1. Consider the vector space $V = \mathbb{R}^3$. Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix} \ \text{and} \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

(a) Determine if $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent.

Ans: The vectors are linearly independent. We can prove this by contradiction: suppose that they are linearly independent and there exist coefficients c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Then, the following set of linear equations must hold.

$$3c_1 - 4c_2 - 2c_3 = 0 \tag{1a}$$
$$c_2 + c_3 = 0 \tag{1b}$$
$$-6c_1 + 7c_2 + 5c_3 = 0 \tag{1c}$$

Equation (1b) tells us $c_2 = -c_3$. Substituting this in (1c), we obtain $-6c_1 - 2c_3 = 0$ or equivalently $c_3 = -3c_1$. Substituting $c_2 = 3c_1$ and $c_3 = -3c_1$ in (1a), we get $3c_1 - 12c_1 + 6c_1 = -3c_1 = 0$, which implies $c_1 = c_2 = c_3 = 0$. Therefore, it is not possible to rewrite any vector as a linear combination of the other two vectors.

(b) Does $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ span V?

Ans: True. Since the vector space $V = \mathbb{R}^3$ has a dimension of three, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for V and, hence, spans V. This can be proved by showing that, given any $\mathbf{x} \in \mathbb{R}^3$, following steps similar to those in part (a), we can find coefficients c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{x}$.

2. Consider the collection of vectors $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ in $\mathbb{R}^3$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \ \text{and} \ \mathbf{b}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Show that $\mathcal{B}$ is a basis for $V = \mathbb{R}^3$.

Ans: Since dim(V)=3, in order to show that $\mathcal{B}$ is a basis for V, it suffices to shows that $\mathcal{B}$ is linearly independent. Following the same steps used in part (a), suppose and there are coefficients c_1, c_2, c_3 such that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}$. Then, the following set of linear equations must hold.

$$c_1 + c_2 + 3c_3 = 0 \tag{2a}$$
$$c_1 + 2c_2 + 2c_3 = 0 \tag{2b}$$
$$c_3 = 0 \tag{2c}$$

Equation (2c) immediately tells us $c_3 = 0$ and we can simplify the first two equations:

$$c_1 + c_2 = 0 \tag{3a}$$
$$c_1 + 2c_2 = 0 \tag{3b}$$

By subtracting (3a) from (3b), we obtain $c_2 = 0$. Substituting this in either equation yields $c_1 = 0$. Therefore, the only solution to the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$ and $\mathcal{B}$ is linearly independent.