Problem 2.8. *Show that given a connected graph $G = (V, E)$, Kruskal's algorithm outputs a T that is both connected and acyclic.*

Solution:
We use the following loop invariant from the text:

$$\text{The edge set } T \cup \{e_{i+1}, \ldots, e_m\} \text{ connects all nodes in } V. \tag{1}$$

Proof. We start from the basis case: before the first iteration, T_0 is the empty set ($i = 0$). Since G is connected, obviously $\{e_1, e_2, \ldots, e_m\} = E$ connects all nodes in V.

Next we prove induction. Assume that, after $i-1$ iterations, $T_{i-1} \cup \{e_i, \ldots, e_m\}$ connects all nodes in V. On iteration i, we have two cases:
Case 1: $T_{i-1} \cup \{e_i\}$ has no cycle, so $T_i = T_{i-1} \cup \{e_i\}$. $T_i \cup \{e_{i+1}, \ldots, e_m\}$ and $T_{i-1} \cup \{e_i, \ldots, e_m\}$ are the same set, e_i has just moved from the "remaining" edges to T. By the hypothesis, the latter edge set connects all nodes in V, so the prior must as well.
Case 2: $T_{i-1} \cup \{e_i\}$ contains a cycle, so $T_i = T_{i-1}$. Consider any two nodes $u, v \in V$. By the hypothesis, there is a path from u to v consisting of edges in $T_{i-1} \cup \{e_i, \ldots, e_m\}$. If e_i is not in this path, then we're done; there is still a path between u and v, as we've only lost access to e_i. If $e_i = (a, b)$ is in this path, we can replace it with another path from a to b; e_i was in a cycle, so another such path necessarily exists.

We have found a path connecting arbitrary u and v in $T_i \cup \{e_i, \ldots, e_m\}$ given that one existed in $T_{i-1} \cup \{e_i, \ldots, e_m\}$, thereby completing the induction step and proving that (1) is a loop invariant.

Clearly, after all $i = m$ iterations, this loop invariant reads "$T_i \cup \{e_{i+1}, \ldots\}$ connects all nodes in V." But e_m was the last edge, so $\{e_{i+1}, \ldots\}$ is the empty set. Therefore, T_m connects all nodes in V. By construction, T_m cannot contain any cycles; any edge which would have completed a cycle was simply not included. So, after m iterations, T connects all nodes in V and is acyclic - T is a spanning tree of G. $\square$