

Third Edition

MECHANICS OF MATERIALS

CHAPTER

4

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Pure Bending

Pure Bending

Pure Bending

Other Loading Types

Symmetric Member in Pure Bending

Bending Deformations

Strain Due to Bending

Beam Section Properties

Properties of American Standard Shapes

Deformations in a Transverse Cross Section

Sample Problem 4.2

Bending of Members Made of Several Materials

Example 4.03

Reinforced Concrete Beams

Sample Problem 4.4

Stress Concentrations

Plastic Deformations

Members Made of an Elastoplastic Material

Example 4.03

Reinforced Concrete Beams

Sample Problem 4.4

Stress Concentrations

Plastic Deformations

Members Made of an Elastoplastic Material

Plastic Deformations of Members With a Single Plane of Symmetry

Residual Stresses

Example 4.05, 4.06

Eccentric Axial Loading in a Plane of Symmetry

Example 4.07

Sample Problem 4.8

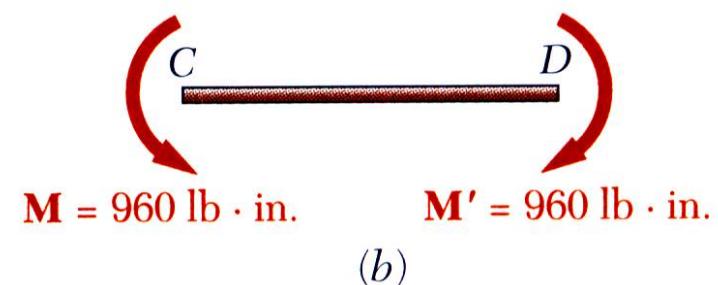
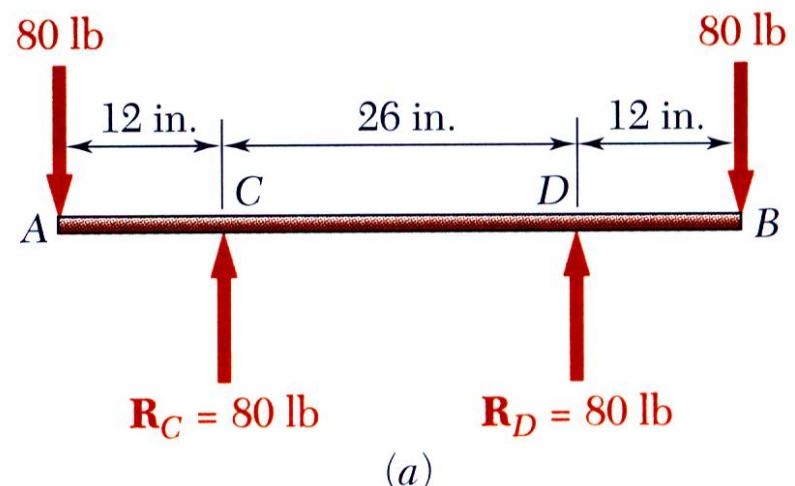
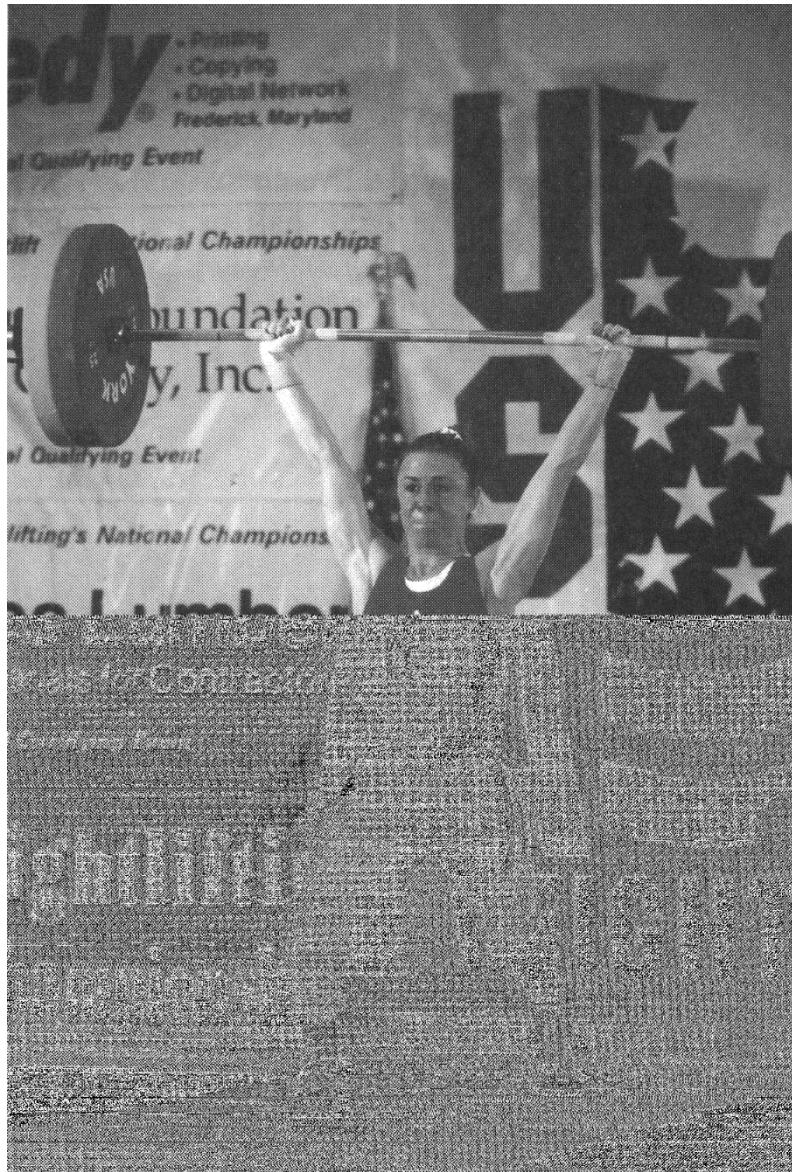
Unsymmetric Bending

Example 4.08

General Case of Eccentric Axial Loading

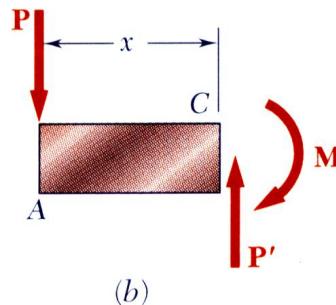
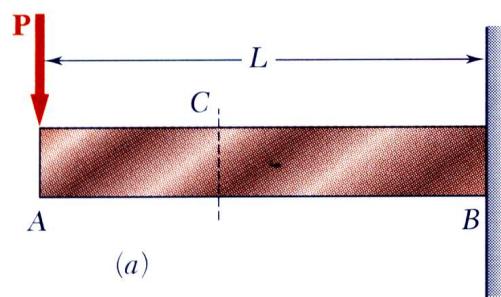
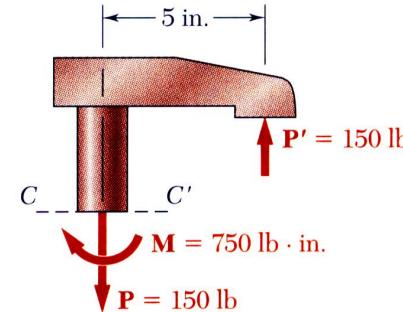
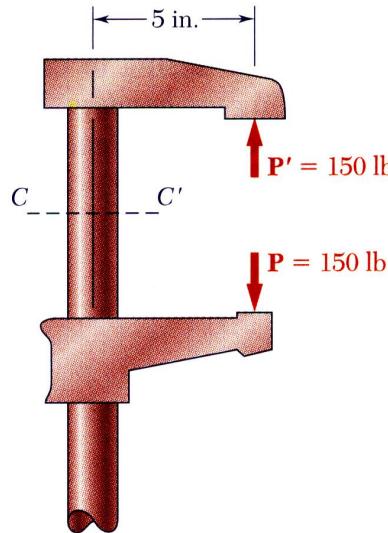


Pure Bending



Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

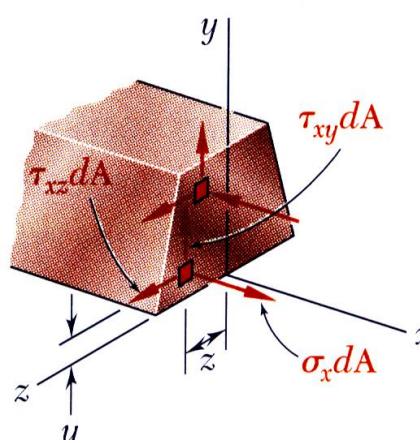
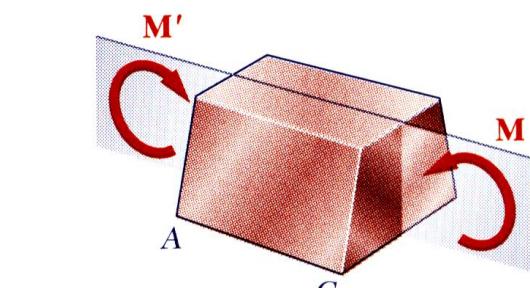
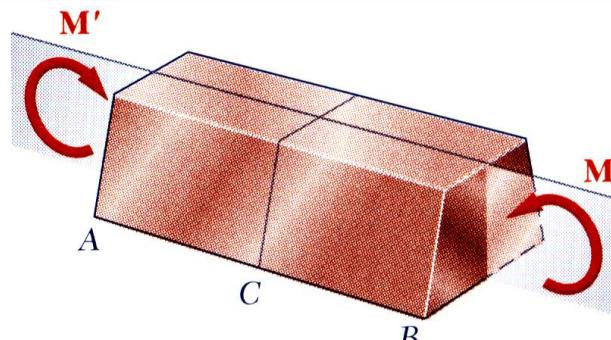
Other Loading Types



- *Eccentric Loading:* Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple
- *Transverse Loading:* Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple
- *Principle of Superposition:* The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.



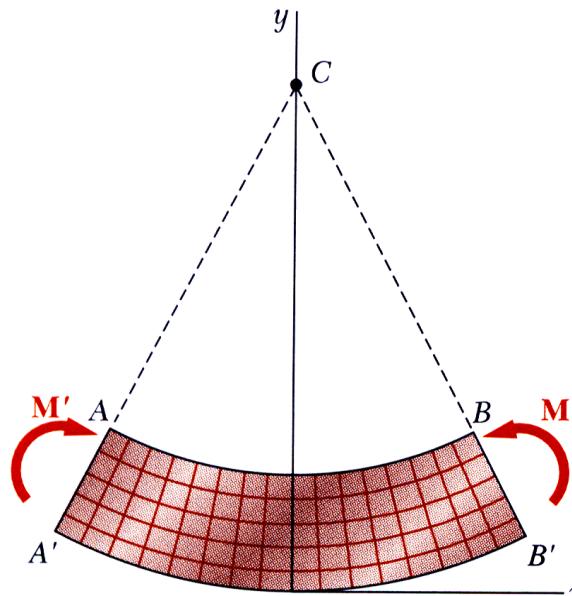
Symmetric Member in Pure Bending



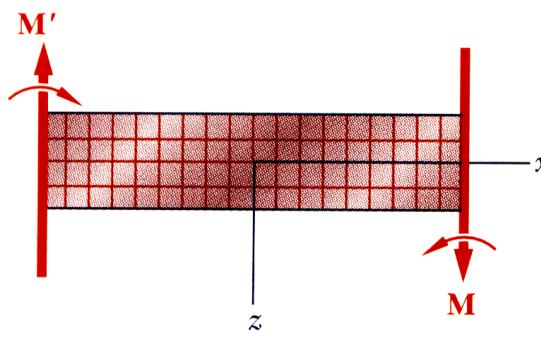
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.
$$\sum \sigma_x dA = 0$$
$$\sum z \sigma_x dA = 0$$
$$\sum -y \sigma_x dA = M$$



Bending Deformations



(a) Longitudinal, vertical section
(plane of symmetry)

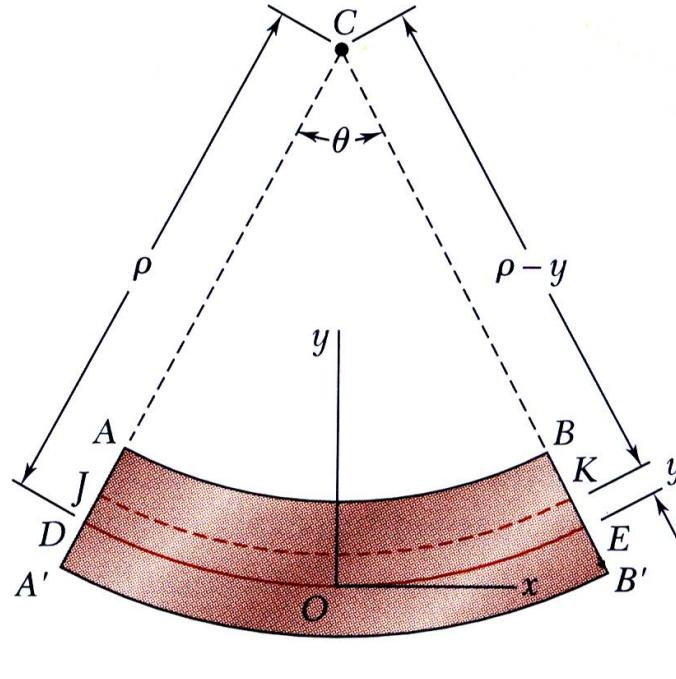


(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending



Consider a beam segment of length L .

After deformation, the length of the neutral surface remains L . At other sections,

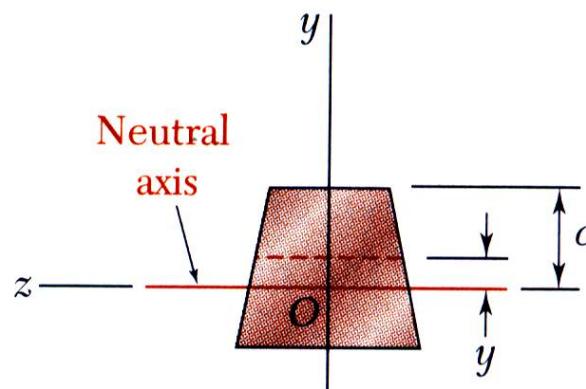
$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad (\text{strain varies linearly})$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$



Stress Due to Bending

- For a linearly elastic material,

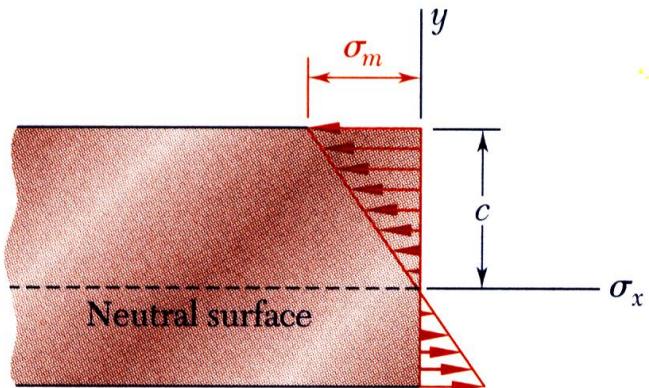
$$\begin{aligned}\sigma_x &= E\varepsilon_x = -\frac{y}{c}E\varepsilon_m \\ &= -\frac{y}{c}\sigma_m \quad (\text{stress varies linearly})\end{aligned}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c} \sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c} \int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y \sigma_x \, dA = \int -y \left(-\frac{y}{c} \sigma_m \right) \, dA$$

$$M = \frac{\sigma_m}{c} \int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$$\text{Substituting } \sigma_x = -\frac{y}{c} \sigma_m$$

$$\sigma_x = -\frac{My}{I}$$

Beam Section Properties

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

I = section moment of inertia

$$S = \frac{I}{c} = \text{section modulus}$$

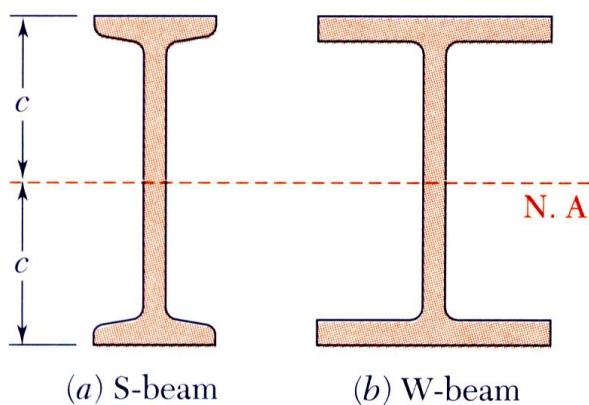
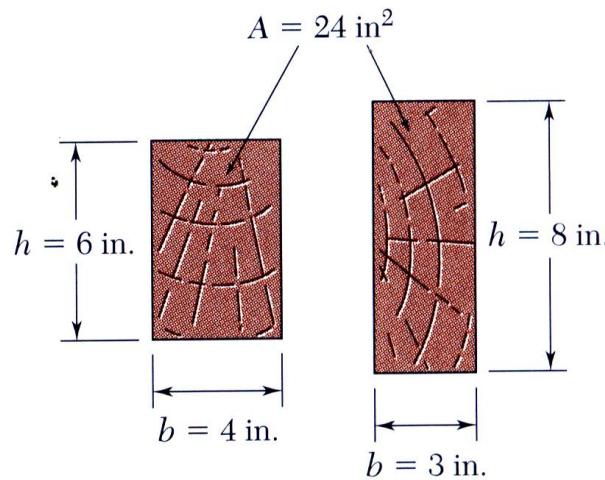
A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^3 = \frac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.



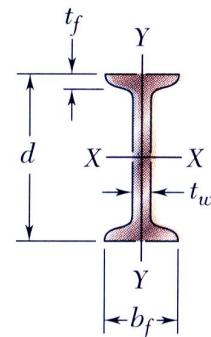
Properties of American Standard Shapes

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Appendix C. Properties of Rolled-Steel Shapes (SI Units)

S Shapes

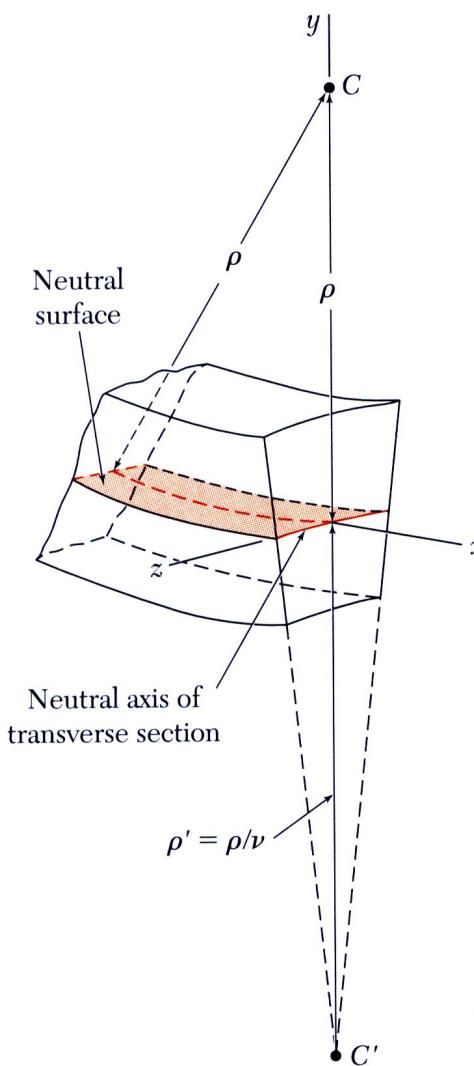
(American Standard Shapes)



Designation†	Flange			Web Thickness t_w , mm	Axis X-X			Axis Y-Y				
	Area A , mm ²	Depth d , mm	Width b_f , mm		I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm		
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0	
	158	20100	622	27.7	15.7	1230	3950	247	32.5	321	39.9	
	149	19000	610	184	22.1	995	3260	229	20.2	215	32.3	
	134	17100	610	181	22.1	938	3080	234	19.0	206	33.0	
	119	15200	610	178	22.1	878	2880	240	17.9	198	34.0	
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9	
	128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
	112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
	98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5	
	81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1	
	64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1



Deformations in a Transverse Cross Section



- Deformation due to bending moment M is quantified by the curvature of the neutral surface

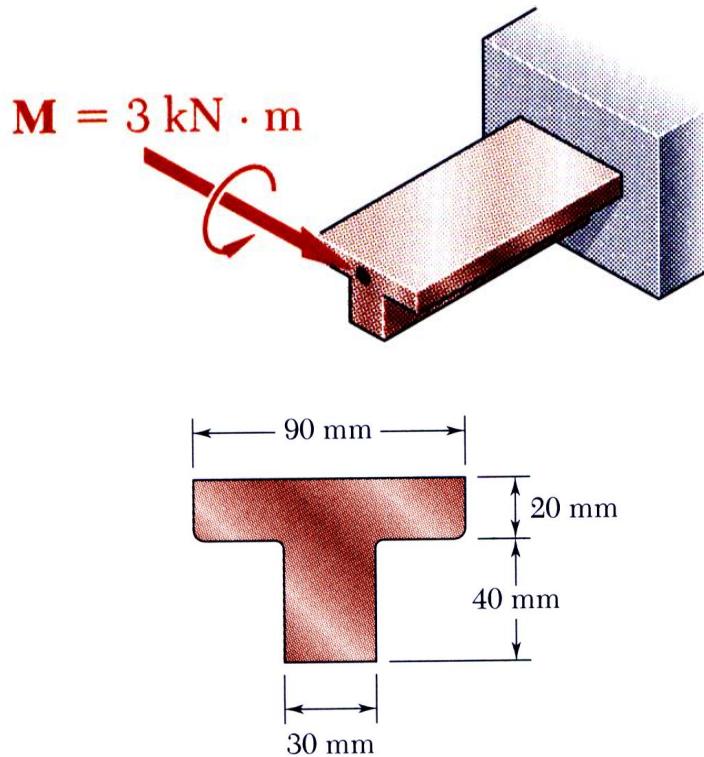
$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec} \frac{Mc}{I}$$
$$= \frac{M}{EI}$$

- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu \varepsilon_x = \frac{\nu y}{\rho} \quad \varepsilon_z = -\nu \varepsilon_x = \frac{\nu y}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature,
 $\frac{1}{\rho'} = \frac{1}{\rho}$ = anticlastic curvature

Sample Problem 4.2



A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E = 165 \text{ GPa}$ and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \quad I_{x'} = \sum (\bar{I} + Ad^2)$$

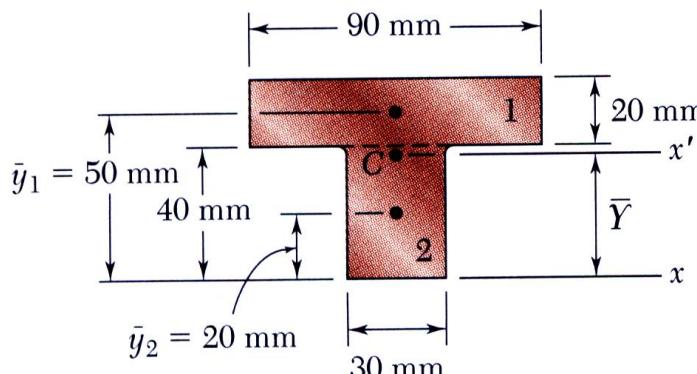
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

Sample Problem 4.2



SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	\bar{y}, mm	$\bar{y}A, \text{mm}^3$
1	$20 \times 90 = 1800$	50	90×10^3
2	$40 \times 30 = 1200$	20	24×10^3
	$\Sigma A = 3000$		$\Sigma \bar{y}A = 114 \times 10^3$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114 \times 10^3}{3000} = 38 \text{ mm}$$

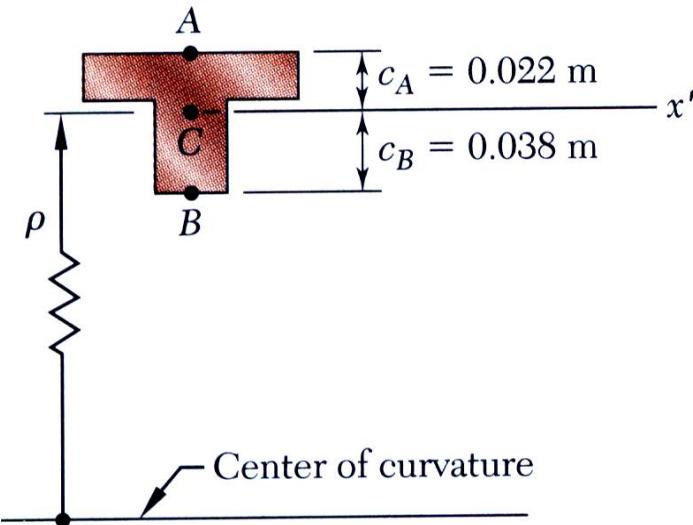
$$I_{x'} = \sum (\bar{I} + Ad^2) = \sum \left(\frac{1}{12}bh^3 + Ad^2 \right)$$

$$= \left(\frac{1}{12}90 \times 20^3 + 1800 \times 12^2 \right) + \left(\frac{1}{12}30 \times 40^3 + 1200 \times 18^2 \right)$$

$$I = 868 \times 10^3 \text{ mm} = 868 \times 10^{-9} \text{ m}^4$$



Sample Problem 4.2



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{Mc_A}{I} = \frac{3 \text{ kN} \cdot \text{m} \times 0.022 \text{ m}}{868 \times 10^{-9} \text{ mm}^4}$$

$$\sigma_A = +76.0 \text{ MPa}$$

$$\sigma_B = -\frac{Mc_B}{I} = -\frac{3 \text{ kN} \cdot \text{m} \times 0.038 \text{ m}}{868 \times 10^{-9} \text{ mm}^4}$$

$$\sigma_B = -131.3 \text{ MPa}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

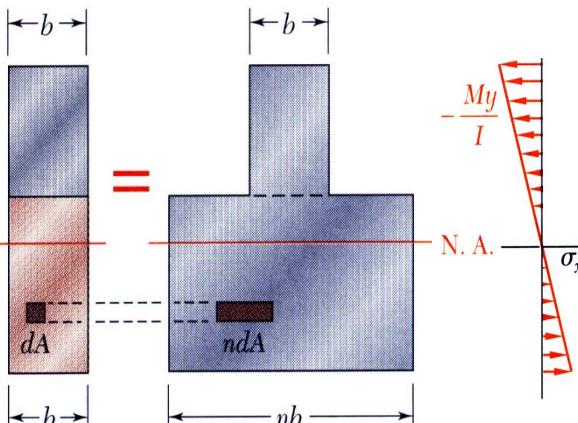
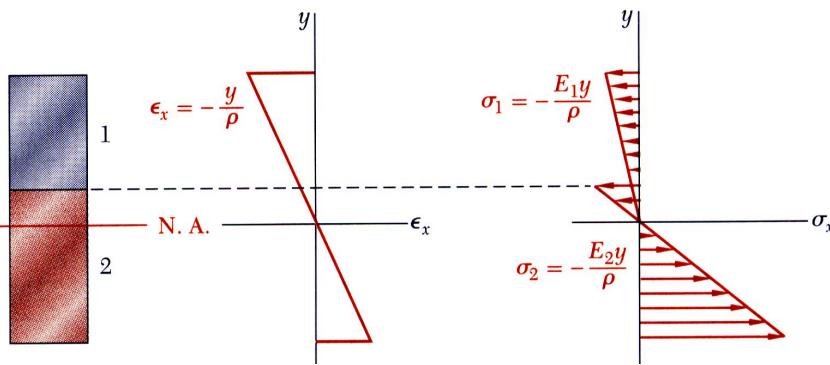
$$= \frac{3 \text{ kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{ m}^4)}$$

$$\frac{1}{\rho} = 20.95 \times 10^{-3} \text{ m}^{-1}$$

$$\rho = 47.7 \text{ m}$$



Bending of Members Made of Several Materials



$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \quad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2 .

- Normal strain varies linearly.

$$\epsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1 \epsilon_x = -\frac{E_1 y}{\rho} \quad \sigma_2 = E_2 \epsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

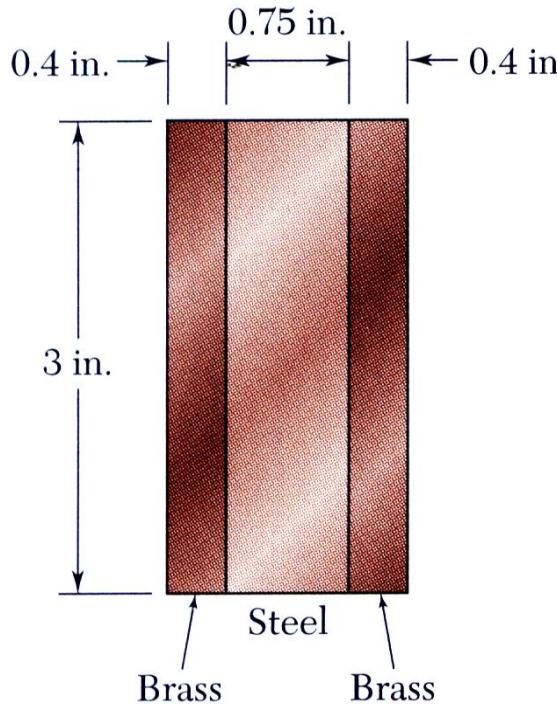
- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho} (n dA) \quad n = \frac{E_2}{E_1}$$

Example 4.03



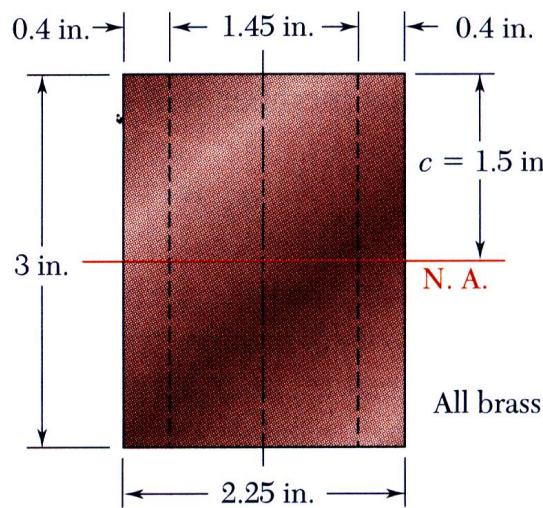
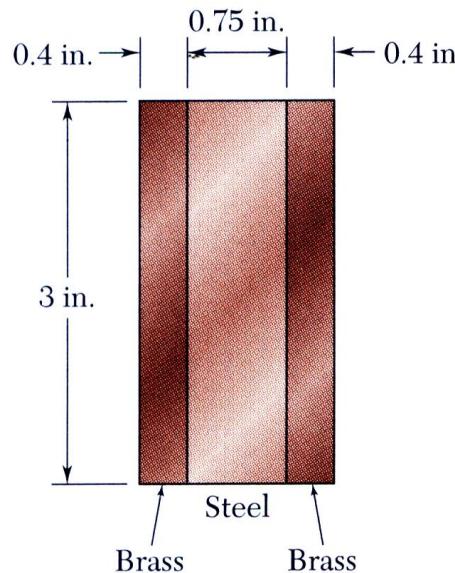
Bar is made from bonded pieces of steel ($E_s = 29 \times 10^6$ psi) and brass ($E_b = 15 \times 10^6$ psi). Determine the maximum stress in the steel and brass when a moment of 40 kip*in is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.



Example 4.03



SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{29 \times 10^6 \text{ psi}}{15 \times 10^6 \text{ psi}} = 1.933$$

$$b_T = 0.4 \text{ in} + 1.933 \times 0.75 \text{ in} + 0.4 \text{ in} = 2.25 \text{ in}$$

- Evaluate the transformed cross sectional properties

$$I = \frac{1}{12} b_T h^3 = \frac{1}{12} (2.25 \text{ in.})(3 \text{ in.})^3 \\ = 5.063 \text{ in}^4$$

- Calculate the maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(40 \text{ kip} \cdot \text{in})(1.5 \text{ in})}{5.063 \text{ in}^4} = 11.85 \text{ ksi}$$

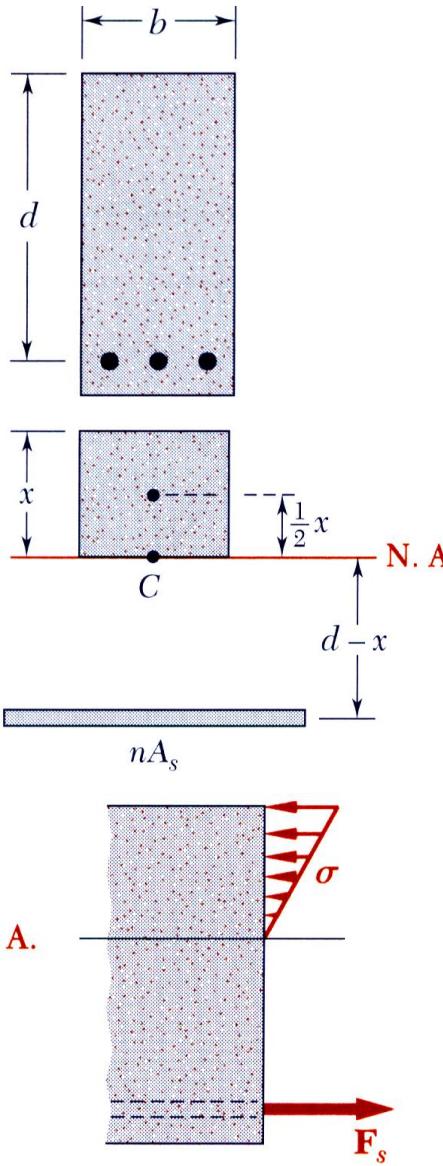
$$(\sigma_b)_{\max} = \sigma_m$$

$$(\sigma_s)_{\max} = n\sigma_m = 1.933 \times 11.85 \text{ ksi}$$

$$(\sigma_b)_{\max} = 11.85 \text{ ksi}$$

$$(\sigma_s)_{\max} = 22.9 \text{ ksi}$$

Reinforced Concrete Beams



- Concrete beams subjected to bending moments are reinforced by steel rods.
- The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load.
- In the transformed section, the cross sectional area of the steel, A_s , is replaced by the equivalent area nA_s where $n = E_s/E_c$.
- To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

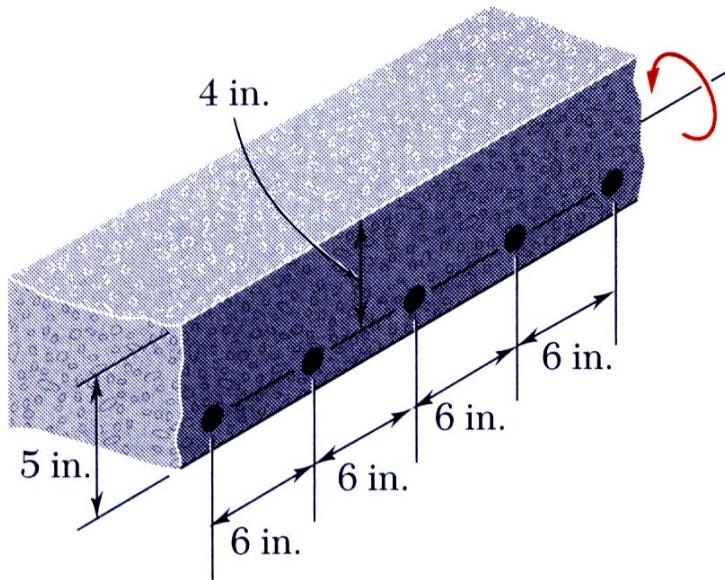
$$\frac{1}{2}bx^2 + nA_sx - nA_sd = 0$$

- The normal stress in the concrete and steel

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \quad \sigma_s = n\sigma_x$$

Sample Problem 4.4



SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.

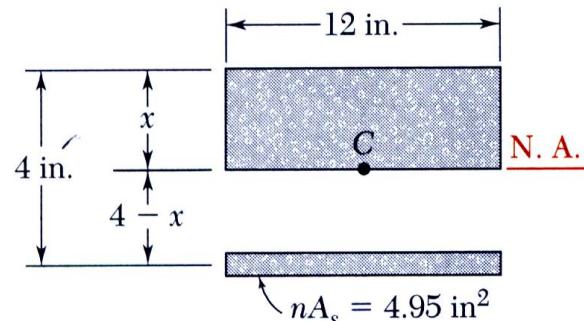
A concrete floor slab is reinforced with 5/8-in-diameter steel rods. The modulus of elasticity is 29×10^6 psi for steel and 3.6×10^6 psi for concrete. With an applied bending moment of 40 kip*in for 1-ft width of the slab, determine the maximum stress in the concrete and steel.



Sample Problem 4.4

SOLUTION:

- Transform to a section made entirely of concrete.



$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6 \text{ psi}}{3.6 \times 10^6 \text{ psi}} = 8.06$$

$$nA_s = 8.06 \times 2 \left[\frac{\pi}{4} \left(\frac{5}{8} \text{ in} \right)^2 \right] = 4.95 \text{ in}^2$$

- Evaluate the geometric properties of the transformed section.

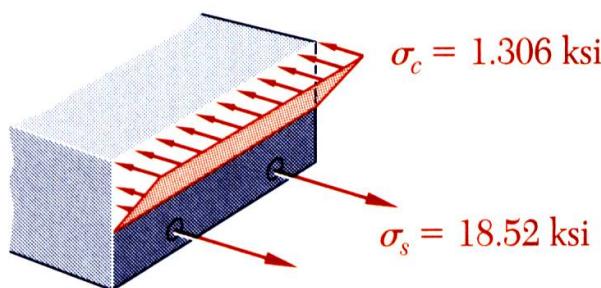
$$12x \left(\frac{x}{2} \right) - 4.95(4 - x) = 0 \quad x = 1.450 \text{ in}$$

$$I = \frac{1}{3}(12 \text{ in})(1.45 \text{ in})^3 + (4.95 \text{ in}^2)(2.55 \text{ in})^2 = 44.4 \text{ in}^4$$

- Calculate the maximum stresses.

$$\sigma_c = \frac{Mc_1}{I} = \frac{40 \text{ kip} \cdot \text{in} \times 1.45 \text{ in}}{44.4 \text{ in}^4}$$

$$\boxed{\sigma_c = 1.306 \text{ ksi}}$$

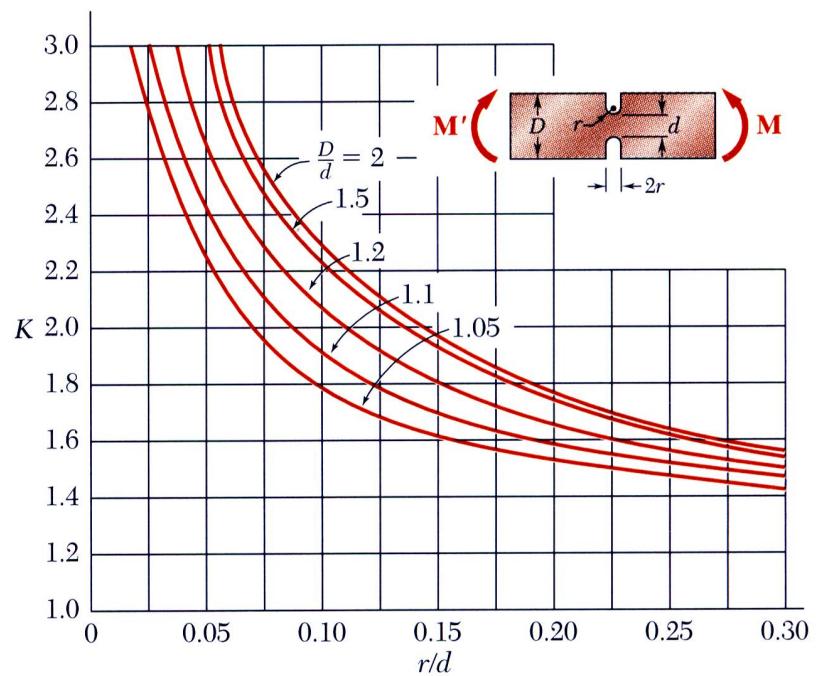
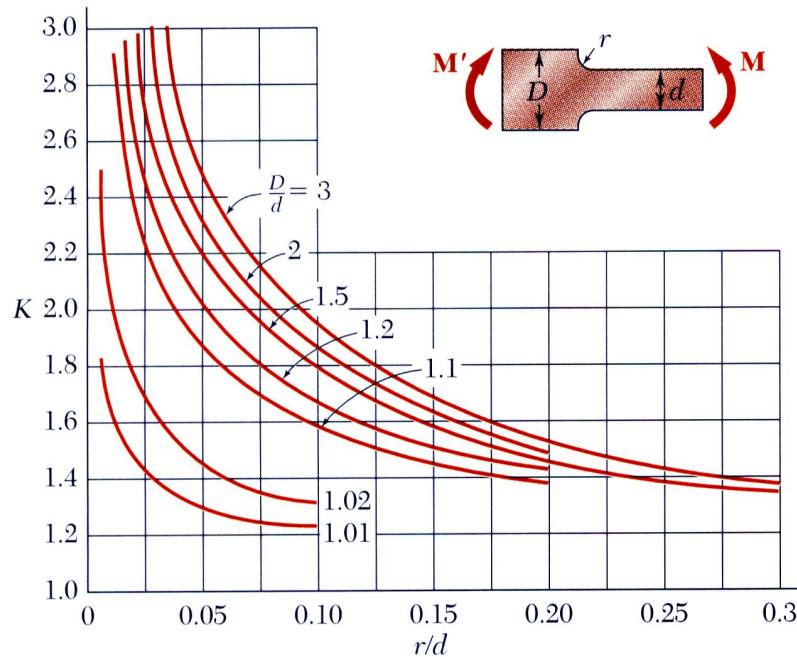


$$\sigma_s = n \frac{Mc_2}{I} = 8.06 \frac{40 \text{ kip} \cdot \text{in} \times 2.55 \text{ in}}{44.4 \text{ in}^4}$$

$$\boxed{\sigma_s = 18.52 \text{ ksi}}$$



Stress Concentrations

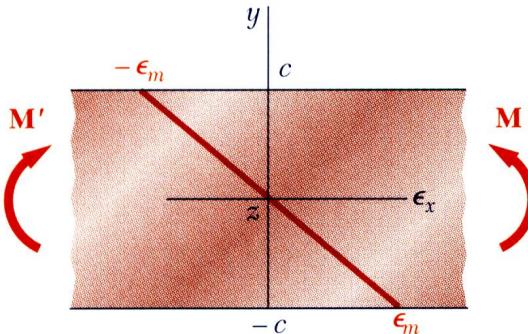


Stress concentrations may occur:

- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

$$\sigma_m = K \frac{Mc}{I}$$

Plastic Deformations

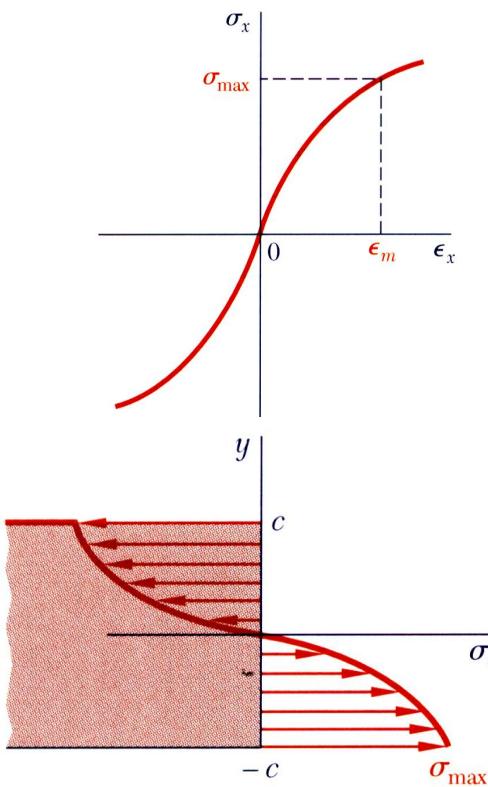


- For any member subjected to pure bending
- $$\epsilon_x = -\frac{y}{c} \epsilon_m$$
- strain varies linearly across the section
- If the member is made of a *linearly elastic material*, the neutral axis passes through the section centroid

$$\text{and } \sigma_x = -\frac{My}{I}$$

- For a material with a nonlinear stress-strain curve, the neutral axis location is found by satisfying

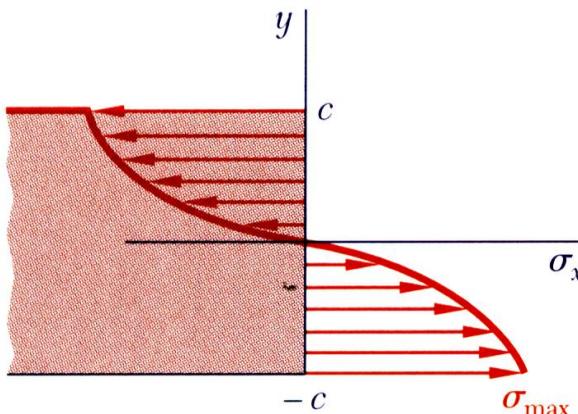
$$F_x = \int \sigma_x dA = 0 \quad M = \int -y \sigma_x dA$$



- For a member with vertical and horizontal planes of symmetry and a material with the same tensile and compressive stress-strain relationship, the neutral axis is located at the section centroid and the stress-strain relationship may be used to map the strain distribution from the stress distribution.

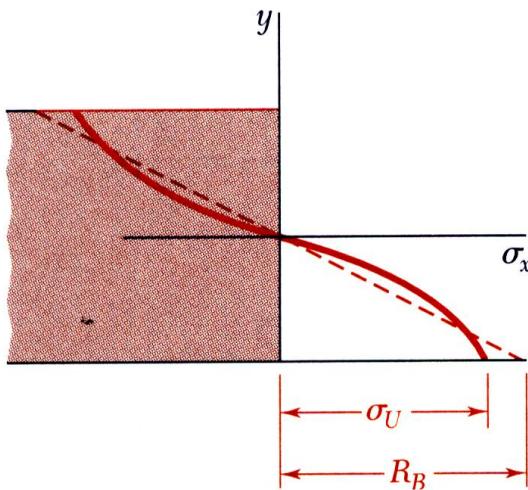


Plastic Deformations



- When the maximum stress is equal to the ultimate strength of the material, failure occurs and the corresponding moment M_U is referred to as the *ultimate bending moment*.
- The *modulus of rupture in bending*, R_B , is found from an experimentally determined value of M_U and a fictitious linear stress distribution.

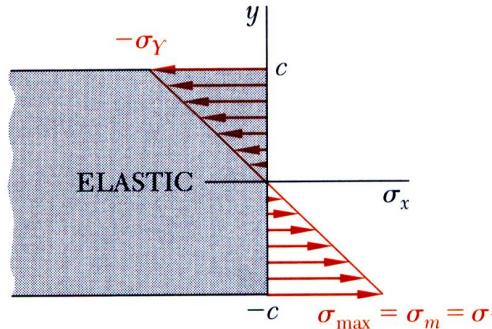
$$R_B = \frac{M_U c}{I}$$



- R_B may be used to determine M_U of any member made of the same material and with the same cross sectional shape but different dimensions.



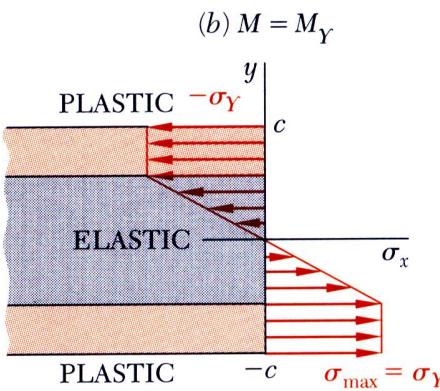
Members Made of an Elastoplastic Material



- Rectangular beam made of an elastoplastic material

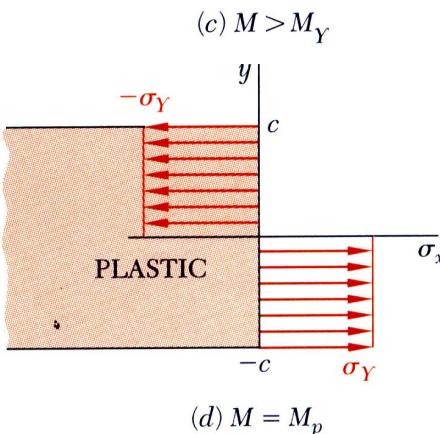
$$\sigma_x \leq \sigma_Y \quad \sigma_m = \frac{Mc}{I}$$

$$\sigma_m = \sigma_Y \quad M_Y = \frac{I}{c} \sigma_Y = \text{maximum elastic moment}$$



- If the moment is increased beyond the maximum elastic moment, plastic zones develop around an elastic core.

$$M = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right) \quad y_Y = \text{elastic core half-thickness}$$

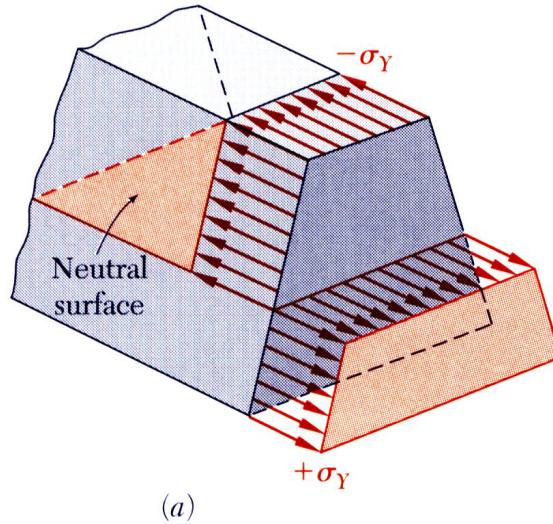


- In the limit as the moment is increased further, the elastic core thickness goes to zero, corresponding to a fully plastic deformation.

$$M_p = \frac{3}{2} M_Y = \text{plastic moment}$$

$$k = \frac{M_p}{M_Y} = \text{shape factor (depends only on cross section shape)}$$

Plastic Deformations of Members With a Single Plane of Symmetry



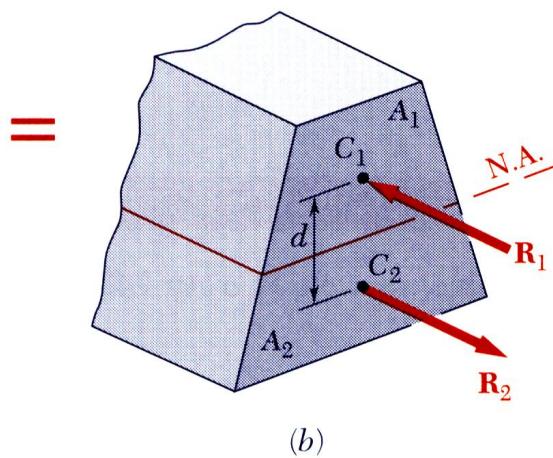
(a)

- Fully plastic deformation of a beam with only a vertical plane of symmetry.
- The neutral axis cannot be assumed to pass through the section centroid.
- Resultants R_1 and R_2 of the elementary compressive and tensile forces form a couple.

$$R_1 = R_2$$

$$A_1 \sigma_Y = A_2 \sigma_Y$$

The neutral axis divides the section into equal areas.

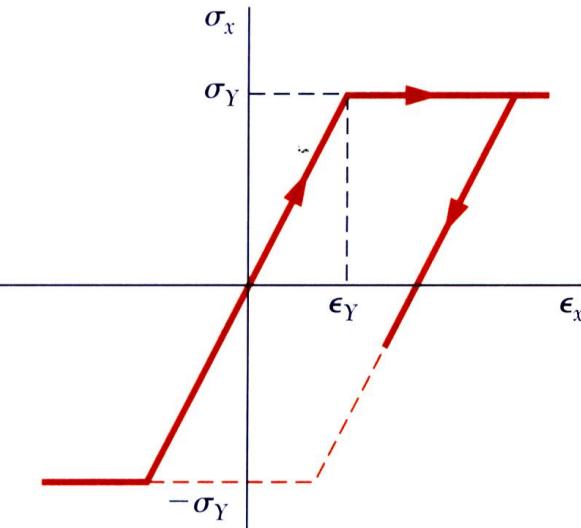


(b)

- The plastic moment for the member,

$$M_p = \left(\frac{1}{2} A \sigma_Y\right) d$$

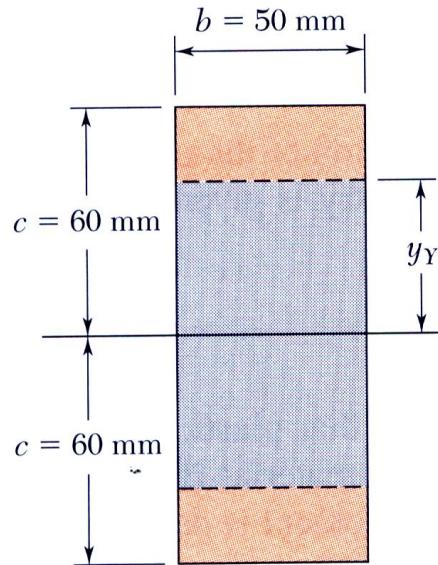
Residual Stresses



- Plastic zones develop in a member made of an elastoplastic material if the bending moment is large enough.
- Since the linear relation between normal stress and strain applies at all points during the unloading phase, it may be handled by assuming the member to be fully elastic.
- Residual stresses are obtained by applying the principle of superposition to combine the stresses due to loading with a moment M (elastoplastic deformation) and unloading with a moment $-M$ (elastic deformation).
- The final value of stress at a point will not, in general, be zero.



Example 4.05, 4.06



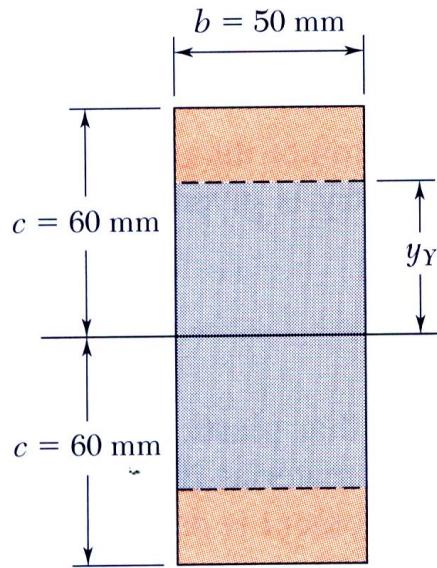
A member of uniform rectangular cross section is subjected to a bending moment $M = 36.8 \text{ kN-m}$. The member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa.

Determine (a) the thickness of the elastic core, (b) the radius of curvature of the neutral surface.

After the loading has been reduced back to zero, determine (c) the distribution of residual stresses, (d) radius of curvature.



Example 4.05, 4.06



- Maximum elastic moment:

$$\frac{I}{c} = \frac{2}{3}bc^2 = \frac{2}{3}(50 \times 10^{-3} m)(60 \times 10^{-3} m)^2 \\ = 120 \times 10^{-6} m^3$$

$$M_Y = \frac{I}{c}\sigma_Y = (120 \times 10^{-6} m^3)(240 MPa) \\ = 28.8 kN \cdot m$$

- Thickness of elastic core:

$$M = \frac{3}{2}M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right)$$

$$36.8 kN \cdot m = \frac{3}{2}(28.8 kN \cdot m) \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right)$$

$$\frac{y_Y}{c} = \frac{y_Y}{60 mm} = 0.666$$

$$2y_Y = 80 mm$$

- Radius of curvature:

$$\varepsilon_Y = \frac{\sigma_Y}{E} = \frac{240 \times 10^6 Pa}{200 \times 10^9 Pa} \\ = 1.2 \times 10^{-3}$$

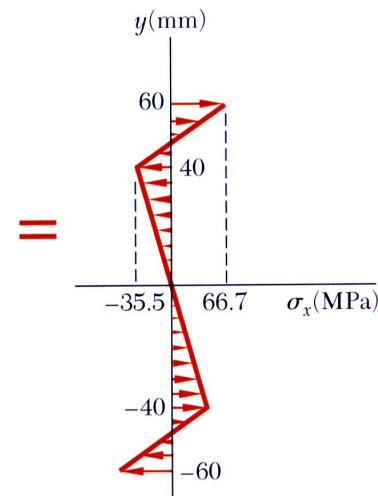
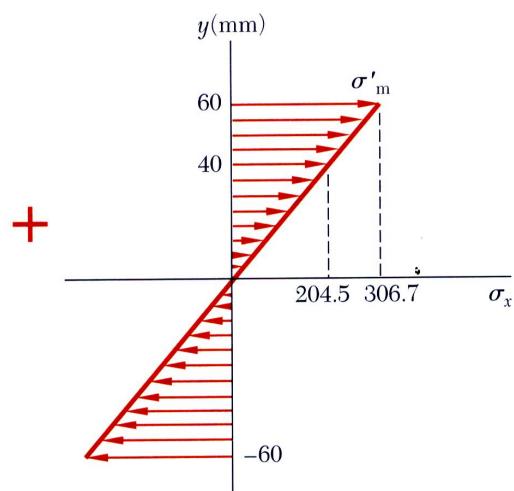
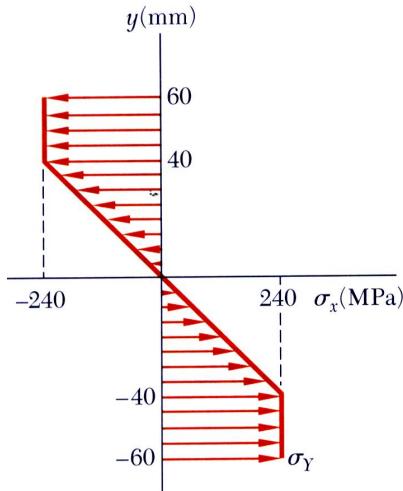
$$\varepsilon_Y = \frac{y_Y}{\rho}$$

$$\rho = \frac{y_Y}{\varepsilon_Y} = \frac{40 \times 10^{-3} m}{1.2 \times 10^{-3}} \\ \rho = 33.3 m$$

$$\rho = 33.3 m$$



Example 4.05, 4.06



- $M = 36.8 \text{ kN}\cdot\text{m}$

$$y_Y = 40 \text{ mm}$$

$$\sigma_Y = 240 \text{ MPa}$$

- $M = -36.8 \text{ kN}\cdot\text{m}$

$$\begin{aligned} \sigma'_m &= \frac{Mc}{I} = \frac{36.8 \text{ kN} \cdot \text{m}}{120 \times 10^6 \text{ m}^3} \\ &= 306.7 \text{ MPa} < 2\sigma_Y \end{aligned}$$

- $M = 0$

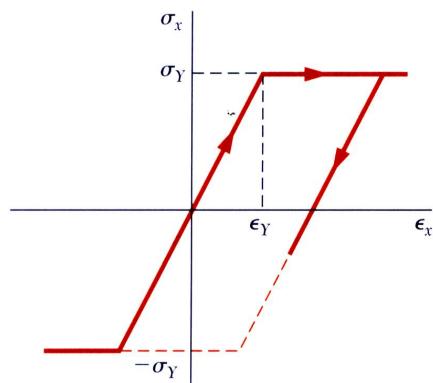
At the edge of the elastic core,

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{-35.5 \times 10^6 \text{ Pa}}{200 \times 10^9 \text{ Pa}}$$

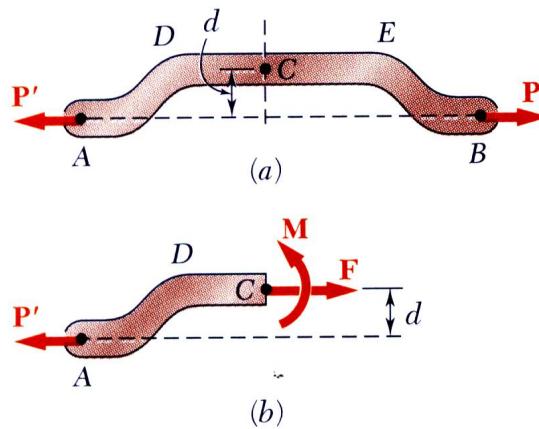
$$= -177.5 \times 10^{-6}$$

$$\rho = -\frac{y_Y}{\epsilon_x} = \frac{40 \times 10^{-3} \text{ m}}{177.5 \times 10^{-6}}$$

$$\rho = 225 \text{ m}$$



Eccentric Axial Loading in a Plane of Symmetry



- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

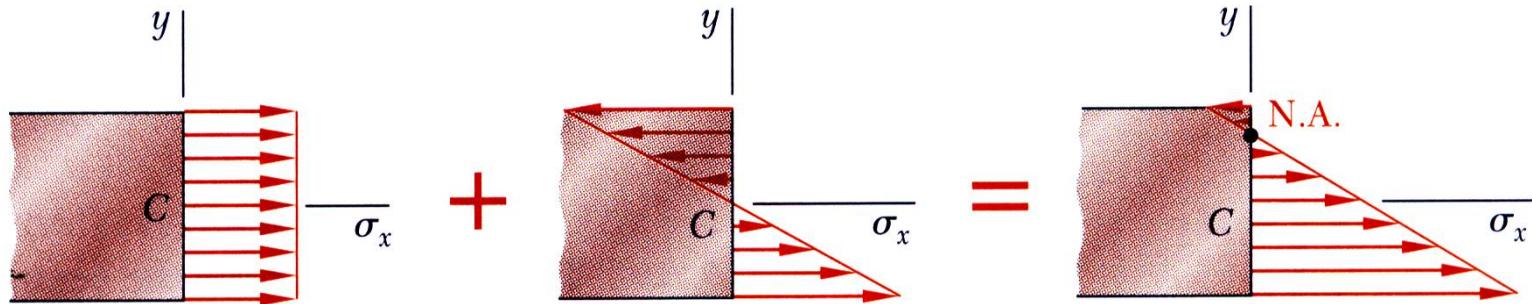
$$= \frac{P}{A} - \frac{My}{I}$$

- Eccentric loading

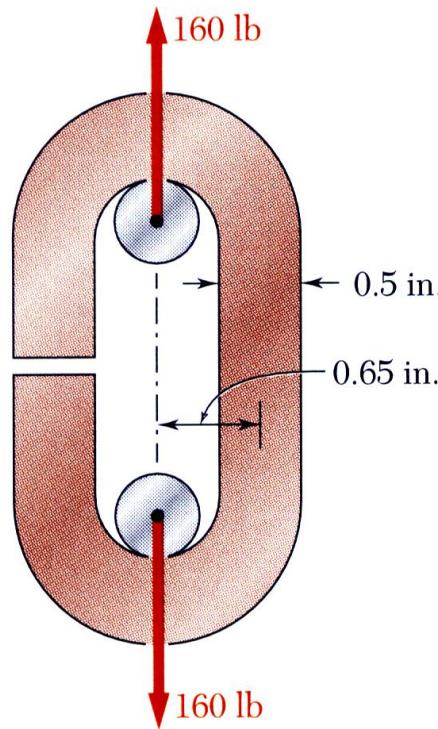
$$F = P$$

$$M = Pd$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.



Example 4.07



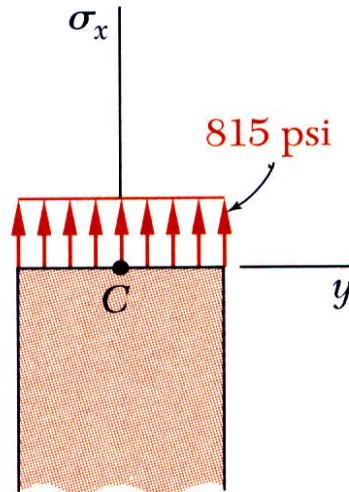
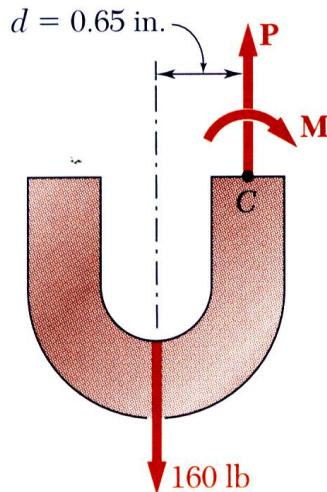
An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

SOLUTION:

- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.



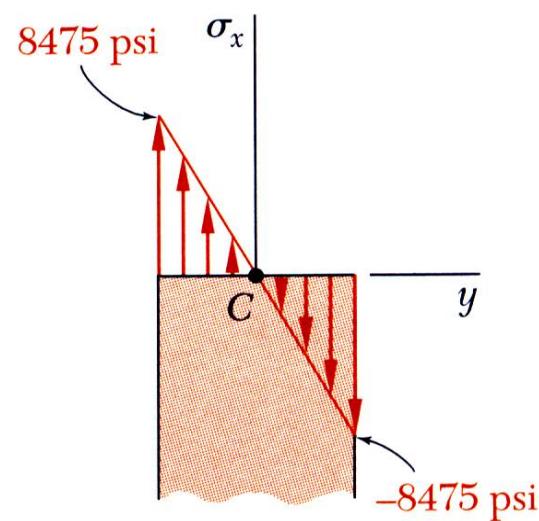
Example 4.07



- Equivalent centric load and bending moment

$$P = 160 \text{ lb}$$

$$M = Pd = (160 \text{ lb})(0.6 \text{ in}) \\ = 104 \text{ lb} \cdot \text{in}$$



- Normal stress due to a centric load

$$A = \pi c^2 = \pi(0.25 \text{ in})^2 \\ = 0.1963 \text{ in}^2$$

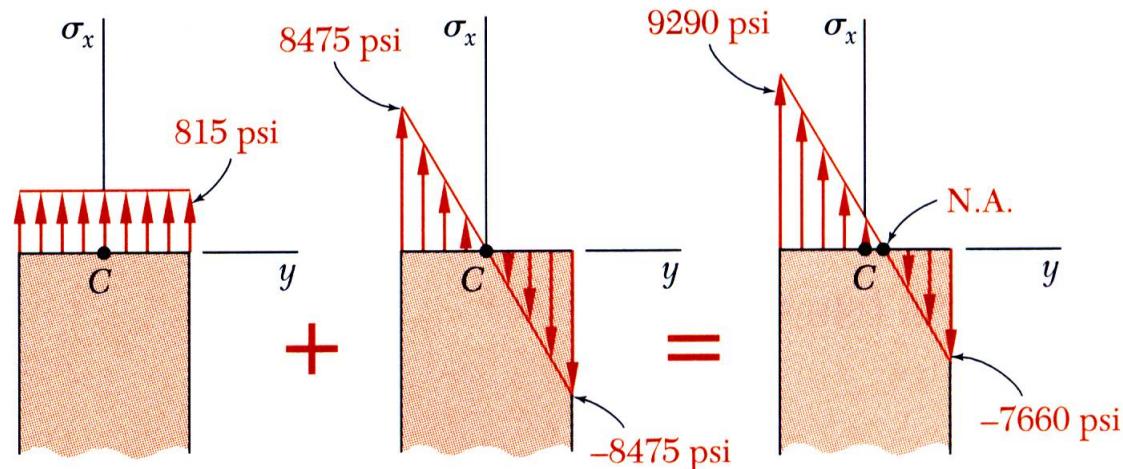
$$\sigma_0 = \frac{P}{A} = \frac{160 \text{ lb}}{0.1963 \text{ in}^2} \\ = 815 \text{ psi}$$

- Normal stress due to bending moment

$$I = \frac{1}{4} \pi c^4 = \frac{1}{4} \pi (0.25)^4 \\ = 3.068 \times 10^{-3} \text{ in}^4$$

$$\sigma_m = \frac{Mc}{I} = \frac{(104 \text{ lb} \cdot \text{in})(0.25 \text{ in})}{3.068 \times 10^{-3} \text{ in}^4} \\ = 8475 \text{ psi}$$

Example 4.07



- Maximum tensile and compressive stresses

$$\begin{aligned}\sigma_t &= \sigma_0 + \sigma_m \\ &= 815 + 8475\end{aligned}$$

$$\boxed{\sigma_t = 9260 \text{ psi}}$$

$$\begin{aligned}\sigma_c &= \sigma_0 - \sigma_m \\ &= 815 - 8475\end{aligned}$$

$$\boxed{\sigma_c = -7660 \text{ psi}}$$

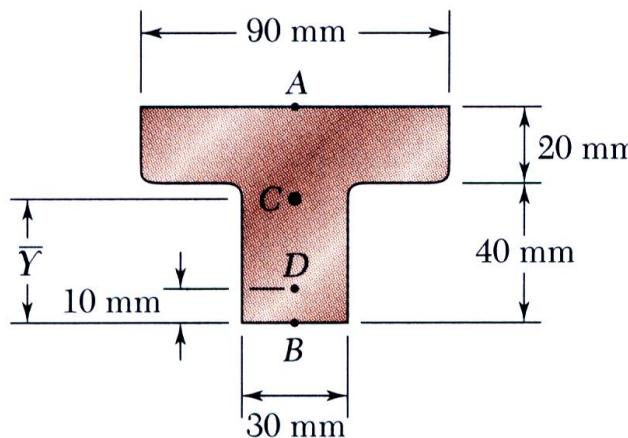
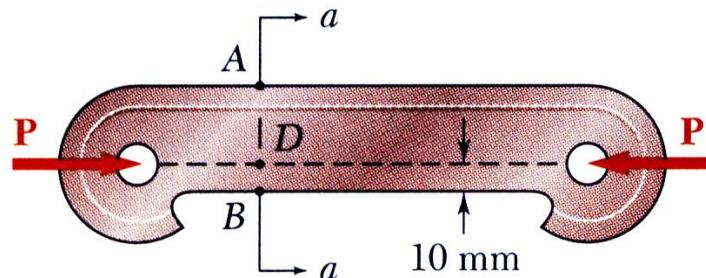
- Neutral axis location

$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A M} I = (815 \text{ psi}) \frac{3.068 \times 10^{-3} \text{ in}^4}{1051 \text{ lb} \cdot \text{in}}$$

$$\boxed{y_0 = 0.0240 \text{ in}}$$

Sample Problem 4.8

Section *a-a*

From Sample Problem 2.4,

$$A = 3 \times 10^{-3} \text{ m}^2$$

$$\bar{Y} = 0.038 \text{ m}$$

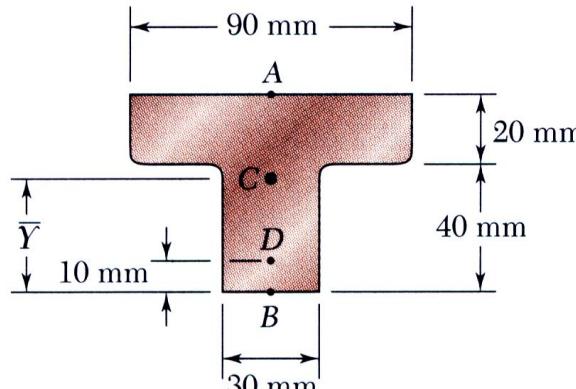
$$I = 868 \times 10^{-9} \text{ m}^4$$

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force P which can be applied to the link.

SOLUTION:

- Determine an equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

Sample Problem 4.8



- Determine an equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028 \text{ m}$$

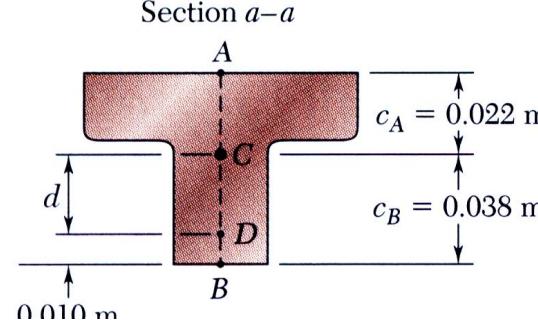
P = centric load

$$M = Pd = 0.028P = \text{bending moment}$$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3 \times 10^{-3}} + \frac{(0.028P)(0.022)}{868 \times 10^{-9}} = +377P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3 \times 10^{-3}} - \frac{(0.028P)(0.022)}{868 \times 10^{-9}} = -1559P$$



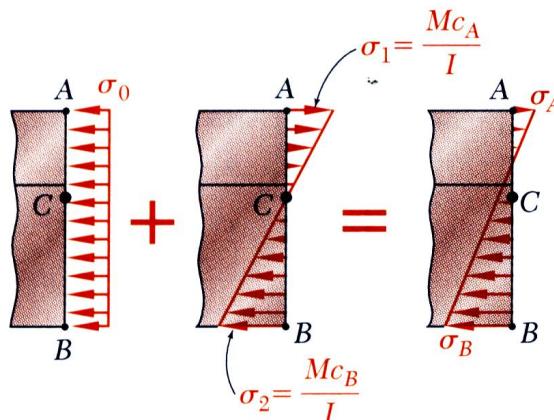
- Evaluate critical loads for allowable stresses.

$$\sigma_A = +377P = 30 \text{ MPa} \quad P = 79.6 \text{ kN}$$

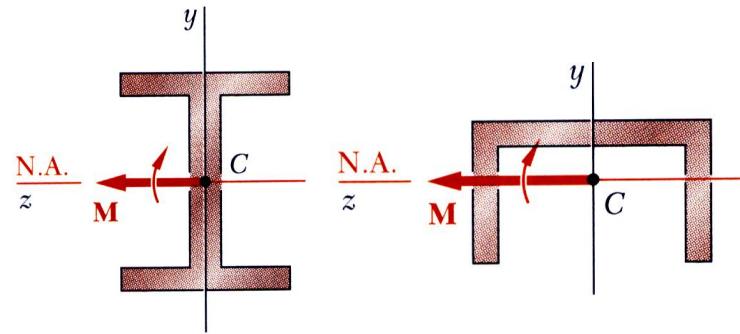
$$\sigma_B = -1559P = -120 \text{ MPa} \quad P = 79.6 \text{ kN}$$

- The largest allowable load

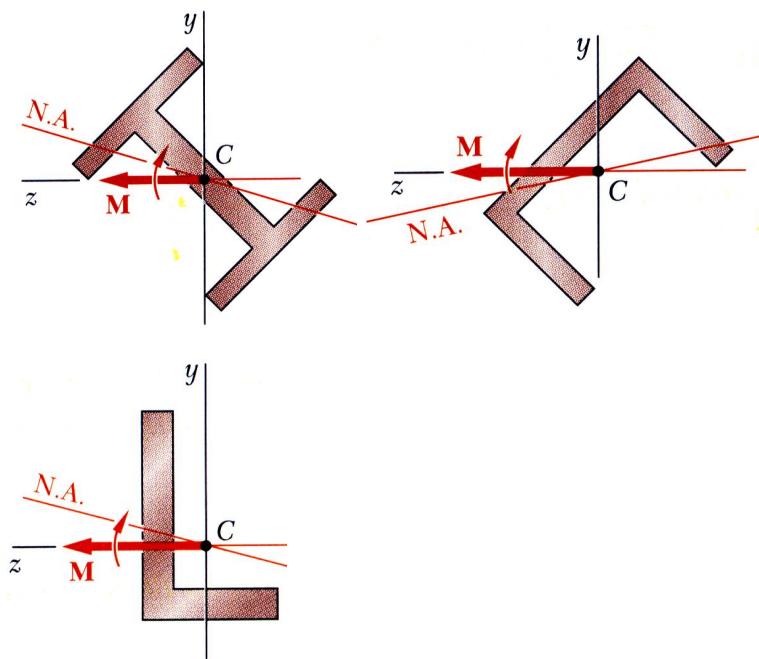
$$P = 77.0 \text{ kN}$$



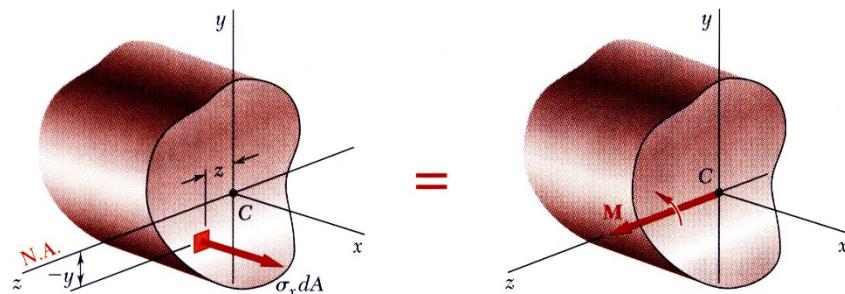
Unsymmetric Bending



- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.



Unsymmetric Bending



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

$$F_x = 0 = M_y \quad M_z = M = \text{applied couple}$$

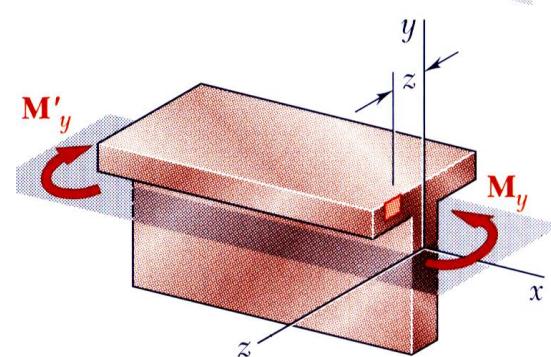
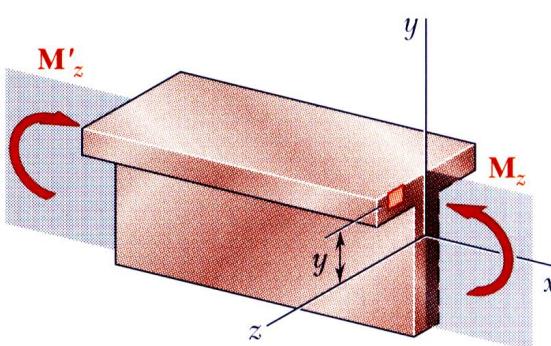
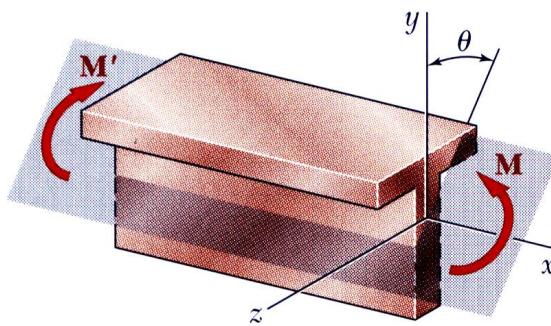
- $0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c} \sigma_m \right) dA$
or $0 = \int y dA$
neutral axis passes through centroid

- $M = M_z = - \int y \left(-\frac{y}{c} \sigma_m \right) dA$
or $M = \frac{\sigma_m I}{c} \quad I = I_z = \text{moment of inertia}$
defines stress distribution

- $0 = M_y = \int z \sigma_x dA = \int z \left(-\frac{y}{c} \sigma_m \right) dA$
or $0 = \int yz dA = I_{yz} = \text{product of inertia}$
couple vector must be directed along a principal centroidal axis



Unsymmetric Bending



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

$$M_z = M \cos \theta \quad M_y = M \sin \theta$$

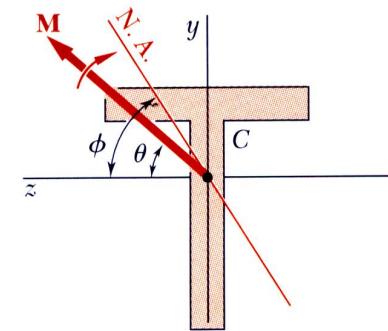
- Superpose the component stress distributions

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

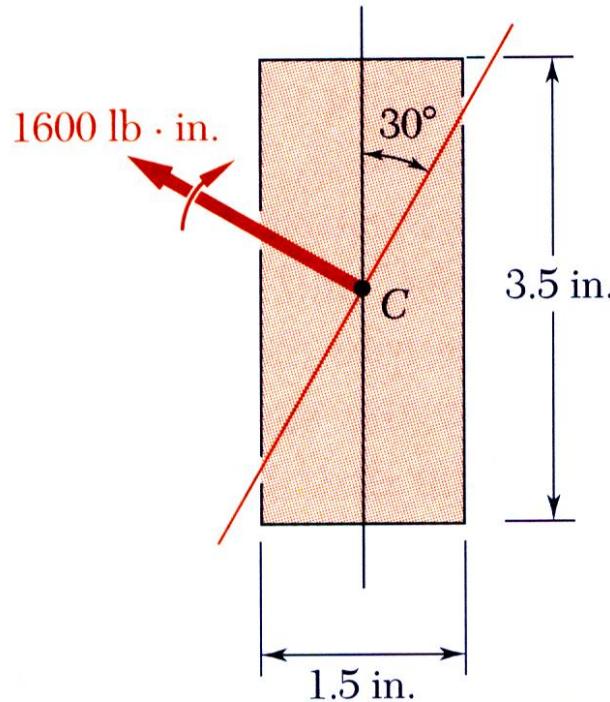
- Along the neutral axis,

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} = -\frac{(M \cos \theta) y}{I_z} + \frac{(M \sin \theta) y}{I_y}$$

$$\tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta$$



Example 4.08



A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

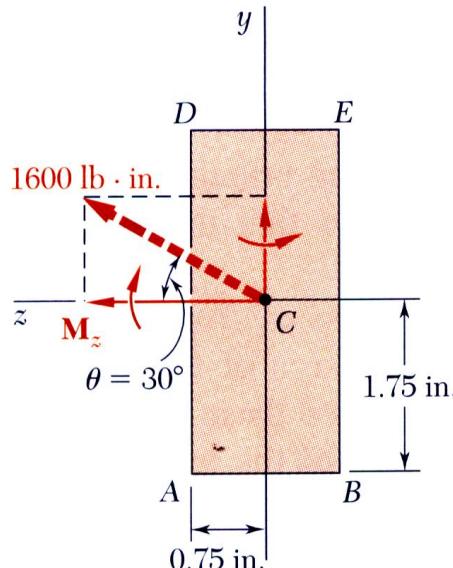
- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.
$$M_z = M \cos\theta \quad M_y = M \sin\theta$$
- Combine the stresses from the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta$$

Example 4.08



- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600 \text{ lb} \cdot \text{in}) \cos 30^\circ = 1386 \text{ lb} \cdot \text{in}$$

$$M_y = (1600 \text{ lb} \cdot \text{in}) \sin 30^\circ = 800 \text{ lb} \cdot \text{in}$$

$$I_z = \frac{1}{12}(1.5 \text{ in})(3.5 \text{ in})^3 = 5.359 \text{ in}^4$$

$$I_y = \frac{1}{12}(3.5 \text{ in})(1.5 \text{ in})^3 = 0.9844 \text{ in}^4$$

The largest tensile stress due to \$M_z\$ occurs along \$AB\$

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386 \text{ lb} \cdot \text{in})(1.75 \text{ in})}{5.359 \text{ in}^4} = 452.6 \text{ psi}$$

The largest tensile stress due to \$M_z\$ occurs along \$AD\$

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800 \text{ lb} \cdot \text{in})(0.75 \text{ in})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}$$

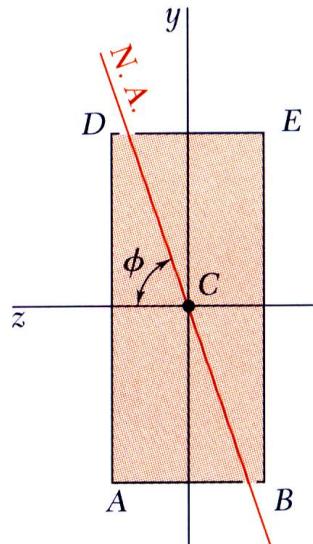
- The largest tensile stress due to the combined loading occurs at \$A\$.

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5$$

$$\sigma_{\max} = 1062 \text{ psi}$$



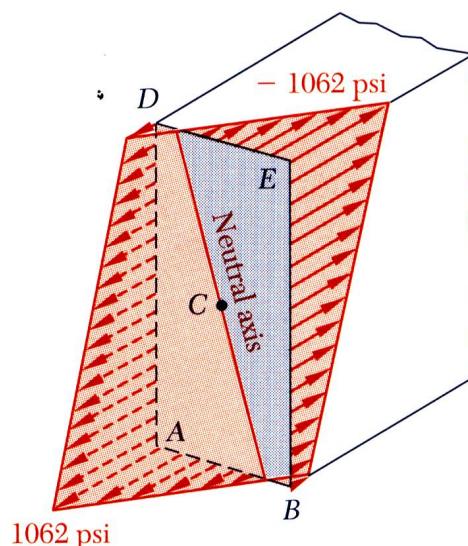
Example 4.08



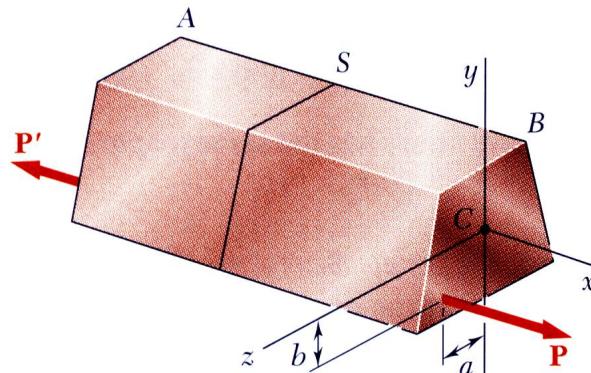
- Determine the angle of the neutral axis.

$$\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30 \\ = 3.143$$

$$\phi = 72.4^\circ$$



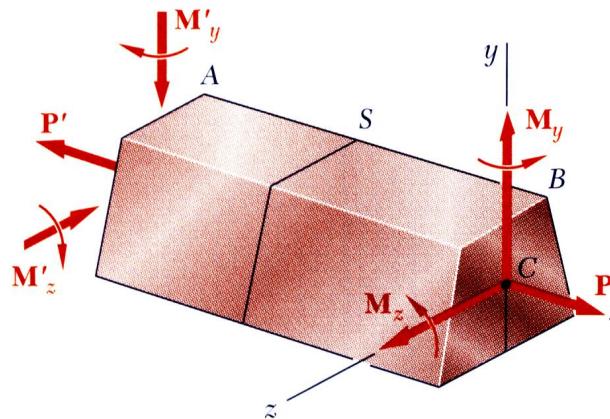
General Case of Eccentric Axial Loading



- Consider a straight member subject to equal and opposite eccentric forces.
- The eccentric force is equivalent to the system of a centric force and two couples.

P =centric force

$$M_y = Pa \quad M_z = Pb$$



- By the principle of superposition, the combined stress distribution is
- If the neutral axis lies on the section, it may be found from

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$

