1) Big omega notation prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

(s) $g(n) \geq c \cdot n^3$

$g(n) = n^3 + 2n^2 + 4n$

For finding constants c and n_0

$n^3 + 2n^2 + 4n \geq cn^3$

Divide both sides with n^3

$$1 + \frac{2n^2}{n^3} + \frac{4n}{n^3} \geq c$$

$$1 + \frac{2}{n} + \frac{4n}{n^2} \geq c$$

Here $\frac{2}{n}$ and $\frac{4}{n^2}$ approaches 0

$1 + 2/n + 4/n^2$

Example $c = 1/2$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2} \Rightarrow 1 + \frac{2}{n} + \frac{4}{n^2} \geq 1 = 1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$$

Thus, $g(n) = n^3 + 2n^2 + 4n$ is indeeded $\Omega(n^3)$

7) Big theta notation:- determine another $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not.

(s) $c_1 \cdot n^2 \leq h(n) \leq c_2 \cdot n^2$

In upper bound $h(n)$ is $O(n^2)$

In lower bound $h(n)$ is $\Omega(n^2)$

upper bound $(O(n^2))$:

$h(n) = 4n^2 + 3n \Rightarrow h(n) \leq 2n^2$

$4n^2 + 3n \leq c_2 n^2$

$4n^2 + 3n \leq 5n^2$

let's - $c_2 = 5$

divide both sides by n^2

$4 + \frac{3}{n} \leq 5$

$h(n) = 4n^2 + 3n$ is $O(n^2)$

lower bound:

$h(n) = 4n^2 + 3n$

$h(n) \geq c_1 n^2$

$4n^2 + 3n \geq c_1 n^2$

let's $c_1 = 4 \Rightarrow 4n^2 + 3n \geq 4n^2$

divide both sides by n^2

$4 + \frac{3}{n} \geq 4$, $h(n) = 4n^2 + 3n$ ($c_1 = 4, n_0 = 1$) is $O(n^2)$

(8) let $F(n) = n^3 - 2n^2 + n$ and $g(n) = n^2$ show whether $F(n) = \Omega g(n)$ is true or false and justify your answer

(S) $F(n) \geq c \cdot g(n)$

substuting $F(n)$ and $g(n)$ into this inequality we get

$n^3 - 2n^2 + n \geq c(-n^2)$

Find c and n_0 holds $n \geq n_0$

$n^3 - 2n^2 + n \geq -cn^2$

$n^3 - 2n^2 + n + cn^2 \geq 0$

$n^3 + (c-2)n^2 \geq 0$

$n^3 + (c-2)n^2 + n \geq 0 \quad (n^3 \geq 0)$

$n^3 + (1-2)n^2 + n = n^3 - n^2 + n \geq 0$

$f(n) = n^3 - 2n^2 + n$ is $\Omega(g(n)) = \Omega(-n^2)$

There fore the statement $F(n) = \Omega g(n)$ is True

9) Determine whether $h(n) = n\log n + n$ is $\Theta(n\log n)$

prove a rigirous proof for your conclusion.

Ⓢ $c_1 n\log n \leq h(n) \leq c_2 n\log n$

Upper bound:-

$h(n) \leq c_2 n\log n$

$h(n) = n\log n + n$

$n\log n + n \leq c_2 n\log n$

divide both sides by $n\log n$

$$1 + \frac{n}{n\log n} \leq 2$$

$$1 + \frac{1}{\log n} \leq c_2 \text{ (simplify)}$$

$$1 + \frac{1}{\log n} \leq 2$$

$(c_2 = 2, n_0 = 2)$

Then $h(n)$ is $O(n\log n)$

lower bound:

$h(n) \geq c_1 n\log n$

$h(n) = n\log n + n$

$n\log n + n \geq c_1 n\log n$

divide both sides by $n\log n$

$$1 + \frac{n}{n\log n} \geq c_1 = 1 + \frac{1}{\log n} \geq 1 \quad (c_1 = 1)$$

$$\frac{1}{\log n} \geq 0 \text{ for all } n \geq 1$$

$F(n)$ is $\Omega(n\log n)$ $(c_1 = 1, n_0 = 1)$

$h(n) = n\log n + n$ is $\Theta(n\log n)$

10) solve the following recurrence relation and find order of growth for solutions $T(n) = 4T(\frac{n}{2}) + n^2, T(1) = 1$

(S) $T(n) = 4T\left(\frac{n}{2}\right) + n^2, T(1) = 1$

$T(n) = aT(n/b) + f(n)$

$a = 4, b = 2, f(n) = n^2$

Applying master theorem

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$f(n) = O(n \log_a b - 1)$ $\left(\overset{F>0}{T(n) = \Theta(n^{\log_b a})}\right)$

$f(n) = \Omega(n \log_b a - 1)$, then $T(n) = f(n)$

calculating $\log_b a$:

$\log_b a = \log_2 4 \simeq 2$

$F(n) = n^2 = \Theta(n^2)$ [comparing $f(n)$ with $n^{\log_b a}$]

$F(n) = \Theta(n^2) = \Theta(n^{\log_b a})$, (case 2)

$F(n) = 4T\left(\frac{n}{2}\right) + n^2$

$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^2 \log n)$

order of growth

$T(n) = 4T\left(\frac{n}{2}\right) + n^2$ with $T(1) = 1$ is $\Theta(n^2 \log n)$