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1. **Problem 1.** Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}.$$

(b) Show that there exist real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that equality holds.

2. **Problem 2.** Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

3. **Problem 3.** In a mathematical competition some competitors are friends. Friendship is mutual. A group of competitors is called a *clique* if each two of them are friends. Given that the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest clique in each room has the same size.

Language: English *Time: 4 hours and 30 minutes*
Each problem is worth 7 points

4. **Problem 4.** In triangle ABC the bisector of angle BCA intersects the circumcircle again at R. The perpendicular bisectors of BC and AC meet at P and Q respectively. Let K and L be the midpoints of BC and AC. Prove that triangles RPK and RQL have the same area.

5. **Problem 5.** Let a and b be positive integers. Show that if

$$4ab - 1 \mid (4a^2 - 1)^2,$$

then $a = b$.

6. **Problem 6.** Let n be a positive integer and consider

$$S = \{(x, y, z) : x, y, z \in \{0, 1, \ldots, n\},\ x + y + z > 0\}.$$

Determine the smallest number of planes whose union contains S but does not contain $(0, 0, 0)$.

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