More on Sets, Relations and Functions

Note that first you should do the exercises from Hammock, and then the challenge problems provided. If you can, then attempt the questions from Munker's topology. Do read the Behnke book, it contains good discussions. While the notation used is a bit outdated, the discussion is very illuminating.

Readings

1. Book of proofs, Hammock — Chapters 11 and 12
2. Naive Set Theory, Halmos — Chapters 7,8,9
3. Topology, Munkres — Sections 6,7
4. Fundamentals of Mathematics, Behnke — Chapter 1 (1-8)

Book Exercises

- Chapter 1 (Hammock) — Section 1.1 (B & C) , Section 1.2 (A) Section 1.3, 1.4, 1.5 , 1.8
- Chapter 11 (Hammock) — Section 11.1 (1-11) , Section 11.2 (1-8 , 10, 13-15) , Section 11.4
- Chapter 12 (Hammock) — Section 12.1 , Section 12.2
- Section 7 (Munkres) — Exercises of section 7

Warm Up Problems

1. Suppose $f : A \to B$ is a function and it is surjective. What can you say about the cardinality of A and B assuming both are finite?
2. Suppose $f : A \to B$ is a function and it is injective. What can you say about the cardinality of A and B assuming both are finite?
3. Suppose $f : A \to B$ is a bijective function. What can you say about cardinality of A and B assuming both are finite?

4. Show that equality is an equivalence relation.
5. Prove De-Morgan's laws for both logical statements and sets.
6. Show that the composition of a function with its inverse is the identity function $Id : A \to A$ defined by $Id(x) = x$

Challenge Problems

1. Composition of functions $f : A \to B$ and $G : B \to C$ is defined as, $g \circ f = g(f(x))$.
   a. Assume f, g are injective.
   b. Assume f, g are surjective.
   c. Assume f, g are bijective.

   In all of these cases, what can you say about the composition of functions?
2. We say that a set is countable if we can list all the elements in that set, X in an ordered manner. That is, we can form a function from $\mathbb{Z}^+ \to X$ such that this function is surjective. That is, one can from an ordered list where each list entry corresponds to an element from X. It is like writing a table as following,

1	x_1
2	x_2
3	x_3

   Which covers all the elements in X. This can be thought of as a map from $\mathbb{Z}^+$ to X. Convince yourself that this map being surjective is a sufficient condition for all the elements to be listed this way. If we can do this for such a set then the set is said to be countable.
   1. If a map $X \to \mathbb{Z}^+$ is injective, then is it sufficient condition for saying that X is countable?
   2. Prove that the set of all natural numbers is countable.
   3. Prove that the set of integers is countable.
   4. Prove that the set of all rational numbers is countable.
3. Show that the cartesian product of two countable sets is also countable.

4. Show that the set of all polynomials of finite degree is countable.
5. Given two finite sets, how many possible functions are possible between them? What about functions from A to B? What about functions from B to A?

Fun Problem

To establish the logical foundations of mathematics, Russel started a program to axiomatize all the mathematics known at the time. In the process he came up with a 200 page long proof for $1+1=2$. Read about Russel's Principia Mathematica. You don't need to read through the entire book (it would not be a very fruitful endeavor) but do read about his program. It is a very important milestone in the history of mathematics.