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# ELUCIDATING PARADOXICAL PURSUITS: A MULTIFACETED EXPLORATION OF THE PRISONER'S DILEMMA

## Abstract

This article delves into the mathematical foundations of the Prisoner's Dilemma, explores certain extensions of the model like the iterated version and extends the discussion to some relevant real-world applications of the problem. By examining the tension between individual rationality and group outcomes, we uncover how mathematics not only models such paradoxes but offers pathways to understanding and resolving them.

## Introduction

Mathematics has been the fundamental tool to model rational decision-making, yet it often reveals paradoxes that challenge our intuitive understanding. The theme "Paradoxical Pursuits" captures this tension perfectly, and in our opinion, no example illustrates this better than the Prisoner's Dilemma, a classic problem studied in Game Theory. At its core, the Prisoner's Dilemma presents two rational individuals whose best logical choices lead to outcomes that are collectively worse than the outcome they would have obtained if they had cooperated. The dilemma captures the essence of paradox that is, rational strategies leading to irrational outcomes.

The simplicity of the scenario which is, a pair of prisoners deciding whether to betray or cooperate, masks deeper mathematical insights. It brings into focus ideas like Nash equilibrium and dominant strategies, showing how real-world players, driven by self-interest, often arrive at decisions that defy collective logic. Although it seems too paradoxical, this dilemma finds real-world applications in economics, politics, and biology.

## The Prisoner's Problem - Understanding the Game Theory behind the Dilemma

The classic case of the Prisoner's Dilemma is a two-player game where each player has two options - Cooperate or Defect.

- **Cooperate**: The player chooses to work with the other player, seeking a mutually beneficial outcome.
- **Defect**: The player chooses to betray or act against the other player, aiming to maximize his/her own benefit even if it harms the other player.

The outcomes are determined by the choices made by both the players independently, represented in a payoff matrix. The dilemma can be understood by analysing this matrix mathematically.

We consider a situation where Player A and Player B decide independently whether to cooperate or defect. The following is a typical payoff matrix for the game:

| | **Player B Cooperates** | **Player B Defects** |
|---|---|---|
| **Player A Cooperates** | (3,3) | (0,5) |
| **Player A Defects** | (5,0) | (1,1) |

Here, the numbers represent the payoffs for Player A and Player B respectively. Since both the players calculate their expected payoffs by assuming what strategy may be considered by the other, apparently it appears to each of them that choosing Defect ensures a higher individual pay off, no matter what strategy has been adopted by the other player. The paradox arises when both players choose to Defect, leading to the **(1, 1) outcome, which is suboptimal compared to mutual cooperation (3, 3).**

Mathematically, the dilemma is driven by the concept of Dominant Strategies. Defection is a dominant strategy for both players because it yields a higher payoff regardless of what the other player does. This leads to a Nash equilibrium at (1, 1), where neither player has an incentive to change their choice. However, this equilibrium is paradoxical because mutual cooperation would have produced a better collective outcome.

The payoff matrix and the analysis of dominant strategies at the individual levels illustrate how mathematical reasoning leads to rational yet collectively inefficient outcomes, highlighting the core paradox of the dilemma.

## The Exchange Game: A Reflection of the Prisoner's Dilemma in Economic Systems

The prisoner’s dilemma problem can be understood in the light of Paradoxical Pursuits through a story discussed by the Stanford encyclopaedia of Philosophy (2019).

Bill has a blue cap but prefers a red one, while Rose has a red cap but wants a blue one. Both would rather have two caps than one and prefer having a cap to none at all. They can either keep their caps or exchange them. This exchange game mirrors the Prisoner's Dilemma: regardless of what Rose does, Bill benefits from keeping his cap, and she benefits if he gives his to her. Similarly, Rose benefits from keeping hers, and Bill benefits if she gives it to him. However, both would be better off if they exchanged caps rather than keeping their own.

This story illustrates how the Prisoner's Dilemma is embedded in our economic system, suggesting that markets designed for mutually beneficial exchanges must overcome or avoid this dilemma.

## Iterated Prisoner's Dilemma: An Extension to the Classical Model

We consider a model where the game is repeatedly played by the same players and eventually, they learn about the behavioural tendencies of the counterparty. As the game is repeated, one player may formulate a strategy that does not follow the regular logical convention of an isolated round. The case of a **Tit for Tat Game** is an example of such a strategy.

Suppose Player A and Player B choose the tit for tat strategy. In the first round, Player A chooses to **cooperate.** In the subsequent round, Player A simply mimics the action of Player B in the previous round.

If both the players cooperate by playing (C, C), then they cooperate forever.

| | **ROUND 1** | **ROUND 2** | **ROUND 3** | …… |
|---|---|---|---|---|
| **PLAYER A** | C | C | C | ….. |
| **PLAYER B** | C | C | C | …… |

This gives the following payoff ($\delta$ *is the* discount factor and a is the payoff per round due to cooperation): $a + a\delta + a\delta^2 + a\delta^3 + \ldots\ldots.,\ a\epsilon\mathbb{N}$.

This is a geometric series that sums to $\frac{a}{1-\delta}$.

If Player B starts to defect (D), then the next round will get affected. There is an alternate occurrence of outcomes between the two players.

| | **ROUND 1** | **ROUND 2** | **ROUND 3** | ……. |
|---|---|---|---|---|
| **PLAYER A** | C | D | C | ……. |
| **PLAYER B** | D | C | D | …….. |

This gives the following payoff: b + e$\delta$ + b$\delta^2$ + e$\delta^3$ + b$\delta^4$ + e$\delta^5$ +……, a sum of two geometric series $\frac{b}{1-\delta^2}+\frac{e\delta}{1-\delta^2}$ (b and e are the payoffs per round due to cooperation and defection respectively; b, e $\epsilon\ \mathbb{N}$ $and\ b > e$)

Collaboration should be expected if payoff of defection is same as that of cooperation.

$$\frac{a}{1-\delta} \geq \frac{b}{1-\delta^2} + \frac{e\delta}{1-\delta^2}$$

where e<b<a.

The above equation finally boils down to,

$$\delta \geq \frac{b-a}{a-e}$$

The players should continue to cooperate if

$$\delta \geq \frac{b-a}{a-e}$$

and should defect if

$$\delta < \frac{b-a}{a-e}$$

We can illustrate this using an example. Let, a = 6, b = 9 and e = 2.

Solving the above inequalities, $\delta \geq \frac{3}{4}$ would imply continue cooperation and $\delta < \frac{3}{4}$ would imply continue defection.

From the perspective of the Prisoner's Dilemma, the analysis reveals that enduring cooperation can be sustained when players consistently follow reciprocal strategies, while defection disrupts this stability, leading to alternating outcomes. The derived conditions underscore the necessity of mutual trust and the alignment of long-term incentives to overcome the inherent conflict between self-interest and collective benefit.

The above strategy is optimal when noise (players deviating from their behavioural rules) does not exist. When noise comes into existence, we re-model this strategy to take into account randomness, allowing for occasional cooperation even after defection. This is known as **Probabilistic Tit for Tat (PTFT)** strategy.

The model of PTFT strategy is shown in the following table-

| Current/Next | Cooperation | Defection |
|---|---|---|
| **Cooperation** | 1-$\lambda\delta$ | $\lambda\delta$ |
| **Defection** | $\delta$ | 1-$\delta$ |

Here $\delta$ is a small number and $\lambda$ is a non-negative integer less than 1. By matrix multiplication of initial set of vectors of a 2×2 game with PTFT transition matrix of 2×2, we show that PTFT is the optimal by adding initial set of vectors to be 1.

$$\frac{1}{1+\lambda}(\lambda\delta) + \frac{\lambda}{1+\lambda}(1-\delta) = \frac{\lambda}{1+\lambda}$$

$$\frac{1}{1+\lambda}(1-\lambda\delta) + \frac{\lambda}{1+\lambda}(\delta) = \frac{1}{1+\lambda}$$

***Source-Scientific and Academic Publishing (2017)***

The initial set of vectors for a 2×2 game is $\pi =[\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}]$ ,where $\pi$ is the row vector denoting the probability of combinations independent of $\delta$. By summing both the values we get the probability value equal to 1.

The generalized initial set of vectors for $2^n \times 2^n$ repeated games will be -

$$\left[\frac{1}{(1+\lambda)^{(n+1)}}, \frac{(n+1)\lambda}{1!\,(1+\lambda)^{(n+1)}}, \frac{(n+1)((n+1)-1)\lambda^2}{2!\,(1+\lambda)^{(n+1)}}, \dots, \frac{\lambda^{(n+1)}}{(1+\lambda)^{(n+1)}}\right]$$

***Source-Scientific and Academic Publishing (2017)***

Hence by generalization of PTFT strategy we get the optimal stage for n≥ 1 for repeated games of $2^n \times 2^n$ rounds.

$$\frac{1}{(1+\lambda)^n} + \frac{n\lambda}{1!\,(1+\lambda)^n} + \frac{n(n-1)\lambda^2}{2!\,(1+\lambda)^n} + \dots + \frac{\lambda^n}{(1+\lambda)^n} = 1$$

***Source-Scientific and Academic Publishing (2017)***

This strategy combines the two psychological concepts of retaliation and forgiveness. This creates a stable environment but keeping in mind human psychology, it is very hard to predict if this strategy will always be followed as every individual will always have the tendency to outsmart the other. Hence even if this gives the most optimal solution it is hard to predict in advance, hence it is a classic paradoxical situation.

## Beyond the Classical Model: Mixed Strategies in Prisoner’s Dilemma

In the traditional Prisoner's Dilemma, players typically choose between two pure strategies—cooperate or defect as there is nothing in between that can possibly exist. However, in real-world scenarios, individuals or entities often employ mixed strategies, where they probabilistically choose between cooperating and defecting based on certain factors.

Mixed strategies allow for a more nuanced approach, reflecting the uncertainty and variability in decision-making. For instance, in scenarios like international trade negotiations or business competition, a player might choose to cooperate with a certain probability while defecting with another, depending on the perceived behaviour of others, past experiences, or expected payoffs. This approach can lead to more complex dynamics where equilibrium is achieved not by strict adherence to a single strategy but by balancing different strategies over time. Mixed strategies, therefore, introduce a level of realism into game theory, acknowledging that real-world decisions are rarely binary and often involve a calculated blend of options.

## Fiscal Conundrums: Extending The Prisoner's Dilemma in Policy Implementation

Although the classic prisoner’s game does not have any identical real-world example as such, the concept of the dilemma can be effectively used to study and analyse many real-world problems. A macroeconomic example of the prisoner's dilemma can be found in the context of Government fiscal policies during an economic downturn. At the time of an economic recession, individual Governments face the choice of implementing expansionary fiscal policies to stimulate economic growth. However, the effectiveness of these policies depends on the actions of other Governments.

In an ideal scenario, if all countries simultaneously adopt expansionary fiscal policies, the global economy would benefit from increased aggregate demand, leading to a potential recovery. However, if one country decides to pursue a more conservative fiscal approach, focusing on austerity measures or budget cuts, it may experience short-term economic stability. However, the global impact could be detrimental.

This situation mirrors the prisoner's dilemma, as each Government must decide whether to **cooperate** by collectively implementing expansionary policies or **defect** by pursuing more conservative measures which might seem more feasible in the short run. If all countries cooperate, needless to say, the global economy can recover more effectively. However, if one or more countries defect and pursue the apparent maximum personal gain, it can hinder the recovery for all nations, resulting in a suboptimal outcome for the broader group of countries.

## Real-World Reflections: Unpacking the Prisoner's Dilemma Beyond Theory

The Prisoner's Dilemma can be observed in numerous real-world scenarios where individual actions lead to collectively suboptimal outcomes. In international relations, arms races exemplify this dilemma, where nations, fearing the other's military advantage, continue to escalate their armament despite the mutual benefits of disarmament. Similarly, trade wars emerge when countries impose tariffs, fearing exploitation by others, leading to economic harm on both sides rather than the mutual gain of free trade.

In environmental conservation, the dilemma surfaces as nations hesitate to reduce emissions, fearing economic disadvantages while hoping others will bear the costs of climate action. This reluctance often results in global environmental degradation. Within the business world, competition often leads to price wars, where companies' lower prices to avoid losing market share, ultimately reducing profits for all involved.

The dilemma is also evident in sports, where athletes may resort to doping, driven by the fear that competitors are doing the same, leading to widespread use despite the risks. In communities, the provision of public goods is hindered by the fear that others may not contribute, resulting in underfunding and lower-quality services.

Hence, the above discussion on certain real-world instances, highlights how the Prisoner's Dilemma shapes critical decisions across various domains and underscores the importance of cooperative strategies in achieving optimal outcomes.

## Insights from the Prisoner's Dilemma Problem

We can certainly conclude that the Prisoner's Dilemma serves as a powerful illustration of the complex interplay between individual rationality and collective outcomes. It reveals that in many scenarios, the pursuit of self-interest, though seemingly logical, can lead to paradoxical and suboptimal results for all involved. This dilemma not only challenges our understanding of decision-making but also underscores the critical need for collaboration and trust in resolving conflicts. Ultimately, the Prisoner's Dilemma reminds us that true progress often lies not in individual gain, but in the pursuit of shared and cooperative solutions.

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