MATHEMATICS

SECTION A

January 30, 2024

1 Vector Algebra

1. Find the magnitude of each of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude such that the angle between them is $60°$ and their scalar product is $\frac{9}{2}$.
2. If θ is the angle between the two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$,find $\sin\theta$.
3. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\overrightarrow{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ and the plane $\overrightarrow{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 5$

2 Trignometry

4. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}\left(\sqrt{-3}\right)$.
5. Prove that: $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right), x \in \left(\frac{-1}{2}, \frac{1}{2}\right)$

3 Functions

6. If $a * b$ denotes the larger of 'a' and 'b' and if $aob = (a * b) + 3$, then write the value of $(5)\,o\,(10)$, where $*$ and o are binary operations.
7. Let $A = \{x \in Z : 0 \leq x \leq 12\}$. Show that $R = \{(a, b) : a, b \in A, \left|a - b\right| \text{ is divisible by } 4\}$ is equivalence relation. Find the set of all element s related to 1. Also write the equivalence class [2].
8. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2+1}, \forall \mathbf{x} \in \mathbb{R}$ is neither one-one nor onto. Also,if $g : \mathbb{R} \to \mathbb{R}$ is defined as $g(x) = 2x - 1$,find $fog(x)$.

4 Matrices

9. If the matrix $A = \begin{pmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{pmatrix}$ is skew symmetric, find the values of 'a' and 'b'.
10. Given $A = \begin{pmatrix} 2 & -3 \\ -4 & 7 \end{pmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.
11. Using properties of determinants,prove that

$$\begin{vmatrix} 1 & 1 & 1 + 3x \\ 1 + 3y & 1 & 1 \\ 1 & 1 + 3z & 1 \end{vmatrix} = 9\left(3xyz + xy + yz + zx\right)$$

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12. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, Fnd the A^{-1}.Use it to solve the system of equations 1

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x - 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

5 Probability

13. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

14. Two numbers are selected at random ($without replacement$) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mea n and variance of X.

15. Suppose a girl throws a die. If she gets 1 or 2,she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die ?

6 Differential Equation

16. Find the differential equation represnting the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

17. Find the particular solution of of the differential equation $e^x \tan y dx + (2 - e^x) \sec^2 y dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$.

18. Find the particular solution of the differential eqa tion $\frac{dy}{dx} + 2y \tan x = \sin x$,given that $y = 0$ when $x = \frac{\pi}{3}$.

7 Integration

19. Evaluate:

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$$

20. Find:

$$\int \frac{2\cos x}{(1 - \sin x)\left(1 + \sin^2 x\right)} d$$

21. Evaluate:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{16 + 9\sin 2x}$$

22. Evaluate:

$$\int_1^3 \left(x^2 + 3x + e^x\right) dx,$$

as the limit of the sum.

23. Using the integration, find the area of the region i n the first quadrant enclosed by the X-axis,the line $y = x$ and the circle $x^2 + y^2 = 32$.

8 Calculus

24. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the material cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

25. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a)strictly increasing,(b)strictly decreasing.

9 Differentiation

26. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

27. If $\left(x^2 + y^2\right)^2 = xy$, find $\frac{dy}{dx}$

28. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$

29. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$

10 Geometry

30. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ?

11 Vectors

31. Find the shortest distance between the lines $\overrightarrow{r} = \left(4\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\overrightarrow{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$.

32. Let $\overrightarrow{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\overrightarrow{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j} - \hat{k}$. find a vector $\overrightarrow{d}$ which is perpendicular to both $\overrightarrow{c}$ and $\overrightarrow{b}$ and $\overrightarrow{d} \cdot \overrightarrow{a} = 21$

12 Conics

33. Find the equations of the tangent and the normal,to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1),where $x_1 = 2$ and $y_1 > 0$

13 Linear Programming

34. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws $'A'$ while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufactures a packet of screws $'B'$. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws $'A'$ at a profit of 70 paise and screws $'B'$ at a profit of|1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit.