

# CSE276C - Integration of Functions

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## Outline

- 1 Introduction
- 2 ODE Introduction
- 3 Runge-Kutta
- 4 Richardson / Burlirsch-Stoer
- 5 Variable Dynamics
- 6 Partial Differential Equations
- 7 Summary

- Interested in integration of function to allow estimation of future value
- Lots of potential applications in robotics
  - Position estimation
  - Path optimization
  - Image restoration
- Consider both end-point and boundary value problems, which anchors the problem

## Introduction - Setting the stage

- We are trying to solve

$$I = \int_a^b f(x) dx$$

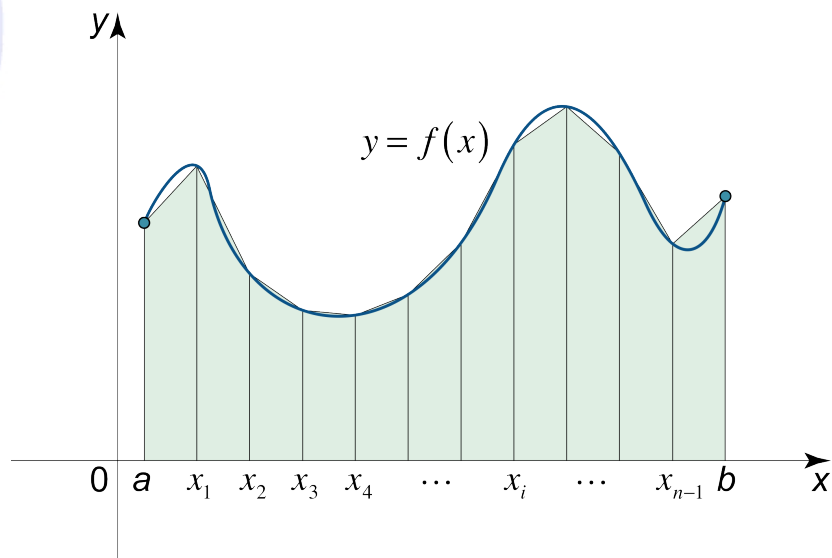
- trying to solve  $I = y(b)$  for the equation

$$\frac{\partial y}{\partial x} = f(x)$$

- with the boundary condition

$$y(a) = 0$$

- Objective to generate a good estimate of  $y(b)$  with a reasonable number of evaluations
- Emphasis on 1D problems, but in most cases generalization is straight forward



## Basic use of Simpson's rule

- Consider equally spaced data points

$$x_i = x_0 + ih \quad i = 0, 1, \dots, N$$

- the function is evaluated at  $x_i$

$$f_i = f(x_i)$$

- The Newton-Cotes rules is then

$$\int_{x_0}^{x_1} f(x) dx = \frac{f_1 + f_0}{2} h + O(f'' h^3)$$

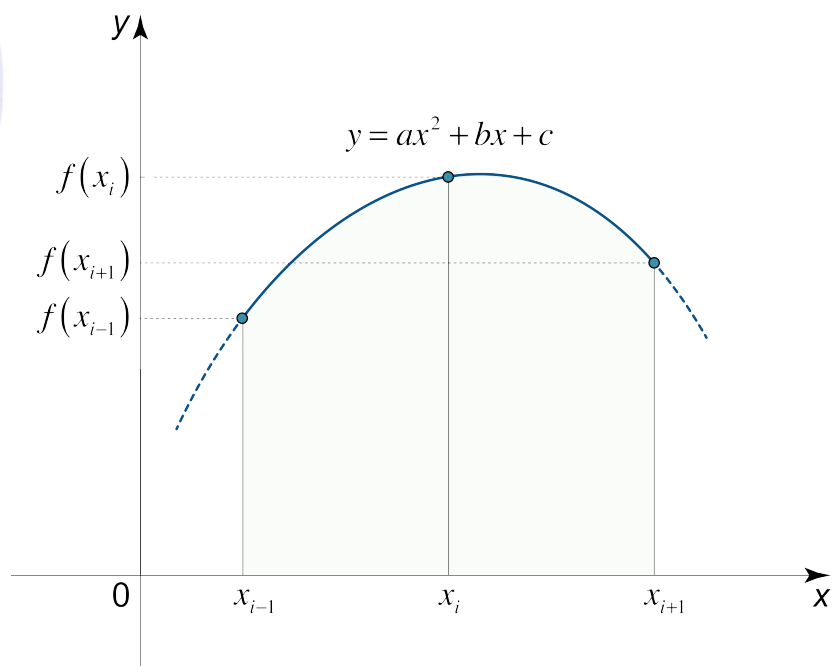
- The Simpson rule is

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} (f_0 + 4f_1 + f_2) + O(h^5 f^{(4)})$$

- which is exact to the 3rd degree
- The Simpson  $\frac{3}{8}$  rule

$$\int_{x_0}^{x_3} f(x) dx = \frac{h}{8} (3f_0 + 9f_1 + 9f_2 + 3f_3)$$

- There are a series of rules for higher order, check literature



## Simpson / Trapezoid Rules

- Clearly the local rules can be chained into a longer evaluation
- $(x_0, x_1), (x_1, x_2), \dots, (x_{N-1}, x_N)$  to get an extended trapezoid form

$$\int_{x_0}^{x_N} f(x) dx = h \left( \frac{1}{2} f_0 + f_1 + f_2 + \dots + f_{N-1} + \frac{1}{2} f_N \right)$$

- The error estimate is

$$O \left( \frac{(x_N - x_0) f''}{N^2} \right)$$

# Trapezoid Rule - Strategy?

- How can you effectively use the trapezoid rule?

# Trapezoid Rule - Strategy?

- How can you effectively use the trapezoid rule?
- Use of a coarse to fine strategy and watch convergence
- This is termed Romberg integration in numerical toolboxes
- In general these methods generate good accuracy for proper functions?

- What is an improper function?

## Handling of improper function

- What is an improper function?
  - 1 Integrand goes to a finite value but cannot be evaluated at a point, such as

$$\frac{\sin x}{x} \text{ at } x = 0$$

- 2 Upper limit is  $\infty$  or lower limit is  $-\infty$
- 3 Has a singularity at a boundary point, e.g.,

$$x^{-1/2} \text{ at } x = 0$$

- 4 Has a singularity within the interval at a known location
- 5 Has a singularity within the interval at an unknown location

- If the value is infinite, e.g.,

$$\int_0^{\infty} x^{-1} dx \text{ or } \int_{-\infty}^{\infty} \cos x dx$$

it is not improper but impossible

# The Euler-Maclaurin Summation Formula

- We can write the basic Simpson's rule as

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{N-1} f(a + kh) + f(b) \right] \\ &\quad - \sum_{k=1}^{N/2} \frac{h^{2k} B_{2k}}{(2k)!} [f^{(2k-1)}(b) - f^{(2k-1)}(a)] \\ &\quad - \sum_{k=0}^{N-1} \frac{h^{2k+1} B_{2k}}{(2k)!} f^{(2k)}(a + kh + \theta h)\end{aligned}$$

- where  $2m$  first derivatives are continuous over  $(a,b)$ .  $h = (a-b)/N$  and  $\theta \in (0,1)$
- So what are the B's?
- They are Bernoulli numbers

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

- example values

$$\begin{aligned}B_0 &= 1 \\ B_2 &= \frac{1}{6} \\ B_4 &= -\frac{1}{30}\end{aligned}$$

- Enables you to compute an estimate of the error for a particular integration
- Other integration functions have similar error functions - decreasing with

## Extended Mid-point Formulation

- In many cases using the mid-point is a valuable alternative

$$\int_{x_0}^{x_{N-1}} f(x)dx = h(f_{1/2} + f_{3/2} + \dots + f_{N-3/2}) + O\left(\frac{1}{N^2}\right)$$

- When combined with the Euler-Maclaurin you get

$$\begin{aligned}\int_{x_0}^{x_{N-1}} f(x)dx &= h(f_{1/2} + f_{3/2} + \dots + f_{N-3/2}) \\ &\quad + \frac{B_2 h^2}{4} (f'_{N-1} - f'_0) + \dots + \frac{B_{2k} h^{2k}}{(2k)!} (f^{(2k)}_{N-1} - f^{(2k)}_0) + \dots\end{aligned}$$

- We can do this recursively to estimate convergence

- A trick for improper integrals is to do variable substitution to eliminate a challenge
- Say one of the values is at  $-\infty$  or  $\infty$  we can substitute

$$\int_a^b f(x) dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt$$

## Variable substitution

- More generally we can do variable substitution as

$$I = \int_a^b f(x) dx = \int_c^d f(x(t)) \frac{dx}{dt} dt$$

- An example is the Schwartz tanh rule

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a) \tanh(t) \quad x \in [a, b] \text{ and } t \in [-\infty, \infty]$$

- where

$$\frac{\partial x}{\partial t} = \frac{1}{2}(b-a) \operatorname{sech}^2(t) = \frac{2}{b-a}(b-t)(t-a)$$

- $\operatorname{sech}()$  converges very rapidly for  $t \rightarrow \infty$  which allows for integration close to singularities



- Sometimes uniform sampling is not ideal
- A Gauss model may be an alternative
- The idea is

$$\int_a^b W(x)f(x)dx \approx \sum_{j=0}^{N-1} W_j f(x_j)$$

- For polynomials this can be an exact approximation
- We can approximate  $f(x)$  with a Gaussian Mixture and choose weights to match

$$f(x) \approx \sum_{k=0}^N W_k N(x|x_k, \sigma_k)$$

## Partitioned / Adaptive Integration

- If you have a function with variable dynamics it makes sense to partition the integration into intervals and use Romberg integration on each interval, i.e.

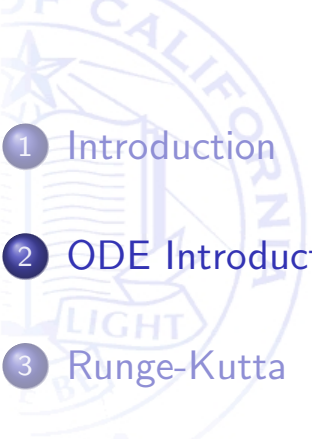
$$\begin{aligned} I &= \int_a^b f(x)dx \\ &= \int_a^m f(x)dx + \int_m^b f(x)dx \end{aligned}$$

- Rule 1 of data analysis understand your data

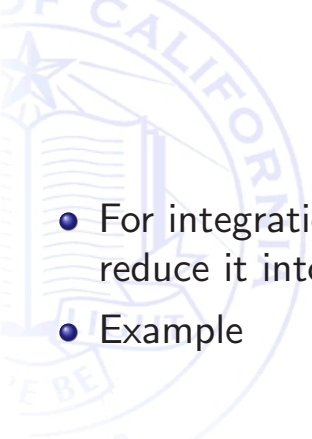
- Simple linear approximations are effective for well-behaved functions
- The order of your approximation can vary according to function complexity
- Using Bernoulli functions we can approximate the estimated error
- Recursive estimation with error monitoring is often effective
- Do a function analysis first to make sure function is proper
- Next we will discuss integration of ODE with standard methods such as Runge-Kutta, Step-size variation, etc.

## Questions

# Questions

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- 1 Introduction
  - 2 ODE Introduction
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## Introduction

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- For integration of a set of ordinary differential equations you can always reduce it into a set of first order differential equations.

- Example

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

- which can be rewritten

$$\begin{aligned}\frac{dy}{dx} &= z(x) \\ \frac{dz}{dx} &= r(x) - q(x)z(x)\end{aligned}$$

- where  $z$  is a new variable

- Consider a simple motion of a mass when actuated by a mass

$$F(u_1) = m \frac{d^2 u_1}{dt^2}$$

- We can rewrite this as

$$\frac{d^2 u_1}{dt^2} = \frac{1}{m} F(u_1)$$

- We can introduce  $u_2 = \frac{du_1}{dt}$  to generate

$$\begin{aligned} \frac{du_1}{dt} &= u_2 \\ \frac{du_2}{dt} &= \frac{1}{m} F(u_1) \end{aligned}$$

OR

$$\frac{du}{dt} = f(u, t) \text{ with } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where

$$f = \begin{pmatrix} u_2 \\ \frac{F(u_1)}{m} \end{pmatrix}$$

## Introduction (cont)

- The generic problem is thus a set of couple 1st order differential equations

$$\frac{dy_i(x)}{dx} = f_i(x_i, y_1, y_2, \dots, y_n)$$

- There are three major approaches:

- 1 Runge-Kutta: Euler type propagation
- 2 Richardson extrapolation / Burlirsch-Stoer: extrapolation type estimation with small step sizes
- 3 Predictor-Corrector: extrapolation with correction.

- Runge-Kutta most widely adopted for “generic” problems. Great if function evaluation is cheap
- Burlirsch-Stoer generates higher precision
- Predictor-Corrector is historically interesting, but rarely used today

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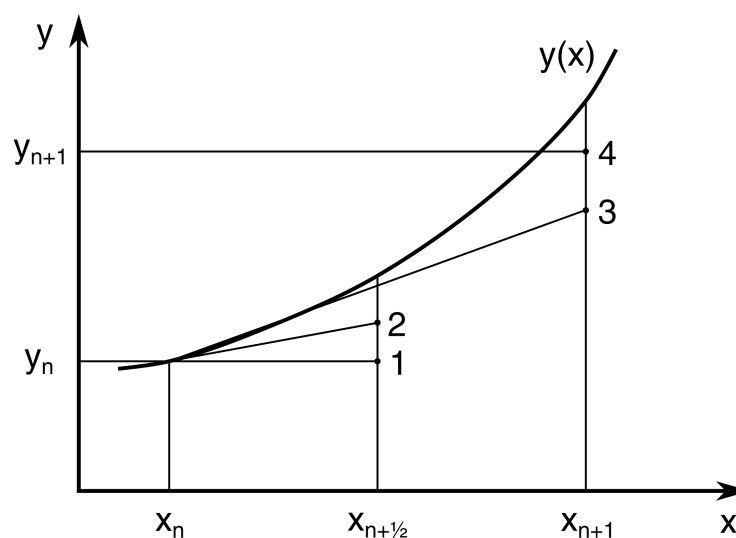
## Runge-Kutta

- The forward Euler method is specified as

$$y_{n+1} = y_n + hf(x_n, y_n)$$

with  $x_{n+1} = x_n + h$

- A problem is that the integration is asymmetric



- We can use a mid-point to get a closer estimate, i.e.,

$$\begin{aligned}k_1 &= hf(x_n, y_n) \\k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\y_{n+1} &= y_n + k_2 + O(h^3)\end{aligned}$$

## 4th order Runge-Kutta

- We can easily extend to richer models. A typical example is the fourth order model

$$\begin{aligned}k_1 &= hf(x_n, y_n) \\k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\k_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \\k_4 &= hf(x_n + h, y_n + k_3) \\y_{n+1} &= y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)\end{aligned}$$

- By far the most frequently used RK method for ODE integration
- Requires four function evaluations for every step

- Could we adjust the step-size?
- Estimation of performance adds an overhead
- What would be an obvious solution?

## Adaptive Runge-Kutta

- Could we adjust the step-size?
- Estimation of performance adds an overhead
- What would be an obvious solution?
  - 1 Do a full step
  - 2 Do a half step
  - 3 Compare (could be recursive)
  - 4 Next
- In general no one goes beyond 5th order Runge-Kutta

- Could we use PI control to track stepsize?

## PI step control of RK

- Could we use PI control to track stepsize?
- How about

$$h_{n+1} = S h_n \text{err}_n^\alpha \text{err}_{n-1}^\beta$$

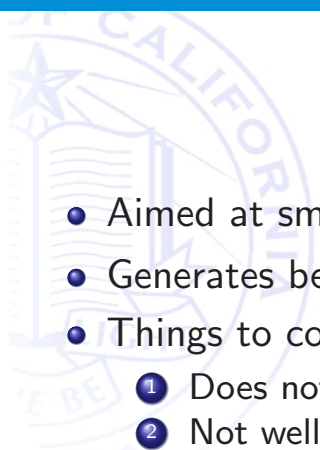
where  $S$  is a scale factor.  $\alpha$  and  $\beta$  are gain factors

- Typical default values  $\alpha = \frac{1}{k} - 0.75\beta$  and  $\beta = \frac{0.4}{k}$  and  $k$  is an integer that designates order of the integrator



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## Richardson Extrapolation / Burlirsch-Stoer

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- Aimed at smooth functions
  - Generates best precision with minimal effort
  - Things to consider
    - 1 Does not do well on functions w. table lookup or interpolation
    - 2 Not well suited for functions with singularities within intg range
    - 3 Not well suited for “expensive” functions
  - The approach is based on three ideas
    - 1 Final answer is based on selection of (adaptive) stepsize just like Romberg
    - 2 Use of rational functions for extrapolation (allow larger  $h$ )
    - 3 Integration method rely on use of even functions
  - Typically the steps size  $H$  is large and  $h$  is 100+ steps

- Consider a modified mid-point strategy

$$x_{n+1} = x_n + H$$

but with sub-steps

$$h = \frac{H}{n}$$

- We can rewrite the integration

$$\begin{aligned} z_0 &= y(x_n) \\ z_1 &= z_0 + hf(x_n, z_0) \\ z_{m+1} &= z_{m-1} + 2hf(x_n + mh, z_n) \quad m = 1, 2, 3, \dots, n-1 \\ y(x_n + H) &= \frac{1}{2}[z_n + z_{n-1} + hf(x + H, z_n)] \end{aligned}$$

- Centered mid-point or centered difference method
- The error can be shown to be

$$y_n - y(x + H) = \sum_{i=0}^{\infty} \alpha_i h^{2i}$$

- The power series implies that we can potentially do less evaluation.

## Burlirsch-Stoer - How good is it?

- Suppose  $n$  is even and  $y_{n/2}$  is the results of half as many steps
- Then

$$y(x + H) = \frac{4y_n - y_{n/2}}{3}$$

- which is accurate to the 4th order as Runge-Kutta but with 2/3 less derivative evaluation?
- How do you choose good step sizes for refinement?

- Suppose  $n$  is even and  $y_{n/2}$  is the results of half as many steps
- Then

$$y(x + H) = \frac{4y_n - y_{n/2}}{3}$$

- which is accurate to the 4th order as Runge-Kutta but with 2/3 less derivative evaluation?
- How do you choose good step sizes for refinement?
- One strategy could be

$$n = 2, 4, 6, 8, 12, 16, 24, 32, \dots \quad n = 2n_{j-2}$$

more recently a suggestion

$$n = 2, 3, 6, 8, 10, 12, 14, \dots \quad n_j = 2(j + 1)$$

## Step size control for Burlirsch-Stoer

- The error estimate can be tabulated as

$$\begin{array}{ccc} T_{00} & & \\ T_{10} & T_{01} & \\ T_{20} & T_{11} & T_{22} \end{array}$$

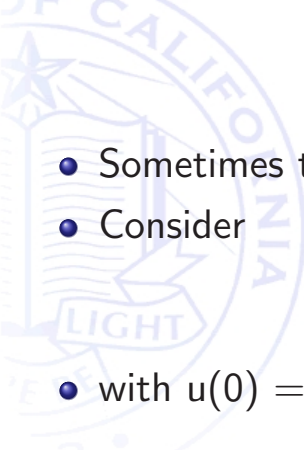
- where  $T_{ij}$  is the Lagrange interpolation of order  $i$  with  $j$  points. The relation between the polynomials is

$$T_{k,j+1} = \frac{2T_{k,j} - T_{k-1,j}}{(n_k/n_{k-j-1})^2 - 1} \quad j = 0, 1, \dots, k-1$$

- Each stepsize starts a new row. The difference  $T_{kk} - T_{kk-1}$  is an error estimate
- We can pre-compute the error estimates and use them to decide on step-size selection

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## Variable Dynamics

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- Sometimes the variable dynamics are very different
  - Consider

$$\begin{aligned}u' &= 998u + 1998v \\v' &= -999u - 1999v\end{aligned}$$

- with  $u(0) = 1$  and  $v(0) = 0$  we can get

$$u = 2y - z \quad v = -y - z$$

We can solve and find

$$\begin{aligned}u &= 2e^{-x} - e^{-1000x} \\v &= -e^{-x} + e^{-1000x}\end{aligned}$$

- The differences in dynamics would generate challenging step sizes

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## Partial Differential Equations

- Huge topics that has its own course - MATH 110/MATH 231 A-C
- Widely used for studies of physical systems - simulation / analysis
- Three main categories
  - 1 Hyperbolic (wave equation)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

where  $v$  is the speed of wave propagation

- 2 Parabolic (diffusion equation)

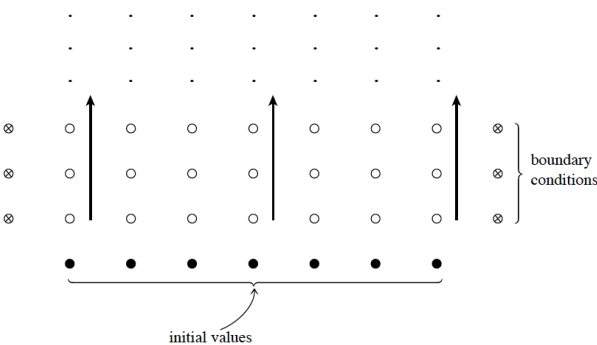
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial u}{\partial x} \right)$$

where  $D$  is the diffusion coefficient

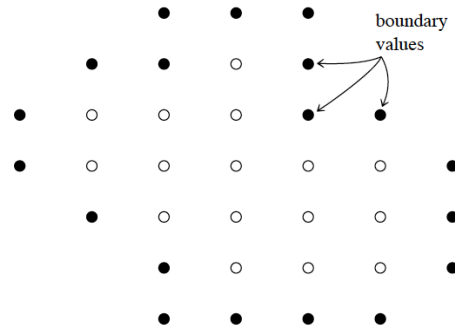
- 3 Elliptic (Poisson equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$

where  $\rho()$  is the source term.



Initial Value

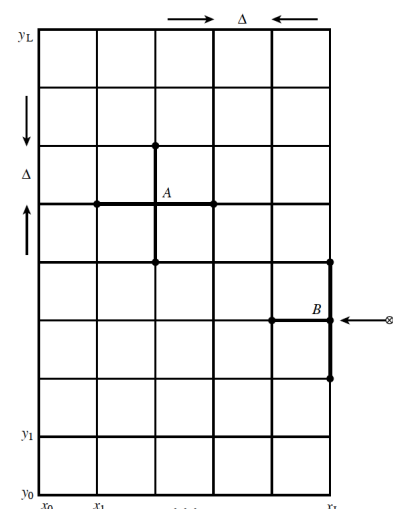


Boundary Value

Source - Numerical Recipes.


## Finite difference calculations

- In most cases grid propagation
- Finite differences is a basic approximation
- Final structure is a sparse matrix
- Numerous models and packages to address



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## Summary

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- We can organize ODEs as a set of coupled 1st order ODEs
  - Runge-Kutta is ideal for “cheap” functions, especially 4th order approximation
  - Buerlirsch-Stoer is ideal for high-accuracy integration
  - It is important to consider the variable dynamics in integration of functions.
  - Adaptive stepsize is often valuable as a way to generate realistic complexity