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Language: English **Day: 1**

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10–22, 2008

Wednesday, July 16, 2008

Problem 1. An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly define B_1, B_2, C_1, C_2. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

Problem 2.

(a) Prove that
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$
for all real numbers $x, y, z \neq 1$ with $xyz = 1$.

(b) Prove that equality holds for infinitely many rational triples.

Problem 3. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

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Time: 4 hours and 30 minutes
Each problem is worth 7 points

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Problem 4. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \geq n$ and $k - n$ even. Let $2n$ lamps be given, each either on or off. Initially all lamps are off. At each step exactly one lamp is switched.

Let N be the number of sequences of k steps resulting in lamps 1 through n on and lamps $n + 1$ through $2n$ off.

Let M be the number of such sequences where lamps $n+1$ through $2n$ are never switched on. Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2. Prove that the common external tangents of ω_1 and ω_2 intersect on a circle tangent to the four given lines.