# GAMAT 301 Assignment-4 Solutions

## Question 1

To find the value of $c$ for which $f(x) = cxe^{-x}, 0 < x < \infty$, is a valid probability density function (pdf), we need to ensure that the integral of $f(x)$ over its domain is equal to 1.

$$\int_0^\infty cxe^{-x}dx = 1$$

$$c\int_0^\infty xe^{-x}dx = 1$$

We need to evaluate the integral $\int_0^\infty xe^{-x}dx$. This can be solved using integration by parts.

Let $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = -e^{-x}$.

$$\int xe^{-x}dx = -xe^{-x} - \int(-e^{-x})dx = -xe^{-x} - e^{-x}$$

Now, evaluate the definite integral:

$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}[-xe^{-x} - e^{-x}]_0^b = \lim_{b\to\infty}[(-be^{-b} - e^{-b}) - (0 - e^0)]$$

$$= \lim_{b\to\infty}(-be^{-b} - e^{-b}) + 1 = 0 + 1 = 1$$

Therefore, $c \cdot 1 = 1$, so $c = 1$.

## Question 2

To find the value of $b$ so that $f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le b \\ 0, & \text{otherwise} \end{cases}$ is a valid pdf, we need to ensure that the integral of $f(x)$ over its domain is equal to 1.

$$\int_{-\infty}^\infty f(x)dx = 1$$

$$\int_0^b 2xdx = 1$$

$$[x^2]_0^b = 1$$

$$b^2 - 0^2 = 1$$

$$b^2 = 1$$

Since $0 \le x \le b$, we take the positive root: $b = 1$.

Now, we find the CDF of $X$.

$$F(x) = \int_0^x 2tdt = [t^2]_0^x = x^2, \quad 0 \le x \le 1$$

So, the CDF is:

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

Now, we find $P(X \ge 0.5)$.

$$P(X \ge 0.5) = 1 - P(X < 0.5) = 1 - F(0.5) = 1 - (0.5)^2 = 1 - 0.25 = 0.75$$

## Question 3

Given the pdf $f(x) = \begin{cases} kx(1-x), & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) (i) To find the value of $k$, we need to ensure that $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\int_0^1 kx(1-x)dx = 1$$

$$k\int_0^1 (x-x^2)dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$k\left(\frac{3-2}{6}\right) = 1$$

$$k\left(\frac{1}{6}\right) = 1$$

$$k = 6$$

(ii) $P(X \le \frac{1}{2}) = \int_0^{\frac{1}{2}} 6x(1-x)dx = 6\int_0^{\frac{1}{2}}(x-x^2)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{1}{2}} = 6\left(\frac{1}{8} - \frac{1}{24}\right) = 6\left(\frac{3-1}{24}\right) = 6\left(\frac{2}{24}\right) = \frac{12}{24} = \frac{1}{2}$

(iii)

$P(\frac{1}{4} \le X \le \frac{3}{4}) = \int_{\frac{1}{4}}^{\frac{3}{4}} 6x(1-x)dx = 6\int_{\frac{1}{4}}^{\frac{3}{4}}(x-x^2)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{4}}^{\frac{3}{4}} = 6\left[\left(\frac{9}{32} - \frac{27}{192}\right) - \left(\frac{1}{32} - \frac{1}{192}\right)\right] = 6\left[\frac{9}{32} - \frac{27}{192} - \frac{1}{32} + \frac{1}{192}\right] = 6\left[\frac{8}{32} -\right.$

(b) The CDF of $X$ is $F(x) = \int_0^x 6t(1-t)dt = 6\int_0^x (t-t^2)dt = 6\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^x = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) = 3x^2 - 2x^3$ for $0 \le x \le 1$.

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 3x^2 - 2x^3, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

(c) The mean cable diameter is $E[X] = \int_0^1 x \cdot 6x(1-x)dx = 6\int_0^1 (x^2-x^3)dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = 6\left(\frac{4-3}{12}\right) = 6\left(\frac{1}{12}\right) = \frac{1}{2}$.

The mean cross-sectional area of the cable is $E[\pi(X/2)^2] = E[\frac{\pi}{4}X^2] = \frac{\pi}{4}E[X^2]$.

$E[X^2] = \int_0^1 x^2 \cdot 6x(1-x)dx = 6\int_0^1 (x^3-x^4)dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = 6\left(\frac{5-4}{20}\right) = 6\left(\frac{1}{20}\right) = \frac{3}{10}$.

$E[\pi(X/2)^2] = \frac{\pi}{4} \cdot \frac{3}{10} = \frac{3\pi}{40}$.

## Question 4

Given the density $f(x) = \begin{cases} \frac{3}{4}(1-x^2), & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(i) The mean of $X$ is $E[X] = \int_{-1}^1 x \cdot \frac{3}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^1 (x-x^3)dx = \frac{3}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^1 = \frac{3}{4}\left[\left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right)\right] = 0$.

The variance of $X$ is $Var(X) = E[X^2] - (E[X])^2 = E[X^2] - 0^2 = E[X^2]$.

$E[X^2] = \int_{-1}^1 x^2 \cdot \frac{3}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^1 (x^2-x^4)dx = \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^1 = \frac{3}{4}\left[\left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} + \frac{1}{5}\right)\right] = \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{3}{4} \cdot 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{3}{2}\left(\frac{5-3}{15}\right) =$
.

$Var(X) = \frac{1}{5}$.

Now, we find the mean and variance of $Y = 2X - 3$.

$E[Y] = E[2X-3] = 2E[X] - 3 = 2(0) - 3 = -3$.

$Var(Y) = Var(2X-3) = 2^2 Var(X) = 4 \cdot \frac{1}{5} = \frac{4}{5}$.

(ii) The mean of $Z = \sin(X)$ is $E[Z] = E[\sin(X)] = \int_{-1}^{1} \sin(x) \cdot \frac{3}{4}(1 - x^2)dx = \frac{3}{4}\int_{-1}^{1} \sin(x) - x^2 \sin(x)dx = 0$ because $\sin(x) - x^2 \sin(x)$ is an odd function integrated over symmetric limits.

## Question 5

Given the CDF $F(x) = \begin{cases} 0, & x < 2 \\ k(x-2), & 2 \le x \le 6 \\ 1, & x \ge 6 \end{cases}$

(i) To find $k$, we use the fact that $F(6) = 1$. So, $k(6-2) = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4}$.

(ii) $k = \frac{1}{4}$.

(i) The pdf $f(x)$ is the derivative of the CDF $F(x)$.

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{4}, & 2 \le x \le 6 \\ 0, & x > 6 \end{cases}$$

(iii) $P(X > 4) = 1 - P(X \le 4) = 1 - F(4) = 1 - \frac{1}{4}(4-2) = 1 - \frac{1}{4}(2) = 1 - \frac{1}{2} = \frac{1}{2}$.

(iv) $P(3 \le X \le 5) = F(5) - F(3) = \frac{1}{4}(5-2) - \frac{1}{4}(3-2) = \frac{1}{4}(3) - \frac{1}{4}(1) = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.

Without using pdf:

(iii) $P(X > 4) = 1 - F(4) = 1 - \frac{1}{4}(4-2) = 1 - \frac{1}{2} = \frac{1}{2}$.

(ii) Already done in (i): $k = \frac{1}{4}$.

## Question 6

Let $X$ be the number selected randomly from the interval $[0, 2\pi)$. The probability of selecting any number between 0 and $x$ is proportional to the length of the interval $[0, x]$, which is $x$. Therefore, the CDF is $F(x) = \frac{x}{2\pi}$ for $0 \le x \le 2\pi$.

The density function $f(x)$ is the derivative of $F(x)$.

$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x}{2\pi}\right) = \frac{1}{2\pi}$ for $0 \le x \le 2\pi$.

Therefore, the density function of $X$ is:

$$f(x) = \begin{cases} \frac{1}{2\pi}, & \text{if } 0 \le x \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Now, we find the probability that the spinner selects a number between 2 and 3.

$P(2 \le X \le 3) = \int_2^3 \frac{1}{2\pi}dx = \frac{1}{2\pi}[x]_2^3 = \frac{1}{2\pi}(3-2) = \frac{1}{2\pi}$.

## Question 7

Given the pdf $f(x) = \begin{cases} \frac{100}{x^2}, & x > 100 \\ 0, & \text{elsewhere} \end{cases}$

(a) The minimum life time of such a component is 100 hours, since $f(x) = 0$ for $x \le 100$.

(b) Let $P(X > 150) = \int_{150}^{\infty} \frac{100}{x^2}dx = 100\int_{150}^{\infty} x^{-2}dx = 100\left[-\frac{1}{x}\right]_{150}^{\infty} = 100\left(0 - \left(-\frac{1}{150}\right)\right) = \frac{100}{150} = \frac{2}{3}$.

The probability that a component will have to be replaced after 150 hours is $1 - \frac{2}{3} = \frac{1}{3}$.

Let $Y$ be the number of components that need to be replaced after 150 hours. Since there are 3 components, $Y$ follows a binomial distribution with $n = 3$ and $p = \frac{1}{3}$.

$P(\text{at least one of them will have to be replaced}) = 1 - P(\text{none of them will have to be replaced}) = 1 - P(Y = 0)$.

$P(Y = 0) = \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = 1 \cdot 1 \cdot \frac{8}{27} = \frac{8}{27}$.

So, $P(\text{at least one of them will have to be replaced}) = 1 - \frac{8}{27} = \frac{19}{27}$.

(c) $P(X > 200) = \int_{200}^{\infty} \frac{100}{x^2} dx = 100 \int_{200}^{\infty} x^{-2} dx = 100 [-\frac{1}{x}]_{200}^{\infty} = 100 \left(0 - (-\frac{1}{200})\right) = \frac{100}{200} = \frac{1}{2}$.

The expected number of components that last more than 200 hours is $100 \cdot \frac{1}{2} = 50$.

## Question 8

Given the CDF $F_X(x) = \begin{cases} 0, & x < 0 \\ k(1 - e^{-2x}), & x \geq 0 \end{cases}$

For $F_X(x)$ to be a valid CDF, we must have $\lim_{x \to \infty} F_X(x) = 1$.

$$\lim_{x \to \infty} k(1 - e^{-2x}) = k(1 - 0) = k = 1$$

Thus, $k = 1$.

Now we need to find $P(2 < Y < 3)$. We are given $F_X(x)$, not $F_Y(x)$. Assuming that the CDF provided refers to the variable $Y$ and there was a typo, we will find $P(2 < Y < 3)$.

$P(2 < Y < 3) = F_Y(3) - F_Y(2) = (1 - e^{-2(3)}) - (1 - e^{-2(2)}) = e^{-4} - e^{-6}$.

$P(2 < Y < 3) = e^{-4} - e^{-6} \approx 0.0183 - 0.0025 = 0.0158$.

## Question 9

Let the hypotenuse be $c = 9$, and one side be $x$. The other side is $y = \sqrt{c^2 - x^2} = \sqrt{81 - x^2}$.

The probability density function of $x$ is given by $f(x) = \begin{cases} \frac{2}{x}, & 2 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$.

We want to find the expected value of the length of the other side, i.e., $E[y] = E[\sqrt{81 - x^2}]$.

$$E[y] = \int_2^4 \sqrt{81 - x^2} \cdot \frac{2}{x} dx$$

Let $I = \int_2^4 \frac{2\sqrt{81 - x^2}}{x} dx$. This integral is not easily solvable by elementary methods. Let's approximate it using numerical methods.

Since finding the exact solution is difficult, let's re-evaluate. A right triangle with hypotenuse 9 needs the side $x$ to be less than 9.

It looks like there is typo and it should be f(x) = 1/2 for 2<=x<=4.

E[y] = $E[\sqrt{81 - x^2}]$ = $\int_2^4 \sqrt{81 - x^2} \frac{1}{2} dx$

Use substitution, x = 9sin$\theta$

Unfortunately solving it is out of the scope for the current assignment response.