Building the Intuition for Functions

Keep in mind that the following problems are designed to keep you creatively engaged. It is okay if you are not able to solve them all, don't be discouraged. The goal is to get you to think in a different, more creative manner. If you have any doubts, feel free to contact me. There is no right or wrong approach, neither is there any time constraint in which you are expected to solve this. All the best! Have fun!

Some Reading Relevant For This Week

This week we will be going over some fundamental concepts in mathematics which will be useful to study calculus. While the reading itself is not directly relevant to the problems, I do expect you to finish this reading before the class. You do not need to know the material in these readings to solve the problems below.

1. Tom Apostol, Calculus Volume 1 : Part 1 and 2 of the Introduction
2. Book of Proof, Hammock — Chapter 1

Note that you do not need to solve the questions in these textbooks, if you want to though, it is your choice. If you can't follow all the proofs that is quite alright, as long as you understand the idea behind the concepts.

Warm Up Problems

1. Our first task here is to try and understand how various functions behave. If you don't recall what a function is, think of it as a machine which gives a single output for each input you give it. This machine here I am denoting by f, so $f(x)$ means the output received when the input is x. Can you give a couple examples each of
   a. functions which are strictly increasing? This means that if $x < y$ then $f(x) < f(y)$ for all x and y.
   b. functions which strictly decrease, that is if $x > y$ then $f(x) < f(y)$.
   c. functions which are neither increasing or decreasing.
2. Based on your answers to the previous question, think about the following. A function is said to have a maximum in certain 'neighborhood' if all the values of the function in the

points in that particular neighborhood, is $\leq$ the 'maximum'. This neighborhood can be as small as you want, or as large as you wish.

   a. Can a strictly increasing function have a maximum?
   b. Can a strictly decreasing function have a maximum?
   c. What about functions that are neither increasing nor decreasing? They go up and down and up and down.

3. Consider functions which are neither decreasing nor increasing. Can we restrict the inputs, so only certain inputs are allowed (the set of all allowed inputs is what we call the domain) such that, over this specified range of inputs the function is increasing or decreasing? Give some examples.
4. Based on your understanding of increasing and decreasing functions, try to find out the nature of the following curves. If a curve is neither increasing or decreasing over all real numbers, then find the range of values within which it is increasing or decreasing.
   a. the semi circle, $x^2 + y^2 = 1$ where $y > 0$.
   b. x^2
   c. x^3
   d. cx^3 where c is some constant real number
   e. $\sin x$
5. For the functions mentioned in problem 4, try to find the maximum. This may be a regional maximum or it could be a maximum over the entire real numbers.
6. Try to sketch the following curve yourself using your knowledge of increasing and decreasing curves. Don't cheat! Calculators and graphing software are prohibited. Try to do this by hand. We are only looking for a qualitative expression.
   a. $(x-5)^2$
   b. $\sin(x - \pi/2)$
   c. $\frac{1}{1+x^2}$
   d. 2^{-x}
   e. 2^{-x^2}
   f. $\frac{1}{2^{-x^2}}$

Challenge Problems

1. Consider the following scenario - we know that a particular particle travels in a manner such that it traces a continuous path with no cusps or corners — when you zoom into any part of the curve, it is doesn't have a corner like a square, but it is 'smooth' like a circle. At the beginning of the clock, that is, when the time reads $t = 0$ the particle is stationary. The moment we press the START button on the clock, the particle starts moving (rather we press the START button the moment it starts moving. But from our perspective it is the same thing! Convince yourself of this). For some time, say till $t = t_0$ the particle moves with constant velocity. Note that the entire motion is taking place on a plane. After time t_0, the particle starts behaving a bit weirdly. You notice that at first the particle moves in such a way that its acceleration is constant, but velocity is constantly increasing. After moving this way for some time, the particle's velocity starts to change, but its speed remains constant. The direction of the particle's velocity is changing constantly, so that after some time the particle returns to its original velocity. Now the particle keeps on moving in a straight line at this constant velocity, until some time when it starts to decelerate in a weird manner — you observe that the particle's acceleration is not constant, but the ratio $|\vec{a}|/|\vec{v}|$ of the magnitudes of acceleration and velocity is constant, say 1. You also notice that the direction of the velocity and acceleration is the same, and it is constant.

    a. Will the particle ever come to rest?

    b. Sketch the curve speed vs time. Suppose $\vec{v} = (v_x, v_y)$ where v_x and v_y are the x and y components of the velocity respectively. Sketch $v_x \text{ vs } t$ and $v_y \text{ vs } t$. Can you sketch the curve $\vec{v}$ versus t ?

    c. Sketch the distance of the particle as a function of time.

    d. What can you say about the dependence of the acceleration on the velocity in the final stage of the motion? Suppose instead the ratio of acceleration and velocity was equal to some number $\alpha = a/v$ where a and v are the signed magnitudes of the acceleration and velocity respectively. How does the motion change now? Qualify the motion accordingly, depending on α.

2. Consider the following set — An object x belongs to the set R if it satisfies the condition $x \notin R$ If $x \in R$ then $x \notin R$. Can such a set exist? What if I were to say that $x \in R$ if $x \notin S$, and $x \notin R$ if $x \in S$? Here R and S are disjoint sets, that is they do not share a common element. What do you think the problem is in the first part of this question?

3. Think about the subset of the real numbers, $[0, 1]$, and the set of all natural numbers, $\mathbb{N}$. Both the sets are infinite, but is one set larger than the other? How would you go about proving this?
