$an^2 + bn + c = \Theta(n^2)$, where $a > 0$. To see this, note that $an^2 + bn + c \leq (a + |b| + |c|)n^2$, for all $n \in \mathbb{N}$, and so $an^2 + bn + c = O(n^2)$. On the other hand, $an^2 + bn + c = a((n + c_1)^2 - c_2)$ where $c_1 = b/2a$ and $c_2 = (b^2 - 4ac)/4a^2$, so we can find a c_3 and an n_0 so that for all $n \geq n_0$, $c_3 n^2 \leq a((n + c_1)^2 - c_2)$, and so $an^2 + bn + c \in \Omega(n^2)$.

Problem 1.2 *Find c_3 and n_0 in terms of a, b, and c. Then prove that for $k \geq 0$, $\sum_{i=0}^{k} a_i n^i = \Theta(n^k)$; this shows the simplfying advantage of the Big O.*

Solution: Clearly,

$$an^2 + bn + c \geq an^2 - |b|n - |c| = n^2(a - |b|/n - |c|/n^2) \tag{1}$$

$|b|$ is finite, so $\exists n_b \in \mathbb{N}$ such that $|b|/n_b \leq a/4$. Similarly, $\exists n_c \in \mathbb{N}$ such that $|c|/n_c^2 \leq a/4$. Let $n_0 = max\{n_b, n_c\}$. For $n \geq n_0$, $a - |b|/n_0 - |c|/n_0^2 \geq a - a/4 - a/4 = a/2$. This, combined with (1), grants:

$$\frac{a}{2} n^2 \leq an^2 + bn + c \tag{2}$$

for all $n \geq n_0$. We need only to assign c_3 the value $a/2$ to complete the proof that $an^2 + bn + c \in \Omega(n^2)$.

Next we deal with the general polynomial with a positive leading coefficient. Let $p(n) = \sum_{i=1}^{k} a_i n^i = n^k \sum_{i=1}^{k} \frac{a_i}{n^{k-i}}$, where $a_k > 0$. Clearly $p(n) \leq n^k \sum_{i=1}^{k} |a_i|$ for all $n \in \mathbb{N}$, so $p(n) = O(n^k)$. Moreover, every a_i is finite, so for each $i \in \mathbb{N}$ such that $0 \leq i \leq k-1$, $\exists n_i$ such that $\frac{a_i}{n^{k-i}} \leq a_k/2k$ for all $n \geq n_i$. Let n_0 be the maximum of these n_i's. $p(n)$ can be rewritten as $n^k(a_k + \sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}})$, so $p(n) \geq n^k(a_k - \sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}})$. We have shown that for $n \geq n_0$, $\sum_{i=0}^{k-1} \frac{a_i}{n^{k-i}} \leq a_k - k(a_k/2k) = a_k/2$, so let $c = a_k/2$. For all $n \geq n_0$, $p(n) \geq (a_k - a_k/2)n^k = cn^k$. Thus, $p(n) = \Omega(n^k)$.

We have shown that $p(n) \in O(n^k)$ and $p(n) \in \Omega(n^k)$, so $p(n) = \Theta(n^k)$. $\square$