MA 102

Linear Algebra

1908 – Cantor (Set)

Vector-Space and Matrices.

★ Set → A collection of well defined and distinct objects of our perception to be looked as a whole.

eg. $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \;\&\; q \neq 0\}$

Irrationals.

$\mathbb{R} = \{ \quad \}$

$\mathbb{C} = \{ \quad \}$

★ Operation : It is a function.

(Relation, Every subset of $A \times A$ where A is a non-empty set)

$A \times A = \{(a, b) \mid a, b \in A\}$.

★ Binary Operation : Let G be a non-empty set. Then a function $f : G \times G \to G$ is said to be a binary operation.

i.e. f is a binary operation on G,

this means $x, y \in G \Rightarrow x f y \in G$

$f(x, y) \in G$.

★ Nagpaul & Jain → Abstract Algebra and its applications. (AMS).
(Book)

★ Linear Algebra of Hoffman & R. Kunze.

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* Closure Property of the set under the given binary operation.

* Binary Operation G $f: G \times G$ $a,b \in G$.

$a f b \in G$ usual addition

Addition ← $a + b \in G$. usual multiplication.

$$N, \oplus(a,b) = 2a + 3b.$$
$$g(a,b) = 2a + 3b.$$

* $f: N \times N \rightarrow N.$ $f(a,b) = |a-b|$ } not binary.

* $f: Z \times Z \rightarrow Z$ $f(a,b) = |a-b|$ } binary.

* $A = \{0,1,2\}$ $f: A \times A \rightarrow A$ $f(a,b) = |a-b|$ } binary.

* $G_1 = \{1,-1\}$. $f: G \times G \rightarrow G$ $f(a,b) = a \times b$ } binary.

$G_2 = \{1, \omega, \omega^2\}$, $G_3 = \{1, -1, i, -i\}$.

$G_n = \{ e^{i2k\pi/n} \mid k = 0, 1, \ldots, n-1 \}$.

(1) Associative Rule → For any non-empty set G and an operation '$*$' on G such that

$$a * (b * c) = (a * b) * c, \text{ for } a,b,c \in G.$$

$*$ is associative in G.

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Date (a,b) 2b 3 Page No.:

$f(a,b) = 2a + 3b.$
$f(a*b, c) = 2(a*b) + 3c$
$= 2(2a+3b) + 3c$

$f(a,(b*c)) = 2a + 3(b*c)$
$= 2a + 3(2b+3c)$

(2) Commutativity $a, b \in G$,

$a * b = b * a$
$a + b = b + a$
$a \times b = b \times a$.

(3) Distributive Law
Let G be a non-empty set and $*$, o be two (binary) operations on G. Then 'o' is said to be distributive over '$*$' if.

left distributivity $a \circ (b * c) = (a \circ b) * (a \circ b)$.
right distributivity $(b * c) \circ a = (b \circ a) * (c \circ a)$.

★ Group →
An algebraic structure $(G, *)$ is said to be a group if the following conditions are satisfied

(i) $*$ is associative in G;
(ii) There exists an identity element in G w.r.t '$*$';
(iii) For each element $x \in G$ inverse exists in G w.r.t '$*$'.

ii) → Identity element i.e. $\exists\, e \in G$ s.t. for any $a \in G$,
$a * e = e * a = a$;

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— Every group is a semigroup.

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an element

iii) → i.e. for each $x \in G$ $\exists$ $y \in G$ s.t.

$$x * y = y * x = e.$$

iv) If $*$ is commutative in G, then this group is called 'Abelian group'.

* 'e' → identity element in Z or Q or IR w.r.t. '+' is 0 - zero. (additive identity).

$x \in G$, x^{-1} → additive inverse in Z or Q or R is $-x = x^{-1}$.

*

in Z or Q or R Multiplicative identity $e = 1$

Multiplicative inverse of an element, $x^{-1} = \frac{1}{x}$

Here, $x \neq 0$.

eg. $* : Z \times Z \to Z$; $a * b = a + b - 1$.

→ $a * e = a + e - 1$

for identity, $a * e = a$

$\Rightarrow a + e - 1 = a$

$\Rightarrow \boxed{e = 1}$.

eg $(Z, \cdot)$ → not a group because inverse does not exist in Z.
(monoid).

* If operation is associative in the set then the algebraic structure is called Semi-group.

* If operation is associative in the set and identity element exists then the algebraic structure is called monoid.

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Q^* or $Q_0 \Rightarrow$ set of all nonzero rationals
R^* or $R_0 \rightarrow$ —— // —— reals.

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eg. $(Q, \cdot) \Rightarrow$ not a group (due to 0).
$(Q^*, \cdot)$ where $Q^* = Q - \{0\}$.
$\Rightarrow$ Abelian group.

eg. $(R, \cdot) \Rightarrow$ not a group
$(R^*, \cdot) \Rightarrow$ Abelian group.

$(Z, +) \Rightarrow$ is an Abelian group.

☆ Multiplicative identity is known as unity element

☆ Elements which are invertible are known as units

$(Z, +, \cdot)$

1. $(Z, +)$ is an abelian group.
2. $(Z, \cdot)$ is a semi-group.
3. For any $a, b, c \in Z$;

$$a\cdot(b+c) = a\cdot b + a\cdot c$$
$$(b+c)\cdot a = b\cdot a + c\cdot a$$

☆ Ring (1,2,3)

Let R be a non-empty set and $*, \circ$ be two binary operations on R. Then $(R, *, \circ)$ is said to be a Ring if -

1. $(R, *)$ is an abelian group.
2. $(R, \circ)$ is a semi-group.
3. For any $a, b, c \in R$
($\circ$ is distributive over $*$ in R (not set of Real))

$$a \circ (b * c) = a \circ b * a \circ c$$
$$(b * c) \circ a = b \circ a * c \circ a.$$

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④ In addition, if R has identity element w.r.t 'o' then R is ring with unity (1,2,3,4)

⑤ In addition, if Every non-zero element in R has inverse in R w.r.t. 'o' then (R, *, o) is a division ring (skew field).
(1,2,3,4,5)

⑥ In addition, if 'o' is commutative in R then ring (R, *, o) is called a field
i.e. a commutative division ring (or skew field is called a field. (1,2,3,4,5,6).

* Field
① (R, *) is an abelian group.
② (R_0, o) is an abelian group.
③ 'o' is distributive over (first operation) '*' in R i.e. $(b * c) o a = b o a * c o a$
$a o (b * c) = a o b * a o c$.
·.d

- In context of Ring,
first operation is called addition
second operation is called multiplication

0 – zero element (additive identity)
1 – unity. (multiplicative identity).

- we can prove, $x \cdot 0 = 0$
$a(b-c) = ab - ac$
using above definitions.

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Vector Space :-

Def^n: let V be a non-empty set and F, a field we define two operations on V as follows :-

(i) $\oplus : V \times V \to V$ i.e. $(x, y) \mapsto x + y$, called addition (or vector addition).

(ii) $\odot : F \times V \to V$ by $(a, x) \mapsto a \cdot x$ called multiplication (or scalar multiplication).

Then V is said to be a vector space over field F if the following conditions are satisfied

(1) $(V, +)$ is an abelian group :-

(2) $a \odot (x \oplus y) = a \odot x \oplus a \odot y$.

(3) $(a + b) \odot x = a \odot x \oplus b \odot x$.

(4) $a \odot (b \odot x) = (a \cdot b) \odot x$

(5) $1 \odot x = x$, for $x, y \in V$; $a, b \in F$ and $1 \in F$, unity element.

* Remarks -

① If V is a vector space over the field F, ~~is~~ we denote this fact by $V(F)$.

② or if Field F is known, then V is a vector space (notation).

③ Vector space is also known as linear space.

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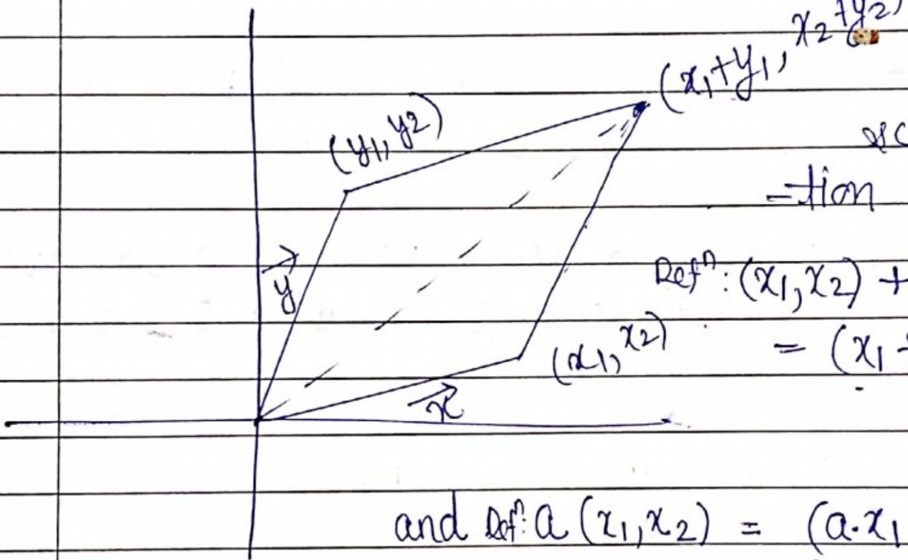
Date ___/___/______ $(F,+,\cdot)$ $V=F$. Page No.: ______

(4) Every field is a vector space over itself.

Ex. 1 $3\hat{i}+4\hat{j}$ $(3,4)$ $R^2 = R\times R = \{(x_1,x_2) \mid x_1,x_2 \in \mathbb{R}\}$

R is a field w.r.t. usual addition '+' and usual multiplication '.'.



We define addition and scalar multiplication by

Def^n: $(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2)$ —— (1).

and Def: $a(x_1,x_2) = (a\cdot x_1, a\cdot x_2)$; — (2)

where $(x_1,x_2),(y_1,y_2)\in R^2$ and $a\in R$.

(1). $+ : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$

$x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$

$(x+y)+z = x+(y+z)$ → Associative

$(0,0)\in\mathbb{R}^2$, $(x_1,x_2)+(0,0) = (x_1,x_2)$ → Identity

$-x = (-x_1,-x_2)\in\mathbb{R}^2$ s.t. → Inverse

$x+(-x) = (x_1,x_2)+(-x_1,-x_2) = (x_1+(-x_1), x_2+(-x_2))$
$= (0,0)$.

Hence, R^2 is an abelian group w.r.t. '+'.

(2). $a(x+y) = a((x_1,x_2)+(y_1,y_2))$
$= a(x_1+y_1, x_2+y_2)$
$= (a(x_1+y_1), a(x_2+y_2))$.

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★ $R^n = \{(x_1, \dots, x_n) \mid x_i \in R,\ i \in [1, n]\}$. ⇒ $R^n(R)$

Date __/__/___ ∴ R^n is a vector space. Page No.: _____

elements of field → scalars.
elements of vector space are called vectors.

Q. Let R be the field of real numbers and $V = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in R\}$. Prove that V is a vector space over the field R under suitable operations.

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$.
& $a(x_1, x_2) = (ax_1, ax_2)$;
$\forall\ (x_1, x_2), (y_1, y_2) \in V,\ a \in R$.

{ Power set → set of all subsets of given set }.
$P(X) : \{A \subseteq X\}$

{ Symmetric difference (Δ) betn two sets
$A, B \in P(X)$, $A \Delta B = (A \cup B) - (A \cap B)$.
$= (A - B) \cup (B - A)$. }.

first operation $A \Delta B$ (addn)
second operation $A \cap B$. (multiplication).

$(P(X), \Delta, \cap)$

$R^n(R)$ $x = (x_1, x_2, \dots, x_n)$; $y = (y_1, y_2, \dots, y_n)$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
$ax = (ax_1, ax_2, \dots, ax_n)$.

Q. Let F be a field and $V = F^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in F,\ 1 \le i \le n\}$. Prove that V is a vector space over the field F.

Same operations as above.

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F is an extension of K.

★ **Subfield**

→ $K \subseteq F$, if F is a field and K is any not empty subset of it then K is called subfield of F if K is a field on same operations.

* Every field is a vector space over subfield.

★ **Subspace** (or vector subspace or linear subspace):

Def^n: A non-empty subset W of a vector space V(F) is said to be a subspace of V if W is itself a vector space over the same field F under the operations of V (restricted to W).

eg. ① . Q(Q) is a vector space. (Every field is a vector space over itself)
R(Q) is also a vector space (Every field is a vector space over its subfield)
∴ Q is a subspace of R over the field Q.

②. R(R) is a vector space.
C(R) is also vector space.
∴ R is a subspace of C over the field R.

③. Set of zero element $\{0\}$.

Let V be a vector space over the field F and 0 be the zero element of V then $\{0\}$ is a vector space over the same field under the same operations.

∴ $\{0\}$ is a subspace of V over the field F.

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★ Every vector space is also a subspace of itself.

★ Every non-zero vector space over the field F has at least two subspaces. {0} and itself.

Let V be a vector space over the field F then

(i) $a.\mathbf{0} = \mathbf{0}$ ($\mathbf{0}$ → zero element of vector space V)

(ii) $0.x = \mathbf{0}$
zero scalar, zero vector. $a \in F$ $x \in V$

(iii) $-(a.x) = (-a)x = a(-x)$ $0 \in F$ $\mathbf{0} \in V$
additive inverse

(iv) $a(x+(-y)) = ax + (-a)y = ax - ay$.

(v) $ax = ay$, $0 \neq a$ $a \in F$ then $x = y$.

(vi) $ax = \mathbf{0} \Rightarrow x = \mathbf{0}$ $(a \neq 0)$.

$\Rightarrow$
$$a + x = b + x$$
$$(a+x) + (-x) = (b+x) + (-x)$$
$$a + (x + (-x)) = b + (x + (-x))$$
$$\Rightarrow a = b.$$

(i) $a.\mathbf{0} = \mathbf{0}$

Proof :- Let $y = a.\mathbf{0} = a.(\mathbf{0}+\mathbf{0})$
$= a.\mathbf{0} + a.\mathbf{0}$

$(\because a.(x+y) = a.x + a.y)$.

$\Rightarrow a.\mathbf{0} + \mathbf{0} = a.\mathbf{0} + a.\mathbf{0}$

(V w.r.t. add^n is abelian gp. there we have additive inverse of each element).

$\Rightarrow -(a.\mathbf{0}) + (a.\mathbf{0} + \mathbf{0}) = (-a.\mathbf{0}) + (a.\mathbf{0} + a.\mathbf{0})$

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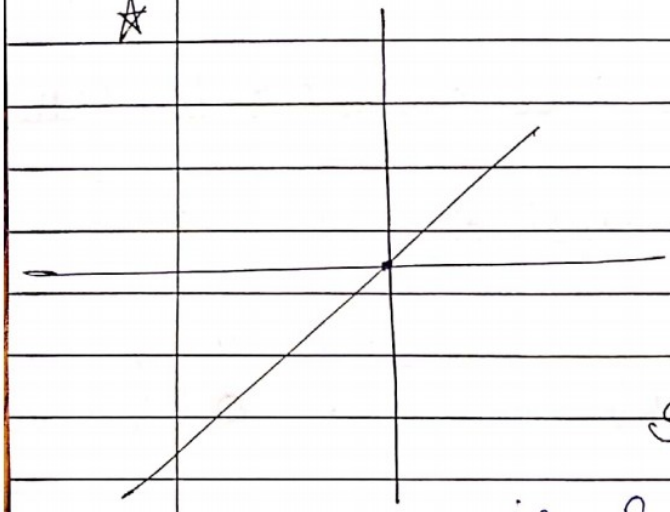
$\Rightarrow ((-a.0)+a.0)+a.0 = ((-a.0)+a.0)+0$

$\Rightarrow \boxed{a.0 = 0}$

☆



$R^2 = \{(a,b) \mid a, b \in R\}$.

$\left\{ \begin{array}{l} S_1 = \{(a,0) \mid a \in R\} \\ S_2 = \{(0,b) \mid b \in R\} \end{array} \right.$

S_1, S_2 are subspace of R^2.

i.e. Every line passing through the origin will be the subspace of given R^2.

$S_3 = \{(a,b) \mid a+2b=0;\ a, b \in R\}$.

☆ Necessary and sufficient conditions to check ~~whether~~ subspace.

① A non-empty subset W of a vector space V over the field F is a subspace if and only if W is closed under vector addition and scalar mul-tiplication of V over the field 'F'.

i.e. (i) $x, y \in W \Rightarrow x+y \in W$;

(ii) $a \in F, x \in W \Rightarrow ax \in W$.

Proof :- let W be a non-empty Subset of V over F.

if condn :- Let W is a subspace V(F).

Since W is a vector space over F.

and in vector space addition is closed.

$\therefore$ for $x, y \in W \Rightarrow x+y \in W$.

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Also scalar multiplication is also closed.

$\therefore\ \forall\ a \in F,\ x \in W \Rightarrow ax \in W.$

only if condⁿ :- Let (i) $\forall\ x, y \in W \Rightarrow x+y \in W$

(ii) $a \in F,\ x \in W \Rightarrow \alpha x \in W.$

So from our assumptions we can conclude that addition and multiplication are closed for W.

Now using these we will prove that W is a vector space over field F.

i.e. $(W, +)$ is an abelian group.

$\Rightarrow$ As addⁿ is associative on V, it will be associative on W also as it is a subset of V.

i.e. for any $x, y, z \in W \Rightarrow x+(y+z) = (x+y)+z.$

$\therefore$ + is associative ~~w.r.t.~~ in W.

$\forall\ a \in F,\ x \in W \Rightarrow a.x \in W.$

Since $1 \in F \Rightarrow -1 \in F$ (because F is a field)

$\therefore\ -1.x \in W \Rightarrow -x \in W.$ ✓ inverse

$\Rightarrow\ x + (-x) \in W$ from (i).

$\Rightarrow\ 0 \in W.$ ✓ identity.

Similar ~~for~~ commutative.

$$\left.\begin{aligned} a.(x+y) &= a.x + a.y. \\ (a +_F b).x &= a.x + b.x \\ a.(b.x) &= (a ._F b).x \\ 1.x &= x. \end{aligned}\right\}\ \text{valid for V over F}$$

are all valid as $x, y \in \cancel{V}$ & W. for W also $\forall\ a, b \in F.$

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$\therefore$ W is a vector space over F.

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(2) The necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace V is that.

$$\boxed{a,b \in F ;\ x,y \in W \ni \ ax+by \in W}.$$

eg $S_1 = \{(a,0) \mid a \in R\}$.

let $x = (a_1, 0)$; $y = (a_2, 0) \in S_1$. & $\alpha \in \mathbb{R}$

then

$$\alpha x + y = \alpha(a_1,0) + (a_2,0).$$

$$= (\alpha a_1, 0) + (a_2, 0).$$

$$= (\underbrace{\alpha a_1 + a_2}_{z}, 0+0)$$

$$= (z, 0) \in S_1.$$

Q. $f : S \to R$.

$V = \{f : S \to R\}$ under operations.

$f, g \in V$; $(f+g)(x) = f(x) + g(x)$; $\forall x \in S$

$\exists\ a \in R, f \in R$; $(af)(x) = af(x)$, $\forall x \in S$.

Prove V as a vector space.

Also prove that $V_e = \{\text{even functions in } V\}$ and $V_o = \{\text{odd functions in } V\}$ are subspaces.

Q. $V = \{a_0 + a_1 x + a_2 x^2 + \quad + a_n x^n \mid a_i \in F \ 1 \le i \le n\}$.

then V is a vector space under,

$$f(x) = a_0 + a_1 x + \quad + a_n x^n$$

$$g(x) = b_0 + b_1 x + \quad + b_m x^m$$

$(m<n)$ $f(x) + g(x) = (a_0+b_0) + (a_1+b_1)x + \ldots + (a_m+b_m)x^m + a_{m+1}x^{m+1} + \ldots + a_n x^n$

$a \in F$, $f(x) \in V$ $\quad a f(x) = (aa_0 + aa_1 x + \ldots + aa_n x^n)$.

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~~Q~~. $C[0,1]$ = set of all continuous real valued functions defined on $[0,1]$, $D[0,1]$.

eg. $\langle 2 \rangle = \{ a.2 \mid a \in \mathbb{Z} \}$
$= \{ \ldots -4, -2, 0, 2, 4, \ldots \}$
set generated by 2

eg. set generated by $(a,0)$

$\langle (a,0) \rangle = S_3 = \{ \alpha(a,0) \mid \alpha \in \mathbb{R} \}$.

* Tutorial

~~VE~~ $(V, \oplus, \otimes)$, F = field.
Conditions for V to be a vector space over F.

$(V, \oplus) \to$ Abelian group.
$\Rightarrow$ B.O. must be closed.
$\Rightarrow \exists e \in V$ s.t. $a + e = a$.
$\Rightarrow (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$
$\Rightarrow \forall u \in V \ \exists -u \in V$, st. $u + (-u) = e = (-u) + u$
$\Rightarrow u + v = v + u$.

$\otimes \quad F \times V \to V$.
$(\alpha, u) \to \alpha \otimes u$.

$a(u \oplus w) = au \oplus aw$
$(a+b)u = au + bu$
$a(bv) = ab(v)$
$1.v = v$.

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Q.1) (i) addⁿ on R^2 $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$

Show additive identity doesn't exist

Let $(e_1, e_2) \in R^2$ be the additive identity of R^2

$\forall \ (a, b) \in \mathbb{R}^2$

$(a, b) + (e_1, e_2) = (a, b)$

$\Rightarrow (a + e_1, 0) = (a, b)$.

$a + e_1 = 0$, $b = 0$

$\times$

$\therefore$ does not exist

(ii) addⁿ on $\mathbb{R}^3$ $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3)$

Let $(e_1, e_2, e_3) \in \mathbb{R}^3$ is additive identity

$(a, b, c) + (e_1, e_2, e_3) = (a, b, c)$.

$(ae_1, be_2, ce_3) = (a, b, c)$.

$ae_1 = a$, $be_2 = b$, $ce_3 = c$

$\Rightarrow e_1 = 1$, $e_2 = 1$, $e_3 = 1$.

$= (1, 1, 1)$. $\therefore$ additive identity exists.

Let (a', b', c') is the inverse of (a, b, c) in R^3.

$(a, b, c) + (a', b', c') = (1, 1, 1)$.

$(aa', bb', cc') = (1, 1, 1)$.

$aa' = 1$, $bb' = 1$, $cc' = 1$.

$\Rightarrow a' = \frac{1}{a}$, $b' = \frac{1}{b}$, $c' = \frac{1}{c}$.

$\therefore$ inverse does not exist for $(a = 0)$.

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4. For any real vector space $(V, \oplus, \otimes)$.
Here $\oplus$ & $\otimes$ are usual operations.

(i) $\alpha \otimes 0 = 0$ for every scalar α.

$\Rightarrow$

~~LH.~~ $\alpha \otimes 0 = \alpha \otimes (0 \oplus 0)$

~~for any vector space $0 \oplus 0 = 0$.~~

$= \alpha \otimes 0 \oplus \alpha \otimes 0$.

$\alpha \otimes 0 = 2\alpha \otimes 0$

$\therefore \boxed{\alpha \otimes 0 = 0}$

(iii) $(-1) \cdot \otimes u = -u, \quad \forall u \in V$.

$\Rightarrow$

$(-1 \otimes u \oplus u = 0)$ we have to prove

L.H.S. $= -1 \otimes u \oplus 1 \otimes u$

$= (-1 \oplus 1) \otimes u$ (Distributive law).

$= 0 \otimes u$

$= 0$

$\therefore \boxed{(-1) \otimes u = -u}$

5. Prove that nonempty subset S of a vector space $(V, \oplus, \otimes)$ is a subspace iff $(\alpha \otimes u) \oplus v \in S$ for all scalars α and $u, v \in S$.

Steps to prove W is a subspace of F.

$(V, \oplus, \otimes)$ ① $W \neq \phi$

first check $e \in W$ or not

additive identity

② If $e \in W$, $u, v \in W$

$u + v \in W$

③ $\alpha \in F$, $u \in W$, $\alpha u \in W$.

(i.e. $\alpha u + v \in W$

for $\alpha \in F$, $u, v \in W$).

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6. $V = C[0,1]$, $F = \mathbb{R}$.
$f, g \in W$
$(f \oplus g)(t) = f(t) + g(t)$.
$(\alpha \otimes f)(t) = \alpha f(t)$.

$(f \oplus 0) = f$

(ii) $S = \{ f \in \cancel{W} V \, . \, f(3/4) + 1 \}$.

AS

$0\left(\frac{3}{4}\right) = 0 \neq 1$.

0 function $\leftarrow$ $0 \notin S$

So, S is not subspace.

★ Set generated by an element :

$<(3,2)> = \{ \alpha(3,2) \mid \alpha \in F \}$.

$= \{ (3\alpha, 2\alpha) \mid \alpha \in F \}$.

$\alpha = a$, $x = (3a, 2a)$ $\alpha = b$, $y = (3b, 2b)$.

$\alpha, \beta \in F$.

★ Check whether $\alpha x + \beta y \in <(3,2)>$.

$\alpha x + \beta y = \alpha(3a, 2a) + \beta(3b, 2b)$
$= (3\alpha a, 2\alpha a) + (3\beta b, 2\beta b)$
$= (3\alpha a + 3\beta b, \; 2\alpha a + \beta b)$.
$= (3(\alpha a + \beta b), 2(\alpha a + \beta b))$

$\alpha a + \beta b$ is the element of given field F.

$\therefore <(3,2)>$ is subspace of $\mathbb{R}^2$.

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* Each element of vector space generates a set which is subspace of given vector space over the given field.

$<S> = <\{(1,0),(0,1)\}>$

let $S = \{(1,0),(0,1)\} \subseteq V = \mathbb{R}^2$.

$$<S> = \{\alpha(1,0) + \beta(0,1) \mid \alpha, \beta \in F\}.$$

linear combination of vectors $(1,0)$ & $(0,1)$

i.e. If $x_1, x_2, x_3, \ldots, x_n$ are n vectors in a vector space $V(F)$, then an expression $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is called the linear combination of the vectors $x_1, x_2, \ldots, x_n$, where $a_1, a_2, \ldots, a_n \in F$.

$$<S> = \{(\alpha, \beta) \mid \alpha, \beta \in F\} = \mathbb{R}^2.$$

* Set generated by two specific elements of a vector space can be equal to the given vector space.

* Prove that intersection of subspaces of a vector space is a subspace
i.e. let W_1 & W_2 be subspaces of a vector space $V(F)$. Then prove that $W_1 \cap W_2$ is a subspace of V.
What do you say about the union of two subspaces of a vector space?

Ans steps. ① $0 \in W_1 \cap W_2$, $W_1 \cap W_2 \neq \phi$.

② $x, y \in W_1 \cap W_2$, $a, b \in F$ then $ax + by \in W_1 \cap W_2$

$x,y \in W_1$ & $x,y \in W_2$

& $a,b \in F$ & $a,b \in F$

$\therefore ax + by \in W_1$ (i) $ax + by \in W_2$ (ii).

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$\therefore$ $W_1 \cap W_2$ is a subspace over $V(F)$.

(II), No, $W_1 \cup W_2$ may not be a subspace

<u>Counter example</u> :

let, $V(R) = \mathbb{R}^2(R)$ $\boxed{F = \mathbb{R}}$

$S_1 = \{(a,0) \mid a \in F\}$

$S_2 = \{(0,b) \mid b \in F\}$

S_1 & S_2 are subspaces over $\mathbb{R}^2(R)$.

$S_1 \cup S_2 = \{(a,0), (0,b) \mid a, b \in F\}$

$a, b \in F$, $x = (1,0)$; $y = (0,1) \in S_1 \cup S_2$.

$\Rightarrow$ $ax + by = a(1,0) + b(0,1)$.

$= (a,b) \notin S_1 \cup S_2$.

$\therefore$ Union of S_1 & S_2 can not be a subspace over $\mathbb{R}^2(R)$.

Union of two subspaces need not be a subspace.

★ ~~Linear sum~~ let W_1 & W_2 be two subspaces of a vector space $V(F)$, then show that $W_1 \cup W_2$ is a subspace of $V(F)$ if and only if

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(One should be contained in the other for union to be a subspace).

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☆ Linear Sum of Subspaces :-

Def^n :- Let W_1 & W_2 be subspaces of a vector space $V(F)$. Then linear sum of W_1 & W_2 is denoted by $W_1 + W_2$ and defined as
$W_1 + W_2 = \{x+y \mid x \in W_1, y \in W_2\}$.

$a, b \in F$, $x = x_1 + y_1$, $y = x_2 + y_2 \in W_1 + W_2$.

$$\begin{aligned} ax+by &= a(x_1+y_1) + b(x_2+y_2) \\ &= (ax_1 + ay_1) + (bx_2 + by_2) \\ &\downarrow \text{Using associativity \& commutativity.} \\ &= \underbrace{(ax_1 + bx_2)}_{W_1} + \underbrace{(ay_1 + by_2)}_{W_2} \\ &\in W_1 + W_2. \end{aligned}$$

$\therefore$ $W_1 + W_2$ is a subspace over $V(F)$.

☆ Direct sum of subspaces :-

Def^n : ~~Let W_1 &~~ $W_1 \oplus W_2$ and each element $x \in W_1 \oplus W_2$ is uniquely written as $x = x_1 + y_1$, where $x_1 \in W_1$, y_1 ~~x_2~~ $\in W_2$.

$$W_1 \oplus W_2 = W_1 + W_2 \text{ with } W_1 \cap W_2 = \{0\}.$$

A vector space ~~ove~~ $V(F)$ is said to be direct sum of the subspaces W_1 and W_2 if $V = W_1 \oplus W_2$

$V(R) = R^2(R)$ $\quad R^2 = S_1 \oplus S_2$; $S_1 \cap S_2 \{(0,0)\}$.
S_1 2nd coordinate zero (a,b)
S_2 1st —//—

Similarly $R^3 = S_1 \oplus S_2 \oplus S_3$
S_1 of form $(a,0,0)$
S_2 —//— $(0,b,0)$
S_3 —//— $(0,0,c)$

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* let $V(F)$ be a vector space and W_1, W_2 be subspaces of V then $V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Proof :- if part :- Let, $V = W_1 \oplus W_2$. — ①

$\therefore$ $V = W_1 + W_2$ (from ①).

Using contradiction, if possible, Let $W_1 \cap W_2 \neq \{0\}$.

then $x \neq 0$; $x \in W_1 \cap W_2 \subseteq V$.

we can write,

$$x = x + 0$$
$$= 0 + x. \Rightarrow x = 0.$$

but as only unique representation is possible for each element $x = 0$.

$\therefore$ Contradiction occurs.

$\therefore$ $W_1 \cap W_2 = \{0\}$.

only if part :- Let $V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

If possible let $x \in V$ $x = x_1 + x_2$ — ①

and $x = y_1 + y_2$ — ②

$x_1, y_1 \in W_1$ & $x_2, y_2 \in W_2$

from ① & ② $\Rightarrow$ $x_1 + x_2 = y_1 + y_2$.

$$(-y_1) + x_1 + x_2 + (-x_2) = (-y_1) + y_1 + y_2 + (-x_2).$$

Additive inverse $\because y_1 \in W_1$ & W_1 is a subspace (vectorspace)

— // — $x_2 \in W_2$ & W_2 — // — (vectorspace).

$$(-y_1) + x_1 + 0 = 0 + y_2 + (-x_2)$$
$$x_1 - y_1 = y_2 - x_2$$ (As add^n is commutative) in vector space.

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$x_1, y_1 \in W_1$ & W_1 is a vectorspace.

$x_1 - y_1 \in W_1$.

Similarly $y_2 - x_2 \in W_2$.

$\Rightarrow \quad x_1 - y_1 = y_2 - x_2 \in W_1 \cap W_2 = \{0\}$.

$\Rightarrow \quad x_1 - y_1 = 0 = y_2 - x_2$

$\Rightarrow \quad x_1 = y_1$ & $y_2 = x_2$.

$\therefore$ ① & ② both representations are similar

$\therefore$ there exists a unique ~~element~~ representation for any $x \in V$.

$\therefore \quad V = W_1 \oplus W_2$.

* $V_n(\mathbb{R}) = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R},\ 1 \leq i \leq n\}$.

$V_n = S_1 \oplus S_2 \oplus \cdots \oplus S_n$.

Linear Span

☆ vector space generated by a subset:

Let $V(F)$ be a vector space and $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of V. Then set generated by S is a subspace of V. This subspace is known as the linear span of S.

Linear span of S is denoted by $[S]$ or $L(S)$ or $\langle S \rangle$. and defined as $L(S) = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in F, x_i \in S\}$.

$\Rightarrow \quad ax + by = \sum_{i=1}^{n} (aa_i) x_i + \sum_{i=1}^{m} (bb_i) y_i \in L(S)$

$x = \sum_{i=1}^{n} a_i x_i, \quad y = \sum_{i=1}^{m} b_i y_i$

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Note: Linear span of S i.e $L(S)$ is the smallest subspace of V containing S. i.e. $L(S) = \bigcap_{i \in I} W_i$;

$$S \subseteq W_1, W_2, \ldots, W_n, \ldots$$
$$S \subseteq \bigcap_{i \in I} W_i$$

Linear Dependence and Independence of the vectors :-

★ let $V(F)$ be a vector space and $x_1, x_2, x_3, \ldots, x_n$ are vectors of V then $x_1, x_2, \ldots, x_n$ are said to be linearly independent if

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$
$$\Rightarrow a_1 = a_2 = \ldots = a_n = 0.$$

★ if $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.
$\Rightarrow$ at least one scalar, say $a_i \neq 0$.
then they are said to be linearly dependent

A set whose vectors are linearly independent is said to be linearly independent set.
—— " —— dependent ——
—— " —— dependent set.

$$\mathbb{R}^2 = \{\overset{x}{(1,0)}, \overset{y}{(0,1)}\},$$

$$ax + by = 0.$$
$$a(1,0) + b(0,1) = (0,0).$$
$$(a,0) + (0,b) = (0,0)$$
$$(a,b) = (0,0). \Rightarrow a = b = 0.$$

linearly independent.

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* linearly dependent → if zero element is there.

* Subset of linearly independent → linearly independent

* Superset of linearly dependent → linearly dependent

eg. $S = \{x_1, \dots, x_n\}$ $x_5 = 0$ linearly indep. in V

$a_1x_1 + a_2x_2 + \dots + a_5x_5 + \dots + a_nx_n = 0.$

may not be equal to zero ∴ linearly dependent.

eg. $\{x\} \subseteq V$; $x \neq 0$.

$ax = 0$

$\Rightarrow (a^{-1})(ax) = a^{-1}0.$

$\Rightarrow (a^{-1}a)(x) = 0.$

$\Rightarrow 1.x = 0$

$\Rightarrow x = 0$ contradicts with $(x \neq 0)$.

eg. let $S = \{x_1, x_2, \dots, x_n\} \subseteq V$.

linearly dependent in V. We have at least one scalar $a_i \neq 0$ among $a_1, a_2, \dots, a_n$ s.t.

$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$, —— (1).

$x \in V$ s.t $x \notin S$ $S_1 = \{x_1, x_2, \dots, x_n, x\}$.

$a_1x_1 + a_2x_2 + \dots + a_nx_n + ax = 0$

if $a = 0$.

S_1 is linearly dependent.

* Let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of a vector space $V(F)$ Then S is linearly dependent iff there exists a vector $x_i, 2 \leq i \leq n$ which is the linear combination of preceding vectors.

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Proof :- S is L.D. $\exists\ a_i \neq 0$ s.t.

if $a_1x_1 + a_2x_2 + \quad + a_ix_i + \quad + a_nx_n = 0 \quad (2 \leq i \leq n)$.

$\Rightarrow$ $-(a_i x_i) = a_1x_1 + a_2x_2 + \dots + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \dots + a_nx_n$.

$\Rightarrow$ $(-a_i)(x_i) = a_1x_1 + a_2x_2 + \dots + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \dots + a_nx_n$.

$\Rightarrow$ $(-a_i^{-1})(-a_i)x_i = (-a_i^{-1})(a_1x_1 + a_2x_2 + \dots + a_nx_n)$

$\Rightarrow$ $1.x_i = (-a_i^{-1})(a_1x_1 + a_2x_2 + \dots + a_nx_n)$,

$\Rightarrow$ $x_i = (-a_i^{-1})(a_1x_1 + a_2x_2 + \dots + a_nx_n)$

$\therefore$

only if $x_i = a_1x_1 + \dots + a_nx_n$

$a_1x_1 + \dots (-1)x_i + \dots + a_nx_n = 0$.

$-1 \neq 0$.

$\therefore$

✱ Basis of the Vector space :-

A non-empty subset B of the vector space $V(F)$ is said to be the basis of V if

(i) B is linearly independent.

(ii) B generates V i.e. $L(B) = V$.

Ex. $V = \mathbb{R}^2(\mathbb{R})$, $B = \{(1,0), (1,1)\}$ lin. ind.

$x \in V$; $(a,b) \in V$.

let $(a,b) = \alpha(1,0) + \beta(1,1)$.

$= (\alpha, 0) + (\beta, \beta)$.

$(a,b) = (\alpha+\beta, \beta)$.

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$b = \beta$; $\alpha = a - \beta$. $\Rightarrow$ $\alpha = a - b$.

$(a,b) = (a-b)(1,0) + b(1,1)$.

$\therefore$ B will be basis of $\mathbb{R}^2$,

eg. $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ will be basis of $\mathbb{R}^3$

eg. $B = \{(1,1,0), (0,1,0), (1,0,0), (0,0,1)\}$.
x_1 x_2 x_3 x_4 .

$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = (0,0,0)$.
is linearly dependent . $\therefore$ B will not be basis of $\mathbb{R}^3$.

* In a basis maximal no. of linearly independent vectors exist . i.e. Basis of a vector space consists maximal no. of independent vectors.

* Tutorial

Q.6) $V = C[0,1]$; $F = \mathbb{R}$

$V = \{f \mid f : [0,1] \to [0,1]$ & f is cont.$\}$.

pointwise add^n $(f \oplus g)(t) = f(t) + g(t)$.
$(\alpha \otimes f)(t) = \alpha f(t)$.

(i) $W = \{f \in V \mid f(1/2) = 0\}$.
Let $\hat{0}$ a zero function $[0,1] \to [0,1]$
$\Rightarrow \hat{0} \in W$.
$\Rightarrow W \neq \emptyset$.
let $f, g \in W \Rightarrow f(1/2) = 0$; $g(1/2) = 0$
$(f \oplus g)(1/2) = f(1/2) + g(1/2) = 0 + 0 = 0$.

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$\Rightarrow f+g \in W.$

$\alpha \in F \Rightarrow (\alpha f)(1/2) = \alpha\,(f(1/2)) = \alpha.(0) = 0$

$\Rightarrow \alpha f \in W \quad \forall\, \alpha \in F,\ f \in W.$

$\therefore$ W is a subspace of V(F).

(iii) $W = \{ f \in V .\ f(3/4) = 1 \}.$

$\Rightarrow$ 0 does not belong to W
$\therefore$ not a subspace.

(iv) $\{ f \in V : f(x) = 0$ only at finite no. of points$\}$.

$\Rightarrow$ not a subspace.

7. $V = R^4,\ F = R,\ R^4 = \{(a,b,c,d) \mid a,b,c,d \in R\}.$

(ix) $S = \{ (a,b,c,d) : a^2 - b^2 = 0 \}.$

As $(0,0,0,0) \in S.$

Let $u, v \in S$; $u = (a,b,c,d),\ v = (a',b',c',d') \in S.$

$\Rightarrow a^2 - b^2 = 0.$
$a'^2 - b'^2 = 0.$

Now, $u+v = (a+a', b+b', c+c', d+d').$

as $(a+a')^2 - (b+b')^2 = 2aa' - 2bb' \neq 0.$

$\Rightarrow u+v \notin S.$

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8. $P = \{\text{set of all polynomials over } \mathbb{R}\}$.

$= \left\{ \sum_{i=1}^{n} a_i x^i \mid a_i \in \mathbb{R} \right\}$.

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$q(x) = b_0 + b_1 x + \cdots + b_m x^m$ $(m<n)$

$p(x) + q(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_m+b_m)x^n + \cdots + a_n x^n$.

$ap(x).\cancel{q(x)} = \cancel{a_0 b_0 - a}\ aa_0 + aa_1 x + \cdots + aa_n x^n$

$p(0) = 0$.

10. St. $W_1, W_2 \subseteq V$ s.t. $W_1 \cup W_2 \subseteq V$.
we have to p.t. either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof:- ∴

let neither $W_1 \nsubseteq W_2$ nor $W_2 \nsubseteq W_1$ is wrong

If $W_1 \nsubseteq W_2 \Rightarrow \exists\ u \in W_2$ s.t. $u \notin W_1$ — ①

$W_2 \nsubseteq W_1 \Rightarrow \exists\ v \in W_1$ St $v \notin W_2$ — ②

Now, As $W_1 \cup W_2$ is a subspace.

$\Rightarrow u+v \in W_1 \cup W_2$.

$\Rightarrow u+v \in W_1$ or $u+v \in W_2$.

If $u+v \in W_2$

So, $v = (u+v) - u$

As, $u \in W_2$ & $u+v \in W_2$.

$\Rightarrow (u+v) - u \in W_2$

$\Rightarrow v \in W_2$.

$\Leftrightarrow$ contradicts with ②.

If $u+v \in W_1$

So, $u = (u+v) - v$

AS, $u+v \in W_1$ & $v \in W_1$

$\Rightarrow u \in W_1 \Leftrightarrow$ contradicts with ①.

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12. Let $S = \{(1,2,3), (1,1,-1), (3,5,5)\}$. Determine which of the following are in $L[S]$.

$Ax = b$.

$|A| \neq 0$ we get sol^n using $X = A^{-1}b$.

consistent → Unique, Infinite

inconsistent → No solution.

(ii) $(1, 1, 0)$. V

$\Rightarrow$ So, V is the Linear $comb^n$ of vectors of S. if $\exists\ (x, y, z) \in \mathbb{R}$ s.t.

$$V = xu_1 + yu_2 + zu_3$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x + y + 3z \\ 2x + y + 5z \\ 3x - y + 5z \end{bmatrix}$$

Check whether solution exists or not if exists, (____) belongs to $L[S]$

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14. (i) $M \subset N \Rightarrow L[M] \subset L[N]$.

let, $u \in L(M)$

$\Rightarrow \quad u = \sum_i a_i u_i$ when $u_i \in M$.

As $M \subset N$

$\Rightarrow \quad u_i \in N$

$\Rightarrow \quad \sum_i a_i u_i \in L(N)$

$\Rightarrow \quad u \in L(N)$

$\Rightarrow \quad L(M) \subset L(N)$.

(ii) M is a subspace of $V \Leftrightarrow L(M) = M$.

For any ~~any~~ set S, $L(S)$ is a subspace of V & $S \subseteq L(S)$.

If condⁿ :- let M is a subspace of V.

$\Rightarrow \quad M \subseteq L(M)$. — ①

Let $x \in L(M) \Rightarrow x = \sum_j a_j u_j$ where $u_j \in M$.

$\therefore$ for M is a subspace

$\Rightarrow \quad \sum_j a_j u_j \in M \Rightarrow x \in M$.

$\Rightarrow \quad L(M) \subseteq M$ — ②

$\Rightarrow \quad L(M) = M$ (from ① & ②).

(iii) $L[L[M]] = L[M]$

~~As $L[M] = M$ as $L(M)$~~

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* No. of elements in a basis of vector space $V(F)$ is said to be the dimension of the vector space.

eg: $V(\mathbb{R}) = \mathbb{R}^2(\mathbb{R})$
$S = \{(1,0), (0,1)\}$ is a basis of V with dimension $\to 2$

$V(\mathbb{R}) = \mathbb{R}^3(\mathbb{R})$.
$S = \{(1,0,0); (0,1,0); (0,0,1)\} \subseteq \mathbb{R}^3(\mathbb{R})$ is a basis of V with dimension $\to 3$.

* **Theorem :-** Let $V(F)$ be a finitely generated vector space. Then a subset $S = \{x_1, x_2, x_3, \ldots x_n\}$ of V is a basis if and only if each $x \in V$ can be expressed uniquely as $x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$; where $a_i \in F$, $1 \leq i \leq n$.

Proof :-

If cond^n :- $x \in V$ then
$$x = a_1x_1 + a_2x_2 + \ldots + a_nx_n \quad \forall a_i \in F$$
(As S is a basis). $\longleftarrow$ (1)

If possible $\exists$ $b_1, b_2, \ldots, b_n \in F$ s.t.
$$x = b_1x_1 + b_2x_2 + \ldots + b_nx_n. \longleftarrow (2).$$

from (1) & (2);
$$(a_1-b_1)x_1 + (a_2-b_2)x_2 + \ldots + (a_n-b_n)x_n = 0.$$
$x_1, x_2, \ldots, x_n \in S$.
As elements of S are linearly independent.

$\therefore$ $a_1-b_1=0$; $a_2-b_2=0$, $\ldots$, $a_n-b_n=0$
$\Rightarrow$ $a_1=b_1$; $a_2=b_2$; $\ldots$; $a_n=b_n$.
As scalars for st. (1) & (2) are same.
for any $x \in V$ we have unique representation.

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only if condition :-

let $x \in V$, $x = a_1x_1 + a_2x_2 + \dots + a_nx_n$, $\forall a_i \in F$

which is a unique representation.

Let, $c_1, c_2, \dots c_n \in F$ s.t.

$c_1x_1 + c_2x_2 + \dots + c_nx_n = 0$.

unique rep. of 0 vector $0x_1 + 0x_2 + \dots + 0x_n = 0$.

$\therefore$ $c_1 = c_2 = \dots = c_n = 0$.

$\therefore$ $x_1, x_2, \dots, x_n$ are linearly independent

$\Rightarrow$ S is a basis of V(F).

☆ Recall $\Rightarrow$ $S = \{x_1, x_2, \dots, x_n\} \subset V$ is L.D.

$\exists$ x_i, $(2 \le i \le n)$ s.t. x_i is the Linear combn of the preceding vectors.

1. ☆ Theorem :- Let $S = \{x_1, \dots, x_n\}$ be a linear independent subset of a finitely generated vector space V(F). Then S can be extended to be a basis of V. (Extension theorem).

2. ☆ Theorem :- Let V(F) be a finitely dimensional vector space and $S = \{x_1, x_2, \dots, x_n\}$ generates V. Then there exists a subset of S which is a basis of V.

3. ☆ Theorem :- Let V(F) be a finite dimensional vector space. Then any two basis have same number of elements.

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1. Proof :- $B \subseteq V$; $B = \{y_1, y_2, \ldots, y_n\}$ generating V.

$S = \{x_1, \ldots, x_n\}$ is linearly independent.

If S generates V then S is itself a basis.
Otherwise $\exists\, x \neq 0 \in V$; which can not be written as Linear Combⁿ of vectors of S.
considering,

$$S_1 = S \cup \{x\} = \{x_1, x_2, \ldots, x_n, x\}$$

is linearly independent (can be proved by considering contradiction).

let S is L.D. $\Rightarrow$ $\underbrace{a_1x_1 + \ldots + a_nx_n}_{=0} + ax = 0$

$\therefore\ ax = 0$

but $x \neq 0 \ \therefore\ a = 0$.

i.e. all scalars are zero.
contradicts with L.D.
$\therefore$ S is L.I.

Now If S_1 generates V then S_1 is basis.
otherwise. $\exists\, y \neq 0 \in V$; which can not be written as L.C. of vectors of S.
considering,

$$S_2 = S \cup \{y\}.$$

continuing till we get S_i basis of V.

if V contains m elements, we will definitely reach to basis in at most $m-n$ steps.

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2. Proof :-

If S is basis of V itself proved

otherwise S is L.D., x_i $2 \le i \le n$

$$x_i = b_1 x_1 + \dots + b_{i-1} x_{i-1}$$

$$x = a_1 x_1 + \dots + a_i(b_1 x_1 + \dots + b_{i-1} x_{i-1}) + \dots + a_n x_n.$$

$$x = (a_1 + a_i b_1) x_1 + (a_2 + a_i b_2) x_2 + \dots + (a_{i-1} + a_i b_{i-1}) x_{i-1} + \dots + a_n x_n.$$

so, on removing x_i we can still generate any element of V.

$$S_1 = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}.$$

if S_1 is basis of V proved.

otherwise S_1 is L.D., x_i

$\downarrow$ continuing till we get S_i basis of V.

if V contains m elements, we will definitely reach to basis. finite.

3 Proof:- $V(F)$ be a finite dimensional vector space.

Let $\dim V = n$ (finite).

So there exists at least one basis having dim n.

Let. $B = \{x_1, x_2, \dots, x_n\}$ be a basis of V.

if possible let B_1 be another basis which has m number of elements

Let, $B_1 = \{y_1, y_2, \dots, y_m\}$ be another basis

~~If.~~ $m \cancel{\neq} n$.

~~Suppose $n < m$~~ $m < n$

$B \subseteq V$; $B_1 \subseteq V$ As B is basis any element of V can be written as L.C. of elements of B. $B_1 \subseteq V$. so, for B_1 same.

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$B' = \{y_1, x_1, x_2, \ldots, x_n\}$. is L.D.
as y_1 can be written as L.C. of other vector.

$B'' = \{y_1, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. is L.I
by removing x_i which was making the set L.D.

if $n \geq m$

Same considering opposite i.e. B_1 as basis and B as subset of V.

then $\Rightarrow$ $m \geq n$

$\therefore$ $\boxed{m = n}$

If $n \leq m$.

In this assumption, by proceeding ahead repeating the above procedure, then elements of B will get exhausted early i.e. some,

$B'''' = \{y_1, y_2, \ldots, y_n\}$

is L.I. but will not generate the vector space V.

$\therefore$ contradiction occurs

$\therefore$ $m \leq n$

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* Dimension of the vector space is invariant

Theorem:- Let W_1, W_2 be two subspaces of a finite dimensional vector space $V(F)$. Then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Proof $W_1 \cap W_2 \subseteq W_1 \subseteq W_1 + W_2 = \{x + 0 \mid x \in W, 0 \in W_2\}$.
$W_1 \cap W_2 \subseteq W_2 \subseteq W_1 + W_2$.

$\Rightarrow W_1 \cap W_2$; Let $B = \{x_1, \ldots, x_r\}$ be a basis of $W_1 \cap W_2$.

$\dim B = r$.

$$B \subseteq W_1 \cap W_2 \subseteq W_1$$
$$\subseteq W_2$$

So, B can be extended to be a basis of W_1 & W_2

$\dim W_1 = m$; $\dim W_2 = n$

$B_1 = \{x_1, \ldots, x_r, y_1, \ldots, y_{m-r}\}$ basis of W_1 $\quad B_2 = \{x_1, \ldots, x_r, z_1, \ldots, z_{n-r}\}$ basis of W_2.

Claim: $B_3 = \{x_1, x_2, \ldots, x_r, y_1, \ldots, y_{m-r}, z_1, \ldots, z_{n-r}\}$ is a basis of $W_1 + W_2$.

Let $z \in W_1 + W_2$; $z = x + y$ where $x \in W_1$ & $y \in W_2$.

$$z = (a_1 x_1 + \ldots + a_r x_r + a_{r+1} y_1 + \ldots + a_m y_{m-r})$$
$$+ (b_1 x_1 + \ldots + b_r x_r + b_{r+1} z_1 + \ldots + b_n y_{n-r})$$

$V = W_1 \oplus W_2$ $\quad W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

$\Rightarrow \dim(W_1 \cap W_2) = 0$.

$\Rightarrow \dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$.

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Isomorphisms ⇒ same form along with one-to-one mapping - eg. congruent triangles.

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① $B = \{x_1, x_2, \ldots x_j, x_i, x_n\}$ is a basis of $V(F)$.

② B is a maximal set w.r.t. Linear independent set in V

③ B is a minimal set which generates the vector space V

☆ Elementary Properties (Transformations).

④ for $\alpha \in F \neq 0$; s.t. $B_1 = \{x_1, \alpha x_2, \ldots, x_n\}$ then B_1 is also basis of $V(F)$.

⑤ $B_2 = \{x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots x_n\}$ is also a basis of $V(F)$. (Order doesn't matter)

⑥ $B_3 = \{x_1, x_2, \ldots, x_i + \alpha x_j, \ldots, x_j, \ldots, x_n\}$ is also a basis of $V(F)$

☆ Linear Transformations (Homomorphisms) :- same form of structures eg. similar triangles.

let U & V be two vector spaces over the same field F. Then a map $T : U \to V$ is said to be a linear transformation if following conditions are satisfied

① $T(x+y) = T(x) + T(y)$ where $x, y \in U$
& $T(\alpha x) = \alpha T(x)$. $\alpha \in F$.

Operations defn on U — Operations defn on V.

equivalent ⇕

① $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ where $x, y \in U$, $\alpha, \beta \in F$

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① Let R and $\mathbb{R}^2$ be two vector spaces over $\mathbb{R}$. We define a map $T: \mathbb{R}^2 \to \mathbb{R}$ by $T(a,b) = a$.

Let (a_1,b_1) & (a_2,b_2) be the two elements of $\mathbb{R}^2$ then

$$T((a_1,b_1)+(a_2,b_2)) = T((a_1+a_2),(b_1+b_2)).$$
$$= a_1+a_2$$
$$= T(a_1,b_1) + T(a_2,b_2).$$

$$T(\alpha(a_1,b_1)) = T(\alpha a_1, \alpha b_1)$$
$$= \alpha a_1$$
$$= \alpha T(a_1,b_1).$$

$\therefore$ $T: \mathbb{R}^2 \to \mathbb{R}$ is a linear Transformation.

② $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(a_1,a_2) = (a_1, a_1-a_2, a_1+a_2)$. over $\mathbb{R}$. Check whether T is linear Transformation or not.

Let (x_1,x_2) & (x_3,x_4) be the two elements of $\mathbb{R}^2$ then

$$T((x_1,x_2)+(x_3,x_4)) = T((x_1+x_3),(x_2+x_4)).$$
$$= (x_1+x_3, (x_1+x_3)-(x_2+x_4), (x_1+x_3)+(x_2+x_4)).$$

$\therefore T(x_1,x_2) + T(x_3,x_4)$
$(x_1, x_1-x_2, x_1+x_2) + (x_3, x_3-x_4, x_3+x_4)$

$$= (x_1, x_1-x_2, x_1+x_2) + (x_3, x_3-x_4, x_3+x_4)$$

$$\therefore \boxed{T((x_1,x_2)+(x_3,x_4)) = T(x_1,x_2) + T(x_3,x_4)}.$$

$$T(\alpha(x_1,x_2)) = T(\alpha x_1, \alpha x_2) = (\alpha x_1, \alpha x_1 - \alpha x_2, \alpha x_1 + \alpha x_2)$$
$$= \alpha(x_1, x_1-x_2, x_1+x_2)$$
$$= \alpha T(x_1,x_2).$$

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$\therefore$ $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear transformation.

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③ Let $\mathbb{C}$ and $V = \mathbb{R}^2$ be two V.S. over $\mathbb{R}$. We define $T : \mathbb{C} \to \mathbb{R}^2$ s.t.

$$T(a+ib) = (a,b), \ \forall \ a+ib \in \mathbb{C}.$$

Verification : $T((a_1+ib_1)+(a_2+ib_2))$

$$= T((a_1+a_2)+i(b_1+b_2))$$
$$= \cancel{((a_1+a_2), i(b_1+b_2))}$$
$$((a_1+a_2), (b_1+b_2))$$
$$= (a_1,b_1)+(a_2,b_2).$$
$$= T(a_1+ib_1)+T(a_2+ib_2).$$

$$T(\alpha(a_1+ib_1)) = T(\alpha a_1 + i\alpha b_1).$$
$$= (\alpha a_1, \alpha b_1)$$
$$= \alpha(a_1,b_1)$$
$$= \alpha T(a_1+ib_1)$$

$\therefore$ Linear Transformation.

$\rightarrow$ (T is one-one onto.)

$$T(a_1+ib_1) = T(a_2+ib_2)$$
$$T(a_1+ib_1) - T(a_2+ib_2) = 0$$
$$\Downarrow$$
$$(0,0) \Rightarrow \mathbb{R}^2.$$

$$T((a_1+ib_1)+(-a_2-ib_2)) = (0,0)$$
$$(a_1-a_2, b_1-b_2) = (0,0)$$

$$\Rightarrow \{ a_1 = a_2 ; b_1 = b_2$$

$\therefore \mathbb{C}(\mathbb{R}) \cong \mathbb{R}^2(\mathbb{R})$. ($\mathbb{R}^2(\mathbb{R}) \cong \mathbb{C}(\mathbb{R})$ is also valid)

$\downarrow$ isomorphic

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* let U,V be vectorspaces over F. $T: U \to V$ is a linear transformation then.

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General Properties

① $T(0) = 0'$, $0, 0'$ are the zero elements of U,V resp.

② $T(-x) = -T(x)$, $x \in U$.

Proof :- $T(0) = T(0.0) = 0.T(0) = 0'$

(By $T(\alpha.x) = \alpha T(x)$)

Proof :- $T(-x) = T(-1.x) = -1.T(x) = -T(x)$

(By $T(\alpha.x) = \alpha T(x)$).

or $\cancel{T(x)} \Rightarrow T(x+(-x)) = T(0) = 0'$

$\Rightarrow T(x) + T(-x) = 0'$ $(T(x+y) = T(x)+T(y))$

$\Rightarrow T(-x) = -T(x)$.

③ One One onto linear Transformation is called an Isomorphism.

* Recall $T: U \to V$ over F

$T(ax+by) = aT(x) + bT(y)$, $x,y \in U$ & $a,b \in F$.

* let V(F) be an n-dimensional vector space Then V is isomorphic to $V_n(F)$.

$V_n(F) = \{(a_1, a_2, \ldots, a_n) \mid a_1, \ldots, a_n \in F\}$.

Proof - Given -

let $B = \{x_1, x_2, \ldots, x_n\}$ is basis of V

$x \in V$; $x = a_1x_1 + \ldots + a_nx_n$,

let $T: V \to V_n$ defined by

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$T(x) = T(a_1x_1 + \ldots + a_nx_n) = (a_1, a_2 \ldots, a_n)$

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where $a_1, a_2, a_3, \ldots, a_n \in F$

* We have to prove T is an isomorphism
  ie (1) T is linear transformation
  (2) T is one-one
  (3) T is onto.

(1) $\Rightarrow$ $T(\alpha x + \beta y) = T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_n y_n)$

$= \cancel{T}(\alpha a_1 + \beta b_1, \ldots, \alpha a_n + \beta b_n)$

$= \alpha(a_1, a_2, \ldots, a_n) + \beta(b_1, b_2, \ldots, b_n)$

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

(2) $\Rightarrow$ $T(x) = T(y)$.

$T(x) - T(y) = 0$.

$(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) - (b_1 x_1 + \ldots + b_n x_n) = 0$

$\Rightarrow (a_1, a_2, \ldots, a_n) - (b_1, \ldots, b_n) = 0$

$\Rightarrow (a_1 - b_1, \ldots, a_n - b_n) = (0, 0, \ldots, 0)$

$\Rightarrow a_1 = b_1 ; a_n = b_n$.

eg. $V(\mathbb{R}) \cong V_n(\mathbb{R})$; $V(\mathbb{C}) \cong V_n(\mathbb{C})$.
are isomorphic.

Q. $V(\mathbb{R}) \cong V_n(\mathbb{C})$? Check.

* $a, b \in \mathbb{Z}$ m is a fixed +ve integer.

(1) $a \equiv b \pmod{m} \Leftrightarrow m \mid a-b$.
(i.e. $a - b = mt$ $t \in \mathbb{Z}$)

This is an equivalence relation.
(Reflexive, Symmetric, Transitive).

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★ Partition
- None of component empty
- ~~Union of all total set~~
- Intersection of any two is ϕ.

★ Partition of equivalence relation ⇒ equivalence classes

① ⇒ Let $a=3$, $m=7$.

$\bar{3} = [3] = \{\ldots, -11, -4, 3, 10, 17, 24, \ldots\}$.

Let $a=3$, $m=3$.

$\bar{6} = \bar{3} = [3] = \{\ldots, -3, 0, 3, 6, 9, \ldots\}$

Let $a = \cancel{10}\ 2$, $m=3$.

$\bar{8} = \bar{2} = [2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$.

Let $a = 1$, $m=3$.

$\bar{7} = \bar{1} = [1] = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$.

Let $a = 5$, $m=3$.

$\bar{5} = [5] = \{\ldots, -4, -1, 2, 3, 5, \ldots\}$

⇒ $\boxed{\bar{2} = \bar{5}}$

★ $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ ⇒ $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$.

$\bar{1} \oplus \bar{2} = \bar{3} = \bar{0}$; $\bar{1}.\bar{2} = \bar{2}$

$\bar{2}.\bar{2} = \bar{4} = \bar{1}$

for both the operations, given set ~~are~~ is abelian group

∴ given set is a field.
$(\mathbb{Z}_3)$

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☆ Z_n is a field if n is prime number.

Quotient $\mathbb{Z}/\langle 2\rangle = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

$\Rightarrow$ $\langle 2\rangle = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$
is a sub-group w.r.t. addition as it is a subset of Z_2.

$a, b \in \langle 2\rangle$; $a-b \in \langle 2\rangle$
$a, b \in \mathbb{Z}$ $W = \langle 2\rangle$.
$a \equiv b \pmod 2 \Leftrightarrow a-b \in \langle 2\rangle$.

If $\mathbb{Z}/\langle 2\rangle$ is group $\Rightarrow$ then it is called quotient group.
-"- ring $\Rightarrow$ -"- ring
-"- vector space $\Rightarrow$ -"- space

$Z_5 = \mathbb{Z}/\langle 5\rangle = \{\bar{5}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$; class of residues $\Rightarrow$ remainders.
$\bar{5} = \bar{0}$

☆ Quotient Space :-

Defn:- Let W be a subspace of a vector space $V(F)$. Then the set
$$V/W = \{x+W \mid x \in V\}$$
is a vector space over the same field F under the operations.

① $(x+W) + (y+W) = (x+y) + W$
where $x+W, y+W \in V/W$.

② $\alpha(x+W) = \alpha x + W$,
$\alpha \in F$, $x+W \in V/W$.

This vector space is known as Quotient Space of V wrt subspace W.

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☆ $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$.

Proof :- $\dim V = n$; $\dim W = m$

$\therefore B = \{x_1, x_2, \ldots, x_m\}$, $m \le n$
basis of W.

$\therefore B_1 = \{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n\}$.
is basis of V

$B_2 = \{x_{m+1} + W, x_{m+2} + W, \ldots, x_n + W\}$.
(As W is a zero element)

☆ Linear Transformation :-
$V(F) \cong V_n(F)$

* Kernel of a linear transformation :-
Defn :- Let U and V be two vector spaces over the same field F and $T : U \to V$, a linear transformation then kernel of T is denoted by $\ker(T)$ and defined as.

$$\ker(T) = \{x \in U \mid T(x) = 0'\} \subseteq U.$$

$\Rightarrow$ is non-empty. as there exists zero element in every vector space.

i.e. $\ker(T)$ is a subspace of U.

Identification $x, y \in \ker(T)$, $T(x) = T(y) = 0$.
$\alpha, \beta \in F$, then

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$
$$= \alpha 0 + \beta 0 = 0.$$
$$\Rightarrow \alpha x + \beta y \in \ker(T)$$

This subspace of U is called null space of T and the dimension of null space (or $\ker(T)$) is called nullity of T. denoted by $\eta(T)$.

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★ Image Set of the Linear Transformation T :-

let U & V be the two vector spaces over field F and $T: U \to V$ be a linear transformation then image set of T is denoted by $Im(T)$ or $R(T)$ and defined as.

$$Im(T) = \{ T(x) \mid x \in U \} \subseteq V.$$

$\Rightarrow$ non-empty

$\Rightarrow$ $T(x), T(y) \in Im(T) \Rightarrow x, y \in U$

$\alpha, \beta \in F$.

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \in V.$$

$\Rightarrow$ $\alpha T(x) + \beta T(y) \in Im(T)$.

(Range Space) $\Rightarrow$ $Im(T)$ is a subspace of V.

★ dimension of Range space i.e. $Im(T)$ is called rank of T. denoted by $\rho(T)$.

★ nullity of T $\Rightarrow$ $\eta(T)$

★ $T: U \to V$.

$\dim U = n$ (finite-dimensional).

then $\dim U = \rho(T) + \eta(T)$.

This result is known as Rank-nullity theorem

Theorem :- Let U and V be two vector spaces over the same field F and $T: U \to V$ be a linear transformation. If U is finite dimen-sional, say $\dim(U) = n$ then

$$\dim U = \rho(T) + \eta(T).$$

$$= \dim(R(T)) + \dim(\ker(T)).$$

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Proof :- Given (statements).

$\ker(T)$ is subspace of U, $\ker(T) \subseteq U$

let $\dim(\ker(T)) = m$.

$\therefore \dim(\ker(T)) = m \leq n$ ($\because$)

Let $B = \{x_1, x_2, \ldots, x_m\}$ be a basis of $\ker(T)$

$\left\{ \begin{array}{l} \text{So, we have to prove } \dim(R(T)) = n - m; \\ \text{i.e. Basis of } R(T) \text{ must contain } n - m \text{ elements} \end{array} \right.$

let $B_1 = \{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n\}$.
be a basis of vector space U
(By using extension theorem).

Claim let $B_2 = \{T(x_{m+1}), T(x_{m+2}), \ldots, T(x_n)\}$,
be a basis of $R(T)$.

Justification

$a_{m+1} T(x_{m+1}) + \ldots + a_n T(x_n) = 0 \in V$

As T is a linear transformation $(\alpha T(x) = T(\alpha x))$
& $(T(x+y) = T(x) + T(y))$

$$\therefore T(\underbrace{a_{m+1} x_{m+1} + \ldots + a_n x_n}_{\in U}) = 0 \in V$$

$\Rightarrow a_{m+1} x_{m+1} + \ldots + a_n x_n \in \ker(T)$.

it can written as linear combn of elements
of B (basis of $\ker(T)$)

we can write

$\therefore a_{m+1} x_{m+1} + \ldots + a_n x_n = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$.

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$\Rightarrow a_1x_1 + a_2x_2 + \dots + a_mx_m + (-a_{m+1})x_{m+1} + \dots + (-a_n)x_n = 0$

$\boxed{0 \in U}$

As $x_1, x_2, \dots, x_n$ are elements of B_1 basis of $U \Rightarrow$ all scalars are zero.

$\therefore a_1 = 0; \dots; a_{m+1} = 0, \dots, a_n = 0.$

~~Let, $T(x) \in R(T)$~~

As $x \in U$, $x = b_1x_1 + \dots + b_nx_n$

$T(x) = T(b_1x_1 + b_2x_2 + \dots + b_mx_m + b_{m+1}x_{m+1} + \dots + b_nx_n).$

$= b_1T(x_1) + b_2T(x_2) + \dots + b_mT(x_m) + b_{m+1}T(x_{m+1}) + \dots + b_nT(x_n)$

As $x_1, \dots, x_m \in \ker(T)$.

$\therefore T(x_1) = T(x_2) \dots = T(x_m) = 0$;

$T(x) = b_{m+1}T(x_{m+1}) + \dots + b_nT(x_n).$

So, each $T(x) \in R(T)$ can be written as linear combination of elements of B_2.

$\therefore B_2$ is basis of $R(T)$

$\therefore \dim(R(T)) = n - m.$

$\therefore \dim(U) = \dim \ker(T) + \dim(R(T))$

$\dim(U) = \eta(T) + \rho(T)$

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☆ $T: U \to V$; $T(x) = 0'$, $\forall x \in U$.
This linear transformation T is called zero linear transformation. It is denoted $\hat{0}$ or 0.

☆ (Zero function ⇒ image of every element is zero).

$$\ker(\hat{0}) = \{x \in U \mid \hat{0}(x) = 0\} = U$$

☆ $T: U \to V$; $T(x) = x$, $\forall x \in U$.
is an Identity Linear Transformation.

☆ $T: U \to U$; $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.
Def^n :- A linear transformation $T: U \to U$ is called a linear operator.

Def^n :- A linear transformation $T: U \to F$ is called a linear functional.
$T(\alpha x + \beta y) = a$ (scalar quantity $\in F$).

☆ $T: U \to V$ is a linear Transformation then $(-T): U \to V$ will also be a linear transformation.

☆ $T_1 + T_2$ is a linear transformation if T_1, T_2 are linear transformation.
$$(T_1 + T_2)(x) = T_1(x) + T_2(x).$$

☆ Product of Linear Transformation :-

U & V be two vector spaces over F & $T_1: U \to U$ & $T_2: U \to U$ be a L.operators Then product of L.T. T_1 & T_2 denoted by T_1T_2 & defined as
$$(T_1T_2)(x) = T_1(T_2(x)), \ \forall x \in U.$$

only one vector space

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$$(f, g \Rightarrow (f \circ g)(x) = f(g(x)))$$

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★ ∴ Product of T_1 & T_2 will be a linear transformation if T_1, T_2 are Linear operators.

eg. $T_1 : V_2(\mathbb{R}) \to V_2(\mathbb{R})$.
defined by.
$T_1(a,b) = (b,a) \quad \forall (a,b) \in V_2$.

$T_2 : V_2(\mathbb{R}) \to V_2(\mathbb{R})$. defined by
$T_2(a,b) = (a,0) \quad \forall (a,b) \in V_2$.

$\underbrace{(T_1 T_2)}_{f}(a,b) = T_1(T_2(a,b)) = T_1(a,0) = (0,a)$.

∴ f is a linear Transformation

$\underbrace{(T_2 T_1)}_{f}(a,b) = T_2(T_1(a,b)) = T_2(b,a) = (0,b)$.

∴ f is a linear Transformation.

∴ $(T_1 T_2)(a,b) \neq (T_2 T_1)(a,b)$
(Just like composition of two functions it is non-commutative).

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Part of Proof of $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

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Claim:- $B_3 = \{x_1, \dots, x_r, y_1, \dots, y_{m-r}, z_1, \dots, z_{n-r}\}$ is a basis of W_1+W_2

Justification:- ① Check that B_3 is linearly independent

Let, $a_1x_1 + \dots + a_rx_r + b_1y_1 + \dots + b_{m-r}y_{m-r} + c_1z_1 + \dots + c_{n-r}z_{n-r} = 0$.
for some scalars $a_1, \dots, a_r, b_1, \dots, b_{m-r}, c_1, \dots, c_{n-r}$.
then.

$(-b_1)y_1 + \dots + (-b_{m-r})y_{m-r} = a_1x_1 + \dots + a_rx_r + c_1z_1 + \dots + c_{n-r}z_{n-r}$
$\in W_1 \cap W_2$

$(-b_1)y_1 + \dots + (-b_{m-r})y_{m-r} = d_1x_1 + \dots + d_rx_r$
for some $d_1, \dots, d_r$ ($\because B$ is basis of $W_1 \cap W_2$).

$\Rightarrow d_1x_1 + \dots + d_rx_r + b_1y_1 + \dots + b_{m-r}y_{m-r} = 0$.
$\Rightarrow d_1 = \dots = d_r = b_1 = \dots = b_{m-r} = 0$ as B_1 is basis of W_1.

$\Rightarrow a_1x_1 + \dots + a_rx_r + c_1z_1 + \dots + c_{n-r}z_{n-r} = 0$.
$\Rightarrow a_1 = \dots = a_r = c_1 = \dots = c_{n-r} = 0$ as B_2 is basis of W_2.
$\therefore$ B_3 is linearly independent

② $\mathrm{Span}(B_3) = W_1+W_2$
Let $z = x+y \in W_1+W_2$, $x \in W_1$ & $y \in W_2$.

$z = (a_1x_1 + \dots + a_rx_r + a_{r+1}y_1 + \dots + a_my_{m-r})$
$+ (b_1x_1 + \dots + b_rx_r + b_{r+1}z_1 + \dots + b_nz_{n-r})$

$z = ((a_1+b_1)x_1 + \dots + (a_r+b_r)x_r + a_{r+1}y_1 + \dots + a_my_{m-r}$
$+ b_{r+1}z_1 + \dots + b_nz_{n-r})$.

i.e. $W_1+W_2 \subseteq \mathrm{Span}(B_3)$.

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Also, $\text{Span}(B_3) \subseteq W_1 + W_2$

$\therefore$ $W_1 + W_2 = \text{span}(B_3)$

$\therefore$ B_3 is basis of $W_1 + W_2$.

$\therefore$ $\dim(W_1+W_2) = r + (m-r) + (n-r)$

$= (m) + (n) - (r)$

$\therefore$ $\boxed{\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)}$

Tutorial. (LI, LD, Basis and Dimension.

(1)(i) Check linear dependence or independence

(e) $\{(x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2}\}$ in P_4

$P_4(\mathbb{R})$

$P_4 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_1, a_2, \ldots a_4 \in \mathbb{R}\}$

$S = \{x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2}\} \subseteq P_4$.

Use $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ s.t.

$\alpha_1 x + \alpha_2(x^3 - x) + \alpha_3(x^4 + x^2) + \alpha_4(x + x^2 + x^4 + \frac{1}{2}) = 0$

$\alpha_4\left(\frac{1}{2}\right) + x(\alpha_1 - \alpha_2 + \alpha_5) + x^2(\alpha_3 + \alpha_5)$
$+ x^3(\alpha_2) + x^4(\alpha_3 + \alpha_5) = 0$

Comp. coefficients on both sides

$\frac{\alpha_4}{2} = 0 \Rightarrow \boxed{\alpha_4 = 0}$; $\alpha_1 - \alpha_2 + \alpha_4 = 0 \Rightarrow \alpha_1 = \alpha_2$

$\alpha_3 + \alpha_4 = 0$ $\boxed{\alpha_2 = 0}$ $\Rightarrow \boxed{\alpha_1 = 0}$

$\boxed{\alpha_3 = 0}$

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(1) (ii) Prove that $S = \{\sin x, \sin 2x, \ldots, \sin nx\}$ is a LI subset of $C[-\pi,\pi]$ for every positive n.

$\Rightarrow$ $C[-\pi,\pi]$ means set of all real valued cont. functions $[-\pi,\pi] \to [-\pi,\pi]$.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

Let $\alpha_1, \ldots, \alpha_n \in F$.

$$\Rightarrow \sum_{i=1}^{n} \alpha_i \sin ix = 0.$$

for any fixed k, $\sum_{i=1}^{n} \alpha_i \sin kx \sin ix = 0$.

Now, integrate for $-\pi$ to π.

$$\sum_{i=1}^{n} \alpha_i \int_{-\pi}^{\pi} \sin kx \sin ix \, dx = 0.$$

$$\Rightarrow \alpha_k \pi = 0 \Rightarrow \boxed{\alpha_k = 0}$$

$\forall\ k = 1, 2, \ldots, n \quad \therefore$ L.I.

Q.2 (i) Prove If u, v and w are L.I. vectors then $u+v$, $v+w$, $w+u$ are L.I. vectors.

Let, $\alpha_1, \alpha_2, \alpha_3 \in F$ s.t.

$\alpha_1(u+v) + \alpha_2(v+w) + \alpha_3(w+u) = 0$

$u(\alpha_1+\alpha_3) + v(\alpha_1+\alpha_2) + w(\alpha_2+\alpha_3) = 0.$

$\Rightarrow$ As u, v, w are L.I.

$\Rightarrow$ $\alpha_1 + \alpha_3 = 0$ $\quad \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$

$\alpha_1 + \alpha_2 = 0$ $\quad \Rightarrow u+v, v+w, u+w$

$\alpha_2 + \alpha_3 = 0$ $\quad$ are L.I. vectors

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2)(ii) S_1, S_2 be subsets of $V(F)$. $(S_1 \subset S_2)$

(i) S_1 is L.D. $\Rightarrow$ S_2 is L.D.

(ii) S_2 is L.I. $\Rightarrow$ S_1 is LI.

2)(iii) S is L.I. subset of V. let $v \in L[S]$

Prove that $\{v\} \cup S$ is L.D set.

$\rightarrow$ let $S = \{v_1, v_2, \ldots, v_k\}$ be L.I. subset of V.

let $\alpha_1, \ldots, \alpha_k, a \in F$.

$v \in L[S]$.

$\Rightarrow \exists\ \alpha_1, \ldots, \alpha_k \in F$ s.t.

$$v = \sum_{i=1}^{k} \alpha_i v_i$$

$\Rightarrow$ $1v - \alpha_1 v_1 - \ldots - \alpha_k v_k = 0$.

$\Rightarrow$ $\alpha_1, \ldots, \alpha_k = 0$.

$1 \neq 0$ $\therefore$ $v \cup S$ is L.D set.

~~$f(v) \rightarrow v$~~ !

~~2(iv)~~ 4.) $B = \{v_1, \ldots, v_m\} \subseteq V(F)$

is basis iff.

① B is L.I.

② $\text{Span}(B) = V$. $|B| = \dim V$.

Check whether following are basis or not.

(i) $B = \{(2,4,0), (0,2,-2)\}$ $V = \mathbb{R}^3$ & $F = \mathbb{R}$.

$\Rightarrow$ $(x,y,z) \in V$ if $L(B) \rightarrow \exists\ \alpha, \beta \in F$ s.t.

$(x,y,z) = \alpha(2,4,0) + \beta(0,2,-2)$

$$M \sim \left[\begin{array}{cc|c} 2 & 0 & x \\ 4 & 2 & y \\ 0 & -2 & z \end{array}\right]$$

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(ii) $\{(6,4,4), (-2,4,2), (0,7,0)\}$; $V = \mathbb{R}^3$, $F = \mathbb{R}$

LI

$\Rightarrow$ let, $x_1, x_2, x_3 \in F$

$\Rightarrow \quad x_1(6,4,4) + x_2(-2,4,2) + x_3(0,7,0) = 0.$

$$\begin{bmatrix} 6 & -2 & 0 \\ 4 & 4 & 7 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = 0$ $\quad \therefore 6(4\times 0) + (-2)(-28)$ -14

if $|A| \neq 0$, LI ✓

$|A| = 0$, LD. $\quad \neq 0$ ~~-28~~

let $v = (x,y,z) \in V \Rightarrow \exists\ \alpha_1, \alpha_2, \alpha_3$ s.t. $\in F$.

$(x,y,z) = \alpha_1(6,4,4) + \alpha_2(-2,4,2) + \alpha_3(0,7,0)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ 4 & 4 & 7 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\left.\begin{aligned} \alpha_1 &= \\ \alpha_2 &= \\ \alpha_3 &= \end{aligned}\right\} \text{form of } x,y,z \quad \begin{aligned} &\cancel{6x-2y}\ \ \frac{x+z}{10} \\ &\cancel{4x}\ \ 3z-2x/10 \\ &y-(2/5)(4z-x)/7 \end{aligned}$$

$\therefore$ for each (x,y,z) $\exists\ \alpha_1 \alpha_2 \alpha_3 \in F$

$\therefore$ span$(B) = V$.

$\therefore$ B is basis of V.

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L.I.

(vi) $\{(1, i, 1+i), (1, i, 1-i), (i, -i, 1)\}$. $V = \mathbb{C}^3, F = \mathbb{C}$

$$\alpha_1(1, i, 1+i) + \alpha_2(1, i, 1-i) + \alpha_3(i, -i, 1) = 0.$$

$\Rightarrow$ $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$

$$\alpha_1 + \alpha_2 + i\alpha_3 = 0$$
$$i\alpha_1 + i\alpha_2 - i\alpha_3 = 0$$
$$(1+i)\alpha_1 + (1-i)\alpha_3 + (1)\alpha_3 = 0.$$

$$\begin{vmatrix} 1 & 1 & i \\ i & i & -i \\ 1+i & 1-i & 1 \end{vmatrix}$$

$$= 1(i + i - i^2)$$
$$-1(i + i + i^2)$$
$$+ i(i - i^2 - i - i^2)$$
$$\neq 0.$$

$\therefore$ L.I set.

$\therefore$ $(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$

$$= \alpha_1(1, i, 1+i) + \alpha_2(1, i, 1-i) + \alpha_3(i, -i, 1)$$

$$\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & i \\ i & i & -i \\ 1+i & 1-i & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$x_1 = \alpha_1 + \alpha_2$; $y_1 = \alpha_3$

$x_2 = 0$; $y_2 = \alpha_1 + \alpha_2 - \alpha_3$

$x_3 = \alpha_1 + \alpha_2 + \alpha_3$; $y_3 = \alpha_1 - \alpha_2$.

$$\alpha_3 = x_3 - x_1$$
$$\alpha_1 = \frac{x_1 + y_3}{2}$$

α_2 = x_3 form of x_i & y_i.

$\therefore$ for each $v \in \mathbb{C}^3$ $\exists$ unique $\alpha_1, \alpha_2, \alpha_3$

$\therefore$ Given set is basis.

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6. Find basis of $P : x - 2y + 3z = 0$ in $\mathbb{R}^3$

$V = \mathbb{R}^3$; $F = \mathbb{R}$.

$W = \{(x,y,z) \in V \mid x - 2y + 3z = 0\} \subseteq \mathbb{R}^3$

$\dim W = \underset{(\mathbb{R}^3)}{3} - 1$ (No of variables which can be expressed in form of other)

$\dim W = 2$.

$3z = 2y - x$.

$x = 1, y = 0 \Rightarrow z = -1/3$.

$x = 0, y = 1 \Rightarrow z = 2/3$.

Claim $B = \{(1, 0, -1/3) ; (0, 1, 2/3)\}$. is a basis (Prove)

Also basis for $P \cap$ (xy plane)

xy plane $W' = \{(x,y,z) \in V \mid z = 0\}$.

Claim $B' = \{(1,0,0) ; (0,1,0)\}$. is basis of W'

$W \cap W' = \{(x,y,z) \in V \mid z = 0 ; x - 2y + 3z = 0\}$

$x - 2y = 0$

$\boxed{x = 2y}$

$\Rightarrow (2, 1, 0)$

$\dim(W \cap W') = 3 - 2 = 1 \Rightarrow B'' = \{(2,1,0)\}$ is basis of $W \cap W'$. (Proved) .

Also basis of plane $P \perp$ to space of vectors.

W

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(7)(i) $\Rightarrow$ Ans: S should be L.D. $|S| = 0$.
for $\underline{a} \nearrow$.

(8) (ix) $W = \{ A \in R^{2\times 2} : A$ is complex Hermitian$\}$

$$\begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix} \in W \quad (\overline{A^T} = A)$$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}.$$

(x) $W = \{ A$ in $R^{m\times n}$: sum of each row $A = 0 \}$

$$\left\{ \begin{matrix} a_{11} & a_{12} & - & - & a_{1n} \\ \vdots & & & & \\ a_{m1} & - & - & - & a_{mn} \end{matrix} \right\}$$

dim (Basis) $= (n-1) \times m$

$$\sum_{j=1}^{n} a_{1j} = 0$$

9(i) R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$
Prove that it is basis.

(10) (ii) $U = \{ P : p(2) = 0 \}$, $W = \{ P : p'(2) = 0 \} \cdot V_4$

$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ ①

$\Rightarrow p(2) = 0$

$\Rightarrow a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 0$.

$\Rightarrow a_0 = -(2a_1 + 4a_2 + - - + 16a_4)$

from ①

$\Rightarrow p(x) = a_1x + a_2x^2 + \dots + a_4x^4 +$
$-(2a_1 + 4a_2 + \dots + 16a_4)$.

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$p(x) = a_1(x-2) + a_2(x^2-4) + a_3(x^3-8) + a_4(x^4-16)$

$B = \{(x-2), (x^2-4), (x^3-8), (x^4-16)\}$.

Check if B is basis

$$\overset{A}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim(U) = 4$

$\therefore |A| \neq 0 \Rightarrow$ L.I.

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$.

$p'(2) = 0$

$\therefore 0 = a_1 + 4a_2 + 12a_3 + 32a_4$.

$\Rightarrow a_1 = -4a_2 - 12a_3 - 32a_4$.

$p(x) = a_0 + (-4a_2 - 12a_3 - 32a_4)x + a_2x^2 + a_3x^3 + a_4x^4$

$p(x) = a_0 + a_2(x^2-4) + a_3(x^3-12) + a_4(x^4-32)$

$B = \{1, x^2-4, x^3-12, x^4-32\}$.

$L(B) = W$.

$\therefore \dim(W) = 4$.

$U \cap W = \{p \mid p(2) = 0, p'(2) = 0\}$

$\therefore \dim(U \cap W) = 3$.

$\dim(U+W) = 4 + 4 - 3 = 5$.

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find Subspaces $S \cap T$, $S + T$.

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11) (ii) - $S = L\left[\{(2,2,-1,2), (1,1,1,-2), (0,0,2,-4)\}\right]$

$T = L\left[\{(2,-1,1,1), (-2,1,3,3), (3,-6,0,0)\}\right]$

$V = \mathbb{R}^4$.

let $(x,y,z,t) \in S$.

$\Rightarrow \quad \alpha_1, \alpha_2, \alpha_3 \in F$ s.t.

$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = (x,y,z,t)$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & -2 & -4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$M \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & x \\ 2 & 1 & 0 & y \\ -1 & 1 & 2 & z \\ 2 & -2 & -4 & t \end{array}\right].$$

$R_2 \to R_2 - R_1$; $R_3 \to 2R_3 + R_1$, $R_4 \to R_4 - R_1$

$$M \sim \begin{bmatrix} 2 & 1 & 0 & x \\ 0 & 0 & 0 & y-x \\ 0 & 3 & 4 & 2z+x \\ 0 & -3 & -4 & t-x \end{bmatrix}.$$

$R_2 \leftrightarrow R_3$, $R_4 \to R_3 + R_4$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & x \\ 0 & 3 & 4 & 2z+x \\ 0 & 0 & 0 & y-x \\ 0 & 0 & 0 & 2z+t \end{array}\right]$$

$\therefore \; y - x = 0 \quad 2z + t = 0$.

$y = x \; ; \; z = -t/2$.

Maharaza $\beta = \left\{ \begin{array}{l} (2,2,-1,2) \\ (1,1,1,-2) \end{array} \right\}$

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☆ $T_1(a,b) = (a,0)$; $T_2(a,b) = (0,b)$; $T_3(a,b) = (b,a)$,

$T_4(a,b) = (a+2, b+3)$ $T_4 : \mathbb{R}^2 \to \mathbb{R}^2$.

Check → $(T_4(\alpha x + \beta y) = \alpha T_4(x) + \beta T_4(y))$

where $x = (a_1, a_2)$; $y = (b_1, b_2)$

→ not a linear transformation.

$T_5(a,b) = (a+b, 0)$.

☆ $T : U(F) \to V(F)$

$$L(U,V) = \{ T : U \to V \mid T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \}.$$

$T_1, T_2 \in L(U,V)$.

$$\left\{ \begin{array}{l} (T_1 + T_2)(x) = T_1(x) + T_2(x) \\ (\alpha T_1)(x) = \alpha T_1(x) \end{array} \right\}$$

☆ $L(U,V)(F)$ is a vector space, $\dim(U) = m$, $\dim(V) = n \Rightarrow \dim(L(U,V)) = mn$.

Th: Let U and V be two finite dimensional vector spaces over the same field F. Let $B = \{x_1, x_2, \ldots, x_n\}$ be a basis of U and $y_1, y_2, \ldots, y_n$ be any n-elements of V. Then there exists a unique linear transformation $T : U \to V$ such that

$$T(x_i) = y_i, \quad 1 \le i \le n.$$

Proof: for any $x \in U$.

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

$T : U \to V$ by $T(x) = T(a_1 x_1 + \ldots + a_n x_n)$

$$= a_1 y_1 + \cdots + a_n y_n$$

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Let $u_1, u_2 \in U$.

$u_1 = b_1x_1 + \dots + b_nx_n$

$u_2 = c_1x_1 + \dots + c_nx_n$.

To check : $\left(T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) \right)$.

$T\left(\alpha(b_1x_1 + \dots + b_nx_n) + \beta(c_1x_1 + \dots + c_nx_n)\right)$

$= T\left((\alpha b_1 + \beta c_1)x_1 + \dots + (\alpha b_n + \beta c_n)x_n \right)$.

$= (\alpha b_1 + \beta c_1)y_1 + \dots + (\alpha b_n + \beta c_n)y_n$

$= \alpha(b_1y_1 + b_2y_2 + \dots + b_ny_n)$
$+ \beta(c_1y_1 + \dots + c_ny_n)$

$= \alpha T(u_1) + \beta T(u_2)$.

So, to prove uniqueness we consider

$S : U \to V$ by $S(x_i) = y_i$.

$S(x) = S(a_1x_1 + \dots + a_nx_n) = a_1S(x_1) + \dots + a_nS(x_n)$

$S(x) = a_1y_1 + \dots + a_ny_n$

$\therefore \quad S(x) = T(x)$.

i.e. S and T should be same.

Hence, proved

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★ $V_n(\mathbb{R}) = \{ (a_1, \ldots, a_n) \mid a_i \in \mathbb{R} \}$.

Date ~~for $\mathbb{R}^2$~~ ; $\{(1,0), (0,1)\}$ is a standard basis of $\mathbb{R}^2$ Page No.

★ $T : U \to V$ one one onto (Isomorphism)
dimU dimV $B = \{x_1, x_2, \ldots, x_n\}$ basis of U.
$\{T(x_1), T(x_2), \ldots T(x_n)\}$ will be basis of V.

for any $y \in V$; $y = T(x) = T(a_1 x_1 + \ldots + a_n x_n)$

$y = a_1 T(x_1) + \ldots + a_n T(x_n)$.

★ $T : U \to V$ over same field F, Prove that T is one-one iff $Ker(T) = \{0\}$.

for one-one $[x, y \in U, \quad T(x) = T(y) \Rightarrow x = y]$.

Proof :- if :- $x \in U$ s.t. $x \in Ker(T) \Rightarrow T(x) = 0 = T(0)$
As T is one-one.
$\Rightarrow x = 0$.
$\therefore Ker(T) = \{0\}$. Hence, proved.

only if :- $Ker(T) = \{0\}$.
(Now prove that T is one-one).

$T(x) = T(y)$
$\Rightarrow T(x) + (-T(y)) = 0$.
$\Rightarrow T(x-y) = 0 \Rightarrow x-y \in ker(T)$
$\Rightarrow x - y = 0$
$\Rightarrow x = y$. Hence proved.

★ $\dim(L(U,V)) = mn$; $\dim U = n$; $\dim V = m$.

Proof :- $B_1 = \{x_1, \ldots, x_n\}$; $B_2 = \{y_1, \ldots, y_n\}$
basis of U basis of V.

$B_3 = \{T_{11}, T_{12}, T_{21}, \ldots, T_{mn}\}$.

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$T_{ij}(x_k) = 0$; when $j \neq k$. $1 \le i \le m$
$= y_i$; $j = k$ $1 \le j, k \le n$

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$T_{11}(x_1) = y_1 \qquad T_{12}(x_1) = 0$

$T_{11}(x_2) = 0 \qquad T_{12}(x_2) = y_1$

$\vdots \qquad\qquad \vdots$

$T_{11}(x_n) = 0 \qquad T_{12}(x_n) = 0$

<u>Claim:</u>- B_3 is a basis element of $L(U,V)$.
(Anyone can prove if he wants)

☆ $T_1, T_2 : V_2(\mathbb{R}) \to V_2(\mathbb{R})$.

$T_1(a,b) = (0,a)$

$T_2(a,b) = (0,b)$.

$(T_1T_2)(a,b) = T_1(T_2(a,b)) = T_1(0,b) = (0,0)$.

Ex. let $V(F) = F[x]$ indeterminate x.

$T_1 : V \to V$ defined by $T_1(f(x)) = \frac{d}{dx}(f(x)) \ \forall f(x) \in V$

$T_2 : V \to V$ defined by $T_2(f(x)) = \int_0^x f(t).dt$

$f(x) = a_0 + a_1x + \_\ \_\ \_\ \_ + a_nx^n$

$a_0, a_1, \_\ \_\ \_, a_n \in F$

T_1, T_2 both are linear transformations.

$T_3(f(x)) = a_0x + \qquad + a_n\frac{x^{n+1}}{n+1}$.

* $\dim(L(U,V)) = \dim U \times \dim V$.

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Singular Matrix det $= 0$

Non-Singular Matrix det $\neq 0$.

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★ A linear Transformation $T: U \to V$ is invertible if there exists another linear transformation S s.t. $TS = ST = I$.

★ A linear Transformation $T: U \to V$ is invertible if T is one one onto.

$T: U \to V$ for $\underset{x \in U}{y} \in V$; $T(x) = y \Rightarrow$

$\exists\ S: V \to U$ s.t. $S(y) = x$

$(T \circ S)(y) = T(S(y)) = T(x) = y = I_V(y)$.

$(S \circ T)(x) = S(T(x)) = S(y) = x = I_U(x)$.

$\therefore$ $T: U \to V$ is invertible.

★ Singular and Non-Singular Linear Transformation :-

Defn :- Let U and V be two vector spaces over the same field F and $T: U \to V$, a linear transformation. Then T is said to be non-singular linear transformation if

→ there does not exist $0 \neq x$ s.t. $T(x) = 0$.

(or) $0 \neq x \in U \Rightarrow T(x) \neq 0$.

(or) $\ker(T) = \{0\}$.

Singular if at least one $0 \neq x \in U$ s.t. $T(x) = 0$.

i.e. $\ker(T)$ contains other elements than 0.

★ $T: U \to V \ni$ T is non-singular iff T is one-one.

Proof :- If $x, y \in U$ $T(x) = T(y)$ $\therefore \ker(T) = \{0\}$

$T(x) - T(y) = 0$ $x - y = 0$

$T(x-y) = 0$ $\therefore x = y$

$x - y \in \ker(T)$ $\therefore T$ is one one

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only if T is one-one.

$T(x) = T(y) \quad x = y.$

$T(x-y) = 0.$

☆ T is a non-singular linear Transformation

$T : U \to V$

if $\{x_1, x_2, \ldots, x_n\}$ is L.I.

then $\{T(x_1), T(x_2), \ldots, T(x_n)\}$ is L.I.

(preservation of L.I).

☆ eg. $T : V_2(\mathbb{R}) \to V_2(\mathbb{R})$.

$T(1,0) = (2,3) \; ; \; T(0,1) = (5,1)$

$T(a,b) = (\quad\quad)$

$\{(1,0) ; (0,1)\}$ forms basis of $V_2(\mathbb{R})$.

$$T(a(1,0) + b(0,1)) = aT(1,0) + bT(0,1)$$

$$= a(2,3) + b(5,1).$$

defn:- $T(a,b) = (2a+5b, 3a+b).$

These should be basis elements.

☆ if $T(5,0) = (2,3)$; $T(7,1) = (5,1)$

then find $T(a,b)$?

☆ $U(F)$ $V(F)$ are finite dimensional

$B_1 = \{x_1, \ldots, x_n\}$ $B_2 = \{y_1, \ldots, y_m\}$.

$T : U \to V$ is a linear transformation.

$V \Rightarrow T(x_1) = a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \ldots + a_{m1}y_m$

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$V \Rightarrow T(x_2) = a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m,$

$V \Rightarrow T(x_3) = a_{13}y_1 + a_{23}y_2 + \cdots + a_{m3}y_m$

$V \Rightarrow T(x_n) = a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m.$

$$\underset{n\times 1}{\begin{bmatrix} T(x_1) \\ T(x_2) \\ \vdots \\ \vdots \\ T(x_n) \end{bmatrix}} = \underset{n\times m}{\begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ & & & & \\ a_{1n} & a_{2n} & & \cdots & a_{mn} \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}$$

A

Matrix representation of linear Transformation.

$$A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ a_{31} & a_{32} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = [T, B_1, B_2] = [T]_{B_1}^{B_2}$$

$y \in V;\ y = \alpha_1 y_1 + \cdots + \alpha_m y_m$

$(\alpha_1, \alpha_2, \cdots, \alpha_m) \in F^m$

Coordinates of y wrt F^m

* Take two Linear Transformations and find their matrix representation.

Note :- Matrix of order $m \times n$, dimension ie. no. of elements in basis would be mn.

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$T : U \rightarrow V$.

$B_1 = \{x_1, \ldots, x_n\}$; $B_2 = \{y_1, \ldots, y_m\}$.

$$\begin{pmatrix} T(x_1) \\ T(x_2) \\ | \\ | \\ T(x_n) \end{pmatrix}_{n\times 1} = \underset{n\times m}{A} \begin{pmatrix} y_1 \\ y_2 \\ , \\ \vdots \\ y_m \end{pmatrix}_{m\times 1}$$

A ⇒ Coeffinient Matrix

$$Z = AY \qquad |A| \neq 0.$$

$$Y = A^{-1}Z$$

$$[T]_{B_1}^{B_2} = A^T_{m\times n}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Guass elimination method.

⇒ upper triangular ⇆ Coefficient matrix to

$$A_{3\times 3} X_{3\times 1} = B_{3\times 1}$$

Guass Jordan method ⇒ coefficient matrix → to → diagonal matrix

Consistent & Inconsistent Systems of Linear Equations

Consistent if rank (A) = rank [A, B]

Coefficient matrix · Augmented matrix

$$P = [C_1 \quad C_2 \quad C_3 \quad B].$$

Augmented matrix C_1, C_2, C_3 column of coefficient matrix

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Rank of a Matrix :-

Defⁿ:- Let A be a matrix of order m×n Then a positive integer $r \le \min(m,n)$ is said to be the rank of A if

(i) $\exists$ at least one minor of order r in A which is non-zero.

(ii) Every minor of order r+1 is zero.

$T: U(F) \to V(F)$ be a linear transformation.
with $B_1 = \{x_1, \ldots, x_n\}$ be basis of U
$B_2 = \{y_1, \ldots, y_m\}$ be basis of V.
$x \in U$ then

$$[T(x)]_{B_2} = [T]_{B_1}^{B_2} [x]_{B_1}$$

Similar Matrices :-

Let A & B are two square matrices then A is said to be similar to B if there exist an invertible matrix P such that

$$B = P^{-1}AP \text{ (or) } PAP^{-1}$$

$$\Rightarrow A = PBP^{-1} \text{ (or) } P^{-1}BP.$$

Th:- Similarity relation on the vector spaces of matrices of order m is ~~similar~~ equivalent relation.

(Reflexive, symmetric, Transitive).

$(A = I^{-1}AI)$ $\begin{pmatrix} A = PBP^{-1} \\ B = P^{-1}AP \end{pmatrix}$

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★ $T : V_2(\mathbb{R}) \to V_2(\mathbb{R})$.

$B = \{(1,0), (0,1)\}$

$I(1,0) = 1(1,0) + 0(0,1)$

$I(0,1) = 0(1,0) + 1(0,1)$.

$$[I]_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

★ $V = P_3[x] \quad D : V(\mathbb{R}) \to V(\mathbb{R})$

$B = \{1, x, x^2, x^3\}$.

$$D(f(x)) = \frac{d}{dx}(f(x)).$$

Coefficient matrix ~~$[D]_B$~~.

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad [D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

★ $T_1(f(x)) = x f(x) \quad ; \quad T_2(f(x)) = \int_0^x f(x)\,dx$.

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☆ Linear Transformation.

☆ $T : V(F) \to V'(F)$

for both vector spaces ~~should~~ field should be same. then T can be a linear transformation if

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ when $x, y \in V$; $\alpha, \beta \in F$.

(1) (i) ⇒ $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(1,0) = (1,1)$; $T(1,1) = (-1,2)$

$B = \{(1,0), (1,1)\}$ is basis of $\mathbb{R}^2$

Claim : B is a basis. (Prove)

let $(x,y) \in \mathbb{R}^2 \Rightarrow \exists\ \alpha, \beta \in \mathbb{R}$.

s.t. $(x,y) = \alpha(1,0) + \beta(1,1)$

$(x,y) = (\alpha+\beta, \beta)$

$\alpha + \beta = x$; $\beta = y$

$T(x,y) = T\big((x-y)(1,0) + (y)(1,1)\big)$

$= (x-y)\, T(1,0) + y\, T(1,1)$

$= (x-y)(1,1) + y(-1,2)$

⇒ $T(x,y) = (x-2y, x+y)$

Find images of vertices of square and check whether they form $\|^{m}$ or not.

☆ For finding $T(x,y)$ given maps i.e. elements should form a basis.

$Ker(T) = \{x \in V \mid T(x) = 0\}$ $T : V \to V'$

$Ker(T) = \{(0,0)\}$, $\dim = 0$

Nulity $T = \dim(Ker(T)) = \eta(T)$

$Im(T) = \{y \in V' \mid \exists\ x \in V,\ T(x) = y\}$.

$\dim(Im(T)) = Rank(T) = \rho(T)$

$\dim V = \rho(T) + \eta(T)$

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3. Find range & null space of following LTS. Also find $\rho(T)$ & $\eta(T)$.

(i) $T: \mathbb{R}^2 \to \mathbb{R}^3$; $T(x_1, x_2) = (3x_1 + x_2, 0, 0)$.

$\ker(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0,0)\}$.

$(3x+y, 0) = (0,0,0)$.

$\ker(T) = \{(x,y) \in \mathbb{R}^2 \mid 3x+y=0\}$

$\boxed{\dim(\ker(T)) = 1}$

$\operatorname{Im} T = \{(x,y,z) \in \mathbb{R}^3 \mid \exists (x_1, x_2) \in \mathbb{R}^2 \Rightarrow T(x_1, x_2) = (x,y,z)\}$.

$(3x_1 + x_2, 0, 0) = (x, y, z)$.

$y = 0$; $z = 0$; $x = 3x_1 + x_2$.

Basis $= \{(1,0,0)\}$

$\therefore$ $\boxed{\dim(\operatorname{Im}(T)) = 1}$

$\therefore$ $\boxed{\dim(\mathbb{R}^2) = \rho(T) + \eta(T)}$

4. Check for LT or not. Find matrix rep. w.r.t. given bases B_1 & B_2.

$T : \underset{(B)}{V} \to V'$

$[T]_B = [T(v_i)] = [T(v_1) \; T(v_2) \; \dots \; T(v_n)]$

$B = \{v_1, v_2, \dots, v_2\}$.

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(ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$;

B_1 & B_2 are standard bases.

(Check for LT)

$B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$T(1,0) = (1,1,0) = \overset{(\alpha)}{1}(1,0,0) + \overset{(\beta)}{1}(0,1,0) + \overset{(\gamma)}{0}(0,0,1)$

$T(0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$.

$$[T(1,0)]_{B_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(0,1)]_{B_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Ans

$$A = [T]_{B_2}^{B_1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3 \times 2}$$

So, A is LT from dim 2 to dim 3.

$A_{m \times n} : F^n \longrightarrow F^m$

6 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ Given, matrix representation is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$B = \{e_1, e_2, e_3\}$ is a std. basis of $\mathbb{R}^3$.

$T(e_1) = 2e_1 + 2e_2 + 0e_3 = (2,2,0)$

$T(e_2) = 0e_1 - 5e_2 + 2e_3 = (0,-5,2)$

$T(e_3) = 0e_1 + 0e_2 + 1e_3 = (0,0,1)$.

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(if $Ker(T) = 0$, T is one one onto
Date __/__/____ i.e. T is invertible)

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$(x,y,z) \in \mathbb{R}^3$; $T(x,y,z) = \alpha e_1 + \beta e_2 + \gamma e_3$

$= \alpha T(e_1) + \beta T(e_2) + \gamma T(e_3)$.

$= x(2,2,0) + y(0,-5,2) + z(0,0,1)$.

$T(x,y,z) = (2x, 2x-5y, 2y+z)$.

$Ker(T) = \{(0,0,0)\}$.

let $(x_1, x_2, x_3) \in \mathbb{R}^3 \Rightarrow \exists (x,y,z) \in \mathbb{R}$

s.t. $T^{-1}(x_1, x_2, x_3) = (x,y,z)$.

$\Rightarrow T(x,y,z) = (x_1, x_2, x_3)$.

$(2x, 2x-5y, 2y+z) = (x_1, x_2, x_3)$.

$\Rightarrow x = \frac{x_1}{2}$; $y = \frac{(x_1 - x_2)}{5}$; $z = x_3 - \frac{2}{5}(x_1 - x_2)$

$$T^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1/2 \\ 1/5(x_1 - x_2) \\ x_3 - 2/5(x_1 - x_2) \end{pmatrix}.$$

$$\Rightarrow [T^{-1}]_B = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/5 & -1/5 & 0 \\ -2/5 & 2/5 & 1 \end{bmatrix}.$$

5. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x_1, x_2, x_3)$
$= (x_1 + x_2, x_1 + 2x_2, 3x_3 + x_2)$.

find T^{-1}

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☆ Echelon Form :-

~~upper triangular matrix~~

eg.
$$\begin{pmatrix} 1 & \otimes & \otimes \\ 0 & 1 & \otimes \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \quad \begin{pmatrix} 1 & \otimes & \otimes & \otimes \\ 0 & 1 & \otimes & \otimes \\ 0 & 0 & 1 & \otimes \\ 0 & 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

$\otimes \Rightarrow$ Any elements (members)

$$\begin{pmatrix} 1 & \otimes & \otimes \\ 0 & 1 & \otimes \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}_{4\times 3}$$

☆ $T : U \to V$

$B_1 = \{x_1, \dots, x_n\}$, $B_2 = \{y_1, \dots, y_m\}$.

$$[T]_{B_1}^{B_2} = \Big(\quad \Big)_{m\times n}$$

$x \in U$, $x = a_1x_1 + \dots + a_nx_n$

$(a_1, \dots, a_n) \Rightarrow$ co-ordinate rep.

eg. Matrix Rep. of L.T.

$D : P_3[x] \longrightarrow P_3[x]$ def by $D(f(x)) = \frac{d}{dx}(f(x))$

$B_1 = \{1, x, x^2, x^3\}$.

$D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$

$D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$

$D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$

$D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$.

Transpose Matrix of Matrix rep. = coeffi. matrix

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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★ Let T and S be two L.T. then

(i) $[T]_{B_1}^{B_2} + [S]_{B_1}^{B_2} = [T+S]_{B_1}^{B_2}$

(ii) $[\alpha T]_{B_1}^{B_2} = \alpha . [T]_{B_1}^{B_2}$; α is scalar.

★ $x \in U$;

$x = \alpha_1 x_1 + \dots + \alpha_n x_n$

Column matrix $[x]_{B_1} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$

[co-ordinate vector for x w.r.t. B_1]

★ $T(x) \in V$;

$T(x) = \beta_1 y_1 + \dots + \beta_n y_m$

Column matrix $[T(x)]_{B_2} = (\beta_1, \beta_2, \dots, \beta_m)^T$.

$$\boxed{[T(x)]_{B_2} = [T]_{B_1}^{B_2} [x]_{B_1}}$$

Resp. Orders of matrices ⇒ $m \times 1$ $m \times n$ $n \times 1$

★ Let $[T]_{B_1}^{B_2} = (a_{ij})_{m \times n}$; $T(x_j) = \sum_{i=1}^{m} a_{ij} y_i$

$T(x_1) = \sum_{i=1}^{m} a_{i1} y_i$.

★ U V W

B_1 B_2 B_3.

$T : U \to V$; $S : V \to W$.

$[T]_{B_1}^{B_2} = A$; $(S \circ T) : U \to W$.

$[S]_{B_2}^{B_3} = B$; $[S \circ T]_{B_1}^{B_3} = DA$, Prove it ?

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★ Similar Linear Transformations :-

Let T & S be two linear transformations if there exists P a invertible linear transformations

s.t.

$$T = PSP^{-1} \quad \text{or} \quad S = P^{-1}TP.$$

★ $T : U \to V$ s.t $0 \neq x \in U$ with the property that $T(x) = \lambda x$, where $\lambda \in F$.

★ Eigen Value and Eigen Vector

Let A be a square matrix over the field. Then a scalar λ is said to be an eigen ~~vector~~ value for the matrix A if there exists $0 \neq x$ such that

$$\boxed{Ax = \lambda x}$$

★ If order of A is n then x is n touple.

$$\Rightarrow \underset{n\times n}{A} \; \underset{n\times 1}{x} = \lambda \underset{n\times 1}{x} \qquad T(x) = Ax.$$

eg. $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (Eigen Value ; Latent Value characteristic Value ; Spectral Val)

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

★ The non-zero vector x is called eigen vector or characteristic vector w.r.t matrix A

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* $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$. Let λ be an eigen value for A, Then $Ax = \lambda x$

$x \neq 0 \in$ Vector space.

Matrix equation $\Rightarrow Ax - \lambda x = 0$. $= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow (A - \lambda I) \cdot x = 0$

This system has non-zero solution only when

$\Rightarrow |A - \lambda I| = 0$ (det $(A - \lambda I)$).

$\Rightarrow$ second degree polynomial in $\lambda = 0$.

* Let $A = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ Steps

(I) Then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (-1)(3+\lambda)(3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 - 9 - 16 = 0$$

$$\Rightarrow \lambda^2 = 25 \Rightarrow \boxed{\lambda = \pm 5}$$

Eigen Values for A are -5 & 5.

(II) Eigen Vector Let $0 \neq x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ Then

$$Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow (1)$$

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for $\lambda = -5$, from eq^n ①

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$8x_1 + 4x_2 = 0$ $\Rightarrow x_1 = a$; $x_2 = -2a$

$4x_1 + 2x_2 = 0$ for $a = 1$;

$2x_1 + x_2 = 0$

$\Rightarrow x_2 = -2x_1$ $\therefore$ $\boxed{x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ is the req. eigen vector.

for $\lambda = 5$, from eq^n ①, we have.

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2x_1 + 4x_2 = 0$ $x_1 = a$; $x_2 = a/2$

$4x_1 - 8x_2 = 0.$ for $a = 1$;

$\Rightarrow x_1 = 2x_2$

$\Rightarrow x_2 = x_1/2$. $\therefore$ $\boxed{x = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}}$ is the req. eigen vector

☆ Eigen Value may be zero but eigen vector will always be non-zero.

☆ Corresponding to any eigen value, we can have more than one eigen vector. but for each eigen vector we have only one eigen value.

$$x = \begin{pmatrix} x_1 \\ -2x_1 + 3x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3x_1 \end{pmatrix}$$

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☆ Inner Product Space :-

Def^n :- Let $U(F)$ be a vector space where $F(\mathbb{R}$ or $\mathbb{C})$ Then a map.

$\langle \cdot, \cdot \rangle : V \times V \longrightarrow F$ is said to be a inner product if the following conditions are satisfied.

(i) $\langle x, x \rangle \geq 0$ for each $x \in U$

(ii) $\langle x, x \rangle = 0 \iff x = 0$.

(iii) $\langle x, y \rangle = \langle y, x \rangle \quad x, y \in U$.

(iv) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
when $\alpha, \beta \in F$ & $x, y, z \in U$.

Then vector space U is called inner product space.

☆ Norm of a Vector :- (Length of a vector).

Def^n :- Let $U(F)$ be a vector ~~space~~. Then a map.
$\| \cdot \| : U \longrightarrow \mathbb{R}$ of $\{\mathbb{R}_+ + \{0\}\}$ is called a norm on U if the following conditions are satisfied.

(i) $\|x\| \geq 0$, for each $x \in U$

(ii) $\|x\| = 0 \iff x = 0$.

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in F$ & $x \in U$.

Triangle inequality (iv) $\|x + y\| \leq \|x\| + \|y\|$; $x, y \in U$

$$\|x\| = +\sqrt{\langle x, x \rangle}$$ (We will prove in next class)

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$\Rightarrow A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I.$

$\Rightarrow A^2 - 2I = 0$ (Matrix Equation)

★ **Cayley - Hamilton Theorem :-**

Every Square matrix satisfies its characteristic equation.

Proof :- Let $A_{n\times n}$ be a square matrix of order n

Its characteristic eq^n is $|A - \lambda I| = 0$

$\Rightarrow a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n = 0$ —— (1).

Here, $a_n = (-1)^n$

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$$A \cdot (\operatorname{adj} A) = |A| I$$

$$(A-\lambda I) \cdot (\operatorname{adj} (A-\lambda I)) = |A-\lambda I| I \quad \text{--- (2)}$$

Let, $\operatorname{Adj}(A-\lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \dots + B_{n-1}\lambda^{n-1}$ --- (3)

where, $B_0, B_1, \dots, B_{n-1}$ are corresponding ~~ma~~ coefficient matrices

After subs. (3) in (2) we get,

$$(A-\lambda I)(B_0 + B_1\lambda + B_2\lambda^2 + \dots + B_{n-1}\lambda^{n-1}) = |A-\lambda I| I$$

$$(A-\lambda I)(B_0 + B_1\lambda + B_2\lambda^2 + \dots + B_{n-1}\lambda^{n-1}) = (a_0 + a_1\lambda + \dots + a_n\lambda^n) I$$

Comp. coefficients on both sides, $a_n = (-1)^n$

$$\left.\begin{aligned}
I &\times (AB_0 = a_0) \\
A &\times (AB_1 - B_0 I = a_1 I) \\
A^2 &\times (AB_2 - B_1 I = a_2 I) \\
&\vdots \\
A^n &\times (AB_{n-1} = (-1)^n I)
\end{aligned}\right\}$$

Adding all above equations we get,

$$a_0 I + a_1 A + a_2 A^2 + \dots + (-1)^n A^n = 0$$

$$a_0 A^{-1} I + a_1 A^{-1} A + a_2 A^{-1} A^2 + \dots + (-1)^n A^{-1} A^n = 0$$

$$a_0 A^{-1} = -(a_1 I + a_2 A + a_3 A^2 + \dots + (-1)^n A^{n-1})$$

$$A^{-1} = \frac{-1}{a_0}(a_1 I + a_2 A + \dots + (-1)^n A^{n-1})$$

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★ R - inner product space $\mathbb{R}(\mathbb{R})$.

Defn:- $\langle x, y\rangle = |xy|$

$\langle x, x\rangle = |x^2| \geq 0$

$= 0 \Leftrightarrow x = 0$.

$\langle x, y\rangle = \langle y, x\rangle$.

$\langle \alpha x + \beta y, z\rangle = |(\alpha x + \beta y) z|$.

$= |\alpha x z + \beta y z|$

$\leq |\alpha x z| + |\beta y z|$.

$\leq |\alpha||xz| + |\beta||yz|$

not a inner product space will be one if $\Rightarrow \langle x, y\rangle = xy$ (i.e. w/o mod)

★ (Triangle inequality)

★ $R^2(\mathbb{R})$ $x = (a_1, b_1)$; $y = (a_2, b_2)$; $z = (a_3, b_3)$.

$\langle x, y\rangle = a_1 a_2 + b_1 b_2$.

(i) $\langle x, x\rangle = a_1^2 + b_1^2 \geq 0$

(ii) $\langle x, x\rangle = a_1^2 + b_1^2 = 0 \Rightarrow a_1 = b_1 = 0$

$\Rightarrow x = 0$

(iii) $\langle x, y\rangle = a_1 a_2 + b_1 b_2$

$= a_2 a_1 + b_2 b_1$ (As a_1, a_2, b_1, b_2 are real)

$= \langle y, x\rangle$

(iv). $\langle \alpha x + \beta y, z\rangle$

$= \langle \alpha(a_1, b_1) + \beta(a_2, b_2), (a_3, b_3)\rangle$

$= \langle (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2), (a_3, b_3)\rangle$

$= \alpha a_1 a_3 + \beta a_2 a_3 + \alpha b_1 b_3 + \beta b_2 b_3$.

$= \alpha(a_1 a_3 + b_1 b_3) + \beta(a_2 a_3 + b_2 b_3)$

$= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

$\therefore$ inner product space.

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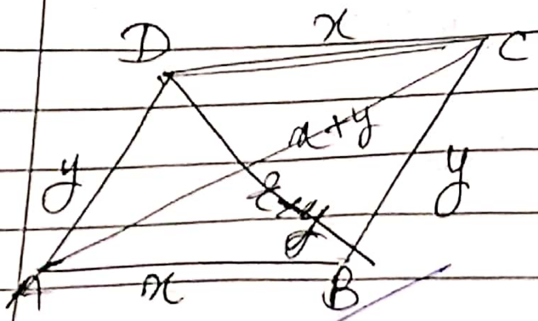


$$AC^2 + BD^2 = 2(AB^2 + BC^2)$$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$\langle x, x\rangle = \|x\|^2$$

$$\|x\| = +\sqrt{\langle x, x\rangle}$$

L.HS $= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

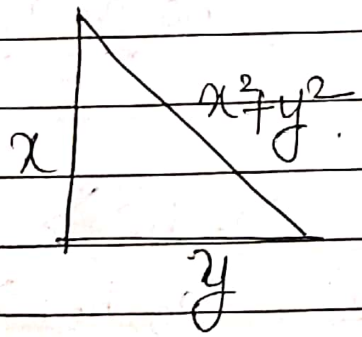
$= \langle x+y, x\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle$

$= \langle x, x\rangle + \cancel{\langle y, x\rangle} + \cancel{\langle y, x\rangle} + \langle y, y\rangle$
$- \cancel{\langle x, y\rangle} + \langle x, x\rangle - \cancel{\langle y, x\rangle} + \langle y, y\rangle$

$= 2\langle x, x\rangle + 2\langle y, y\rangle$

$= 2\|x\|^2 + 2\|y\|^2$

$= 2(\|x\|^2 + \|y\|^2)$



$\Rightarrow$ $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\langle x, y\rangle = 0$

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Eigen Value and Eigen Vectors (Tut)

$T: V(F) \to V(F)$ be a L.O.
Then $\lambda \in F$ in eigen vector if $\exists\ 0 \neq u \in V$.
s.t. $\boxed{Tu = \lambda u}$

$\Leftrightarrow (T - \lambda I)u = 0 \quad u \neq 0$.
$\Leftrightarrow \ker(T - \lambda I) \neq \{0\}$.
$\Leftrightarrow \eta(T - \lambda I) \geq 1$.
$\Leftrightarrow T - \lambda I$ is singular.

$Tu = \lambda u$.
$\Leftrightarrow Au = \lambda u$.
$\Leftrightarrow (A - \lambda I)x = 0$.
$\Leftrightarrow \eta(A - \lambda I) \geq 1 \Rightarrow |A - \lambda I| = 0$.

(1). (ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(If the given matrix is ~~upper triangular/~~ ~~lower triangular/~~ diagonal matrix then its eigen values are diagonal elements)

Let λ is an eigen value of A & x is the corresponding eigen vector. $x = \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times 1}$.

$\Rightarrow Ax = \lambda x$
$\Rightarrow (A - \lambda I)x = 0$.
$\Rightarrow |A - \lambda I| = 0$,

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1$

A is any matrix ; Det $|A|$ = Product of eigen values of A
$Tr(A)$ = Sum of eigen values of A

$Ax = x$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 = x_1 \Rightarrow x_2 = 0.$
$x_2 = x_2$
$x_3 = x_3$

$\Rightarrow$ eigen vector $= \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$

Q(i) $\Rightarrow$ λ is eigen value of A $\Rightarrow \lambda \neq 0$.
suppose u is eigen vector corresponding to λ.

$\Rightarrow Au = \lambda u.$
$\Rightarrow A^{-1}(Au) = A^{-1}\lambda u$
$\Rightarrow (A^{-1}A)u = (A^{-1}\lambda)u$
$\Rightarrow \lambda(A^{-1}u) = Iu$
$\Rightarrow A^{-1}u = \frac{1}{\lambda}u.$

$\Rightarrow \boxed{A^{-1}u = \lambda^{-1}u}$

$\therefore \lambda^{-1}$ is an eigenvalue of A^{-1}.

(ii) St. If A and P be both $n \times n$ matrices and P be nonsingular, then verify that A and $P^{-1}AP$ have the same eigen values.

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Proof :- Suppose $B = P^{-1}AP$.

$$C_B(\lambda) = |P^{-1}AP - \lambda I|$$
$$= |P^{-1}(A-\lambda I)P|$$
$$= |P^{-1}|\,|A-\lambda I|\,|P|$$
$$= |A-\lambda I|$$
$$= C_A(\lambda).$$

(iii) Prove that eigen values of a real symmetric matrices are all real.

Proof :- A is symmetric. $\Rightarrow A^T = A$.
Let $0 \neq u \in \mathbb{C}^n$ be the eigen vectors corresponding to eigen values $\lambda \in \mathbb{C}$.

$$\Rightarrow Au = \lambda u \quad \text{—— (1)}.$$

Let $\bar{u}$ & $\bar{\lambda}$ be complex conjugates.
(we have to prove, $\lambda = \bar{\lambda}$).

$$\Rightarrow \overline{Au} = \overline{\lambda u}$$
$$\Rightarrow A\bar{u} = \bar{\lambda}\bar{u} \quad (\because A \text{ is real})$$

Now,
$$\lambda \bar{u}^T u = \bar{u}^T \lambda u.$$
$$= \bar{u}^T A u$$
$$= (A^T\bar{u})^T u.$$
$$= (A\bar{u})^T u$$
$$= (\bar{\lambda}\bar{u})^T u.$$

$$= \bar{\lambda}\,\bar{u}^T u$$
$$\Rightarrow (\lambda - \bar{\lambda})\,\bar{u}^T u = 0.$$
$$\Rightarrow \lambda = \bar{\lambda}$$

Hence proved (i.e λ is real).

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(iv) Consider $A^T = -A$, else similar to (iii)

(v) Prove that eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

$\Rightarrow$ Let, A be the given matrix $(A^T = A)$

$\lambda_1 \neq \lambda_2$ are eigen values of A &
u_1 & u_2 are corresponding eigen vectors.

$\Rightarrow \quad Au_1 = \lambda_1 u_1 \quad \& \quad Au_2 = \lambda_2 u_2$.

(we have to prove $u_1^T u_2$ or $u_2^T u_1 = 0$).

$$\Rightarrow \quad \lambda_1 u_1 = Au_1$$

$$\begin{aligned}
\Rightarrow \quad u_2^T(\lambda_1 u_1) &= u_2^T(Au_1) \\
&= (A^T u_2)^T u_1 \\
&= (Au_2)^T u_1 \\
&= (\lambda_2 u_2)^T u_1 \\
&= \lambda_2 u_2^T u_1
\end{aligned}$$

$$\Rightarrow \quad (\lambda_1 - \lambda_2)(u_2^T u_1) = 0.$$

$$\Rightarrow \quad u_2^T u_1 = 0.$$

$\Rightarrow$ u_1 & u_2 are orthogonal to each other.

(vii) Let A and B matrices of order n.
Show AB & BA have same eigen values.

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Proof :- Case I : Let A is non-singular.

$$BA = A^{-1}A\,BA$$
$$= A^{-1}(AB)A$$

⇒ AB and BA are similar matrices.
⇒ AB and BA have same eigen values.

Case II : Let A & B both are singular.
⇒ AB & BA are singular.
⇒ AB & BA both have 0 as an eigen value

Let $0 \neq \lambda$ be an eigen ~~vector~~ value of AB & $0 \neq u$ be corresponding eigen vector.

$$\Rightarrow \boxed{ABu = \lambda u.} \quad \text{—(1)}$$

Let

$$Bu = v \Rightarrow Av = \lambda u.$$

$$BAv = BABu = B((AB)u)$$
$$= B\lambda u$$
$$= \lambda v$$
$$\Rightarrow \boxed{BAv = \lambda v} \quad \text{—(2)}$$

So, from (1) & (2) AB and BA have same eigen value (they may different eigen vectors).

* Find P s.t. $P^{-1}AP$ is diagonal where A is

(i) $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$

$\Rightarrow$ $|A - \lambda I| = 0$.

$\Rightarrow \begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 0, 4, 6$.

*. How to find eigen vector $\Rightarrow$
for $\lambda = 4$

$\Rightarrow (A - \lambda I) x = 0$

$\Rightarrow \begin{bmatrix} 0 & -2 & 0 \\ -2 & -2 & -2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

$\Rightarrow x_1 + x_2 + x_3 = 0 \rightarrow \boxed{x_1 + x_3 = 0}$

$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 0$; $X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ for $\lambda = 6$; $X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

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$$\Rightarrow \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

is the req. matrix.

4 Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; A^{-1} & A^4 by Caley-Hamilton Theorem.

$$\Rightarrow \quad C_A(\lambda) = (1-\lambda)^3.$$
$$= -\lambda^3 + 1 - 3\lambda + 3\lambda^2.$$

By CHT, $-A^3 + I - 3A + 3A^2 = 0.$

$$I = A^3 + 3A - 3A^2.$$
$$I = A(A^2 + 3I - 3A).$$

$$A^{-1}.I = A^2 + 3I - 3A$$
$$\underline{A^{-1} = A^2 + 3I - 3A}.$$

$$A.I = A^4 + 3A^2 - 3A^3$$
$$A = A^4 + 3A^2 - 3(I - 3A + 3A^2)$$

$$A = A^4 + 3A^2 - 3I + 9A - 9A^2$$

$$\Rightarrow \quad \underline{A^4 = 6A^2 - 8A + 3I}.$$

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