$an^2 + bn + c = \Theta(n^2)$, where a > 0. To see this, note that $an^2 + bn + c \le (a + |b| + |c|)n^2$, for all $n \in \mathbb{N}$, and so $an^2 + bn + c = O(n^2)$. On the other hand, $an^2 + bn + c = a((n + c_1)^2 - c_2)$ where $c_1 = b/2a$ and $c_2 = (b^2 - 4ac)/4a^2$, so we can find a c_3 and an n_0 so that for all $n \ge n_0$, $c_3n^2 \le a((n + c_1)^2 - c_2)$, and so $an^2 + bn + c \in \Omega(n^2)$.

Problem 1.2 Find c_3 and n_0 in terms of a, b, and c. Then prove that for $k \geq 0$, $\sum_{i=0}^{k} a_i n^i = \Theta(n^k)$; this shows the simplfying advantage of the Big O.

Solution:

Clearly,

$$an^{2} + bn + c \ge an^{2} - |b|n - |c| = n^{2}(a - |b|/n - |c|/n^{2})$$
 (1)

|b| is finite, so $\exists n_b \in \mathbb{N}$ such that $|b|/n_b \leq a/4$. Similarly, $\exists n_c \in \mathbb{N}$ such that $|c|/n_c^2 \leq a/4$. Let $n_0 = \max\{n_b, n_c\}$. For $n \geq n_0$, $a - |b|/n_0 - |c|/n_0^2 \geq a - a/4 - a/4 = a/2$. This, combined with (1), grants:

$$\frac{a}{2}n^2 \le an^2 + bn + c \tag{2}$$

for all $n \ge n_0$. We need only to assign c_3 the value a/2 to complete the proof that $an^2 + bn + c \in \Omega(n^2)$.

Next we deal with the general polynomial with a positive leading coefficient. Let $p(n) = \sum_{i=1}^k a_i n^i = n^k \sum_{i=1}^k \frac{a_i}{n^{k-i}}$, where $a_k > 0$. Clearly $p(n) \le n^k \sum_{i=1}^k |a_i|$ for all $n \in \mathbb{N}$, so $p(n) = O(n^k)$. Moreover, every a_i is finite, so for each $i \in \mathbb{N}$ such that $0 \le i \le k-1$, $\exists n_i$ such that $\frac{a_i}{n^{k-i}} \le a_k/2k$ for all $n \ge n_i$. Let n_0 be the maximum of these n_i 's. p(n) can be rewritten as $n^k(a_k + \sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}})$, so $p(n) \ge n^k(a_k - \sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}})$. We have shown that for $n \ge n_0$, $\sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}} \le a_k - k(a_k/2k) = a_k/2$, so let $c = a_k/2$. For all $n \ge n_0$, $p(n) \ge (a_k - a_k/2)n^k = cn^k$. Thus, $p(n) = \Omega(n^k)$.

We have shown that $p(n) \in O(n^k)$ and $p(n) \in \Omega(n^k)$, so $p(n) = \Theta(n^k)$.