Problem 2.13. *Suppose that edge e_1 has a smaller cost than any of the other edges in graph G; that is, $c(e_1) < c(e_i)$, for all $i > 1$. Show that there is at least one MCST for G that includes e_1.*

Proof. First, note that if we give G to Kruskal's, with the edges in the order of their indices as opposed to sorted by weight, the resulting tree will include e_1 - a cycle cannot be formed with the first (or second) edge. Therefore, there is necessarily a spanning tree T_1 of G such that $e_1 \in T_1$.

For contradiction, assume that there is a MCST T_2 such that $e_1 \notin T_2$. By the Exchange Lemma, there is an e_2 in T_2 such that $T_3 = T_2 \cup \{e_1\} - \{e_2\}$ is a spanning tree. But $c(e_1) < c(e_2)$, so $c(T_3) < c(T_2)$; that is, T_2 is not a minimum cost spanning tree. We've found our contradiction; there cannot be a MCST that does not contain e_1. Clearly, there must me at least one minimum cost spanning tree. We've shown that it contains e_1. $\square$