GAMAT301 Assignment-2 Solutions

Question 1

Ten fair coins are thrown simultaneously. Find the probability of getting at least 7 heads?

Let X be the number of heads when 10 fair coins are thrown. This is a binomial distribution problem.

- Number of trials, $n = 10$.
- Probability of success (getting a head), $p = 0.5$ (since the coin is fair).
- The number of failures (getting a tail), $1 - p = 0.5$.

The probability mass function (PMF) for a binomial distribution is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

We need to find the probability of getting at least 7 heads, which means $P(X \geq 7)$.

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

Calculate each term:

For $P(X = 7)$:

$$P(X = 7) = \binom{10}{7}(0.5)^7(0.5)^{10-7} = \binom{10}{7}(0.5)^7(0.5)^3 = \frac{10!}{7!3!}(0.5)^{10}$$

$$P(X = 7) = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}(0.5)^{10} = 120 \times 0.0009765625 = 0.1171875$$

For $P(X = 8)$:

$$P(X = 8) = \binom{10}{8}(0.5)^8(0.5)^{10-8} = \binom{10}{8}(0.5)^8(0.5)^2 = \frac{10!}{8!2!}(0.5)^{10}$$

$$P(X = 8) = \frac{10 \times 9}{2 \times 1}(0.5)^{10} = 45 \times 0.0009765625 = 0.0439453125$$

For $P(X = 9)$:

$$P(X = 9) = \binom{10}{9}(0.5)^9(0.5)^{10-9} = \binom{10}{9}(0.5)^9(0.5)^1 = \frac{10!}{9!1!}(0.5)^{10}$$

$$P(X = 9) = 10 \times 0.0009765625 = 0.009765625$$

For $P(X = 10)$:

$$P(X = 10) = \binom{10}{10}(0.5)^{10}(0.5)^{10-10} = \binom{10}{10}(0.5)^{10}(0.5)^0 = 1 \times 0.0009765625 = 0.0009765625$$

Summing these probabilities:

$$P(X \geq 7) = 0.1171875 + 0.0439453125 + 0.009765625 + 0.0009765625$$

$$P(X \geq 7) = 0.171875$$

The probability of getting at least 7 heads is approximately 0.1719.

Question 2

A fair die is rolled 5 times. What is the probability of observing a 2 or 3 at least twice? What is the expected number of times the die shows 2 or 3?

Part 1: Probability of observing a 2 or 3 at least twice

Let X be the number of times a 2 or 3 is observed in 5 rolls. This is a binomial distribution problem.

- Number of trials, $n = 5$.
- A "success" is rolling a 2 or a 3. There are 2 favorable outcomes out of 6 possible outcomes (1, 2, 3, 4, 5, 6).
- Probability of success, $p = 2/6 = 1/3$.
- Probability of failure, $1 - p = 1 - 1/3 = 2/3$.

We need to find $P(X \geq 2)$. It's easier to calculate $1 - P(X < 2)$, which is $1 - (P(X = 0) + P(X = 1))$.

Using the binomial PMF $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$:

For $P(X = 0)$:

$$P(X = 0) = \binom{5}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^5 = 1 \times 1 \times \frac{2^5}{3^5} = \frac{32}{243}$$

For $P(X = 1)$:

$$P(X = 1) = \binom{5}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4 = 5 \times \frac{1}{3} \times \frac{16}{81} = \frac{80}{243}$$

Now, calculate $P(X \geq 2)$:

$$P(X \geq 2) = 1 - (P(X = 0) + P(X = 1))$$
$$P(X \geq 2) = 1 - \left(\frac{32}{243} + \frac{80}{243}\right)$$
$$P(X \geq 2) = 1 - \frac{112}{243}$$
$$P(X \geq 2) = \frac{243 - 112}{243} = \frac{131}{243}$$

As a decimal, $131/243 \approx 0.53909$.

Part 2: Expected number of times the die shows 2 or 3

For a binomial distribution, the expected value $E[X]$ is given by:

$$E[X] = np$$

Given $n = 5$ and $p = 1/3$:

$$E[X] = 5 \times \frac{1}{3} = \frac{5}{3}$$

The expected number of times the die shows 2 or 3 is $5/3 \approx 1.6667$.

Question 3

Four dice are rolled. What is the probability that at most one 6 appears?

Let X be the number of 6's observed when four dice are rolled. This is a binomial distribution problem.

- Number of trials, $n = 4$.
- A "success" is rolling a 6.
- Probability of success, $p = 1/6$.
- Probability of failure, $1 - p = 5/6$.

We need to find the probability that at most one 6 appears, which means $P(X \le 1)$.

$$P(X \le 1) = P(X = 0) + P(X = 1)$$

Using the binomial PMF $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$:

For $P(X = 0)$:

$$P(X = 0) = \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 = 1 \times 1 \times \frac{5^4}{6^4} = \frac{625}{1296}$$

For $P(X = 1)$:

$$P(X = 1) = \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 = 4 \times \frac{1}{6} \times \frac{125}{216} = \frac{4}{6} \times \frac{125}{216} = \frac{2}{3} \times \frac{125}{216} = \frac{250}{648} = \frac{125}{324}$$

To sum them, we need a common denominator, which is 1296:

$$P(X = 1) = \frac{125 \times 4}{324 \times 4} = \frac{500}{1296}$$

Now, sum the probabilities:

$$P(X \le 1) = P(X = 0) + P(X = 1) = \frac{625}{1296} + \frac{500}{1296} = \frac{1125}{1296}$$

As a decimal, $1125/1296 \approx 0.868055$.

Question 4

An urn contains 10 black and 8 white balls. Four balls are drawn from it and let X denote the number of black balls drawn. Find the probability distribution of X when the balls are drawn (i) with replacement (ii) without replacement.

Total number of balls in the urn, $N = 10 \text{ (black)} + 8 \text{ (white)} = 18$.

Number of black balls, $K = 10$.

Number of white balls, $N - K = 8$.

Number of balls drawn, $n = 4$.

X is the number of black balls drawn. X can take values $0, 1, 2, 3, 4$.

(i) With replacement

When balls are drawn with replacement, each draw is an independent Bernoulli trial. Thus, X follows a Binomial distribution.

- Number of trials, $n = 4$.
- Probability of success (drawing a black ball), $p = \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{10}{18} = \frac{5}{9}$.
- Probability of failure (drawing a white ball), $1 - p = 1 - \frac{5}{9} = \frac{4}{9}$.

The PMF is $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Let's calculate $P(X = k)$ for $k = 0, 1, 2, 3, 4$:

- $P(X = 0) = \binom{4}{0}\left(\frac{5}{9}\right)^0\left(\frac{4}{9}\right)^4 = 1 \times 1 \times \frac{4^4}{9^4} = \frac{256}{6561}$
- $P(X = 1) = \binom{4}{1}\left(\frac{5}{9}\right)^1\left(\frac{4}{9}\right)^3 = 4 \times \frac{5}{9} \times \frac{4^3}{9^3} = 4 \times \frac{5}{9} \times \frac{64}{729} = \frac{20 \times 64}{9 \times 729} = \frac{1280}{6561}$
- $P(X = 2) = \binom{4}{2}\left(\frac{5}{9}\right)^2\left(\frac{4}{9}\right)^2 = 6 \times \frac{5^2}{9^2} \times \frac{4^2}{9^2} = 6 \times \frac{25}{81} \times \frac{16}{81} = \frac{6 \times 25 \times 16}{6561} = \frac{2400}{6561}$
- $P(X = 3) = \binom{4}{3}\left(\frac{5}{9}\right)^3\left(\frac{4}{9}\right)^1 = 4 \times \frac{5^3}{9^3} \times \frac{4}{9} = 4 \times \frac{125}{729} \times \frac{4}{9} = \frac{4 \times 125 \times 4}{6561} = \frac{2000}{6561}$
- $P(X = 4) = \binom{4}{4}\left(\frac{5}{9}\right)^4\left(\frac{4}{9}\right)^0 = 1 \times \frac{5^4}{9^4} \times 1 = \frac{625}{6561}$

The probability distribution of X with replacement is:

k	$P(X = k)$	**Approximate Value**
0	$\frac{256}{6561}$	0.0390
1	$\frac{1280}{6561}$	0.1951
2	$\frac{2400}{6561}$	0.3658
3	$\frac{2000}{6561}$	0.3048
4	$\frac{625}{6561}$	0.0952

(ii) Without replacement

When balls are drawn without replacement, X follows a Hypergeometric distribution.

- Total number of balls, $N = 18$.
- Number of black balls, $K = 10$.
- Number of balls drawn, $n = 4$.

The PMF is given by:

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

First, calculate $\binom{N}{n} = \binom{18}{4}$:

$$\binom{18}{4} = \frac{18 \times 17 \times 16 \times 15}{4 \times 3 \times 2 \times 1} = 18 \times 17 \times \frac{16}{24} \times 15 = 18 \times 17 \times \frac{2}{3} \times 15 = 18 \times 17 \times 10 = 3060$$

Now, calculate $P(X = k)$ for $k = 0, 1, 2, 3, 4$:

- $P(X = 0) = \frac{\binom{10}{0}\binom{8}{4}}{\binom{18}{4}} = \frac{1 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}}{3060} = \frac{1 \times 70}{3060} = \frac{70}{3060} = \frac{7}{306}$
- $P(X = 1) = \frac{\binom{10}{1}\binom{8}{3}}{\binom{18}{4}} = \frac{10 \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}}{3060} = \frac{10 \times 56}{3060} = \frac{560}{3060} = \frac{56}{306} = \frac{28}{153}$

- $P(X=2) = \frac{\binom{10}{2}\binom{8}{2}}{\binom{18}{4}} = \frac{\frac{10\times9}{2\times1}\times\frac{8\times7}{2\times1}}{3060} = \frac{45\times28}{3060} = \frac{1260}{3060} = \frac{126}{306} = \frac{7}{17}$
- $P(X=3) = \frac{\binom{10}{3}\binom{8}{1}}{\binom{18}{4}} = \frac{\frac{10\times9\times8}{3\times2\times1}\times8}{3060} = \frac{120\times8}{3060} = \frac{960}{3060} = \frac{96}{306} = \frac{16}{51}$
- $P(X=4) = \frac{\binom{10}{4}\binom{8}{0}}{\binom{18}{4}} = \frac{\frac{10\times9\times8\times7}{4\times3\times2\times1}\times1}{3060} = \frac{210\times1}{3060} = \frac{210}{3060} = \frac{21}{306} = \frac{7}{102}$

The probability distribution of X without replacement is:

k	$P(X=k)$	**Approximate Value**
0	$\frac{70}{3060} = \frac{7}{306}$	0.0229
1	$\frac{560}{3060} = \frac{28}{153}$	0.1830
2	$\frac{1260}{3060} = \frac{7}{17}$	0.4118
3	$\frac{960}{3060} = \frac{16}{51}$	0.3137
4	$\frac{210}{3060} = \frac{7}{102}$	0.0686

Question 5

A binomial random variable has mean 12 and variance 3. Write down its pmf.

For a binomial random variable $X \sim B(n,p)$, the mean and variance are given by:

- Mean: $E[X] = np$
- Variance: $Var(X) = np(1-p)$

We are given:

- $np = 12$ (Equation 1)
- $np(1-p) = 3$ (Equation 2)

Substitute Equation 1 into Equation 2:

$$12(1-p) = 3$$

$$1-p = \frac{3}{12}$$

$$1-p = \frac{1}{4}$$

$$p = 1 - \frac{1}{4}$$

$$p = \frac{3}{4}$$

Now, substitute the value of p back into Equation 1 to find n:

$$n\left(\frac{3}{4}\right) = 12$$

$$n = 12 \times \frac{4}{3}$$

$$n = 4 \times 4$$

$$n = 16$$

So, the binomial random variable is $X \sim B(16, 3/4)$.

The probability mass function (PMF) is:

$$P(X=k) = \binom{16}{k}\left(\frac{3}{4}\right)^k\left(1-\frac{3}{4}\right)^{16-k}$$

$$P(X=k) = \binom{16}{k}\left(\frac{3}{4}\right)^k\left(\frac{1}{4}\right)^{16-k}$$

for $k = 0, 1, 2, \ldots, 16$.

Question 6

A multiple-choice exam has 6 problems, each with 3 alternatives to choose from. What is the probability that a candidate will get 4 or more correct answers by just guessing?

Let X be the number of correct answers by just guessing. This is a binomial distribution problem.

- Number of problems (trials), $n = 6$.
- Each problem has 3 alternatives, so the probability of guessing a correct answer for one problem is $p = 1/3$.
- Probability of guessing an incorrect answer, $1 - p = 2/3$.

We need to find the probability that a candidate will get 4 or more correct answers, i.e., $P(X \geq 4)$.

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$$

Using the binomial PMF $P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$:

For $P(X=4)$:

$$P(X=4) = \binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{6-4} = \frac{6 \times 5}{2 \times 1} \times \frac{1}{3^4} \times \frac{2^2}{3^2} = 15 \times \frac{1}{81} \times \frac{4}{9} = \frac{15 \times 4}{729} = \frac{60}{729}$$

For $P(X=5)$:

$$P(X=5) = \binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^{6-5} = 6 \times \frac{1}{3^5} \times \frac{2^1}{3^1} = 6 \times \frac{1}{243} \times \frac{2}{3} = \frac{12}{729}$$

For $P(X=6)$:

$$P(X=6) = \binom{6}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^{6-6} = 1 \times \frac{1}{3^6} \times 1 = \frac{1}{729}$$

Summing these probabilities:

$$P(X \geq 4) = \frac{60}{729} + \frac{12}{729} + \frac{1}{729} = \frac{60+12+1}{729} = \frac{73}{729}$$

As a decimal, $73/729 \approx 0.100137$.

Question 7

The mean and variance of a binomial random variable X are 16 and 8 respectively. Find P(X=0) and P(X=1).

For a binomial random variable $X \sim B(n, p)$, we have:

- Mean: $E[X] = np = 16$ (Equation 1)
- Variance: $Var(X) = np(1-p) = 8$ (Equation 2)

Substitute Equation 1 into Equation 2:

$$16(1-p) = 8$$

$$1 - p = \frac{8}{16}$$

$$1 - p = \frac{1}{2}$$

$$p = 1 - \frac{1}{2}$$

$$p = \frac{1}{2}$$

Now, substitute the value of p back into Equation 1 to find n:

$$n\left(\frac{1}{2}\right) = 16$$

$$n = 16 \times 2$$

$$n = 32$$

So, the binomial random variable is $X \sim B(32, 1/2)$.

The probability mass function (PMF) is $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

We need to find $P(X = 0)$ and $P(X = 1)$.

For $P(X = 0)$:

$$P(X = 0) = \binom{32}{0} \left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^{32-0}$$

$$P(X = 0) = 1 \times 1 \times \left(\frac{1}{2}\right)^{32}$$

$$P(X = 0) = \frac{1}{2^{32}}$$

$$P(X = 0) = \frac{1}{4,294,967,296} \approx 2.328 \times 10^{-10}$$

For $P(X = 1)$:

$$P(X = 1) = \binom{32}{1} \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{32-1}$$

$$P(X = 1) = 32 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^{31}$$

$$P(X = 1) = 32 \times \left(\frac{1}{2}\right)^{32}$$

$$P(X = 1) = \frac{32}{2^{32}} = \frac{2^5}{2^{32}} = \frac{1}{2^{27}}$$

$$P(X=1) = \frac{1}{134,217,728} \approx 7.451 \times 10^{-9}$$

Question 8

In a binomial distribution consisting of 6 independent trials, probabilities of 1 and 2 successes are 0.28336 and 0.0506 respectively. Find the mean and variance of the distribution.

Let X be the number of successes in n trials. This is a binomial distribution $X \sim B(n,p)$.

- Number of trials, $n = 6$.
- We need to find p.

Given probabilities:

- $P(X=1) = \binom{6}{1}p^1(1-p)^{6-1} = 6p(1-p)^5 = 0.28336$ (Equation 1)
- $P(X=2) = \binom{6}{2}p^2(1-p)^{6-2} = 15p^2(1-p)^4 = 0.0506$ (Equation 2)

To find p, divide Equation 2 by Equation 1 (assuming $p \neq 0$ and $1-p \neq 0$):

$$\frac{P(X=2)}{P(X=1)} = \frac{15p^2(1-p)^4}{6p(1-p)^5} = \frac{0.0506}{0.28336}$$

$$\frac{15p}{6(1-p)} = \frac{0.0506}{0.28336}$$

$$\frac{5p}{2(1-p)} = 0.17857354969649986$$

Let's use the fraction $0.0506/0.28336$ directly to avoid rounding errors:

$$5p = 2(1-p) \times \frac{0.0506}{0.28336}$$

$$5p = \frac{0.1012}{0.28336}(1-p)$$

$$5p = 0.3571470993929997(1-p)$$

$$5p = 0.3571470993929997 - 0.3571470993929997p$$

$$5p + 0.3571470993929997p = 0.3571470993929997$$

$$5.3571470993929997p = 0.3571470993929997$$

$$p = \frac{0.3571470993929997}{5.3571470993929997}$$

$$p = 0.06666666666666667 \approx \frac{1}{15}$$

Let's verify $p = 1/15$ with $P(X=1) = 6p(1-p)^5$:

$$P(X=1) = 6\left(\frac{1}{15}\right)\left(1-\frac{1}{15}\right)^5 = \frac{6}{15}\left(\frac{14}{15}\right)^5 = \frac{2}{5} \times \frac{14^5}{15^5}$$

$$P(X=1) = 0.4 \times \frac{537824}{759375} = 0.4 \times 0.708256 = 0.2833024$$

This is very close to 0.28336, so $p = 1/15$ is a reasonable approximation for the true probability p. For practical purposes, $p = 1/15$ will be used.

Now we find the mean and variance using $n = 6$ and $p = 1/15$.

- Mean: $E[X] = np$
- Variance: $Var(X) = np(1-p)$

Calculate Mean:

$$E[X] = 6 \times \frac{1}{15} = \frac{6}{15} = \frac{2}{5} = 0.4$$

Calculate Variance:

$$Var(X) = 6 \times \frac{1}{15} \times \left(1 - \frac{1}{15}\right) = \frac{2}{5} \times \frac{14}{15} = \frac{28}{75}$$

As a decimal, $28/75 \approx 0.3733$.

Question 9

A block of 100 bits is to be transmitted over a binary channel with a bit-error probability 0.001. What is the probability that 3 or more bits are received in error?

Let X be the number of bits received in error. This is a binomial distribution problem.

- Number of bits (trials), $n = 100$.
- Probability of a bit error (success), $p = 0.001$.
- Probability of no bit error (failure), $1 - p = 0.999$.

We need to find the probability that 3 or more bits are received in error, i.e., $P(X \geq 3)$.

It's easier to calculate $1 - P(X < 3)$, which is $1 - (P(X = 0) + P(X = 1) + P(X = 2))$.

Using the binomial PMF $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$:

For $P(X = 0)$:

$$P(X = 0) = \binom{100}{0} (0.001)^0 (0.999)^{100} = 1 \times 1 \times (0.999)^{100}$$

$$(0.999)^{100} \approx 0.904792$$

For $P(X = 1)$:

$$P(X = 1) = \binom{100}{1} (0.001)^1 (0.999)^{99} = 100 \times 0.001 \times (0.999)^{99}$$

$$P(X = 1) = 0.1 \times (0.999)^{99}$$

$$(0.999)^{99} = (0.999)^{100}/0.999 \approx 0.904792/0.999 \approx 0.905697$$

$$P(X = 1) \approx 0.1 \times 0.905697 = 0.0905697$$

For $P(X = 2)$:

$$P(X = 2) = \binom{100}{2} (0.001)^2 (0.999)^{98} = \frac{100 \times 99}{2 \times 1} \times (0.001)^2 \times (0.999)^{98}$$

$$P(X = 2) = 4950 \times 0.000001 \times (0.999)^{98}$$

$$(0.999)^{98} = (0.999)^{100}/(0.999)^2 \approx 0.904792/0.998001 \approx 0.906606$$

$$P(X=2) \approx 4950 \times 0.000001 \times 0.906606 = 0.00495 \times 0.906606 \approx 0.0044877$$

Now, calculate $P(X \geq 3)$:

$$P(X \geq 3) = 1 - (P(X=0) + P(X=1) + P(X=2))$$

$$P(X \geq 3) \approx 1 - (0.904792 + 0.0905697 + 0.0044877)$$

$$P(X \geq 3) \approx 1 - 0.9998494$$

$$P(X \geq 3) \approx 0.0001506$$

Alternatively, using Poisson approximation for $n = 100, p = 0.001$. $\lambda = np = 100 \times 0.001 = 0.1$.

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$P(X=0) = \frac{e^{-0.1}(0.1)^0}{0!} = e^{-0.1} \approx 0.904837$$

$$P(X=1) = \frac{e^{-0.1}(0.1)^1}{1!} = 0.1e^{-0.1} \approx 0.1 \times 0.904837 = 0.0904837$$

$$P(X=2) = \frac{e^{-0.1}(0.1)^2}{2!} = \frac{0.01}{2}e^{-0.1} = 0.005e^{-0.1} \approx 0.005 \times 0.904837 = 0.004524185$$

$$P(X \geq 3) = 1 - (0.904837 + 0.0904837 + 0.004524185) = 1 - 0.999844885 \approx 0.000155115$$

The problem asks for the probability using the binomial distribution. The values are very close. Using precise binomial calculations: $P(X=0) = (0.999)^{100} \approx 0.904792138$
$P(X=1) = 100 \times 0.001 \times (0.999)^{99} \approx 0.1 \times 0.905696535 \approx 0.090569654$
$P(X=2) = 4950 \times (0.001)^2 \times (0.999)^{98} \approx 0.00495 \times 0.906601837 \approx 0.004487739$
$P(X \geq 3) = 1 - (0.904792138 + 0.090569654 + 0.004487739) = 1 - 0.999849531 \approx 0.000150469$

Question 10

Six fair dice are thrown 729 times. How many times do you expect at least three dice to show 1 or 2?

This problem involves two stages:

1. Calculate the probability of the event "at least three dice show 1 or 2" in a single throw of six dice.
2. Calculate the expected number of times this event occurs over 729 throws.

Stage 1: Probability for a single throw of six dice

Let Y be the number of dice that show 1 or 2 when six fair dice are thrown. This is a binomial distribution.

- Number of trials (dice), $n = 6$.
- A "success" is a die showing 1 or 2. There are 2 favorable outcomes out of 6.
- Probability of success, $p = 2/6 = 1/3$.
- Probability of failure, $1 - p = 2/3$.

We need to find $P(Y \geq 3)$.

$$P(Y \geq 3) = P(Y=3) + P(Y=4) + P(Y=5) + P(Y=6)$$

Using the binomial PMF $P(Y=k) = \binom{n}{k}p^k(1-p)^{n-k}$:

For $P(Y=3)$:

$$P(Y=3) = \binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 = 20 \times \frac{1}{27} \times \frac{8}{27} = \frac{160}{729}$$

For $P(Y=4)$:

$$P(Y=4) = \binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = 15 \times \frac{1}{81} \times \frac{4}{9} = \frac{60}{729}$$

For $P(Y=5)$:

$$P(Y=5) = \binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1 = 6 \times \frac{1}{243} \times \frac{2}{3} = \frac{12}{729}$$

For $P(Y=6)$:

$$P(Y=6) = \binom{6}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0 = 1 \times \frac{1}{729} \times 1 = \frac{1}{729}$$

Summing these probabilities to get P_{event}:

$$P_{event} = P(Y \geq 3) = \frac{160}{729} + \frac{60}{729} + \frac{12}{729} + \frac{1}{729} = \frac{160+60+12+1}{729} = \frac{233}{729}$$

Stage 2: Expected number of times the event occurs over 729 throws

The number of times the dice are thrown is $N_{throws} = 729$.

The probability of the event in a single throw is $P_{event} = \frac{233}{729}$.

The expected number of times an event occurs in a series of independent trials is $N_{throws} \times P_{event}$.

$$E[\text{Number of events}] = 729 \times \frac{233}{729} = 233$$

You expect at least three dice to show 1 or 2 exactly 233 times.

Question 11

An expert shot hits a target 95% of the time. If the expert shoots 15 times in succession, find the probability that (i) he misses the target only once (ii) he misses the target only at the last shot.

Let H denote a hit and M denote a miss. We have:

- Probability of a hit, $p = 0.95$.
- Probability of a miss, $1 - p = 0.05$.
- Number of shots, $n = 15$.

(i) He misses the target only once

Let X be the number of misses in 15 shots. This follows a binomial distribution $X \sim B(n, 1-p)$.

- Number of trials, $n = 15$.
- Probability of success (a miss), $p_m = 0.05$.
- Probability of failure (a hit), $p_h = 0.95$.

We need to find $P(X = 1)$.

$$P(X=1) = \binom{15}{1} (0.05)^1 (0.95)^{15-1}$$

$$P(X=1) = 15 \times 0.05 \times (0.95)^{14}$$

$$P(X=1) = 0.75 \times (0.95)^{14}$$

$$(0.95)^{14} \approx 0.48767$$

$$P(X=1) \approx 0.75 \times 0.48767 = 0.3657525$$

The probability that he misses the target only once is approximately 0.3658.

(ii) He misses the target only at the last shot

This describes a specific sequence of events: the first 14 shots are hits, and the 15th shot is a miss.

The shots are independent, so the probability of this specific sequence is the product of individual probabilities:

$$P(\text{H, H, ..., H, M}) = P(H \text{ on 1st shot}) \times \ldots \times P(H \text{ on 14th shot}) \times P(M \text{ on 15th shot})$$

$$P(\text{14 hits, then 1 miss}) = (0.95)^{14} \times (0.05)^1$$

$$P(\text{14 hits, then 1 miss}) = (0.95)^{14} \times 0.05$$

$$(0.95)^{14} \approx 0.48767$$

$$P(\text{14 hits, then 1 miss}) \approx 0.48767 \times 0.05 = 0.0243835$$

The probability that he misses the target only at the last shot is approximately 0.0244.

Question 12

A boy throws stones at a target and the probability of hitting the target is $2/3$. What is the probability that his 10th throw is the fifth hit?

This is a negative binomial distribution problem or a sequence of Bernoulli trials.

- Probability of hitting the target (success), $p = 2/3$.
- Probability of missing the target (failure), $1 - p = 1/3$.

We want the 5th hit to occur on the 10th throw. This means:

1. Among the first 9 throws, there must be exactly $5 - 1 = 4$ hits.
2. The 10th throw must be a hit.

Let Y be the number of hits in the first 9 throws. This follows a binomial distribution $Y \sim B(9, 2/3)$.

The probability of 4 hits in the first 9 throws is $P(Y=4)$:

$$P(Y=4) = \binom{9}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{9-4}$$

$$P(Y=4) = \binom{9}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^5$$

$$\binom{9}{4} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 9 \times 2 \times 7 = 126$$

$$P(Y=4) = 126 \times \frac{2^4}{3^4} \times \frac{1^5}{3^5} = 126 \times \frac{16}{81} \times \frac{1}{243}$$

$$P(Y=4) = \frac{126 \times 16}{81 \times 243} = \frac{2016}{19683}$$

Now, the probability that the 10th throw is a hit is $p = 2/3$.

The probability that his 10th throw is the fifth hit is the product of these two probabilities:

$$P(\text{5th hit on 10th throw}) = P(Y=4) \times p$$

$$P(\text{5th hit on 10th throw}) = \frac{2016}{19683} \times \frac{2}{3}$$

$$P(\text{5th hit on 10th throw}) = \frac{4032}{59049}$$

We can simplify this fraction by dividing by common factors. $4032 = 16 \times 252 = 16 \times 4 \times 63 = 64 \times 9 \times 7. 59049 = 3^{10}$. $4032 = 2^6 \times 3^2 \times 7. 59049 = 3^{10}$.

$$P(\text{5th hit on 10th throw}) = \frac{2^6 \times 3^2 \times 7}{3^{10}} = \frac{2^6 \times 7}{3^8} = \frac{64 \times 7}{6561} = \frac{448}{6561}$$

As a decimal, $448/6561 \approx 0.06828$.

Question 13

The probability that an electric component manufactured by a firm is defective is 0.01. The produced items are sent to the market in packets of 10. In a consignment of 1000 such packets how many can be expected to contain (i) exactly two defectives (ii) at most two defectives?

Let X be the number of defective components in a single packet of 10 items. This is a binomial distribution.

- Number of items in a packet (trials), $n = 10$.
- Probability that a component is defective (success), $p = 0.01$.
- Probability that a component is not defective (failure), $1 - p = 0.99$.

There are 1000 packets in the consignment.

(i) Expected number of packets with exactly two defectives

First, calculate the probability that a single packet has exactly two defectives, $P(X=2)$:

$$P(X=2) = \binom{10}{2} (0.01)^2 (0.99)^{10-2}$$

$$P(X=2) = \binom{10}{2} (0.01)^2 (0.99)^8$$

$$\binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45$$

$$P(X=2) = 45 \times (0.01)^2 \times (0.99)^8$$

$$P(X=2) = 45 \times 0.0001 \times (0.99)^8$$

$$(0.99)^8 \approx 0.92274469$$

$$P(X=2) \approx 45 \times 0.0001 \times 0.92274469 \approx 0.004152351$$

The expected number of packets with exactly two defectives in 1000 packets is:

$$E[\text{Packets with 2 defectives}] = 1000 \times P(X = 2)$$

$$E[\text{Packets with 2 defectives}] \approx 1000 \times 0.004152351 \approx 4.152$$

So, approximately 4 packets are expected to contain exactly two defectives.

(ii) Expected number of packets with at most two defectives

First, calculate the probability that a single packet has at most two defectives, $P(X \leq 2)$:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

We already calculated $P(X = 2) \approx 0.004152351$.

For $P(X = 0)$:

$$P(X = 0) = \binom{10}{0}(0.01)^0(0.99)^{10} = 1 \times 1 \times (0.99)^{10}$$

$$(0.99)^{10} \approx 0.904382075$$

For $P(X = 1)$:

$$P(X = 1) = \binom{10}{1}(0.01)^1(0.99)^9 = 10 \times 0.01 \times (0.99)^9$$

$$P(X = 1) = 0.1 \times (0.99)^9$$

$$(0.99)^9 = (0.99)^{10}/0.99 \approx 0.904382075/0.99 \approx 0.913517247$$

$$P(X = 1) \approx 0.1 \times 0.913517247 \approx 0.091351725$$

Summing these probabilities:

$$P(X \leq 2) \approx 0.904382075 + 0.091351725 + 0.004152351$$

$$P(X \leq 2) \approx 0.999886151$$

The expected number of packets with at most two defectives in 1000 packets is:

$$E[\text{Packets with at most 2 defectives}] = 1000 \times P(X \leq 2)$$

$$E[\text{Packets with at most 2 defectives}] \approx 1000 \times 0.999886151 \approx 999.886$$

So, approximately 1000 packets are expected to contain at most two defectives (which makes sense since the defect rate is very low).

Question 14

The probability that a component is acceptable is 0.93. Ten components are picked at random. What is the probability that (i) at least 9 are acceptable (ii) at most 3 are acceptable?

Let X be the number of acceptable components out of 10. This is a binomial distribution.

- Number of components (trials), $n = 10$.
- Probability that a component is acceptable (success), $p = 0.93$.
- Probability that a component is not acceptable (failure), $1 - p = 0.07$.

(i) At least 9 are acceptable

We need to find $P(X \geq 9)$.

$$P(X \geq 9) = P(X = 9) + P(X = 10)$$

Using the binomial PMF $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$:

For $P(X = 9)$:

$$P(X = 9) = \binom{10}{9} (0.93)^9 (0.07)^{10-9} = 10 \times (0.93)^9 \times (0.07)^1$$

$$(0.93)^9 \approx 0.52103447$$

$$P(X = 9) \approx 10 \times 0.52103447 \times 0.07 \approx 0.364724129$$

For $P(X = 10)$:

$$P(X = 10) = \binom{10}{10} (0.93)^{10} (0.07)^{10-10} = 1 \times (0.93)^{10} \times 1$$

$$(0.93)^{10} \approx 0.48398205$$

$$P(X = 10) \approx 0.48398205$$

Summing these probabilities:

$$P(X \geq 9) \approx 0.364724129 + 0.48398205$$

$$P(X \geq 9) \approx 0.848706179$$

The probability that at least 9 components are acceptable is approximately 0.8487.

(ii) At most 3 are acceptable

We need to find $P(X \leq 3)$.

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

Given that $p = 0.93$ (acceptable) and $1 - p = 0.07$ (not acceptable), we can also define Y as the number of not acceptable components. $Y \sim B(10, 0.07)$.

If $X \leq 3$ (at most 3 acceptable), then $Y \geq 7$ (at least 7 not acceptable).

Let's calculate directly $P(X \leq 3)$:

For $P(X = 0)$:

$$P(X = 0) = \binom{10}{0} (0.93)^0 (0.07)^{10} = (0.07)^{10} \approx 2.82475249 \times 10^{-12}$$

For $P(X = 1)$:

$$P(X = 1) = \binom{10}{1} (0.93)^1 (0.07)^9 = 10 \times 0.93 \times (0.07)^9$$

$$(0.07)^9 \approx 4.0353607 \times 10^{-11}$$

$$P(X = 1) \approx 10 \times 0.93 \times 4.0353607 \times 10^{-11} \approx 3.75288545 \times 10^{-10}$$

For $P(X = 2)$:

$$P(X=2) = \binom{10}{2}(0.93)^2(0.07)^8 = 45 \times (0.93)^2 \times (0.07)^8$$

$$(0.93)^2 = 0.8649$$

$$(0.07)^8 \approx 5.764801 \times 10^{-10}$$

$$P(X=2) \approx 45 \times 0.8649 \times 5.764801 \times 10^{-10} \approx 2.247657 \times 10^{-8}$$

For $P(X=3)$:

$$P(X=3) = \binom{10}{3}(0.93)^3(0.07)^7 = 120 \times (0.93)^3 \times (0.07)^7$$

$$(0.93)^3 \approx 0.804357$$

$$(0.07)^7 \approx 8.23543 \times 10^{-9}$$

$$P(X=3) \approx 120 \times 0.804357 \times 8.23543 \times 10^{-9} \approx 7.9405 \times 10^{-7}$$

Summing these very small probabilities:

$$P(X \le 3) \approx 2.82 \times 10^{-12} + 3.75 \times 10^{-10} + 2.25 \times 10^{-8} + 7.94 \times 10^{-7}$$

$$P(X \le 3) \approx 0.00000000282 + 0.000000375 + 0.0000225 + 0.000794$$

$$P(X \le 3) \approx 0.00081687782$$

The probability that at most 3 components are acceptable is extremely small, approximately 0.000817. This result is expected because the probability of an item being acceptable is very high (0.93), so getting only 3 or fewer acceptable items out of 10 is very unlikely.

Question 15

In an examination a candidate has to answer 15 multiple choice questions, each of which has 4 choices for the answer. He knows the correct answer to 10 of these questions and for the remaining 5 questions he chooses an answer randomly. Let X be the total number of correct answers he gives.

(a) Find the pmf of X

(b) What is the probability that he answers 13 or more questions correctly?

(c) What are the mean and variance of the number of correct answers he gives?

The candidate answers 15 multiple choice questions. Each question has 4 choices, so the probability of guessing correctly is $1/4 = 0.25$.

The total number of correct answers X can be broken down into two parts:

1. Answers to known questions: He knows the correct answer to 10 questions, so these 10 are always correct.
2. Answers to guessed questions: For the remaining $15 - 10 = 5$ questions, he guesses randomly.

Let Y be the number of correct answers from the 5 guessed questions. Y follows a binomial distribution.

- Number of guessed questions (trials), $n_g = 5$.
- Probability of guessing correctly, $p_g = 1/4 = 0.25$.
- Probability of guessing incorrectly, $1 - p_g = 3/4 = 0.75$.

So, $Y \sim B(5, 0.25)$.

The total number of correct answers X is $X = 10 + Y$.

(a) Find the pmf of X

Since Y can take values $0, 1, 2, 3, 4, 5$, X can take values $10 + 0 = 10, 10 + 1 = 11, \dots, 10 + 5 = 15$.

The PMF of X is given by $P(X = k) = P(10 + Y = k) = P(Y = k - 10)$ for $k \in \{10, 11, 12, 13, 14, 15\}$.

Using the binomial PMF for Y: $P(Y = j) = \binom{5}{j}(0.25)^j(0.75)^{5-j}$.

- For $k = 10$ ($j = 0$): $P(X = 10) = P(Y = 0) = \binom{5}{0}(0.25)^0(0.75)^5 = 1 \times 1 \times (0.75)^5 = 0.2373046875$
- For $k = 11$ ($j = 1$): $P(X = 11) = P(Y = 1) = \binom{5}{1}(0.25)^1(0.75)^4 = 5 \times 0.25 \times (0.75)^4 = 5 \times 0.25 \times 0.31640625 = 0.3955078125$
- For $k = 12$ ($j = 2$): $P(X = 12) = P(Y = 2) = \binom{5}{2}(0.25)^2(0.75)^3 = 10 \times (0.25)^2 \times (0.75)^3 = 10 \times 0.0625 \times 0.421875 = 0.263671875$
- For $k = 13$ ($j = 3$): $P(X = 13) = P(Y = 3) = \binom{5}{3}(0.25)^3(0.75)^2 = 10 \times (0.25)^3 \times (0.75)^2 = 10 \times 0.015625 \times 0.5625 = 0.087890625$
- For $k = 14$ ($j = 4$): $P(X = 14) = P(Y = 4) = \binom{5}{4}(0.25)^4(0.75)^1 = 5 \times (0.25)^4 \times 0.75 = 5 \times 0.00390625 \times 0.75 = 0.0146484375$
- For $k = 15$ ($j = 5$): $P(X = 15) = P(Y = 5) = \binom{5}{5}(0.25)^5(0.75)^0 = 1 \times (0.25)^5 \times 1 = 0.0009765625$

The PMF of X is:

k	$P(X = k)$
10	0.2373046875
11	0.3955078125
12	0.263671875
13	0.087890625
14	0.0146484375
15	0.0009765625

(b) What is the probability that he answers 13 or more questions correctly?

We need to find $P(X \ge 13)$. This means $P(Y \ge 3)$.

$$P(X \ge 13) = P(X = 13) + P(X = 14) + P(X = 15)$$

$$P(X \ge 13) = P(Y = 3) + P(Y = 4) + P(Y = 5)$$

$$P(X \ge 13) = 0.087890625 + 0.0146484375 + 0.0009765625$$

$$P(X \ge 13) = 0.103515625$$

The probability is approximately 0.1035.

(c) What are the mean and variance of the number of correct answers he gives?

For the binomial variable $Y \sim B(5, 0.25)$:

- Mean of Y: $E[Y] = n_g p_g = 5 \times 0.25 = 1.25$.
- Variance of Y: $Var(Y) = n_g p_g(1 - p_g) = 5 \times 0.25 \times (1 - 0.25) = 5 \times 0.25 \times 0.75 = 0.9375$.

Now, for the total number of correct answers $X = 10 + Y$:

Mean of X:

$$E[X] = E[10 + Y] = E[10] + E[Y] = 10 + 1.25 = 11.25$$

Variance of X:

$$Var(X) = Var(10 + Y) = Var(Y)$$

(Since adding a constant does not change the variance.)

$$Var(X) = 0.9375$$

Question 16

A firm sells four items randomly selected from a large lot that is known to contain 12% defectives. Find the probability that among the four items sold (i) none is defective (ii) at most one is defective.

Suppose that the items are sold for Rs. 500 each. The firm has a policy of buying back defective items at double the original price of the item. Find the probability that the net gain of the firm in selling the four items is at least Rs. 1000. What is the expected net gain of the firm?

Let D be the number of defective items among the four items sold. Since the items are selected from a large lot, we can model this as a binomial distribution.

- Number of items sold (trials), $n = 4$.
- Probability of an item being defective (success), $p = 0.12$.
- Probability of an item being non-defective, $1 - p = 0.88$.

So, $D \sim B(4, 0.12)$.

(i) None is defective

We need to find $P(D = 0)$.

$$P(D = 0) = \binom{4}{0}(0.12)^0(0.88)^{4-0}$$

$$P(D = 0) = 1 \times 1 \times (0.88)^4$$

$$(0.88)^4 \approx 0.59969536$$

The probability that none of the four items is defective is approximately 0.5997.

(ii) At most one is defective

We need to find $P(D \leq 1)$.

$$P(D \leq 1) = P(D = 0) + P(D = 1)$$

We already have $P(D = 0) \approx 0.59969536$.

For $P(D = 1)$:

$$P(D = 1) = \binom{4}{1}(0.12)^1(0.88)^{4-1}$$

$$P(D = 1) = 4 \times 0.12 \times (0.88)^3$$

$$(0.88)^3 \approx 0.681472$$

$$P(D = 1) \approx 4 \times 0.12 \times 0.681472 = 0.48 \times 0.681472 \approx 0.32710656$$

Summing these probabilities:

$$P(D \leq 1) \approx 0.59969536 + 0.32710656 = 0.92680192$$

The probability that at most one item is defective is approximately 0.9268.

Probability that the net gain of the firm in selling the four items is at least Rs. 1000

Each item is sold for Rs. 500. Total revenue from selling 4 items is $4 \times 500 = 2000$ Rs.

Defective items are bought back at double the original price, i.e., $2 \times 500 = 1000$ Rs per defective item.

Let G be the net gain. If D is the number of defective items, the buy-back cost is $1000D$.

$$G = 2000 - 1000D$$

We want to find $P(G \geq 1000)$:

$$2000 - 1000D \geq 1000$$
$$1000 \geq 1000D$$
$$1 \geq D$$

So, we need to find $P(D \leq 1)$. This is the same probability calculated in part (ii).

$$P(G \geq 1000) = P(D \leq 1) \approx 0.9268$$

The probability that the net gain is at least Rs. 1000 is approximately 0.9268.

Expected net gain of the firm

The expected value of the number of defective items is $E[D] = np = 4 \times 0.12 = 0.48$.

The expected net gain is $E[G] = E[2000 - 1000D]$.

$$E[G] = E[2000] - E[1000D]$$
$$E[G] = 2000 - 1000E[D]$$
$$E[G] = 2000 - 1000 \times 0.48$$
$$E[G] = 2000 - 480$$
$$E[G] = 1520$$

The expected net gain of the firm is Rs. 1520.

Question 17

The management of a hospital plans to install a sufficient number of independent but identical backup generators to operate critical systems should electricity go out. The probability that each of the selected brand of generator would work correctly when called upon is 0.80. Find the minimum number of generators that the hospital should install to ensure that the probability is not less than 0.99 that at least one of the generators will work correctly in an emergency.

Let N be the number of generators installed. Let X be the number of generators that work correctly.

- Probability that a single generator works correctly, $p = 0.80$.
- Probability that a single generator fails, $1 - p = 0.20$.

The number of working generators X follows a binomial distribution $X \sim B(N, 0.80)$.

We want to find the minimum N such that the probability that at least one generator works is not less than 0.99.

$$P(X \geq 1) \geq 0.99$$

It's easier to work with the complement probability:

$$1 - P(X < 1) \geq 0.99$$
$$1 - P(X = 0) \geq 0.99$$
$$P(X = 0) \leq 1 - 0.99$$
$$P(X = 0) \leq 0.01$$

The probability that zero generators work correctly is given by the binomial PMF for $k = 0$:

$$P(X = 0) = \binom{N}{0}(0.80)^0(0.20)^{N-0}$$
$$P(X = 0) = 1 \times 1 \times (0.20)^N$$
$$P(X = 0) = (0.20)^N$$

So, we need to solve the inequality:

$$(0.20)^N \leq 0.01$$

To solve for N, take the logarithm of both sides (using base 10 or natural logarithm):

$$N \log(0.20) \leq \log(0.01)$$

Note that $\log(0.20)$ is a negative value. When dividing by a negative number, we must reverse the inequality sign.

$$\log(0.20) \approx -0.69897$$
$$\log(0.01) = -2$$
$$N \geq \frac{\log(0.01)}{\log(0.20)}$$
$$N \geq \frac{-2}{-0.69897}$$
$$N \geq 2.86135$$

Since N must be an integer (number of generators), and it must be at least 2.86135, the minimum integer value for N is 3.

Let's check this:

- If $N = 2$: $P(X \geq 1) = 1 - (0.20)^2 = 1 - 0.04 = 0.96$ (which is less than 0.99).
- If $N = 3$: $P(X \geq 1) = 1 - (0.20)^3 = 1 - 0.008 = 0.992$ (which is greater than or equal to 0.99).

The minimum number of generators that the hospital should install is 3.

Question 18

The probability that each engine in an airplane will work without failure while in flight is p, independent of other engines. The airplane will have a successful flight if at least 50% of its engines remain operational. For what values of p is a four-engine plane preferable to a two-engine plane?

Let p be the probability that a single engine works correctly. The engines work independently.

Two-engine plane

- Number of engines, $n_2 = 2$.
- Successful flight if at least 50% of engines work: $2 \times 0.5 = 1$ engine.
- Let X_2 be the number of working engines. $X_2 \sim B(2, p)$.
- Probability of success for a two-engine plane: $P(X_2 \geq 1) = 1 - P(X_2 = 0)$.
- $P(X_2 = 0) = \binom{2}{0} p^0 (1-p)^2 = (1-p)^2$.
- So, $P(\text{success, 2-engine}) = 1 - (1-p)^2$.

Four-engine plane

- Number of engines, $n_4 = 4$.
- Successful flight if at least 50% of engines work: $4 \times 0.5 = 2$ engines.
- Let X_4 be the number of working engines. $X_4 \sim B(4, p)$.
- Probability of success for a four-engine plane: $P(X_4 \geq 2) = 1 - (P(X_4 = 0) + P(X_4 = 1))$.
- $P(X_4 = 0) = \binom{4}{0} p^0 (1-p)^4 = (1-p)^4$.
- $P(X_4 = 1) = \binom{4}{1} p^1 (1-p)^3 = 4p(1-p)^3$.
- So, $P(\text{success, 4-engine}) = 1 - \left((1-p)^4 + 4p(1-p)^3\right)$.

Condition for four-engine plane to be preferable

A four-engine plane is preferable if its probability of success is greater than that of a two-engine plane:

$$P(\text{success, 4-engine}) > P(\text{success, 2-engine})$$

$$1 - \left((1-p)^4 + 4p(1-p)^3\right) > 1 - (1-p)^2$$

Subtract 1 from both sides and multiply by -1 (reversing the inequality sign):

$$(1-p)^4 + 4p(1-p)^3 < (1-p)^2$$

We consider the cases for p. As p is a probability, $0 \leq p \leq 1$.

Case 1: $p = 1$ (engines always work). $P(\text{success, 2-engine}) = 1 - (1-1)^2 = 1$.
$P(\text{success, 4-engine}) = 1 - ((1-1)^4 + 4(1)(1-1)^3) = 1$. They are equal, so 4-engine is not strictly preferable.

Case 2: $p = 0$ (engines never work). $P(\text{success, 2-engine}) = 1 - (1-0)^2 = 0$.
$P(\text{success, 4-engine}) = 1 - ((1-0)^4 + 4(0)(1-0)^3) = 1 - 1 = 0$. They are equal, so 4-engine is not strictly preferable.

Case 3: $0 < p < 1$ (so $1 - p \neq 0$). We can divide the inequality by $(1-p)^2$ without changing the sign:

$$\frac{(1-p)^4 + 4p(1-p)^3}{(1-p)^2} < 1$$

$$(1-p)^2 + 4p(1-p) < 1$$

Factor out $(1-p)$ from the left side:

$$(1-p)\left((1-p) + 4p\right) < 1$$

$$(1-p)(1+3p) < 1$$

Expand the left side:

$$1 + 3p - p - 3p^2 < 1$$

$$1 + 2p - 3p^2 < 1$$

Subtract 1 from both sides:

$$2p - 3p^2 < 0$$

Factor out p:

$$p(2-3p)<0$$

Since we assumed $0<p<1$, p is positive. Therefore, for the product $p(2-3p)$ to be negative, the term $(2-3p)$ must be negative:

$$2-3p<0$$

$$2<3p$$

$$p>\frac{2}{3}$$

Combining with $0<p<1$, the values of p for which a four-engine plane is preferable are $2/3<p<1$.