# MATHEMATICS

## SECTION A

January 26, 2024

1. Find the magnitude of each of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having yhe same magnitude such that the angle between them is $60°$ and their scalar product is $\frac{9}{2}$.
2. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}\left(\sqrt{-3}\right)$.
3. If $a * b$ denotes the larger of '$a$' and '$b$' and if $aob = (a * b) + 3$, then write the value of $(5)\, o\, (10)$, where $*$ and $o$ are binary operations.
4. If the matrix $A = \begin{pmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{pmatrix}$ is skew symmetric, find the values of '$a$' and '$b$'.
5. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.
6. If $\theta$ is the angle between the two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$,find $\sin\theta$.
7. Find the diffrential equation represnting the family of curves $y = ae^{bx+5}$, where $a$ and $b$ are arbitrary constants.
8. Evaluate:
$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$$
9. The total cost $C(x)$ associated with the production of $x$ units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the material cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
10. Differntiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to $x$.
11. Given $A = \begin{pmatrix} 2 & -3 \\ -4 & 7 \end{pmatrix}$, compute $A^{-1}$ and show that $2A^{-1} = 9I - A$.
12. Prove that: $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), x \in \left(\frac{-1}{2}, \frac{1}{2}\right)$
13. Two numbers are selected at random ($withoutreplacement$) from the first five positive integers. Let $X$ denote the larger of the two numbers obtained. Find the mean and variance of $X$.
14. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ?
15. Find the equations of the tangent and the normal,to the curve $16x^2 + 9y^2 = 145$ at the point $(x_1, y_1)$,where $x_1 = 2$ and $y_1 > 0$.
16. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is $(a)$strictly increasing,$(b)$strictly decreasing.

17. If $\left(x^2+y^2\right)^2 = xy$, find $\frac{dy}{dx}$

18. If $x = a\left(2\theta - \sin 2\theta\right)$ and $y = a\left(1 - \cos 2\theta\right)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$

19. If $y = \sin\left(\sin x\right)$, prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$

20. Find the particular solution of of the differential equation $e^x \tan y dx + \left(2 - e^x\right)\sec^2 y dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$.

21. Find the particular solution of the differential eqation $\frac{dy}{dx} + 2y\tan x = \sin x$,given that $y = 0$ when$x = \frac{\pi}{3}$.

22. Find the shortest distance between the lines $\overrightarrow{r} = \left(4\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\overrightarrow{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$.

23. Find:

$$\int \frac{2\cos x}{\left(1-\sin x\right)\left(1+\sin^2 x\right)}dx$$

24. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die ?

25. Let $\overrightarrow{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\overrightarrow{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j} - \hat{k}$. find a vector $\overrightarrow{d}$ which is perpendicular to both $\overrightarrow{c}$ and $\overrightarrow{b}$ and $\overrightarrow{d} \cdot \overrightarrow{a} = 21$

26. Using properties of determinants,prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9\left(3xyz + xy + yz + zx\right)$$

.

27. Using the integration, find the area of the region in the first quadrant enclosed by the X-axis,the line $y = x$and the circle $x^2 + y^2 = 32$.

28. Let $A = \{x \in Z : 0 \leq x \leq 12\}$. Show that $R = \{(a,b) : a,b \in A, \left|a-b\right|$ is divisible by 4$\}$ is equivalence relation. Find the set of all elememtsrelated to 1. Also write the equivalence class [2].

29. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f\left(x\right) = \frac{x}{x^2+1}, \forall \mathbf{x} \in \mathbb{R}$ is neither one-one nor onto. Also,if $g : \mathbb{R} \to \mathbb{R}$ is defined as $g\left(x\right) = 2x - 1$,find $fog\left(x\right)$.

30. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\overrightarrow{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ and the plane $\overrightarrow{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 5$.

31. A factory manufactures two types of screws $A$ and $B$, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws $'A'$ while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws $'B'$. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws $'A'$ at a profit of 70 paise and screws $'B'$ at a profit of|1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit.

32. Evaluate:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{16 + 9\sin 2x}$$

33. Evaluate:

$$\int_1^3 \left(x^2 + 3x + e^x\right) dx,$$

as the limit of the sum.

34. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, Fnd the $A^{-1}$.Use it ton solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x - 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$