Srinidhi A
ID: COMETFWC053

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1. Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Prove that for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}.$$

(b) Show that equality can occur.

2. Problem 2. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line through A. Suppose that ℓ intersects DC at F and BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

3. Problem 3. In a mathematical competition some competitors are friends. Friendship is mutual. A group is called a clique if each two of them are friends. Given that the largest clique has even size, prove that competitors can be arranged into two rooms so that the largest clique in each room has the same size.

Language: English Time: 4 hours 30 minutes Each problem is worth 7 points

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4. Problem 4. In triangle ABC the bisector of angle BCA meets the circumcircle again at R. The perpendicular bisectors of BC and AC meet at P and Q. Let K and L be the midpoints of BC and AC. Prove that triangles RPK and RQL have the same area.

5. Problem 5. Let a and b be positive integers. Show that if

$$4ab - 1 \mid (4a^2 - 1)^2,$$

then $a = b$.

6. Problem 6. Let

$$S = \{(x, y, z) : x, y, z \in \{0, 1, \ldots, n\},\ x + y + z > 0\}.$$

Determine the smallest number of planes whose union contains S but does not contain $(0, 0, 0)$.

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