Sets and Logic

Readings

1. Book of Proofs, Hammock — Chapter 1 and Chapter 2
2. Calculus Volume 1, Tom Apostol — Part 2 of the Introduction

Exercises From Books

- The following exercises from Hammock chapter 2 — Exercises for sections 2.1 — 2.7 , Section 2.9.
- Questions 1-7 in Exercises $I2.5$ in the second part of the Introduction of Apostol's book.

Challenge Problems

1. Consider the following statement — P : "all swans are white". Is this statement true or false? Write this statement in symbolic logic. Suppose you are negating this statement. What would the truth table of the negation of this statement, $\sim P$ look like? Can you write the negation of this statement in symbolic logic?

Note : Do the first part of this question based on what was covered in the lecture. In Hammock's book, he talks about the negation of statements involving quantifiers. However I would like you do this exercise without reading the section 2.10

Hint : Think about the meaning of the statement P. Suppose you are a scientist. You set out to determine via empirical observations whether or not this statement is true. Is there any observation that would disprove this statement? Would this observation be the negation of this statement? Based on this, can you write the negation of the statement P in symbolic logic?

Once you have given enough thought to this question and you have an answer read through the section 2.10 of the Hammock book. Solve exercises of the section 2.10

2. We discussed that proofs are essentially logical deductions of various statements. As such it is important to understand how to draw correct logical inferences.

    a. Consider the following situation. You are given that $P \implies Q$ is true and that P is true. What can you deduce about Q? Write an example using English where this deduction holds true.

    b. You are given that $P \implies Q$ is true and that Q is false (in other words, Q doesn't happen and $\sim Q$ is true.) What can you say about P in this case?

    c. You are given that $P \vee Q$ is true. and that P isn't True. That is, $\sim P$ is true. What can you infer about the truth value of Q in this case?

Once you are done with this exercise read through the section 2.10 of the Hammock book. Learn the formal names of the operations mentioned above.

3. In the lecture we discussed that the operation 'or' is ambiguous in the sense that depending on the context, in English, we use it in either the inclusive sense or the exclusive sense. How does our definition of the operation $\vee$ remove this ambiguity?

This takes care of the logic part. We now have a precise language in which we can communicate mathematics. Let us now turn to sets. In the upcoming lecture we will be formalizing the notion of sets and functions.

1. Consider the following two sets —

$$S = \{x | x \text{ is odd and } \ x \text{ is a prime number}\}$$
$$T = \{x | x \text{ is a non even prime number}\}$$

    Write down the first few elements of both of these sets explicitly. What do you notice? Would you call these sets as being 'equal' ? What if instead T was the set of all prime numbers, instead of the set of all non-even prime numbers. Would S and T be equal now?

2. Let us now formalize the idea of equality of sets. Write a description of what you intuitively understand by the equality of sets. Let us introduce a definition. We say that S is a subset of T if every element of S is an element of T. Write this condition in symbolic logic.

a. Is T a subset of the set T for any arbitrary set T? Prove your answer.

b. Suppose that for any two sets S and T, we have that S is a subset of T and that T is a subset of S. What can you infer about the elements of S and the elements of T? Would you say that these two sets are equal? Why? Does your answer to this question match the intuitive description that you gave for the equality of sets in questions 1? If not, then give an explanation for why the above criterion doesn't necessarily give equality of sets.

3. One way to think about a subset S of the set T is to imagine taking T and removing elements one by one until you get the set S. Now consider the set $\{\}$. This set is what we call the empty set, it doesn't contain any elements. Is it possible to arrive at the empty set by removing elements one by one from any set T? Does this mean that the empty set is a subset of any set?

4. Write down sets with $1, 2, 3, 4$ elements each. Write all the possible subsets of each of these sets. Do you notice a pattern? How many subsets does each of these sets have? Based on this can you make a guess, or a conjecture as we call it in mathematics about the number of subsets any set has?

5. Can you prove your answer in question 4? We will meet this again later in the course. We will see two proofs one using counting argument and the other using the principle of mathematical induction.

6. Suppose sets S and R are finite, that is they only have finite elements each. Suppose that $S \subset R$. What can you say about the number of elements in S and the number of elements in R? What if S and R are equal? Can you prove that their cardinality — the number of elements they contain — are also equal?

7. Consider the set $R = \{a, b\}$ and the set $S = \{R\}$. The set S contains only one element and that element is the set R. Are the sets S and R equal?

8. Consider the empty set $\{\}$. We denote this set by ϕ. Consider the set which contains the empty set, let's call it $S = \{\{\}\} = \{\phi\}$. Are the sets S and ϕ equal?

9. Read through the section on Russel's paradox in the Hammock's book. Is there a solution to this paradox? Write about how we removed this paradox

from arising in our mathematical works with the reference to the discussion that took place in class.