1. Consider the collection of vectors $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ in $\mathbb{R}^3$, where

$$\mathbf{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix} \ \text{ and } \ \mathbf{b}_3 = \begin{bmatrix}3\\2\\1\end{bmatrix}.$$

   (a) Compute the dot product $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ to show that they are not orthogonal.

   Ans: The dot product $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 1 \cdot 1 + 1 \cdot 2 = 3 \neq 0$. Thus, it is not an orthogonal basis.

   (b) Construct an orthogonal basis $\mathcal{B}'$ from $\mathcal{B}$, using the Gram-Schmidt process. Verify that the basis vectors $\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3$ are orthogonal.

   Ans: After performing the Gram-Schmidt process, we get

$$\mathbf{b}'_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \ \mathbf{b}'_2 = \mathbf{b}_2 - \frac{\langle \mathbf{b}_2, \mathbf{b}'_1 \rangle}{\|\mathbf{b}'_1\|^2}\mathbf{b}'_1 = \begin{bmatrix}-0.5\\0.5\\0\end{bmatrix}$$

   and

$$\mathbf{b}'_3 = \mathbf{b}_3 - \frac{\langle \mathbf{b}_3, \mathbf{b}'_1 \rangle}{\|\mathbf{b}'_1\|^2}\mathbf{b}'_1 - \frac{\langle \mathbf{b}_3, \mathbf{b}'_2 \rangle}{\|\mathbf{b}'_2\|^2}\mathbf{b}'_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

   We can verify that the new basis is orthogonal by showing the inner product between any pair of distinct vectors is equal to zero. It is clear $\langle \mathbf{b}'_1, \mathbf{b}'_3 \rangle = \langle \mathbf{b}'_2, \mathbf{b}'_3 \rangle = 0$. Finally, $\langle \mathbf{b}'_1, \mathbf{b}'_2 \rangle = 1(-0.5) + 1 \cdot 0.5 + 0 \cdot 0 = 0$.

2. Consider a 3×3 matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 4\\3 & 5 & 7\\4 & 6 & 6\end{bmatrix}.$$

   (a) Compute the rank of matrix $\mathbf{A}$. Is $\mathbf{A}$ invertible?

   Ans: One can verify that $\mathbf{A}_{\cdot 3} = -6\mathbf{A}_{\cdot 1} + 5\mathbf{A}_{\cdot 2}$ and $\mathbf{A}_{\cdot 1}$ and $\mathbf{A}_{\cdot 2}$ are linearly independent. Thus, the rank of $\mathbf{A}$ is two. Since the rank is strictly smaller than the number of rows (or columns), the matrix is not invertible.

   (b) Find the null space $\mathcal{N}(\mathbf{A})$.

   Ans: From the rank theorem, the dimension of the null space $\mathcal{N}(\mathbf{A})$ is one. By solving the following set of linear equations $\mathbf{A}\mathbf{x} = \mathbf{0}$, we can show that

$$\mathcal{N}(\mathbf{A}) = \left\{ a \begin{bmatrix}6\\-5\\1\end{bmatrix} \ : \ a \in \mathbb{R} \right\}.$$

3. Recall that two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are similar if there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Consider a basis $\mathcal{B}' = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \mathbf{b}_2 = \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

(a) Given $\mathbf{x} \in \mathbb{R}^2$, write $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$, i.e., $\mathbf{x} = x_1'\mathbf{b}_1 + x_2'\mathbf{b}_2$.

Ans: First,

$$\mathbf{e}_1 = \langle \mathbf{b}_1, \mathbf{e}_1\rangle \mathbf{b}_1 + \langle \mathbf{b}_2, \mathbf{e}_1\rangle \mathbf{b}_2 = \frac{\sqrt{3}}{2}\mathbf{b}_1 + \frac{-1}{2}\mathbf{b}_2$$

and

$$\mathbf{e}_2 = \langle \mathbf{b}_1, \mathbf{e}_2\rangle \mathbf{b}_1 + \langle \mathbf{b}_2, \mathbf{e}_2\rangle \mathbf{b}_2 = \frac{1}{2}\mathbf{b}_1 + \frac{\sqrt{3}}{2}\mathbf{b}_2$$

Since $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, we get

$$\mathbf{x} = \frac{\sqrt{3}x_1}{2}\mathbf{b}_1 + \frac{-x_1}{2}\mathbf{b}_2 + \frac{x_2}{2}\mathbf{b}_1 + \frac{\sqrt{3}x_2}{2}\mathbf{b}_2 = \left(\frac{\sqrt{3}x_1}{2} + \frac{x_2}{2}\right)\mathbf{b}_1 + \left(\frac{-x_1}{2} + \frac{\sqrt{3}x_2}{2}\right)\mathbf{b}_2.$$

Note that we can get this expression using $\mathbf{x} = \langle \mathbf{x}, \mathbf{b}_1\rangle \mathbf{b}_1 + \langle \mathbf{x}, \mathbf{b}_2\rangle \mathbf{b}_2$ as well, where $\langle \mathbf{x}, \mathbf{b}_i\rangle \mathbf{b}_i$, $i = 1, 2$, is the projection of $\mathbf{x}$ onto the subspace spanned by the basis vector $\mathbf{b}_i$.

(b) Let $\mathbf{x}' = [x_1' \; x_2']^T$, where x_1' and x_2' are from part (a). Find a matrix $\mathbf{P} \in \mathbb{R}^{2\times 2}$ such that $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

Ans: From part (a), it is clear

$$\mathbf{P} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

(c) Suppose

$$\mathbf{B} = \begin{bmatrix} 2.25 & -0.433 \\ -0.433 & 2.75 \end{bmatrix}.$$

Find a matrix $\mathbf{A}$ similar to $\mathbf{B}$ satisfying $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Ans: First

$$\mathbf{P}^{-1} = \left(\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - \frac{-1}{2}\frac{1}{2}\right)\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Then,

$$\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(d) Find the eigenvalues and right eigenvectors of $\mathbf{B}$, and show that $\mathbf{b}_1$ and $\mathbf{b}_2$ are right eigenvectors of $\mathbf{B}$ with the eigenvalues given by the diagonal elements of $\mathbf{A}$.

Ans: The characteristic polynomial of $\mathbf{B}$ is equal to

$$p_{\mathbf{B}}(\lambda) = (\lambda - 2.25)(\lambda - 2.75) - 0.433 \times 0.433 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Thus, the two roots are $\lambda_1 = 2$ and $\lambda_2 = 3$, which are the eigenvalues of $\mathbf{B}$. With these eigenvalues and solving $\mathbf{Bx} = \lambda_i \mathbf{x}$, $i = 1, 2$, we obtain the eigenvectors $\mathbf{v}_i$, $i = 1, 2$, which are $\mathbf{v}_1 = \mathbf{b}_1$ and $\mathbf{v}_2 = \mathbf{b}_2$. For example, to find $\mathbf{v}_1$, we solve the following set of linear equations:

$$2.25x_1 - 0.433x_2 = 2x_1 \tag{1a}$$

$$-0.433x_1 + 2.75x_2 = 2x_2 \tag{1b}$$

From (1b), we get $-0.433x_1 = -0.75x_2$ or, equivalently, $x_1 = \sqrt{3}x_2$.

(e) Compute the determinant of $\mathbf{B}$ and verify that the determinant equals the product of the eigenvalues.

Ans: The determinant of $\mathbf{B}$ is equal to $b_{1,1}b_{2,2} - b_{2,1}b_{1,2} = 2.25 \cdot 2.75 - (-0.433)(-0.433) = 6$, which is equal to 2×3.