

# CSE276C - Polynomial Interpolation and Approximation

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## Outline


- 1 Introduction
- 2 Uniform approximation
- 3 Chebyshev Approximation
- 4 Truncated Power Series
- 5 Summary

# Introduction

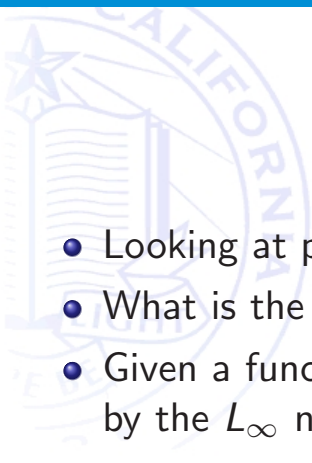
- Last time we spoke about direct use of data point / simple models
- What if we want an explicit functional approximation to data?
- Approximating a function/data by a class of simpler functions
- Two main motivations
  - ① Decomposition of a complicated function into constituent simpler functions to simplify further work
  - ② Recover a function from partial or noisy information
- Applications:
  - ① Signal compression / reconstruction (Fourier would be an example)
  - ② Data fitting (line, plane, manifold, ...)
  - ③ Recovery of a model say CAD recovery

## Material

- Numerical Recipes: Chapter 3.4-3.5
- Numerical Renaissance: Chapter 5

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## Uniform approximation by polynomials

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- Looking at polynomial again
  - What is the best uniform approximation?
  - Given a function  $f: [a, b] \rightarrow \mathbb{R}$  and a polynomial  $p$  we can measure the error by the  $L_\infty$  norm, i.e.,

$$\|f - p\|_\infty = \max_{a < x < b} |f(x) - p(x)|$$

- A good approximation is one where the norm is small
- Remember Weierstrass' theorem.

# Polynomial approximation

- Lets restrict the degree of the polynomial -  $n$
- Lets set  $\pi_n$  be all the polynomials degree at most  $n$
- Let uniform distance of  $f$  from  $\pi_n$  be the smallest error achievable using polynomials from  $\pi_n$  denoted by

$$d(f, \pi_n) = \min_{p \in \pi_n} \|f - p\|_{\infty}$$

- How can we make it happen?

## Polynomial approximation - getting help

- We have a theorem:
  - A function  $f$  continuous in  $[a, b]$  has exactly one best solution from  $\pi_n$
  - The polynomial  $p \in \pi_n$  of  $f$  across  $[a, b]$  iff
  - there are  $n+2$  point  $a \leq x_0 \leq \dots \leq x_n + 1 \leq b$  such that

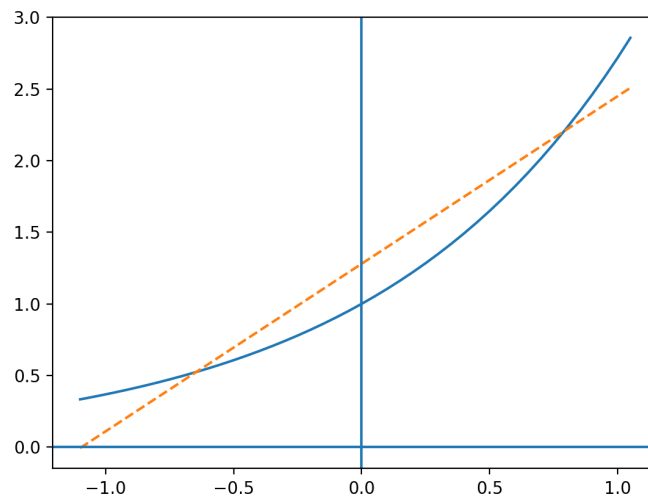
$$(-1)^i [f(x_i) - p(x_i)] = \epsilon \|f - p\|_{\infty}$$

where  $\epsilon = \text{signum}[f(x_0) - p(x_0)]$

- By alternating signs at  $n+2$  points the different between  $f$  and  $p$  is precisely equal to the  $L_{\infty}$

# Putting theorem to work

- Can we use the theorem to build a strategy?
- Lets consider  $f(x) = e^x$  on  $[-1, 1]$
- What would be the best 1st order approximation, i.e.,  $\pi_1$



## Fitting the line

- So we have three points
- $x_0 = -1$ ,  $x_1 = ?$  and  $x_2 = 1$
- at which the error is  $f(x) = p(x)$
- So what is  $x_1$ ?

# Fitting the line

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- So what is  $x_1$ ?
- we can write  $p(x) = a + bx$
- We can compute the error at the three points:

$$\begin{aligned}e(x_0) &= f(x_0) - p(x_0) = f(-1) - p(-1) = \frac{1}{e} - a + b \\e(x_1) &= f(x_1) - p(x_1) = e^{x_1} - a + bx_1 \\e(x_2) &= f(x_2) - p(x_2) = f(1) - p(1) = e - a - b\end{aligned}$$

- Given  $e(x_0) = e(x_2)$

$$\begin{aligned}\frac{1}{e} - a + b &= e - a - b \\2b &= e - \frac{1}{e} \\b &= 1.1752\end{aligned}$$

The slope is equal to the average change

## Fitting the line (cont)

- How do we find  $a$ ?
- The difference (positive / negative) should be symmetric
- The error function should at an extrema at  $x_0, x_1, x_2$  but with alternate signs
- $e(x) = f(x) - p(x) = e^x - a - bx$  so
- $e'(x) = e^x - b \Rightarrow e^{x_1} - b = 0$
- $x_1 = \ln b$
- $x_1 \approx 0.16144$

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- $x_1 \approx 0.16144$
- $e(x_1) = -e(x_2) \Rightarrow e^{x_1} - a - bx_1 = -e + a + b$
- $a = \frac{e - bx_1}{2} \approx 1.2643$
- $p(x) \approx 1.2643 + 1.1752x$
- The maximum error would be  $e(x_1) = \|f(x_1) - p(x_1)\|_\infty \approx 0.2788$

## Approximation - Discussion

- Example showed a way to construct a solution.
- What if we did not know the appropriate  $n$ ?
- If we make  $n$  too small there is a lack of fit
- If we make  $n$  too large the fit will be poor (too much wiggle)
- Could we estimate  $d(f, \pi_n)$ ?
- Maybe not, but a lower bound might be possible

# Divided Differences

- Slight detour
- Divided differences are frequently used to compute coefficients in interpolation polynomials.
- Recursive formulation. Given a set of data points  $(x_0, y_0), \dots, (x_k, y_k)$

$$[y_v, \dots, y_{v+j}] = \frac{[y_{v+1}, \dots, y_{v+j}] - [y_v, \dots, y_{v+j-1}]}{x_{v+j} - x_v}$$

and

$$[y_v] = y_v \quad v \in \{0, \dots, k\}$$

- The recursive formulation is computationally effective
- The first few terms

$$\begin{aligned} [y_0] &= y_0 \\ [y_0, y_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\ [y_0, y_1, y_2] &= \frac{[y_1, y_2] - [y_0, y_1]}{x_2 - x_0} = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} \\ &= \frac{\frac{y_2 - y_1}{(x_2 - x_1)(x_2 - x_0)} - \frac{y_1 - y_0}{(x_1 - x_0)(x_2 - x_0)}}{1} \end{aligned}$$

## Estimating a lower bound

- Assume we have a function  $f : [a, b] \rightarrow \mathbb{R}$
- We will use divided differences to compute bounds
- Lets assume we have three points  $x_0, x_1, x_2$  as  $p$  is linear

$$p[x_0, x_1, x_2] = 0$$

i.e. the gradient does not vary

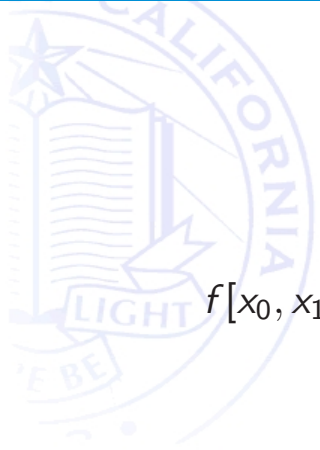
- we can also write

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

so



## Estimating lower bound (cont.)

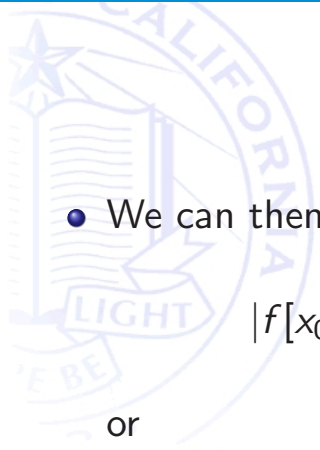

$$\begin{aligned}f[x_0, x_1, x_2] &= f[x_0, x_1, x_2] - p[x_0, x_1, x_2] \\&= (f - p)[x_0, x_1, x_2] \\&= \frac{f(x_0) - p(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1) - p(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2) - p(x_2)}{(x_2 - x_0)(x_2 - x_1)} \\&= \frac{f(x_0) - p(x_0)}{w'(x_0)} + \frac{f(x_1) - p(x_1)}{w'(x_1)} + \frac{f(x_2) - p(x_2)}{w'(x_2)}\end{aligned}$$

where

$$w'(x) = (x - x_0)(x - x_1)(x - x_2)$$

## Estimating lower bound (cont.)

- We can then estimate a bound


$$|f[x_0, x_1, x_2]| \leq \|f - p\|_\infty \left( \frac{1}{|w'(x_0)|} + \frac{1}{|w'(x_1)|} + \frac{1}{|w'(x_2)|} \right)$$

or

$$\|f - p\|_\infty \geq \frac{|f[x_0, x_1, x_2]|}{\frac{1}{|w'(x_0)|} + \frac{1}{|w'(x_1)|} + \frac{1}{|w'(x_2)|}}$$

- the polynomial on left hand side is arbitrary so  $d(f, \pi_n) = \min_{p \in \pi_n} \|f - p\|_\infty$
- right hand side is purely based on  $f$  and three points, so we can estimate the value

# Back to our example

- Lets use  $f(x) = e^x$  in the interval  $[-1, 1]$ .
- Pick say -1, 0, 1 as our points

$$f[x_0, x_1, x_2] = \frac{1}{2}f(-1) - f(0) + \frac{1}{2}f(1)$$

and

$$\frac{1}{|w'(x_0)|} + \frac{1}{|w'(x_0)|} + \frac{1}{|w'(x_0)|} = \frac{1}{2} + 1 + \frac{1}{2} = 2$$

thus

$$d(f, \pi_1) \geq \frac{f(-1) - 2f(0) + f(1)}{4}$$

- the bound is then  $d(f, \pi_1) = 0.2715$ , which is not too far away from 0.2788 that was achieved.
- the lower bounds says that we cannot estimate  $e^x$  much better than .3 in the interval -1,1 with a linear approximation, which is very valuable.

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# Chebyshev polynomials

- Chebyshev polynomials are sequences of polynomials that are defined recursively.
- The first kind of a Chebyshev polynomial is denoted  $T_N(x)$  and given by

$$T_N(x) = \cos(n \arccos x)$$

looks trigonometric but can be used to general polynomials. I.e

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \text{ (as } \cos(2\theta) = 2\cos^2(\theta) - 1) \\ T_3(x) &= 4x^3 - 3x \\ T_{N+1}(x) &= 2xT_N(x) - T_{N-1}(x), \text{ for } n \geq 1 \end{aligned}$$

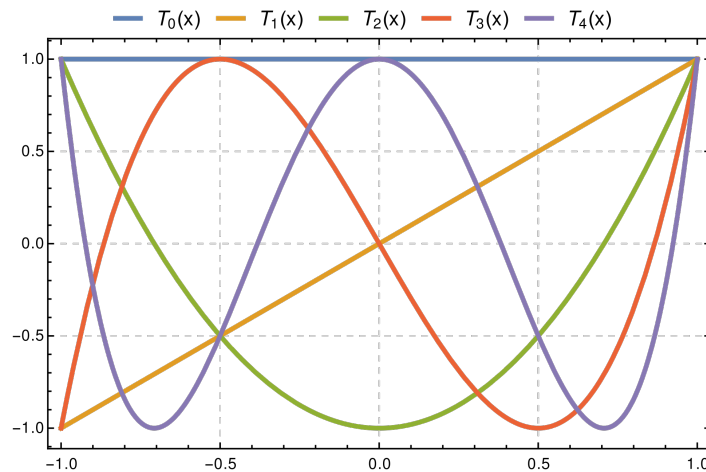
## Chebyshev Polynomials

- The polynomials are orthogonal over the interval  $[-1, 1]$  over a weight of  $(1 - x^2)^{-1/2}$  so that

$$\int_{-1}^1 \frac{T_i(x) T_j(x)}{\sqrt{1 - x^2}} dx = \begin{cases} 0 & i \neq j \\ \frac{\pi}{2} & j = i \neq 0 \\ \pi & i = j = 0 \end{cases}$$

# Chebyshev Polynomials

- The polynomial  $T_N(x)$  has  $N$  zeros in the interval  $[-1, 1]$  at the points  $x = \cos\left(\frac{\pi(k+\frac{1}{2})}{N}\right)$  for  $k \in 0, \dots, N-1$
- There is a similar set of extrema at  $x = \cos\left(\frac{\pi k}{N}\right)$



## Chebyshev Approximation

- For periodic functions.  $f(x)$ , over the interval  $[-1, 1]$  an  $N$  coefficient approximation is

$$\begin{aligned} c_j &= \frac{2}{N} \sum_{k=0}^{N-1} f(x_k) T_j(x_k) \\ &= \frac{2}{N} \sum_{k=0}^{N-1} f\left(\cos \frac{\pi(k+\frac{1}{2})}{N}\right) \cos \frac{\pi(k+\frac{1}{2})}{N} \end{aligned}$$

- The approximation is then

$$f(x) \approx p(x) = \left[ \sum_{k=1}^{N-1} c_k T_k(x) \right] - \frac{1}{2} c_0$$

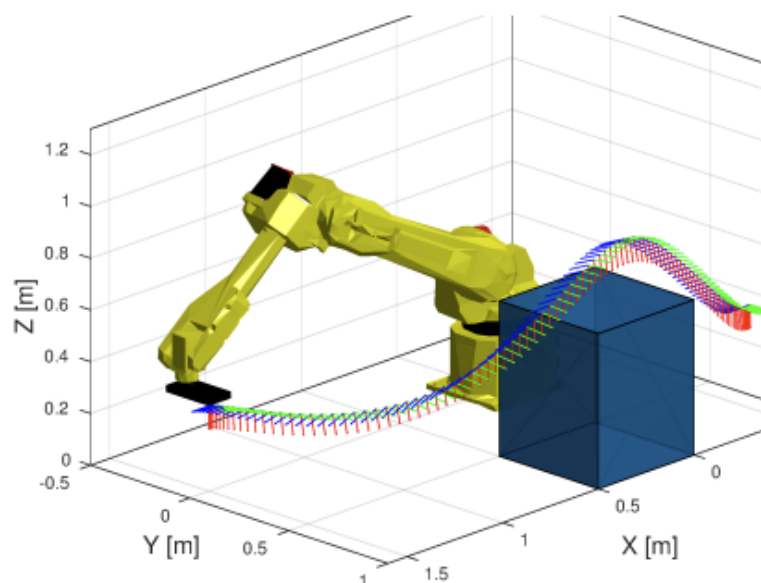
- which is an exact match in terms of zero crossings
- the errors are uniformly distributed over  $[-1, 1]$

- If the domain is different from  $[-1, 1]$  the variable can be changed from  $[a, b]$

$$y = \frac{x - \frac{1}{2}(b - a)}{\frac{1}{2}(b - a)}$$

the approximated can be mapped forward / back as needed

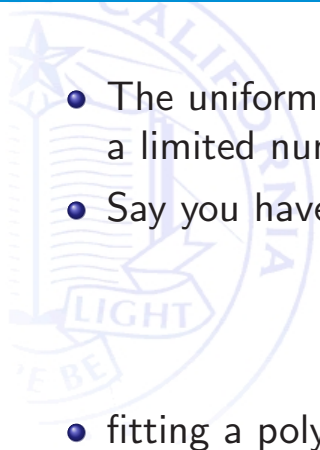
## Example of using Chebyshev Points for Control



(a) Test case 1


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## Truncated Power Series

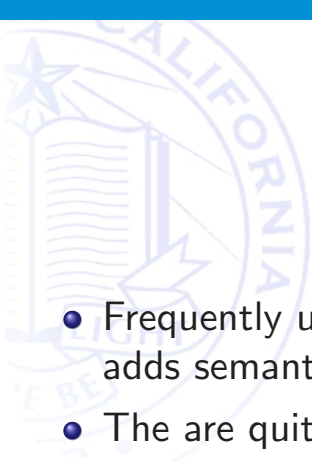
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- The uniform error of the Chebyshev functions/series implies that one can use a limited number of terms
  - Say you have a series

$$f(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$$

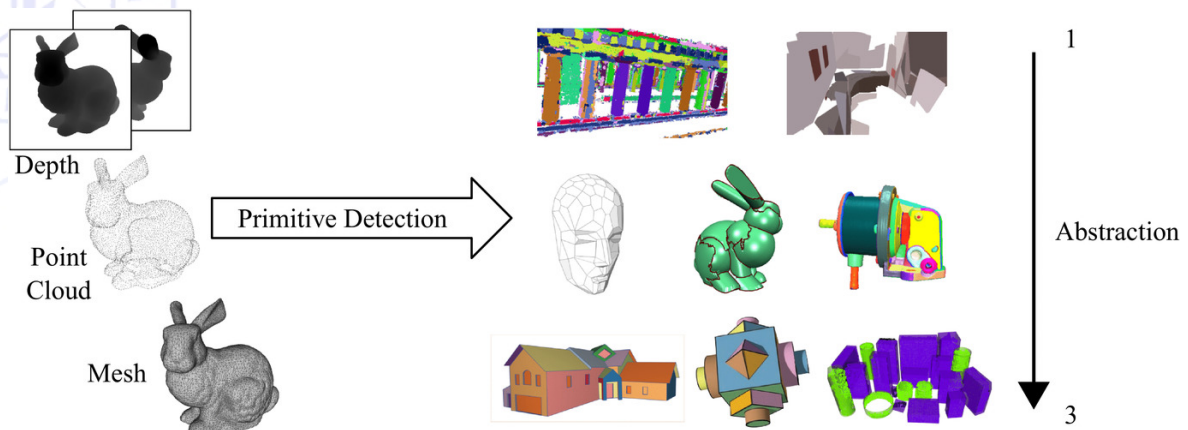
- fitting a polynomial function and trying to achieve  $\epsilon < 10^{-9}$  would require more than 30 terms
- If we use a Chebyshev approximation
  - 1 Compute enough terms to have  $\epsilon < T$  across series
  - 2 Change variable to  $[-1, 1]$
  - 3 Find Chebyshev series that satisfy error
  - 4 Truncate series using  $c_k T_k(x)$  as an estimated error residual
  - 5 Convert back to polynomial form
  - 6 Convert back to original coordinate range
- For the example the reduction is from 30 to 9 terms

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## Functional approximation and interpolation

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- Frequently using a functional approximation is much more effective and it adds semantic information (a class) to the data approximation
  - There are quite a few functional approximation forms
  - Giving a few examples from polynomial,  $\pi_n$ , form to periodic function
  - A key consideration is what domain knowledge is available to guide model selection

# Small example



## Questions

# Questions