Quantum Groups and Noncommutative Algebras in Physics

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Contents

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Abstract

The purpose of this project was to gain experience with the algebraic framework used to understand quantum groups. First, mathematical prerequisites such as Hopf algebras, Lie algebras, Lie bialgebras, Poisson Lie groups were understood by means of calculations and literature study. After that some elementary aspects of quantum groups were studied, as well as application of quantum groups in the construction of noncommutative spacetimes. For the latter, a paper [1] on quantum groups and noncommutative spacetimes was used which was the main goal of the project. Overall, an experience in mathematical framework of quantum groups as well as recent scientific literature was gained. The project also provided necessary background and the skill set to read and understand research literature as well as write an efficient and professional scientific report.

1 Introduction

The project started on 12th May, 2025, and lasted for 7 weeks, until 30th June, 2025. This project introduced mathematical literature of quantum groups, applications on quantum groups in mathematical physics, also allowed for independent problem solving and improvement of scientific communication skills. In this section a rough outline of this report is given which includes the various things that I have learned during my internship period.

I started this project with a very basic understanding of the underlying mathematics. I was familiar with group theory and linear algebra but had to pick up a lot of topics along the way, such as tensor products, tensor algebras, definition of algebras and coalgebras, Lie algebras, Lie bialgebras, universal enveloping algebras, Poisson Lie groups, Hopf algebras, and quantum groups. A significant challenge in the initial weeks was to get acquainted with these new algebraic structures and definitions. I also had to learn how to read and draw commutative diagrams. I have provided a comprehensive overview of these topics in Section 2 of this report.

During the initial stage of the project, various tasks of computational nature were assigned to me. A detailed overview of these tasks as well as some other important calculations is given in Section 3. During the first two weeks this included exploring examples of Hopf algebras and Lie bialgebras. Numerous exercises were given to gain experience with the axioms of these algebraic structures, examples such as Sweedler's example and $\mathfrak{sl}_2$ were studied. These calculations provided me with the necessary experience required to better understand the paper by Ballesteros et al [1], as well as perform calculations to confirm the results within that paper. Further details on the tasks performed can be found in Section 3 of this report.

For the purposes of this project, the resources and readings I was assigned were primarily from the book 'Foundations of Quantum Group Theory' by Majid [2], as well as books on quantum groups by Chari, Pressley [3], and Kassel [4]. I used Majid as a secondary resource (Chapters 1, 2, and 8), the primary resource being the notes provided by Dr. Pachol during our weekly meetings. However, to expand my limited background in algebra, I also used many other books. I gained further understanding of tensor products and algebras from the book 'Multilinear Algebra' by Greub [5] (chapters 1 and 3); Lie groups, algebras, and universal enveloping algebras of Lie algebras from the book by Humphreys [6] (chapter 1 and 5); a masters thesis [7] was particularly helpful to understand the mathematical prerequisites. Some other resources were also utilized such as [8], [9], [10], [11] to further understand and refresh some topics.

Once learning of the mathematical prerequisites was completed, I started reading the paper [1]. A detailed analysis of the paper can be found in Section 4 of this report.

Learning about quantum groups was certainly a beautiful experience for me; it introduced me to numerous new ideas, and under the guidance of Dr. Pachol, I was able to get an exposure to recent research in the field of mathematical physics. I have written a detailed description of my experience in Section 5 of this report.

2 Description of Quantum Groups

In this section, I include mathematical prerequisites needed for understanding quantum groups. This section also serves the secondary purpose of providing a detailed description of the various mathematical objects that were encountered during the project. We will begin by reviewing some of the concepts that will be used frequently throughout this report.

2.1 Algebras, Bialgebras & Hopf Algebras

Definition 2.1. An associative algebra is a vector space A over some field k, equipped with a bilinear mapping $\mu : A \times A \longrightarrow A$ which is compatible with addition and scalar multiplication. More explicitly, an unital associative algebra over a field k is a triple (A, μ, ι) such that A is a vector space along with the linear maps $\mu : A \times A \longrightarrow A$ called multiplication map and $\iota : k \to A$ called the unit map, such that the following conditions are satisfied,

(A1) $\quad \mu(\mu \otimes id) = \mu(id \otimes \mu)$

(A2) $\quad \mu(\iota \otimes id) = id = \mu(id \otimes \iota)$

Here, (A1) and (A2) express the conditions of associativity and unit elements respectively.

Note that using the properties of tensor products allows to express the bilinearity and the compatibility of addition of the multiplication map μ. Alternatively, an associative algebra can also be defined as following: an algebra $(A, \cdot, +; k)$ over a field k is a ring $(A, \cdot, +)$ and an action (scalar multiplication) of k on A that is compatible with both the product and addition structure of this ring. Equivalence of these conditions is easy to verify. Let us now take a look at the definition of a Lie algebra. Comprehensive overview of Lie algebras and their properties can be found in [6].

Definition 2.2. Let L be a vector space over a finite dimensional field k, with an operation $L \times L \longrightarrow L$ denoted as $(x, y) \mapsto [x, y]$ called the Lie bracket of x and y. L is called a Lie algebra over k if the following conditions are satisfied,

(L1) $[\ ,\ ]$ is a bilinear map

(L2) $[x, x] = 0 \quad \forall x \in L$

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$

It is trivial to check that conditions (L1) and (L2) give us anticommutativity, $[x, y] = -[y, x]$. In particular, anticommutativity implies (L2) whenever the characteristic of the underlying field isn't 2. Another important concept related to algebras is that of a derivation,

Definition 2.3. A derivation of an associative algebra, A, is a linear map $\delta : A \longrightarrow A$ such that it is satisfies the Leibniz condition,

$$\delta(ab) = \delta(a)b + a\delta(b)$$

if the algebra A is also a Lie algebra then we can verify that the commutator of two derivations $[\delta, \delta']$ is also a derivation. As such the set of all derivations of A forms a Lie subalgebra of the Lie algebra of endomorphisms of A, $\mathfrak{gl}(A)$, which is denoted as $\mathrm{Der}(A)$. One of the most important derivations is known as the adjoint action. An adjoint action is an endomorphism such that if $x \in A$ then $y \mapsto [x, y]$. This map is denoted as ad_x. The map $L \longrightarrow \mathrm{Der}(L)$ which sends x to its adjoint action, ad_x is known as the adjoint representation and plays an important role in Lie theory (see section 1.3 of [6]). Note that the adjoint action also satisfies the following property,

$$\mathrm{ad}_x(yz) = \mathrm{ad}_x(y)z + y\mathrm{ad}_x(z)$$

which follows from the properties of the Lie bracket. In the discussion that follows, we are interested in how the adjoint action behaves over tensor products. We begin be recalling the how representations act over tensor spaces. In particular, if g, h are Lie algebras and π_1, π_2 are representations of g and h acting on U and V respectively, then the tensor product of π_1 and π_2 denoted as $\pi_1 \otimes \pi_2$ is a representation of $g \otimes h$ acting on $U \otimes V$. It is defined as,

$$\pi_1 \otimes \pi_2(x, y) = \pi_1(x) \otimes id + id \otimes \pi_2(y)$$

Applying this to the adjoint representation we get the extension of the adjoint action to tensor spaces defined as follows,

$$\mathrm{ad}_x(z) = \mathrm{ad}_x(z) \otimes 1 + 1 \otimes \mathrm{ad}_x(z) = [x \otimes 1 + 1 \otimes x, z]$$

where g is a Lie algebra and $z \in g \otimes g$. We have thus derived the extension of the adjoint action over tensor spaces. We shall make use of this repeatedly in the following sections. Having discussed unital associative algebras, we can now introduce the idea of coalgebras. First notice that axioms (A1) and (A2) of an unital associative algebra could be represented using the following commutative diagrams,

$$\begin{tikzcd}
& A \otimes A \arrow[dr, "\mu"] & \\
A \otimes A \otimes A \arrow[ur, "\mu \otimes id"] \arrow[dr, "id \otimes \mu"'] & & A \\
& A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A \arrow[r, "\iota \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "id \otimes \iota"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

We can form a dual to this algebra by reversing all the arrows in the diagrams above, while keeping the nodes unchanged. This gives us the coalgebra structure on A. More formally,

Definition 2.4. A co-unital, coassociative coalgebra over a field k is a triple (A, Δ, ϵ) where A is a vector space along with the linear maps $\Delta : A \longrightarrow A \otimes A$ and $\epsilon : A \longrightarrow k$ such that they fulfil the following conditions,

(C1) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

(C2) $(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta$

Linear maps Δ, ϵ are called the coproduct and counit maps respectively, whereas the axioms (C1) and (C2) are referred to as the coassocitivity and counit axioms respectively. In terms of commutative diagrams, the axioms above have the same structure as the diagrams for associative algebra, with the arrows reversed and the product and counit maps being replaced with the coproduct and counit maps respectively. The diagrams are as following,

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\Delta \otimes id"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A \otimes A \otimes A \\ & A \otimes A \arrow[ur, "id \otimes \Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[dr, "\cong"'] & A \otimes A \arrow[l, "\epsilon \otimes id"'] \arrow[r, "id \otimes \epsilon"] & A \otimes k \arrow[dl, "\cong"] \\ & A \arrow[u, "\Delta"'] & \end{tikzcd}$$

Definition 2.5. A quintuple $(H, \cdot, \iota, \Delta, \epsilon)$ is said to be a bialgebra if $(H, \cdot, \iota)$ is an associative unital algebra, and (H, Δ, ϵ) is a coassociative counital coalgebra such that the compatibility between the two structures is provided by requiring the coproduct and counit maps to be algebra homomorphisms,

$$\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y), \quad \Delta(1) = 1 \otimes 1, \qquad \epsilon(x \cdot y) = \epsilon(x) \cdot \epsilon(y), \quad \epsilon(1) = 1$$

Here we are ensuring compatibility of the two structures by relating the product map $\cdot$ and the coproduct map Δ. In particular, instead of requiring Δ, ϵ to be algebra homomorphisms we also could have replaced this condition by requiring $\cdot, \iota$ to be coalgebra homomorphisms. It is easy to see that these two conditions are equivalent. We are now in a position to define Hopf algebras.

Definition 2.6. Formally, a Hopf algebra is a six-tuple $(H, \mu, \iota, \Delta, \epsilon, S)$ such that (H, μ, ι) is an algebra and (H, Δ, ϵ) is a coalgebra such that Δ, ϵ are algebra homomorphisms and $S : H \longrightarrow H$ satisfies the following condition,

$$\mu \circ (id \otimes S) \circ \Delta = \iota \circ \epsilon = \mu \circ (S \otimes id) \circ \Delta$$

The above axiom is known as the antipode axiom and the map $S : H \longrightarrow H$ is known as the antipode mapping. Note that the antipode map is an anti-homomorphism. We say that a Hopf algebra H is commutative if $\mu(x, y) =$

$\mu(y, x)$. We say that a Hopf algebra H is cocommutative if $\Delta = \tau \circ \Delta$ where τ is the flip map defined as $\tau(x \otimes y) = y \otimes x$. In fact if the Hopf algebra is commutative or cocommutative then $S^2 = id$ holds.

Definition 2.7. Let $(g, [,])$ be a Lie algebra. Lie bialgebras structures (g, δ) are defined as follows, where δ is called the cobracket or the cocommutator map which satisfies antisymmetry and the cocycle condition, which are defined as,

(LB1) $\delta(x) \in \wedge^2 g, \forall x \in g$

(LB2) $\delta([x, y]) = \mathrm{ad}_x(\delta(y)) - \mathrm{ad}_y(\delta(x))$

(LB3) $\sum cyc(\delta \otimes id) \circ \delta(x) = 0$

Here, $\wedge^2 g$ denotes the second exterior power of g which can also be thought of as the equivalence class of $x \otimes y - y \otimes x$. In other words, $\wedge^2 g = (g \otimes g)/\langle x \otimes y + y \otimes x \rangle$ where $\langle x \otimes y + y \otimes x \rangle$ denotes the linear span of $x \otimes y + y \otimes x$.

Note that now the extension of the adjoint action to the tensor space can be utilized. Thus, $\forall X, Y \in g$, we have,

$$\delta([X, Y]) = [X \otimes 1 + 1 \otimes X, \delta(Y)] + [\delta(X), Y \otimes 1 + 1 \otimes Y] \tag{1}$$

This will be used throughout the paper [1] as well. An alternate way to define Lie bialgebra structures has been given in the paper, where we require that the 'modified' cocyle condition above is satisfied and the dual map $\delta^* : g^* \wedge g^* \longrightarrow g^*$ is a Lie bracket on g^*, where g^* denotes the dual space of g.

2.2 Poisson-Lie Groups & Yang Baxter Equation

In this subsection we shall summarise the relationship between Poisson-Lie groups, Lie bialgebras and the Yang-Baxter equation. A detailed description can be found in [3] (chapters 1 and 2).

Definition 2.8. Let M be a smooth manifold of finite dimension m. Then a Poisson structure on M is a $\mathbb{R}$-bilinear map

$$\{,\}_M : \mathbb{C}^\infty(M) \times \mathbb{C}^\infty(M) \longrightarrow \mathbb{C}^\infty(M)$$

which is called the Poisson bracket which satisfies the following conditions,

1. $\{f_1, f_2\} = -\{f_2, f_1\}$

2. $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$

3. $\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + \{f_1, f_3\}f_2$

$\forall f_1, f_2, f_3 \in \mathbb{C}^\infty(M)$

Note that the properties (1) and (2), the skew-symmetry and Jacobi identity respectively, together they mean that the Poisson bracket is a Lie bracket on $\mathbb{C}^\infty(M)$. The third property, known as the Leibniz rule means that $\forall f \in \mathbb{C}^\infty(M)$ the map $g \mapsto \{g, f\}$ is a derivation of $\mathbb{C}^\infty(M)$.

Definition 2.9. A smooth map $F : N \longrightarrow M$ between Poisson manifolds is called a Poisson map if it preserves the Poisson brackets of M, N.

$$\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N$$

for all $f_1, f_2 \in \mathbb{C}^\infty(M)$.

Definition 2.10. A Poisson-Lie (PL) group is a Lie group and a Poisson manifold such that the two structures are compatible in the following sense,

(1) A Poisson-Lie group is a Lie group G which has the Poisson structure $\{,\}$ such that the multiplication map $\mu : G \times G \longrightarrow G$ of the group G is a Poisson map.

(2) a homomorphism $\Phi : G \longrightarrow H$ of Poisson-Lie groups is a homomorphism of Lie groups that is also a Poisson map.

Note that Lie bialgebras are infinitesimal analogues of Poisson-Lie groups, just how Lie algebras are infinitesimal analogues of Lie groups.

Definition 2.11. A 1-cocyle of g with values in $g \otimes g$ is called a coboundary if

$$\delta(X) = \mathrm{ad}_X(r) = [X \otimes 1 + 1 \otimes X, r]$$

for some $r \in g \otimes g$ and $\forall X \in g$. Note that a Lie bialgebra (g, δ) is called a coboundary Lie Bialgebra if δ is a coboundary. Here we are using the extension of adjoint action to tensor product spaces as was described in Section 2.1.

We have the following result concerning Lie bialgebras (which we state without proof, which can be found in [3]). Let g be a Lie algebra and let $r \in g \otimes g$. The map δ defined as follows,

$$\delta(X) = \mathrm{ad}_X(r)$$

is a cocommutator of the Lie bialgebra structure on g if and only if the following conditions are met,

1. $r_{12} + r_{21}$ is a g-invariant element of $g \otimes g$

2. $[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ is a g-invariant element of $g \otimes g \otimes g$.

We call $[[r, r]]$ the Schouten bracket and we use the notation

$$
\begin{aligned}
r_{12} &= r^{ab} X_a \otimes X_b \otimes 1 \\
r_{13} &= r^{ab} X_a \otimes 1 \otimes X_b \\
r_{23} &= r^{ab} 1 \otimes X_a \otimes X_b
\end{aligned}
$$

It is obvious to see that if $[[r, r]] = 0$ then the second condition above is met. We call this equation the Classical Yang-Baxter Equation (CYBE). A solution to this equation is known as the classical r-matrix. Now, a coboundary Lie bialgebra structure that arises from a solution to the CYBE is said to be quasitriangular. The equation

$$[[r, r]] = -\omega$$

is known as the Modified Classical Yang-Baxter Equation (MYCBE) and $\omega \in g^{\otimes 3}$ is a fixed element that is invariant under the adjoint action of g.

2.3 Quantum Groups and Deformations

We are now in a position to give the definition of quasitriangular Hopf algebras which provide one example of quantum groups.

Definition 2.12. A quasitriangular bialgebra or Hopf algebra is a pair (H, R) where H is a bialgebra or a Hopf algebra and $R \in H \otimes H$ is invertible and obeys,

1. $(\Delta \otimes id)R = R_{13}R_{23}$

2. $(id \otimes \Delta) = R_{13}R_{12}$

3. $\tau \circ \Delta h = R(\Delta h)R^{-1}, \quad \forall h \in H$

where τ is the transposition map and $R = \sum R^{(1)} \otimes R^{(2)}$ the notation is,

$$R_{ij} = \sum 1 \otimes \cdots \otimes R^{(1)} \otimes 1 \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$$

which is an element of $H \otimes H \otimes \cdots \otimes H$ which has R in the ith and jth factors.

Definition 2.13. Let H be a bialgebra or a Hopf algebra. Then a 1-cocycle is an invertible element $\chi \in H$ such that,

$$\chi \otimes \chi = \Delta \chi$$

A 2-cocyle is an invertible element $\chi \in H \otimes H$ such that

$$(1 \otimes \chi)(id \otimes \Delta)\chi = (\chi \otimes 1)(\Delta \otimes id)\chi$$

In Section 4 of this report, we shall consider quantum groups obtained by 'twists'. Twisting is a process that gives us a new Hopf algebra structure from an existing one. More formally,

Definition 2.14. Let (H, R) be a quasitriangular Hopf algebra (quantum group) and let χ be a counital 2-cocycle. Then there's a new quasitriangular Hopf algebra (H_χ, R_χ) defined by the same algebra and counit maps and

$$\Delta_\chi h = \chi(\Delta h)\chi^{-1}, \qquad R_\chi = \chi_{21} R \chi^{-1}, \qquad S_\chi h = U(Sh)U^{-1}$$

for all $h \in H$. Here, $U = \sum \chi^{(1)}(S\chi^{(2)})$ and is invertible.

Another important concept is that of Hopf algebra deformations. Essentially, we introduce a non-zero parameter, say, q, such that new relations dependent on this parameter also form a Hopf algebra. One of the seminal examples is the universal enveloping algebra of $\mathfrak{sl}_2$, $U(\mathfrak{sl}_2)$. The deformation parameter q defines a noncommutative algebra generated by 1 and $X_+, X_-, q^{H/2}, q^{-H/2}$ with the relations,

$$q^{\pm H/2} q^{\mp H/2} = 1, \qquad q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm, \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

which is the deformation of $U(\mathfrak{sl}_2)$ denoted as $U_q(\mathfrak{sl}_2)$. This forms a Hopf algebra with

$$\begin{aligned} &\Delta q^{\pm H/2} = q^{\pm H/2} \otimes q^{\pm H/2} && \Delta X_\pm = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm \\ &\varepsilon q^{\pm H/2} = 1 && \varepsilon X_\pm = 0 \\ &S X_\pm = -q^{\pm 1} X_\pm && S q^{\pm H/2} = q^{\mp H/2} \end{aligned}$$

Note that we usually write $q = e^t$. We work with q being close to zero, such as $q = e^\hbar$. Then the deformation parameter can be thought of as a formal power series. Substituting this expression of q in the relations above, the generators $H, X_\pm$ also can be thought of as formal power series. Over $\mathbb{C}[[t]]$ we can regard $H, X_\pm$ as the generators then the Hopf algebra is quasitriangular with

$$R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]!} (q^{\frac{H}{2}} X_+ \otimes q^{-\frac{H}{2}} X_-)^n q^{\frac{n(n-1)}{2}}$$

where,

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = [n][n-1]\cdots[1]$$

2.4 Relationship Between Drinfel'd Double (DD) Lie Groups, Lie Bialgebras and Quantum Deformations

We shall now explore the relationship between the concepts that we discussed in the preceding section. Some more discussion regarding Lie bialgebras is necessary in order to understand the relationship between Lie bialgebras and quantum deformations.

Let us consider a basis for the Lie bialgebra g. Suppose it is given by[1],

$$[X_i, X_j] = C^k_{ij} X_k$$

then any cocommutator can be written in the form,

$$\delta(X_i) = f^{jk}_i X_j \wedge X_k \tag{2}$$

where f^{jk}_i are the structure constants which are given by,

$$[\hat{\xi}^j, \hat{\xi}^k] = f^{jk}_i \hat{\xi}^i \tag{3}$$

where $\langle \hat{\xi}^j, X_k \rangle = \delta^j_k$. Also, if G is a group of isometries of given spacetime then X_i will be the Lie algebra generators and $\hat{\xi}^j$ would be the local coordinates of the group. This will be important in our calculations in Section 4.2.

Additionally, in some cases it is found that the 1-cocycle δ is the coboundary, which allows us to explicitly describe the cocommutator map of the Lie bialgebra in terms of r-matrix.

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r] \qquad \forall X \in g \tag{4}$$

where r is the classical r-matrix given by, $r = r^{ab} X_a \wedge X_b$ which is the solution of the following MYCBE,

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]] = 0, \forall X \in g$$

[1] note that in the following discussion we are utilizing the Einstein summation convention of suppressing summation over repeated indices

We also mention the coisotropy condition which is as follows: Let H be a subgroup of the group G. Then we say that the 1-cocycle δ fulfils the coisotropy condition if the following holds,

$$\delta(h) \subset h \wedge g, \qquad h = \mathrm{Lie}(h), \qquad g = \mathrm{Lie}(G) \tag{5}$$

We are now in a position to discuss how Lie bialgebras can be alternatively described as Drinfel'd double (DD) Lie algebras. These Lie algebras are given by the relations,

$$[X_i, X_j] = C_{ij}^k X_k, \quad [\hat{\xi}^j, \hat{\xi}^k] = f_i^{jk} \hat{\xi}^i, \quad [\hat{\xi}^i, X_j] = C_{jk}^i \hat{\xi}^k - f_j^{ik} X_k \tag{6}$$

and the corresponding Lie group $D(G)$ is endowed with a PL structure which is generated by the canonical classical r-matrix,

$$r = \sum_i x^i \otimes X_i = 0 \tag{7}$$

which is a solution to the CYBE, $[[r, r]] = 0$.

We will use the definition of Lie bialgebras that was introduced in Section 2.1, where the cobracket is given by (1). It is possible to prove that every quantum group G_z (where the quantum deformation parameter is $z = \ln q$) is in one-to-one correspondence with a PL group G, where the latter is in a one-to-one correspondence with a Lie bialgebra structure [12]. In other words, there is a bijective mapping from the set of all quantum groups to the set of Lie bialgebras (g, δ). This bijection between PL structures on Lie groups and Lie bialgebras only exists for connected and simply connected Lie groups G.

Note that quantum groups provide us with a description of quantum algebras, which are Hopf algebra deformations $U_z(g)$ of the universal enveloping algebra $U(g)$. There is a Hopf algebra structure to this deformation given by the coassociative coproduct map $\Delta_z : U_z(g) \longrightarrow U_z(g) \otimes U_z(g)$ along with the counit ϵ and the antipode mapping γ. The first order deformation in z of the coproduct map is given by,

$$\Delta_z = \Delta_0 + z\delta + o[z^2]$$

where δ is the Lie bialgebra cocommutator map and Δ_0 is the primitive coproduct for $U(g)$ defined as

$$\Delta_0 = X \otimes 1 + 1 \otimes X$$

In this way, we see the clear relationship between quantum groups and Lie bialgebras. Therefore the problem of classifying quantum deformations is equivalent to classifying Lie bialgebra structures.

Additionally, if a given even-dimensional Lie algebra is a DD one, i.e., it can be expressed in the form (6) then the quantum deformation associated to the classical r-matrix (7) is known as the DD deformation. Thus, a coboundary Lie bialgebra would correspond to a particular solution to the MCYBE. Therefore, the classification of coboundary Lie bialgebras and as such the quantum deformations of DD type, corresponds to that of finding solutions (inequivalent under automorphisms) to the MCYBE.

3 Tasks

3.1 An Overview of the Various Tasks Performed

A number of tasks were provided to me by Dr. Pachol as a medium to test my understanding of the material we had discussed during our weekly meetings. These tasks were mainly of computational nature where I had to verify axioms for a certain example, or compute some relations, eg the cobrackets for a particular Lie bialgbra. There were a total of 15 tasks that I performed, broadly, these tasks included,

1. Working with Lie algebras: Verifying commutation rules for the Casimir element, checking the relations between differential representations of some Lie algebras.
2. Working with Hopf algeras: Verifying the axioms of coalgebra and antipode mappings for various examples such as the algebra of functions over a group, tensor algebra, Sweedler's example.
3. Exercises on universal enveloping of Lie algebras: Acquiring some experience with the construction of the universal enveloping algebra of a Lie algebra, and verifying certain relations.
4. Working with Lie bialgebras: Similarly to Hopf algebras, here the goal was to familiarize myself with the axioms of Lie bialgebras and working out some examples through explicit calculations, such as verifying the cocyle condition for $\mathfrak{sl}_2$.

3.2 Some Important Calculations

This subsection contains examples of some of the tasks that were carried out. This will also demonstrate how to use the definitions that are included in Section 2.

We consider some important examples of Hopf algebras, such as **Sweedler's example**. This is the Hopf algebra generated by $\{1, g, x, gx\}$ with the following relations,

$$g^2 = 1, \qquad x^2 = 0, \qquad xg = -gx$$

with the coalgebra structure being given by the following relations,

$$\begin{aligned} \Delta(g) &= x \otimes g, \qquad \varepsilon(g) &&= 1, \\ \Delta(x) &= x \otimes 1 + g \otimes x, \qquad \varepsilon(x) &&= 0 \end{aligned}$$

and the antipode is given by $S(g) = g$ and $S(x) = -gx$. Using this structure it is straightforward to prove that this is indeed a Hopf algebra i.e. the axioms (A1), (A2), (C1), (C2) and the antipode axiom are satisfied. Not only that, but this also happens to be a noncommutative and noncocommutative Hopf algebra. Noncommutativity is easy to see from the defining relations. To see the noncocommutativity we need to show that $\Delta \neq \tau \circ \Delta$. This can be seen from the following calculations,

$$\begin{aligned} \tau \circ \Delta(x) &= \tau(x \otimes 1 + g \otimes x) \\ &= \tau(x \otimes 1) + \tau(g \otimes x) \\ &= 1 \otimes x + x \otimes g \\ &\neq x \otimes 1 + g \otimes x \\ &= \Delta(x) \end{aligned}$$

Other important examples of Hopf algebras include the algebra of functions over a group, tensor algebra, the universal enveloping algebra of a Lie algebra. In fact any Lie algebra may be endowed with a Hopf algebra structure by considering the universal enveloping algebra of that Lie algebra. We will focus on some calculations on the example of the $\mathfrak{sl}_2$ Lie algebra. The Lie brackets of this Lie algebra are given by the relations,

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H$$

where the generators $\{H, Y, X\}$ of $\mathfrak{sl}_2$ can be represented in matrix form:

$$H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

One can equipp $\mathfrak{sl}_2$ with a Lie bialgebra structure by defining the cocommutator on the generators as following,

$$\delta(H) = 0, \qquad \delta(X) = \frac{1}{2}(X \wedge H), \qquad \delta(Y) = \frac{1}{2}(Y \wedge H).$$

It is easy to see that the first axiom for the Lie bialgebra (LB1), the anti-commutativity of the cocommutator map is satisfied, that is,

$$\delta(A) + \tau(\delta(A)) = 0, \quad \forall A \in \{H, X, Y\}$$

We will first check the third axiom (LB3) for Lie bialgebras. Clearly, for the generator H this does hold true. The other two generators are of the form $\delta(A) = \frac{1}{2}A \wedge H$ where $A \in \{X, Y\}$. Consider the following calculation,

$$\begin{aligned}
(\delta \otimes id)(\delta(A)) &= \frac{1}{2}(\delta(A) \wedge H) \\
&= \frac{1}{2} \cdot \frac{1}{2}(A \otimes H - H \otimes A) \wedge H \\
&= \frac{1}{4}(A \otimes H \otimes H - H \otimes A \otimes H - H \otimes A \otimes H + H \otimes H \otimes A) \\
&= \frac{1}{4}(A \otimes H \otimes H - 2H \otimes A \otimes H + H \otimes H \otimes A) \\
\implies \sum_{cyc}(\delta \otimes id)(\delta(A)) &= \sum_{cyc} \frac{1}{4}(A \otimes H \otimes H - 2H \otimes A \otimes H + H \otimes H \otimes A) \\
&= A \otimes H \otimes H - 2H \otimes A \otimes H + H \otimes H \otimes A \\
&\quad + A \otimes H \otimes H + H \otimes A \otimes H - 2H \otimes H \otimes A \\
&\quad - 2A \otimes H \otimes H + H \otimes A \otimes H + H \otimes H \otimes A \\
&= 0.
\end{aligned}$$

Thus (LB3) is verified for the generator $A \in \{X, Y\}$. Since the we know that $\delta(H) = 0$ clearly, the condition (LB3) is satisfied. We can now prove the (LB2) condition. Using the relation $\mathrm{ad}_X(Y) = [X, Y]$ and (1) we get,

$$\begin{aligned}
&\mathrm{ad}_X(\delta(H)) = \mathrm{ad}_Y(\delta(H)) = 0 && \mathrm{ad}_H(\delta(X)) = X \wedge H \\
&\mathrm{ad}_X(\delta(Y)) = \mathrm{ad}_Y(\delta(X)) = X \wedge Y && \mathrm{ad}_H(\delta(Y)) = Y \wedge H
\end{aligned}$$

from this we can clearly see that,

$$\begin{aligned}
\mathrm{ad}_H(\delta(X)) - \mathrm{ad}_X(\delta(H)) &= X \wedge H - 0 = \delta(2X) = \delta([H, X]) \\
\mathrm{ad}_H(\delta(Y)) - \mathrm{ad}_Y(\delta(H)) &= Y \wedge H - 0 = \delta(2Y) = \delta([H, Y]) \\
\mathrm{ad}_X(\delta(Y)) - \mathrm{ad}_Y(\delta(X)) &= X \wedge Y - X \wedge Y = 0 = \delta(H) = \delta([X, Y])
\end{aligned}$$

Thus (LB2) is verified. We can now repeat the calculations but this time by using the relation (1). Then we get,

$$\begin{aligned}
[\delta(H), X \otimes 1 + 1 \otimes X] &= [\delta(H), Y \otimes 1 + 1 \otimes Y] = 0 \\
[\delta(X), Y \otimes 1 + 1 \otimes Y] &= Y \wedge X \\
[X \otimes 1 + 1 \otimes X, \delta(Y)] &= -Y \wedge X \\
[H \otimes 1 + 1 \otimes H, \delta(X)] &= X \wedge H \\
[H \otimes 1 + 1 \otimes H, \delta(Y)] &= H \wedge Y
\end{aligned}$$

where Lemma (4.1) has been used to simplify calculations. Therefore we can verify the relations,

$$\begin{aligned}
\delta([X, Y]) &= \delta(H) = 0 = Y \wedge X - Y \wedge X \\
&= [\delta(X), Y \otimes 1 + 1 \otimes Y] + [X \otimes 1 + 1 \otimes X, \delta(Y)] \\
\delta([H, X]) &= X \wedge H = [H \otimes 1 + 1 \otimes H, \delta(X)] + [\delta(Y), H \otimes 1 + 1 \otimes H] \\
\delta([H, Y]) &= H \wedge Y = [H \otimes 1 + 1 \otimes H, \delta(Y)] + [\delta(H), Y \otimes 1 + 1 \otimes Y]
\end{aligned}$$

this proves (LB2). With this, we conclude this section.

4 Paper on Quantum Groups and Noncommutative Spacetimes

In this section we will build upon the tools recalled in Section 2 and 3, and explore the material presented in the research paper [1], understanding of which was the primary aim of the internship. Section 4.1 gives an overview of the ideas presented in the paper, whereas section 4.2 dives into calculations.

4.1 Main Ideas of the Paper

The paper is a review of paper from a conference proceeding, which summarized some of the results that have been recently obtained in the field of mathematical physics, pertaining to the construction of noncommutative spacetimes from quantum groups. The paper is divided into three sections; the first section introduces the motivation for the paper while the second section provides a review of how noncommutative spacetimes are constructed using quantum groups. The third and final section provides the explicit construction of $(2+1)$ AdS_ω, $(2+1)$ twisted κ-AdS_ω and $(3+1)$ AdS_ω noncommutative spacetimes. These constructions are made by utilizing the bijective correspondence between a quantum group and Lie bialgebra structure as well as the fact that each Lie bialgebra structure can be alternatively expressed as a Drinfel'd Double (DD) Lie algebra. Since (A)dS and Poincaré algebras in $(2+1)$ and $(3+1)$ dimensions, all 1-cocycles δ are coboundaries; the problem of classifying all quantum deformations of these Lie algebras is equivalent to finding all non-isomorphic solutions of the modified classical Yang-Baxter equation.

In the introduction, authors describe three different ways in which noncommutative spacetimes are introduced - i) first one being the example of the nonrelativistic quantum mechanics where noncommutativity is controlled by the fundamental constant $\hbar$. ii) second example is obtained from the $(3+1)$ AdS Lie algebra where the noncommutativity is controlled by the cosmological constant. iii) the third example is one arising from certain approaches to quantum gravity by means of quantum groups, where the noncomutativity of spacetime is controlled by the parameter, l_P, Planck length. The authors focus on the latter of these three, where they summarize the recent results obtained in the field concerning the construction of noncommutative AdS spacetimes.

I will begin by first defining the terms presented in the research paper [1] as well as providing a comprehensive overview of the underlying ideas; and then discuss the construction of the noncommutative spacetimes.

4.2 Calculations of the Paper

We can now get to the heart of the paper, the third section; where one learns how noncommutative spacetimes are constructed based on the solutions of the MCYBE. We start by recalling two Lie algebras,

- $(3+1)$ **Anti de Sitter Algebra** is given by,

$$\begin{array}{lll} [J_i, J_j] = \epsilon_{ijk} J_k & [J_i, P_j] = \epsilon_{ijk} P_k & [J_i, K_j] = \epsilon_{ijk} K_k \\ [P_i, P_j] = -\omega \epsilon_{ijk} J_k & [P_i, K_j] = -\delta_{ij} P_0 & [K_i, K_j] = -\epsilon_{ijk} J_k \\ [P_0, P_i] = \omega K_i & [P_0, K_i] = -P_i & [P_0, J_i] = 0 \end{array} \tag{8}$$

  Here, $i, j = 1, 2, 3$, $\epsilon_{123} = 1$ and $\omega = -\Lambda$. Here, Λ is the cosmological constant and J_i, P_0, P_i, K_i denote the generators of rotations, time translation, space translations and boosts respectively. This algebra (8) is called AdS_ω Lie algebra.

- $(2+1)$ **$\mathbf{AdS}_\omega$ Algebra :** The algebra (8) is a $(3+1)$ algebra, when we consider only $(2+1)$ dimensions, the relation (8) reduces to the following,

$$\begin{array}{lll} [J,P_0]=0 & [J,P_i]=\epsilon_{ij}P_j & [J,K_i]=\epsilon_{ij}K_j \\ [P_0,K_i]=-P_i & [P_i,K_j]=-\delta_{ij}P_0 & [K_1,K_2]=-J \\ [P_0,P_i]=\omega K_i & [P_1,P_2]=-\omega J & \end{array} \tag{9}$$

where $i,j=1,2$ and $\epsilon_{12}=1$. We define this as the $(2+1)$ AdS_ω Lie algebra.

We shall make use of both (8) and (9) in the next section.

Before reproducing the calculations of the paper, let us derive a lemma which will make our calculations significantly easier.

Lemma 4.1. $[X\otimes 1+1\otimes X, Y\wedge Z]=[X,Y]\wedge Z+Y\wedge[X,Z]$

Proof.

$$\begin{aligned}
[X\otimes 1+1\otimes X, Y\wedge Z] =& [X\otimes 1+1\otimes X, Y\otimes Z-Z\otimes Y]\\
=&[X\otimes 1+1\otimes X, Y\otimes Z]-[X\otimes 1+1\otimes X, Z\otimes Y]\\
=&[X\otimes 1, Y\otimes Z]+[1\otimes X, Y\otimes Z]-[X\otimes 1, Z\otimes Y]-[1\otimes X, Z\otimes Y]\\
=&(X\otimes 1)(Y\otimes Z)-(Y\otimes Z)(X\otimes 1)\\
&+(1\otimes X)(Y\otimes Z)-(Y\otimes Z)(1\otimes X)\\
&-(X\otimes 1)(Z\otimes Y)+(Z\otimes Y)(X\otimes 1)\\
&-(1\otimes X)(Z\otimes Y)+(Z\otimes Y)(1\otimes X)\\
=&(XY\otimes Z-YZ\otimes Z)+(Y\otimes XZ-Y\otimes ZX)\\
&-(XZ\otimes Y-ZX\otimes Y)-(Z\otimes XY-Z\otimes YX)\\
=&[X,Y]\otimes Z+Y\otimes[X,Z]-[X,Z]\otimes Y-Z\otimes[X,Y]\\
=&[X,Y]\wedge Z+Y\wedge[X,Z]
\end{aligned}$$

□

Now that we have this lemma, we shall refer to it repeatedly throughout the calculations in the following subsections.

4.2.1 Noncommutative Spacetime from $(2+1)$ anti-de Sitter Lie Algebra

For the case of $(2+1)$ AdS_ω, the r-matrix is found to be,

$$r_1 := \eta J\wedge K_1 - \tfrac{1}{2}(-J\wedge P_0 - K_2\wedge P_1 + K_1\wedge P_2) \tag{10}$$

where $\Lambda = -\eta^2 = -\omega < 0$. Using (4) and the linearity of the Lie bracket we get,

$$\begin{aligned}\delta(X) =&[X \otimes 1 + 1 \otimes X, \eta J \wedge K_1] \\ &+ \tfrac{1}{2}[X \otimes 1 + 1 \otimes X, J \wedge P_0] \\ &+ \tfrac{1}{2}[X \otimes 1 + 1 \otimes X, K_2 \wedge P_1] \\ &- \tfrac{1}{2}[X \otimes 1 + 1 \otimes X, K_1 \wedge P_2]\end{aligned}$$

Utilizing the lemma 4.1, we get,

$$\begin{aligned}\delta(X) = \quad & \eta[X, J] \wedge K_1 + \eta J \wedge [X, K_1] + \tfrac{1}{2}[X, J] \wedge P_0 + \tfrac{1}{2} J \wedge [X, P_0] \\ & + \tfrac{1}{2}[X, K_2] \wedge P_1 + \tfrac{1}{2} K_2 \wedge [X, P_1] - \tfrac{1}{2}[X, K_1] \wedge P_2 - \tfrac{1}{2} K_1 \wedge [X, P_2]\end{aligned} \tag{11}$$

We can now use equation (11) to calculate the cocommutators for each of the generators using the Lie algebra relations that are given by equations (9). The results are,

$$\begin{aligned}&\delta(J) = -\eta J \wedge K_2 \quad \delta(K_1) = -\eta K_2 \wedge K_1 \quad \delta(K_2) = 0 \\ &\delta(P_0) = P_1 \wedge P_2 + \eta P_1 \wedge J - \eta^2 K_1 \wedge K_2 \\ &\delta(P_1) = P_0 \wedge P_2 + \eta P_0 \wedge J - \eta P_2 \wedge K_1 + \eta^2 K_1 \wedge J \\ &\delta(P_2) = P_1 \wedge P_0 + \eta P_1 \wedge K_1 - \eta^2 J \wedge K_2\end{aligned} \tag{12}$$

It is worth noting that from (12) one can see that the coisotropy condition (5) is satisfied. Which requires that there exists a sub Lie algebra, h of (9) such that $\forall x \in h, \delta(x) \in h \wedge g$ where g is the $(2+1)$ anti-de Sitter Lie algebra. One can see the sub Lie algebra h generated by $\{K_1, K_2, J\}$ fulfils the requirement.

We now turn our attention back to calculating the noncommutative spacetime. We first need to calculate the structure constants. This will help us determine the relations of the coordinate generators of the noncommutative spacetime. Utilising (2) and the duality of momenta and coordinates, (3) we get the following relations,

$$\begin{aligned}&f_0^{01} = f_0^{02} = f_1^{01} = f_1^{01} = f_1^{12} = f_2^{02} = f_2^{12} = 0 \\ &f_0^{12} = 1 \qquad f_1^{02} = 1 \qquad f_2^{01} = -1\end{aligned} \tag{13}$$

Using (13) and (3) to calculate the Lie algebra commutative relations between the coordinate generators of spacetime gives us,

$$[\hat{x}_0, \hat{x}_1] = -\hat{x}_2 \qquad [\hat{x}_0, \hat{x}_2] = \hat{x}_1 \qquad [\hat{x}_1, \hat{x}_2] = \hat{x}_0$$

This finishes the construction of the $(2+1)$ noncommutative spacetime from $(2+1)$ AdS_ω. The relations we have obtained above tell us the relations of the Lie algebra to the first order. In general, the relations for the Poisson noncommutative spacetime is given by,

$$\{x_0, x_1\} = -\frac{\tanh \eta x_2}{\eta}\gamma(x_0, x_1) = -x_2 + o[\eta] \tag{14}$$

$$\{x_0, x_2\} = \frac{\tanh \eta x_1}{\eta}\gamma(x_0, x_1) = x_1 + o[\eta] \tag{15}$$

$$\{x_1, x_2\} = \frac{\tan \eta x_0}{\eta}\gamma(x_0, x_1) = x_0 + o[\eta] \tag{16}$$

Here, $\gamma(x_0, x_1) = \cos \eta x_0 (\cos \eta x_0 \cosh \eta x_1 + \sinh \eta x_1)$We can derive the relations on the right hand side of each of the equations (13), (14) and (15) by expanding the Taylor series. First note that,

$$\begin{aligned} f(x_0, x_1) &:= \cos \eta x_0 \cosh \eta x_1 + \sinh \eta x_1 \\ &= (1 - o[\eta])(1 + o[\eta]) + o[\eta] \\ &= 1 - o[\eta] + o[\eta] + o[\eta]o[\eta] + o[\eta] \\ &= 1 + o[\eta] \end{aligned} \tag{17}$$

Therefore from equation (17) and the Taylor series expansion of cos we get,

$$\gamma(x_0, x_1) = \cos \eta x_0 f(x) = (1 + o[\eta])(1 + o[\eta]) = 1 + o[\eta]$$

which can be used along with the Taylor series expansions of tanh and tan to calculate the following relations,

$$\begin{aligned} \{x_0, x_1\} &= -\frac{\tanh \eta x_2}{\eta}\gamma(x_0, x_1) \\ &= -\frac{1}{\eta}(\eta x_2 + o[\eta^2])(1 + o[\eta]) \\ &= -x_2 + o[\eta] \end{aligned}$$

$$\begin{aligned} \{x_0, x_2\} &= \frac{\tanh \eta x_1}{\eta}\gamma(x_0, x_1) \\ &= \frac{1}{\eta}(\eta x_1 + o[\eta^2])(1 + o[\eta]) \\ &= -x_0 + o[\eta] \end{aligned}$$

$$\begin{aligned} \{x_1, x_2\} &= \frac{\tan \eta x_0}{\eta}\gamma(x_0, x_1) \\ &= \frac{1}{\eta}(\eta x_0 + o[\eta^3])(1 + o[\eta]) \\ &= x_0 + o[\eta] \end{aligned}$$

This proves the second equality in the equations (14), (15) and (16).

4.2.2 Noncommutative Spacetime from $(2+1)$ twisted κ-AdS$_\omega$

Let us now take a look at the twisted case, where we have two deformation parameters, z and ϑ instead of just one like in the previous section. This time the r-matrix is given by,

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2) + \vartheta J \wedge P_0 \tag{18}$$

Using the linearity of the Lie bracket and lemma 4.1, the cocommutator map takes the following general form,

$$\begin{aligned}\delta(X) =&[X \otimes 1 + 1 \otimes X, r] \\ =&[X \otimes 1 + 1 \otimes X, z(K_1 \wedge P_1 + K_2 \wedge P_2) + \vartheta J \wedge P_0] \qquad (19)\\ =&z[X, K_1] \wedge P_1 + zK_1 \wedge [X, P_1] \\ &+ z[X, K_2] \wedge P_2 + zK_2 \wedge [X, P_2] \\ &+ \vartheta[X, J] \wedge P_0 + \vartheta J \wedge [X, P_0] \qquad (20)\end{aligned}$$

We can now use equation (18) along with the Lie algebra relations in (9) to derive the cocommutator relations. They are as following,

$$\begin{aligned}\delta(J) &= \delta(P_0) = 0 && (21)\\ \delta(P_1) &= z(P_1 \wedge P_0 - \omega K_2 \wedge J) + \vartheta(P_0 \wedge P_2 + \omega K_1 \wedge K) && (22)\\ \delta(P_2) &= z(P_2 \wedge P_0 + \omega K_1 \wedge J) - \vartheta(P_0 \wedge P_1 - \omega K_2 \wedge J) && (23)\\ \delta(K_1) &= z(K_1 \wedge P_0 + P_2 \wedge J) + \vartheta(P_0 \wedge K_2 - P_1 \wedge J) \\ \delta(K_2) &= z(K_2 \wedge P_0 - P_1 \wedge J) - \vartheta(P_0 \wedge K_1 + P_2 \wedge J)\end{aligned}$$

The coisotropy condition (5) is once again satisfied by choosing the sub-Lie algebra h to be the one generated by J and P_0.

Once again, utilising the duality of momenta and coordinates mentioned in (3) along with equations (2) and (19), we can calculate the structure constants, which are,

$$\begin{aligned}&f_0^{01} = f_0^{02} = f_0^{12} = 0 \\ &f_1^{01} = -z \quad f_1^{02} = \vartheta \quad f_1^{12} = 0 \\ &f_2^{01} = -\vartheta \quad f_2^{02} = -z \quad f_2^{12} = 0\end{aligned}$$

Therefore the first order noncommutative spacetime relations are,

$$
\begin{aligned}
[\hat{x}_1, \hat{x}_2] &= \sum_{i=0}^{2} f_i^{12} \hat{x}_i = 0 \\
[\hat{x}_0, \hat{x}_1] &= \sum_{i=0}^{2} f_i^{01} \hat{x}_i = f_0^{01} \hat{x}_0 + f_1^{01} \hat{x}_1 + f_2^{01} \hat{x}_2 = -z\hat{x}_1 - \vartheta \hat{x}_2 \\
[\hat{x}_0, \hat{x}_2] &= \sum_{i=0}^{2} f_i^{02} \hat{x}_i = f_0^{02} \hat{x}_0 + f_1^{02} \hat{x}_1 + f_2^{02} \hat{x}_2 = \vartheta \hat{x}_1 - z\hat{x}_2
\end{aligned}
$$

Which concludes our calculations for this section.

4.2.3 Noncommutative Spacetime from $(3+1)$ AdS$_\omega$

We now shift our attention to the Lie algebra given by the equation (8). We would like to find the commutation relations for the corresponding noncommutative spacetime. For this, we only require the cocommutator map of the momenta generators, P_i. Here, the r-matrix is given by,

$$r_3 := z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega} J_1 \wedge J_2) + \vartheta J_3 \wedge P_0 \tag{24}$$

Something worth mentioning here is that there is a typo in Section 3.3 of the paper [1]. If we take a look at the preprint of this paper, or the paper that they have cited in reference to these calculations [13] , one observes that the r-matrix is different. The r-matrix given in the paper [1] is,

$$r_3 := z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega} J_3 \wedge J_1) + \vartheta P_0 \wedge J_2 \tag{25}$$

As we can clearly see, the rotation generators J_2 and J_3 are swapped in equations (25) and (24). As such, the convention chosen in (25) is that of a non-right handed coordinate system. In this report, we shall perform calculations using the r-matrix given in (24).

To calculate the noncommutative spacetime relations, we adopt a process similar to that of deriving (12), utilizing the Lemma 4.1 allows us to derive the following expression for the cocommutator map,

$$
\begin{aligned}
\delta(X) =&[X \otimes 1 + 1 \otimes X, r_3] \\
=&z[X, K_1] \wedge P_1 + zK_1 \wedge [X, P_1] \\
&+ z[X, K_2] \wedge P_2 + zK_2 \wedge [X, P_2] \\
&+ z[X, K_3] \wedge P_3 + zK_3 \wedge [X, P_3] \\
&+ z\sqrt{\omega}[X, J_1] \wedge J_2 + z\sqrt{\omega}J_1 \wedge [X, J_2] \\
&+ \vartheta[X, J_3] \wedge P_0 + \vartheta J_3 \wedge [X, P_0]
\end{aligned} \tag{26}
$$

Utilizing the above cocommutator map along with the Lie algebra relations in (8) we obtain the following cocommutator maps for the generators of momenta,

$$
\begin{aligned}
\delta(P_0) =&0 \\
\delta(P_1) =&z(P_1 \wedge P_0 - \omega J_3 \wedge K_2 + \sqrt{\omega}J_1 \wedge P_3) - \vartheta(P_2 \wedge P_0 + J_3 \wedge K_1) \\
\delta(P_2) =&z(P_2 \wedge P_0 - \omega J_3 \wedge K_1 + \omega J_1 \wedge K_3 + \sqrt{\omega}J_2 \wedge P_3) \\
&+ \vartheta(P_1 \wedge P_0 + \omega J_3 \wedge K_1) \\
\delta(P_3) =&z(P_3 \wedge P_0 - \omega J_1 \wedge K_2 + \omega J_2 \wedge K_1 - \sqrt{\omega}J_1 \wedge P_1 - \sqrt{\omega}J_2 \wedge K_2) \\
&+ \omega\vartheta K_3 \wedge J_2
\end{aligned}
$$

Note that we have only calculated the cocommutator associated with P_i as these are dual to the coordinates for spacetime relations. Utilising this duality along with the equation (2) we can calculate the structure constants,

$$
f_1^{01} = f_2^{03} = f_3^{03} = -z \qquad f_1^{02} = \vartheta \qquad f_2^{01} = -\vartheta
$$

with all the remaining ones being zero. Now we calculate the spacetime commutative relations using (3).

$$
\begin{aligned}
[\hat{x}_0, \hat{x}_1] &= \sum_{i=0}^{3} f_i^{01}\hat{x}_i = f_0^{01}\hat{x}_0 + f_1^{01}\hat{x}_1 + f_2^{01}\hat{x}_2 + f_3^{01}\hat{x}_3 = -z\hat{x}_1 - \vartheta\hat{x}_2 \\
[\hat{x}_0, \hat{x}_2] &= \sum_{i=0}^{3} f_i^{01}\hat{x}_i = f_0^{02}\hat{x}_0 + f_1^{02}\hat{x}_1 + f_2^{02}\hat{x}_2 + f_3^{02}\hat{x}_3 = \vartheta\hat{x}_1 - z\hat{x}_2 \\
[\hat{x}_0, \hat{x}_3] &= \sum_{i=0}^{3} f_i^{03}\hat{x}_i = f_0^{03}\hat{x}_0 + f_1^{03}\hat{x}_1 + f_2^{03}\hat{x}_2 + f_3^{03}\hat{x}_3 = -z\hat{x}_3 \\
[\hat{x}_a, \hat{x}_b] &= 0 \qquad\qquad \forall a, b = 1, 2, 3
\end{aligned}
$$

It is worth mentioning that the typo in the r-matrix given in the paper [1] doesn't extend to the spacetime relations, as the spacetime relations we have calculated above are precisely the ones which appear in the paper.

Following this description of the noncommutative spacetime obtained from $(3+1)$ AdS_ω, the paper concludes with a brief note on the Casimir operators associated with the AdS_ω from (8). Here, a major problem is to find the explicit form of the deformed version of this Casimir. Two different Casimir are mentioned, one arising from the Killing-Cartan form, and another one that is fourth-order invariant, with the explicit form of the former being given. The paper concludes by mentioning open problems in the field along with recent advancements in that direction.

5 An Experience With Quantum Groups

In the initial stages this project was certainly very overwhelming, I was learning about objects that I had never seen before through books that I wasn't fully able to follow. Even though at times I felt like giving up, I stayed determined to understand the subject.The material was challenging in just the right aspects, I constantly felt drawn towards it. I haven't felt this engrossed into a topic for quite a long time, although it may sound like an exaggeration, it is quite truthful to say that I hadn't felt this alive in quite some time. I was engaged in a manner I hadn't experienced in years.

Academically, of course, I got to learn a lot. All those things have been mentioned throughout this report. But more than that, this internship taught me a lot of things about my study habits themselves. One of the primary reason that during the initial two weeks the content was so overwhelming was due to my perfectionist attitude towards my studies. Prior to this, I was the student who would study every tiny aspect in excruciating detail, to the extent that I would often find myself chasing tangents that were never really relevant to what I was learning. I realised that while this approach did help me understand certain concepts and motivation for topics better, it was far from practical. Once I realised this, I made myself keep certain questions in a black-box and return to them later. I was no longer reading through an entire textbook of Lie algebras just to understand the motivation behind the definition; rather I learned to take certain definitions 'on faith' as being useful and progress with what was most relevant to me at the time. Learning only the ideas and tools directly relevant to me was an important skill I acquired. I could always return to this black-box later on and try to answer those questions. This made learning new things significantly easier and the project a lot less overwhelming. This was certainly the important learning experience I had from this internship, often not chasing tangents is the best way to learn something even though at times definitions do feel unmotivated. In fact, once I was confident with the basics I returned to this black-box and was able to answer many questions on my own.

Abandoning this perfectionist approach was essential to the main purpose of the project, which was understanding the research paper. When I was reading the paper, there were many concepts that weren't familiar to me, as such I only focused on the main ideas of the paper. This taught me how to read scientific papers as well. I am pretty sure that had I not abandoned my initial attitudes towards studies, I would've tried to go through every citation to understand the material, which of course wasn't necessary at all.

Another thing I realised is how important it is to revisit the topics you have learned later on. At the end of the internship when I revisited my notes I was able to appreciate the relationship between various concepts far better than before which helped me understand the material much better. Counter-intuitively, not chasing perfectionism made it easier to understand the material and made my experience far more enriching. These are some of the learnings that I intend to implement in my future studies as well. After the duration of the internship period I also went through certain sections of the textbooks and read about the things that I found interesting at a first glance. In particular, I would like to explore the applications of quantum groups to noncommutative geometry and knot theory.

Overall, this project was incredibly beneficial for me in a lot of ways. I got to learn a lot academically but also a lot about professional scientific communication. While I was familiar with certain aspects of LaTeX before, during this project I learned a lot more packages such as tikz and beamer. I am incredibly grateful to Dr. Pachol for providing me opportunity as well as a nurturing environment where I could learn all these things.

6 Conclusion

In conclusion, the underlying mathematical framework of quantum groups was studied, applications of quantum groups in the construction of $(2+1)$ AdS and Poincaré spacetimes were studied by using paper [1] as a reference.

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