha < \dim F$. Then by Frostman's Theorem 42, (since $\mathcal{H}^\alpha(F) > 0$), there is a finite measure σ on F such that $\iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{|t-s|^\alpha} < \infty$. In order to use Energy Criterion (Theorem 40) to show lower bound of $\dim B(F)$, we need the corresponding probability measure μ. For our claim, choose $\mu(A) = \int_F \mathbb{1}_A(B_t)d\sigma(t)$ (occupation measure) and $\lambda = 2\alpha < \min\{d, 2\dim F\}$. Then, by Tonelli's theorem

$$\begin{aligned}
\mathbb{E}\left(\iint_{B(F)\times B(F)} \frac{d\mu(x)d\mu(y)}{|x-y|^\lambda}\right) &= \mathbb{E}\left(\iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{|B_t - B_s|^\lambda}\right) \\
&= \iint_{F\times F} \mathbb{E}\frac{1}{|B_t - B_s|^\lambda} d\sigma(t)d\sigma(s) \\
&= \mathbb{E}|B_1|^{-\lambda} \iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{|t-s|^{\frac{\lambda}{2}}} \quad (\because B_t - B_s \stackrel{d}{=} \sqrt{t-s}B_1) \\
&< \infty \quad (\because B_1 \sim N(0,1) \text{ and Previous Result}).
\end{aligned}$$

We can conclude using Energy Criterion (Theorem 40) now, letting $\lambda \to \min\{d, 2\dim F\}$. □

However the null set may depend on F as the lower bound proof suggests. But Kaufman showed for BM^d with $d \ge 2$, the null set can be independently chosen of F.

Theorem 44 (Kaufman 1969)**.** *Let $(B_t)_{t\ge 0}$ be a BM^d, $d \ge 2$ and $E \subset [0,1]$ a Borel set. Then*

$$\mathbb{P}[\dim B(E) = 2\dim E \ \forall\ E \in \mathcal{B}(0,1)] = 1.$$

Corollary 45. Let $(B_t)_{t\ge 0}$ be a BM^d, $d \ge 2$. Then

$$\dim B^{-1}(A) \le \min\left\{1, \frac{1}{2}\dim A\right\} \text{ a.s. for all } A \in \mathcal{B}(\mathbb{R}^d). \tag{78}$$

In particular, the zero set $\dim B^{-1}(\{0\}) = 0$ a.s.

Corollary 46 (Space-filling nature of two-dimensional Brownian Motion)**.** For BM^2, $\dim B([s,t]) = 2$ for all $0 \le s < t \le 1$.

A similar result holds for "graph" of Brownian Motion, which uses similar techniques like Theorem 43 to prove.

Theorem 47 (Taylor 1955)**.** *Let $(B_t)_{t\ge 0}$ be a BM^d and $F \subset [0,1]$ a closed set. Then*

$$\dim GrB(F) = \min\left\{2\dim F, \dim F + \frac{1}{2}d\right\} = \begin{cases} \dim F + \frac{1}{2}, & d = 1, \dim F \ge \frac{1}{2} \\ 2\dim F, & d = 1, \dim F \le \frac{1}{2} \\ 2\dim F, & d \ge 2 \end{cases} \ a.s. \tag{79}$$

10.4 An open problem: Percolation Dimension

When looking more into the applications of Hausdorff dimension in Brownian Motion, I encountered an interesting concept named **Percolation Dimension** of a fractal, which roughly quantifies the shortest path size between two distinct points of it (See [2]). For Brownian Trace, the percolation dimension is non-trivial so it seems to be an additional tool for analyzing such random fractals. It is shown that in four or more dimensions, the percolation dimension is 2, and in three dimension, the percolation dimension $\in (1,2)$. The Percolation Dimension of a two-dimensional Brownian Motion is conjectured to be equal to 1 by Burdzy and still is an open problem (See [6]).

11 Application: Regularity of Weak Solutions of PDE

11.1 Examples

Consider various partial differential equations given below. ($\Omega \subset \mathbb{R}^d$ open and bounded, $g \subset H^1(\Omega)$).

1. *(Poisson Problem)*

$$-\nabla^2 u = f \text{ in } \Omega, \quad u = g \text{ for } \partial\Omega \text{ (or, } u - g \in H_0^1(\Omega)). \tag{80}$$

2. *(Homogeneous Dirichlet Problem)*

$$-\nabla^2 u + u = f \text{ in } \Omega, \quad u = g \text{ for } \partial\Omega. \tag{81}$$

   where ν is the outward unit normal vector to ∂U

3. *(Generalized Elliptic Equation)*

$$-\boldsymbol{\nabla} \cdot (A\boldsymbol{\nabla})u + cu = f \text{ in } \Omega, \quad u = g \text{ for } \partial\Omega. \tag{82}$$

   where A is symmetric, elliptic, bounded and c is non-negative, bounded.

The solution $u \in C^2(\Omega)$ satisfying these equations are called **classical solutions** corresponding to the equations.

11.2 General Strategy to Solve: Weak Solutions

The classical solutions of these partial differential equation can be analyzed by the following steps.

Step 1: Corresponding to each of the previous equations, there is a variational formulation $\mathcal{V}(u,v) = \int_\Omega fv \ \forall v \in H^1(\Omega)$ (or H_0^1 depnding on the problem) (by density, enough to assume $\forall v \in C^1(\bar{\Omega})$) and corresponding solution $u \in H_0^1$ is called **weak solution**. Every classical solution is a weak solution too.

Step 2: The existence and uniqueness of the weak solution $u \in H^1(\Omega)$ can be shown and obtained by minimization of a variational equation. To be precise, $\mathcal{M}(v) \coloneqq \frac{1}{2}\mathcal{V}(v,v) - \int_\Omega fv$ and $u = \arg\inf_{v \in H^1(\Omega)} \mathcal{M}(v)$.

Step 3: The regularity of the solution is analyzed.

Step 4: Recover the classical solution.

The step 3 is of our interest where the theory of Sobolev Spaces come into play.

11.3 Variational Problem

The corresponding variational formulations are mentioned below

1. *(Poisson Problem)*
$$\mathcal{V}(u,v) = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v. \tag{83}$$

2. *(Homogeneous Dirichlet Problem)*
$$\mathcal{V}(u,v) = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + \int_\Omega uv. \tag{84}$$

3. *(Generalized Elliptic Equation)*
$$\mathcal{V}(u,v) = \int_\Omega \boldsymbol{\nabla} u \cdot (A\boldsymbol{\nabla} v) + \int_\Omega cuv. \tag{85}$$

11.4 Regularity of Solutions

Typically, two types of analysis arise naturally when analyzing the regularity of solution, namely the regularity behaviour of the function itself and an estimate for the solution. For example, an estimate for Poisson problem can be acquired in the process as following,

$$||u||_{W^{1,2}(\Omega)} \leq C_{|\Omega|,d}\big(||g||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)}\big). \tag{86}$$

Also since the solution exists in H^1, the regularity of solution in general as well as specific to problem can be analyzed. For example, the theorems of section 8.2 can be used. An example is given for $d \geq 3$. We know by Theorem 39 that the solution has a representative that is 2-quasicontinuous. By the quasicontinuity, except the zero 2-capacity set A, the representative is continuous. By the relation $\mathcal{L}^d(A) \leq C\,\mathrm{Cap}_2(A)^{\frac{d}{d-2}}$ (Refer Section 9.1), $\mathcal{L}^d(A) = 0$, i.e., the representative is continuous a.e. Since $2^* = \frac{2d}{d-2} \geq 2$, so we can directly conclude from Theorem 39 and monotonicity of L^p norms that the solution matches the representative a.e., so the solution is continuous a.e.

References

[1] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer New York, 2010. ISBN: 9780387709147. URL: https://books.google.co.in/books?id=1bFCAAAAQBAJ.

[2] Krzysztof Burdzy. "Percolation dimension of fractals". In: *Journal of Mathematical Analysis and Applications* 145.1 (1990), pp. 282–288. ISSN: 0022-247X. DOI: https://doi.org/10.1016/0022-247X(90)90447-N. URL: https://www.sciencedirect.com/science/article/pii/0022247X9090447N.

[3] R. Durrett. *Probability: Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010. ISBN: 9781139491136. URL: https://books.google.co.in/books?id=evbGTPhuvSoC.

[4] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition*. Textbooks in Mathematics. CRC Press, 2015. ISBN: 9781482242393. URL: https://books.google.co.in/books?id=e3R3CAAAQBAJ.

[5] K.J. Falconer. *The Geometry of Fractal Sets*. Cambridge Tracts in Mathematics. Cambridge University Press, 1985. ISBN: 9780521337052. URL: https://books.google.co.in/books?id=-Kwp-GrimCIC.

[6] Chad Fargason. "Percolation Dimension of Brownian Motion in R^3". In: *Electronic Communications in Probability* 3.none (1998), pp. 51–63. DOI: 10.1214/ECP.v3-993. URL: https://doi.org/10.1214/ECP.v3-993.

[7] H. Federer. *Geometric Measure Theory*. Classics in Mathematics. Springer Berlin Heidelberg, 2014. ISBN: 9783642620102. URL: https://books.google.co.in/books?id=jld-BgAAQBAJ.

[8] J. Jost. *Partial Differential Equations*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781461448099. URL: https://books.google.co.in/books?id=t0kLl5lxum0C.

[9] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010. ISBN: 9781139486576. URL: https://books.google.co.in/books?id=e-TbA-dSrzYC.

[10] R.L. Schilling, L. Partzsch, and B. Böttcher. *Brownian Motion: An Introduction to Stochastic Processes*. De Gruyter Textbook. De Gruyter, 2014. ISBN: 9783110307306. URL: https://books.google.co.in/books?id=Ph7oBQAAQBAJ.

[11] D.B. West. *Introduction to Graph Theory*. Introduction to Graph Theory. Prentice Hall, 1996. ISBN: 9780132278287. URL: https://books.google.co.in/books?id=HuvuAAAAMAAJ.

We relied heavily on [4] for Section 1,2,3,4,5,6,7 and 9. For more details and further references in those sections, we consulted [7] and [5]. We referred [9] and [10] for Section 10. Finally, we principally used [1] and [8] for the functional space descriptions and solution of partial differential equations discussion in Section 8 and 11. For MaxFlow-MinCut Theorem (used in Section 10), we refer to [11]. For discussion on Percolation Dimension, we have consulted [2] and [6]. Finally, for Probability Theory, we mainly referred to [3].