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Thirteenth International Olympiad, 1977

1977/1.

Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$:

If $a_1, a_2, ..., a_n$ are arbitrary real numbers, then

$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n) + \cdots + (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \geq 0$

1977/2.

Consider a convex polyhedron P_1 with nine vertices $A_1, A_2, ..., A_9$; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to$A_i(i = 2, 3, ..., 9)$. Prove that at least two of the polyhedra $P_1, P_2, ..., P_9$ have an interior point in common.

1977/3.

Prove that the set of integers of the form $2^k - 3(k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.

1977/4.

All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges $BCCD, DA$, respectively. Prove:

(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.

(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely manyshortest polygonal paths, their common length being $2AC \sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

1977/5.

Prove that for every natural number m, there exists a finite set S of points in a plane with the following property: For every point A in S, there are exactly m points in S which are at unit distance from A.

1977/6.

Let $A = (a_{ij})(i, j = 1, 2, ..., n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the i^{th} row and the j^{th} column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

Fourteenth International Olympiad, 1978

1978/1.

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

1978/2.

Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

1978/3.

Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1.)$

1978/4.

Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{array}{c}
(x_2^1 - x_3x_5)(x_2^2 - x_3x_5) \leq 0 \\
(x_2^2 - x_4x_1)(x_2^3 - x_4x_1) \leq 0 \\
(x_2^3 - x_5x_2)(x_2^4 - x_5x_2) \leq 0 \\
(x_2^4 - x_1x_3)(x_2^5 - x_1x_3) \leq 0 \\
(x_2^5 - x_2x_4)(x_2^1 - x_2x_4) \leq 0
\end{array}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers

1978/5.

Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation $f(x+y) + f(x-y) = 2f(x)g(y)$ for all x, y. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

1978/6.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

Fifteenth International Olympiad, 1979

1979/1.

Point O lies on line g; $\vec{OP_1}, \vec{OP_2}, ..., \vec{OP_n}$ are unit vectors such that points $P_1, P_2, ..., P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd,

$$|\vec{OP_1} + \vec{OP_2} + \cdots + \vec{OP_n}| \geq 1$$

Here $|\vec{OM}|$ denotes the length of vector $\vec{OM}$.

1979/2.

Determine whether or not there exists a finite set M of points in space not lying in the same plane such that, for any two points A and B of M, one can select two other points C and D of M so that lines AB and CD are parallel and not coincident.

1979/3.

Let a and b be real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

1979/4.

A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path shouid he follow in order to travel the least possible distance and still accomplish his mission?

1979/5.

G is a set of non-constant functions of the real variable x of the form

$$f(x) = ax + b,\ a \text{ and } b \text{ are real numbers,}$$

and G has the following properties:

(a) If f and g are in G, then $g \circ f$ is in G; here $(g \circ f)(x) = g[f(x)]$.
(b) If f is in G, then its inverse f^{-1} is in G; here the inverse of $f(x) = ax + b$ is $f^{-1}(x) = (x - b)/a$.
(c) For every f in G, there exists a real number x_f such that $f(x_f) = x_f$.

Prove that there exists a real number k such that $f(k) = k$ for all f in G.

1979/6.

Let $a_1, a_2, ..., a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, ..., b_n$ for which

(a) $a_k < b_k$ for $k = 1, 2, ..., n$,
(b) $q < \frac{b_k+1}{b_k} < \frac{1}{q}$ for $k = 1, 2, ..., n - 1$,
(c) $b_1 + b_2 + \cdots + b_n < \frac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.