More on Functions and Relations and Sets

Readings

1. Topology (Munkres) — Sections 2 and 3

Textbook Exercises

1. Munkres — Section 2 exercises
2. Munkres — Section 3 exercises

Challenging Problems

1. Consider the relation of modulo n. a, b are said to be congruent modulo n, $a \equiv b \pmod{n}$ if a and b leave the same remainder when divided by n. Prove the following properties,
   a. $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$.
   b. Show that congruence modulo n is an equivalence relation.
2. Consider a quadratic equation of the form $x^2 + bx + c = 0$ and α, β are roots of this equation then find the relationship between b, c and α, β.
3. Derive the quadratic formula. Follow the following steps,
   a. Can you solve the equation $x^2 = a$?
   b. What about $(x - a)^2 = b$?
   c. Can you rewrite a monic quadratic polynomial in the form $(x - c)^2 = a$?
   d. Use the observations to these questions to derive a quadratic formula.
4. Generalize the above observations to any monic polynomial. The relations between the coefficients and the roots of the polynomial. These relations

are called Vieta's relations.

5. Solve the equation $\sqrt{x} = x - 2$.
6. Use a process similar to the question 3 to derive the cubic formula — this is a formula for the roots of the polynomial of the form $x^3 + px^2 + qx + r = 0$.
7. Suppose $y_n = \sqrt{x + \sqrt{x + \cdots + \sqrt{x}}}$ where the square root is taken n times. Find the value of y_∞.
8. A function is called odd if it satisfies $f(-x) = -f(x)$. It is called even if it satisfies $f(x) = f(-x)$.
   a. Write some examples of odd and even functions.
   b. Can we rewrite any arbitrary function as the sum of odd and even functions? Prove your claim.
9. We define cantor set by the following algorithm. We define $c_0 = [0, 1]$. Define $c_1 = [0, 1/3] \cup [2/3, 1]$. In other words, for every iteration we remove the middle third (open interval) from each closed interval to obtain c_{n+1}.
   a. Calculate c_2, c_3, c_4, c_5
   b. Suppose you are given c_n. When forming c_{n+1} how many intervals do we remove? How much length do we remove (the set $[0, 1]$ has length one) ?
   c. We define the cantor set as following, $C = \cap_{n=1}^{\infty} c_n$. What is the length of this set?
   d. Is the cantor set countable? Convince yourself that this set is non empty.