Simulation of Stochastic processes And Valuation of Vanilla Options

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- *Simulation of Random Variables*
- *Simulation of Geometric Brownian Motion*
- *Simulation of Square-root Diffusion*
- *Simulation of Stochastic Volatility*
- *Simulation of Jump Diffusion*
- *Valuation of European Options*
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Simulation of Random Variables

Simulating future index level in Black-Scholes-Merton setup

$$S_T = S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z)$$

The variables and parameters have the following meaning:

S_T: Index levelat date T

r: Constant riskless short rate

σ: Constant volatility (= standard deviation of returns) of S

z: Standard normally distributed random variable

Simulation of Geometric Brownian Motion

A stochastic process is a sequence of repeated simulations of a random variable.

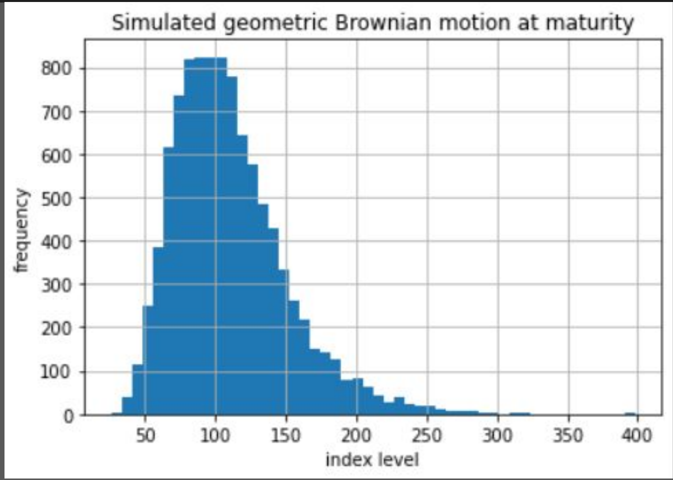
Consider the Black-Scholes-Merton model in its dynamic form. Z_t is a standard Brownian motion. The SDE is called a geometric Brownian motion.

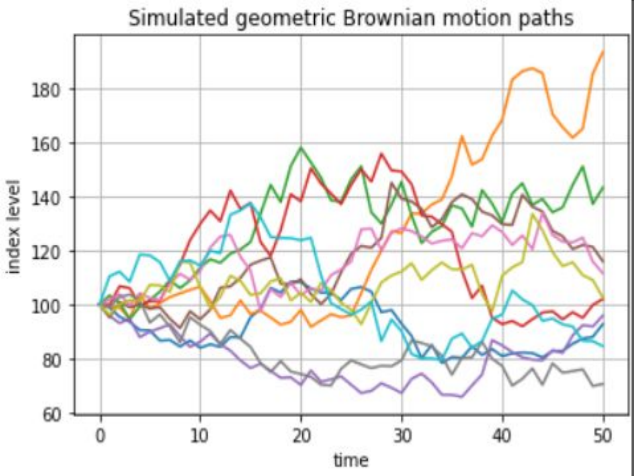
Stochastic differential equation in Black-Scholes-Merton setup

$$dS_t = rS_t dt + S_t dZ_t$$

Simulating index levels dynamically in Black-Scholes-Merton setup

$$S_t = S_{t-\Delta t}\exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z$$





Simulation of Square-root Diffusion

Another important class of financial processes is mean-reverting processes, which are used to model short rates or volatility processes, for example. A popular and widely used model is the square-root diffusion, as proposed by Cox, Ingersoll, and Ross (1985).

Stochastic differential equation for square-root diffusion

$$dx_t = \kappa(\theta - x_t)dt + \sigma\sqrt{x_t}dZ_t$$

The variables and parameters have the following meaning:

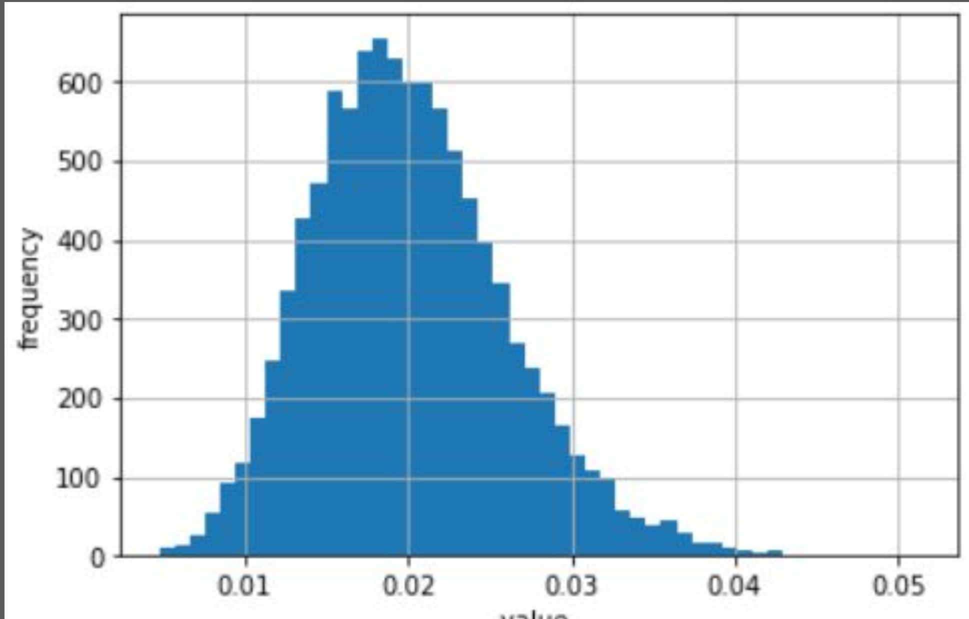
x_t: Process level at date t, κ: Mean-reversion factor, θ: Long-term mean of the process, σ: Constant volatility parameter, Z: Standard Brownian motion.

It is well known that the values of xt are chi-squared distributed. However, as stated before, many financial models can be discretized and approximated by using the normal distribution (i.e., a so-called Euler discretization scheme). While the Euler scheme is exactfor the geometric Brownian motion, it is biased for the majority of other stochastic processes. Even if there is an exact scheme available — one for the square-root diffusion will be presented shortly — the use of an Euler scheme might be desirable due to numerical and/or computational reasons. Defining s ≡ t – Δt and x+ ≡ max(x,0), The following Equation presents such an Euler scheme. This particular one is generally called full truncation in the literature (cf. Hilpisch (2015)).

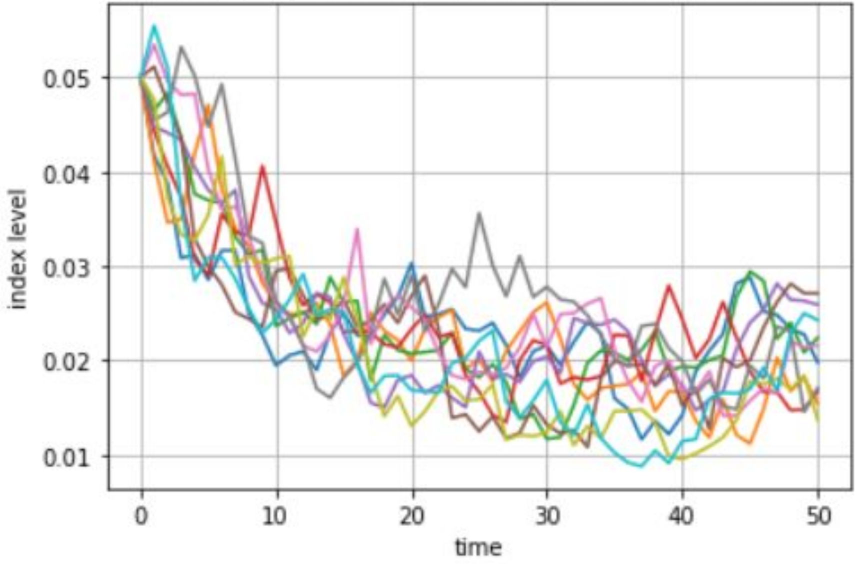
Euler discretization for square-root diffusion

$$\tilde{x}_t = \tilde{x}_s + \kappa(\theta - \tilde{x}_s{}^+)\Delta t + \sigma\sqrt{\tilde{x}_s{}^+}\sqrt{\Delta t}z_t$$

$$x_t = \tilde{x}_s{}^+$$



Simulated square-root diffusion at maturity (Euler scheme)



Simulation of Stochastic volatility

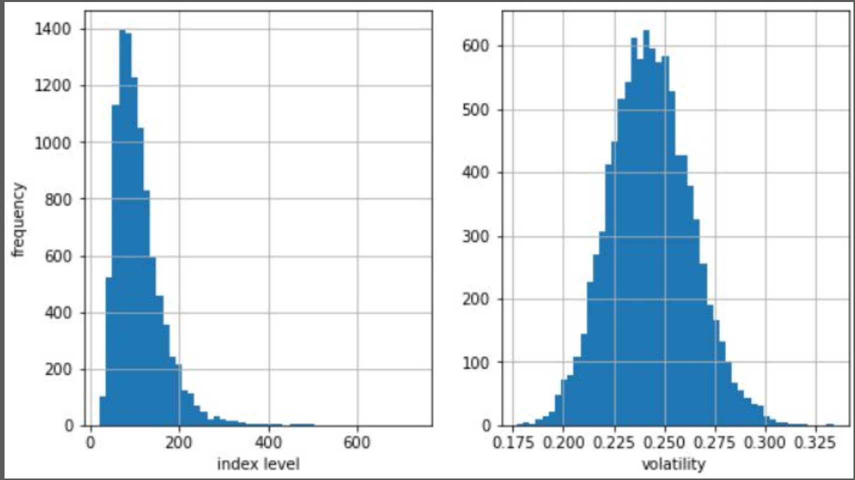
One of the major simplifying assumptions of the Black-Scholes-Merton model is the constant volatility. However, volatility in general is neither constant nor deterministic; it is stochastic. Therefore, a major advancement with regard to financial modeling was achieved in the early 1990s with the introduction of so-called stochastic volatility models. One of the most popular models that fall into that category is that of Heston (1993), which is presented below.

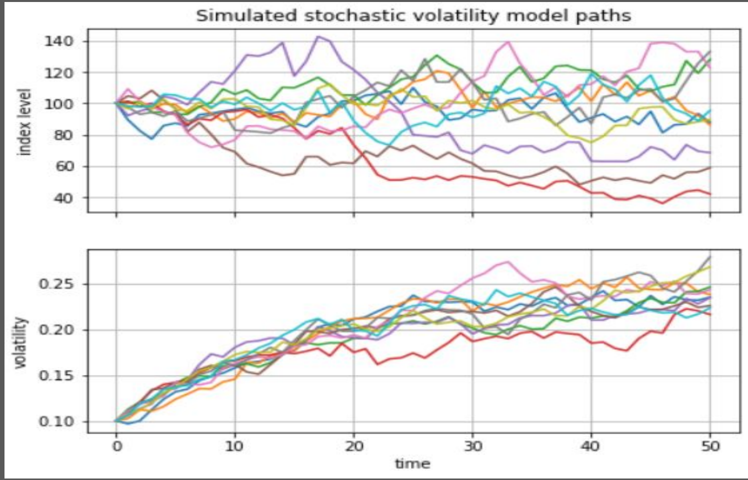
Stochastic differential equations for Heston stochastic volatility model

$$dS_t = rS_t dt + \sqrt{v_t} S_t dZ_t^1$$

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v \sqrt{v_t} dZ_t^2$$

$$dZ_t^1 dZ_t^2 = \rho$$





Simulation of Jump-Diffusion

Stochastic volatility and the leverage effect are stylized (empirical) facts found in a number of markets. Another important stylized empirical fact is the existence of jumps in asset prices and, for example, volatility. In 1976, Merton published his jump diffusion model, enhancing the Black-Scholes-Merton setup by a model component generating jumps with log-normal distribution. The risk-neutral SDE is presented in Equation below.

Stochastic differential equation for Merton jump diffusion model

$$dS_t = (r - r_J)S_t dt + S_t dZ_t + J_t S_t dN_t$$

For completeness, here is an overview of the variables' and parameters' meaning:

S_t: Index level at date t, r: Constant riskless short rate

$r_J \equiv \lambda \cdot (e^{\mu_J + \delta^2/2} - 1)$: Drift correction for jump to maintain risk neutrality

σ: Constant volatility of S, dZ_t: Standard Brownian motion, J_t: Jump at date t with distribution ...

- $\ldots log(1 + J_t) \approx N(log(1 + \mu_J - \frac{\delta^2}{2}, \delta^2)$ with

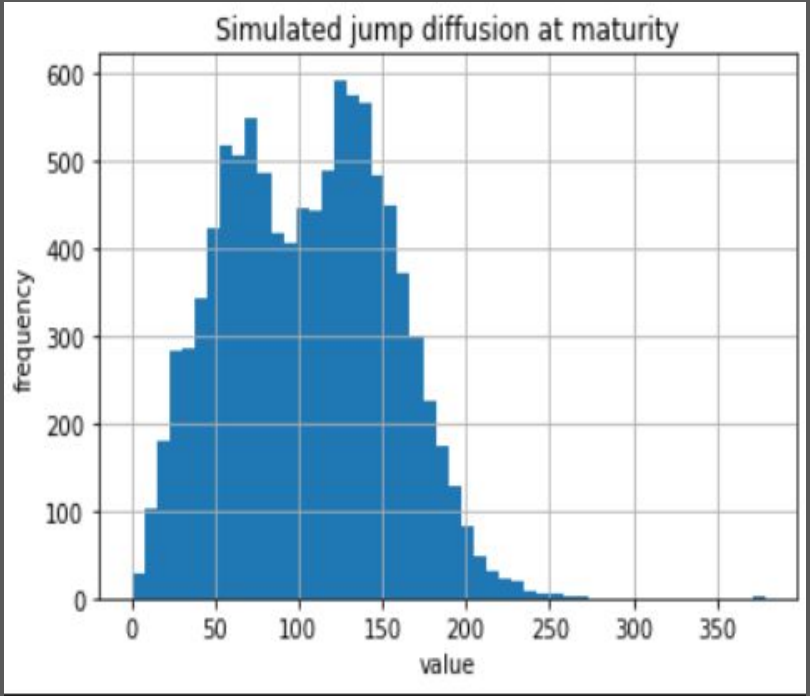
N as the cumulative distribution function of a standard normal random variable

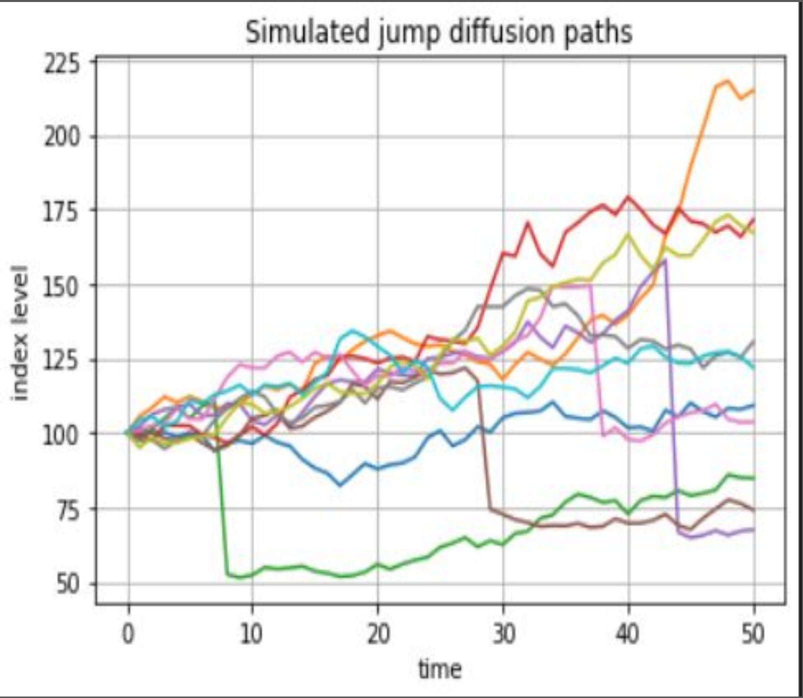
N_t: Poisson process with intensity

Below Equation presents an Euler discretization for the jump diffusion where the z_t^n are standard normally distributed and the y_t are Poisson distributed with intensity.

Euler discretization for Merton jump diffusion model

$$S_t = S_{t-\Delta t}(e^{(r - r_J - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} z_t^1} + (e^{\mu_J + \delta z_t^2} - 1)y_t)$$


Simulated jump diffusion at maturity
frequency
value
0
100
200
300
400
500
600
50
150
250
350


Simulated jump diffusion paths
index level
time
50
75
100
125
150
175
200
225
0
10
20
30
40

Valuation of European Options

The payoff of a European call option on an index at maturity is given by h(ST) ≡ max(ST – K,0), where ST is the index level at maturity date T and K is the strike price. Given a, or in complete markets the, risk-neutral measure for the relevant stochastic process (e.g., geometric Brownian motion), the price of such an option is given by the formula in Equation below.

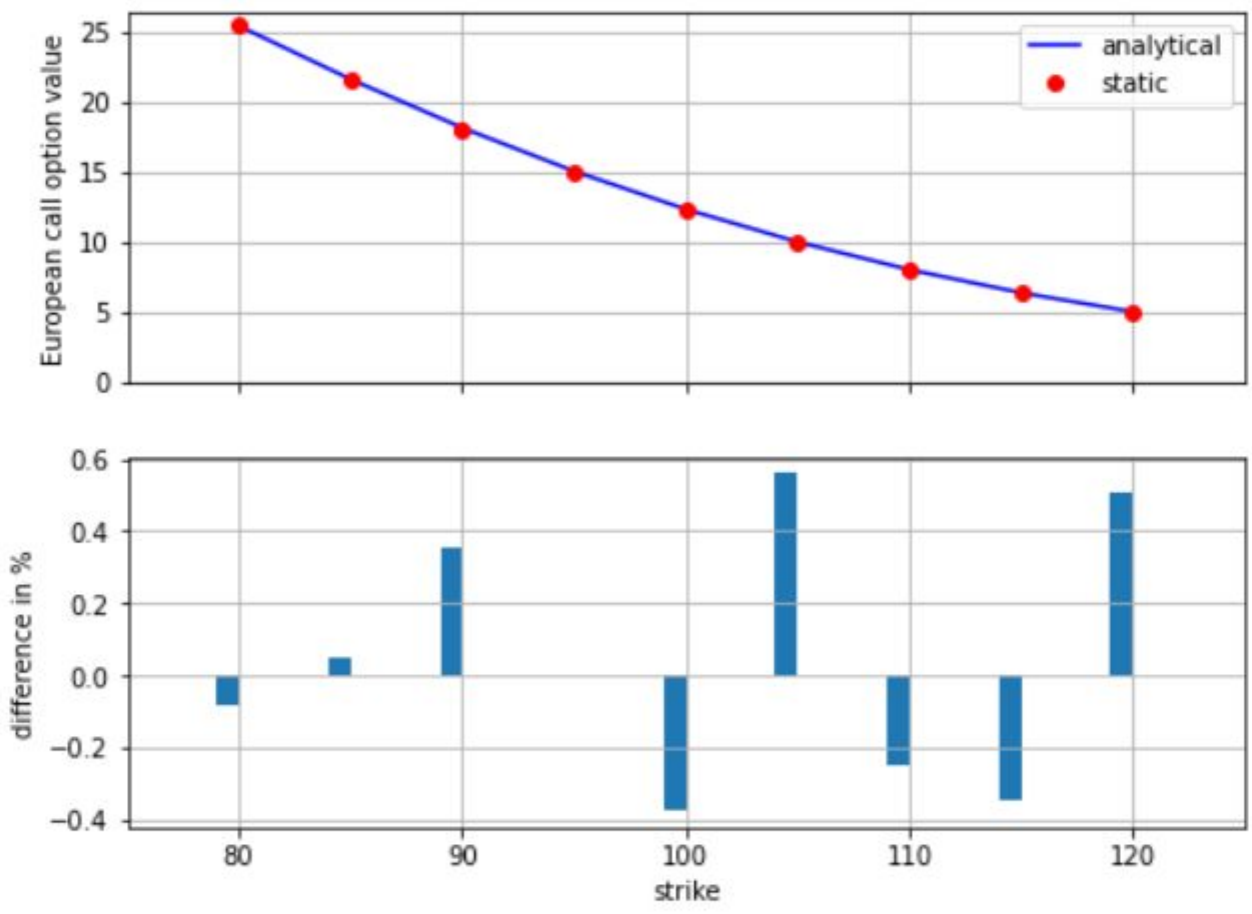
Pricing by risk-neutral expectation

$$C_0 = e^{rT} E_0^Q(h(S_T)) e^{rT} \int_0^\infty h(s)q(s)ds$$

Below Equation provides the respective Monte Carlo estimator for the European option, where is the ith simulated index level at maturity.

Risk-neutral Monte Carlo estimator

$$\widetilde{C_0} = e^{rT} \frac{1}{I} \sum_{i=1}^{I} h(\tilde{S}_T^i)$$



European call option value
analytical
static
25
20
15
10
5
0
difference in %
0.6
0.4
0.2
0.0
−0.2
−0.4
80
90
100
110
120
strike

Valuation of American Options

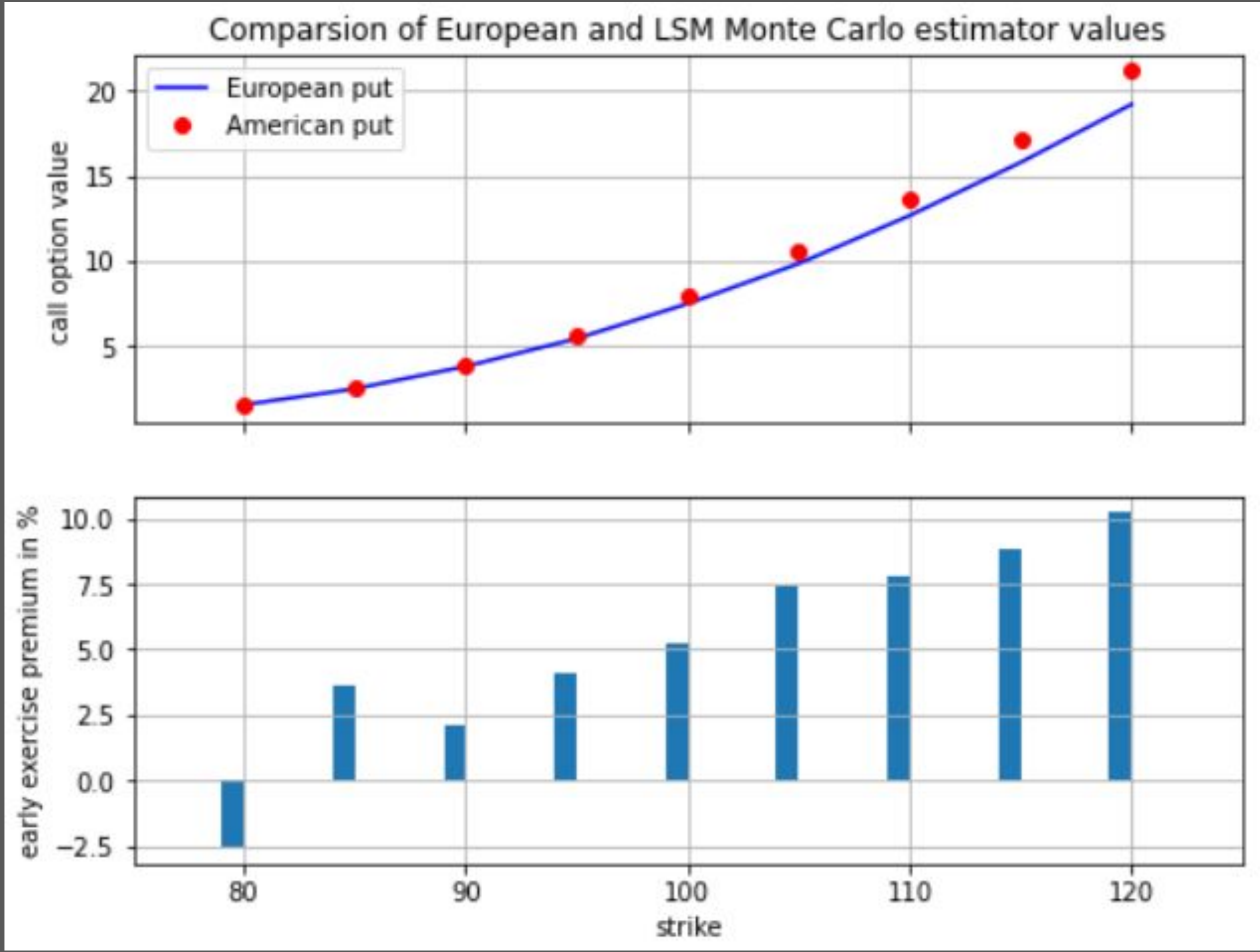
The valuation of American options is more involved compared to European options. In this case, an optimal stopping problem has to be solved to come up with a fair value of the option. Below Equation formulates the valuation of an American option as such a problem. The problem formulation is already based on a discrete time grid for use with numerical simulation. In a sense, it is therefore more correct to speak of an option value given Bermudan exercise. For the time interval converging to zero length, the value of the Bermudan option converges to the one of the American option.

American option prices as optimal stopping problem

$$V_0 = \sup_{\tau \in \{0, \Delta t, 2\Delta t, \ldots, T\}} e^{rT} E_0^Q(h(S_T))$$

Least-squares regression for American option valuation

$$\min_{\alpha_{1,t}, \ldots, \alpha_{D,t}} \frac{1}{I} \sum_{i=1}^{I} (Y_{t,i} - \sum_{d=1}^{D} \alpha_{d,t} \cdot b_d(S_{t,i}))$$


Comparsion of European and LSM Monte Carlo estimator values
European put
American put
call option value
20
15
10
5
early exercise premium in %
10.0
7.5
5.0
2.5
0.0
−2.5
80
90
100
110
120
strike

~ End ~