AI1103 ASSIGNMENT-2

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Question 22:

A carpenter has 90 ,80 and 50 running feet respectively of teak wood, plywood and rosewood which is used to produce product A and product B. Each unit of product A requires 2, 1 and 1 running feet and each unit of product B requires 1, 2 and 1 running feet of teak wood,plywood and rosewood respectively. If product A is sold for ₹ 48 per unit and product B is sold for ₹ 40 per unit, how many units of product A and product B should be produced and sold by the carpenter, in order to obtain the maximum gross income?

Formulate the above as a Linear Programming Problem and solve it, indicating clearly the feasible region in the graph.

Solution:

Let x be the number of units of product A produced and y be the number of units of product B produced and I be the maximum gross income. From the given information, the problem can be formulated as

$$I = \max_{x,y} 48x + 40y \tag{1}$$
$$2x + y \leq 90 \tag{2}$$
$$x + 2y \leq 80 \tag{3}$$
$$x + y \leq 50 \tag{4}$$

which can be expressed in vector form as

$$P = \max_{\mathbf{x}} \begin{pmatrix} 48 & 40 \end{pmatrix} \mathbf{x} \tag{5}$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 90 \\ 80 \\ 50 \end{pmatrix} \tag{6}$$
$$\mathbf{x} \succeq \mathbf{0} \tag{7}$$

1) *Graphical solution:*
From the Fig.1, the feasible region is a pentagon with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 45 \\ 0 \end{pmatrix}, \begin{pmatrix} 40 \\ 10 \end{pmatrix}, \begin{pmatrix} 20 \\ 30 \end{pmatrix}, \begin{pmatrix} 0 \\ 40 \end{pmatrix} \tag{8}$$

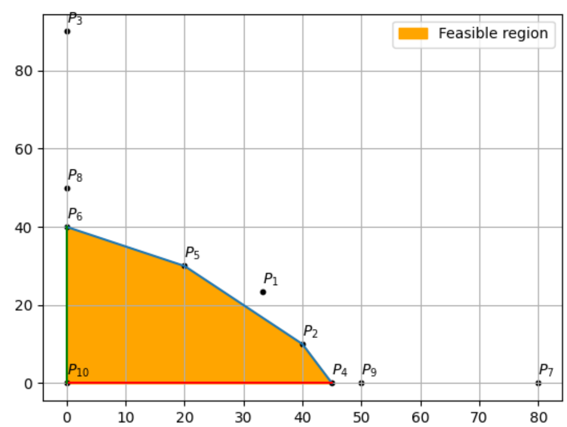

Fig. 1. $P_6P_5P_2P_4P_{10}$ is the feasible region

with respective gross income

$$\begin{pmatrix} 48 & 40 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{9}$$
$$\begin{pmatrix} 48 & 40 \end{pmatrix} \begin{pmatrix} 45 \\ 0 \end{pmatrix} = 2160 \tag{10}$$
$$\begin{pmatrix} 48 & 40 \end{pmatrix} \begin{pmatrix} 40 \\ 10 \end{pmatrix} = 2320 \tag{11}$$
$$\begin{pmatrix} 48 & 40 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = 2160 \tag{12}$$
$$\begin{pmatrix} 48 & 40 \end{pmatrix} \begin{pmatrix} 0 \\ 40 \end{pmatrix} = 1600 \tag{13}$$

Thus, the carpenter should produce 40 units of product A and 10 units of product B for maximum gross income.

2) *Lagrange Multipliers:*

The given problem is expressed in the form

$$P = -\min_{\mathbf{x}} \begin{pmatrix}48 & 40\end{pmatrix} \mathbf{x} \tag{14}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 90 \\ 80 \\ 50 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

The Lagrangian, defined as the linear combination of the loss function and the constraints is defined as

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\lambda}) = &- \begin{pmatrix}48 & 40\end{pmatrix} \mathbf{x} \\ &+ \lambda_1 \left[\begin{pmatrix}2 & 1\end{pmatrix} \mathbf{x} - 90\right] \\ &+ \lambda_2 \left[\begin{pmatrix}1 & 2\end{pmatrix} \mathbf{x} - 80\right] + \lambda_3 \left[\begin{pmatrix}1 & 1\end{pmatrix} \mathbf{x} - 50\right] \\ &+ \lambda_4 \left[\begin{pmatrix}-1 & 0\end{pmatrix} \mathbf{x}\right] \\ &+ \lambda_5 \left[\begin{pmatrix}0 & -1\end{pmatrix} \mathbf{x}\right] \end{aligned} \tag{16}$$

Taking the derivative

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = 0, \tag{17}$$

we obtain

$$2\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 = 48 \tag{18}$$

$$\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_5 = 40 \tag{19}$$

$$2x_1 + x_2 = 90 \tag{20}$$

$$x_1 + 2x_2 = 80 \tag{21}$$

$$x_1 + x_2 = 50 \tag{22}$$

$$x_1 = 0 \tag{23}$$

$$x_2 = 0 \tag{24}$$

It is obvious that $x_1 = 0, x_2 = 0$ are infeasible. Hence, considering λ_4, λ_5 as the inactive multipliers, And also from equations (20),(21),(22) we will not get unique values of x_1 and x_2 . So, there has to be a inactive multiplier in λ_1, λ_2 and λ_3 .

On solving equations (18) and (19) we get

$$\lambda_1 - \lambda_2 = 8 \tag{25}$$

From which we can imply that λ_1 can't be the inactive multiplier , because if it is then we get the value for λ_2 as a negative value .

So , either λ_2 or λ_3 is the inactive multiplier.

a) λ_3 *is inactive multiplier:*
So, considering λ_1 and λ_2 as active multipliers, the equations from (18) to (24) can be expressed as

$$\begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 48 \\ 40 \\ 90 \\ 80 \end{pmatrix} \tag{26}$$

yielding the solution as

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 100/3 \\ 70/3 \\ 56/3 \\ 32/3 \end{pmatrix} \tag{27}$$

But , the solution does not satisfy the constraint (4) , so this is not the correct case.

b) λ_2 *is inactive multiplier:*
So, considering λ_1 and λ_3 as active multipliers, the equations from (18) to (24) can be expressed as

$$\begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 48 \\ 40 \\ 90 \\ 50 \end{pmatrix} \tag{28}$$

yielding the solution as

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 40 \\ 10 \\ 8 \\ 32 \end{pmatrix} \tag{29}$$

This solution satisfies all the constraints , so we can say this is the optimal solution i.e,

$$\mathbf{x} = \begin{pmatrix} 40 \\ 10 \end{pmatrix} \tag{30}$$

The number of product A and product B should be produced and sold by carpenter , in order to obtain the maximum gross income are 40 and 10 respectively.