SECTION 1

CLASSICAL MECHANICS

Chapter - 1
motion

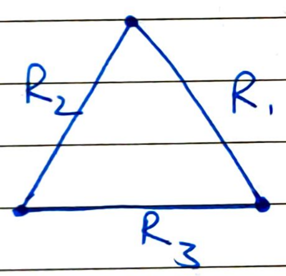
→ **DEGREES OF FREEDOM** →

- Degrees of freedom are calculated as

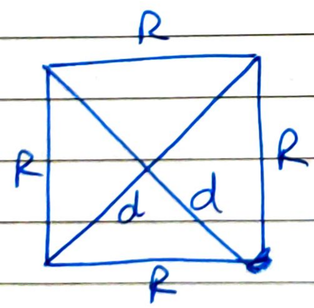
  $3n -$ number of constraints

- by constraint we mean, that dist. b/w 2 particles must be constant, or we could put an ~~dista~~ angle constraint.

- Examples-

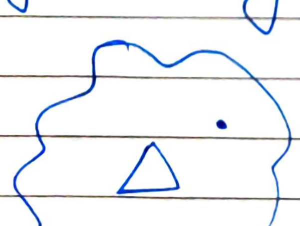
i) 

- now R_1, R_2 & R_3 are fixed $\therefore$ 3 constraints
  $\therefore$ dof $= 9-3=6$

ii) 

now we have square. ~~do~~ & 6 constraints.
$\therefore$ dof $= 12-6=6$

iii) Rigid body



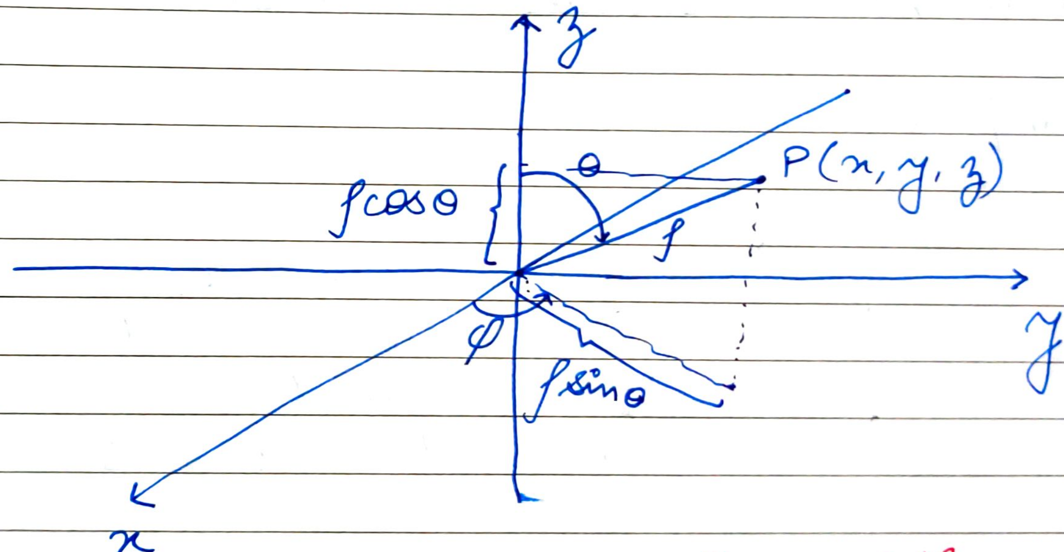
we know that Δ has 6 dof so we draw a Δ in rigid body. Now we fix a point in body. To define a point, we have to take its dist from 3 pts on Δ to

$\therefore$ we have 3 constraints. Also we have 3 new cartesian coordinates are added $\therefore$ 3 dof are added but 3 are subtracted as well due to constraints $\therefore$

$\therefore$ rigid body has 6 dof.

→ COORDINATE SYSTEM →

a) Spherical polar coordinates

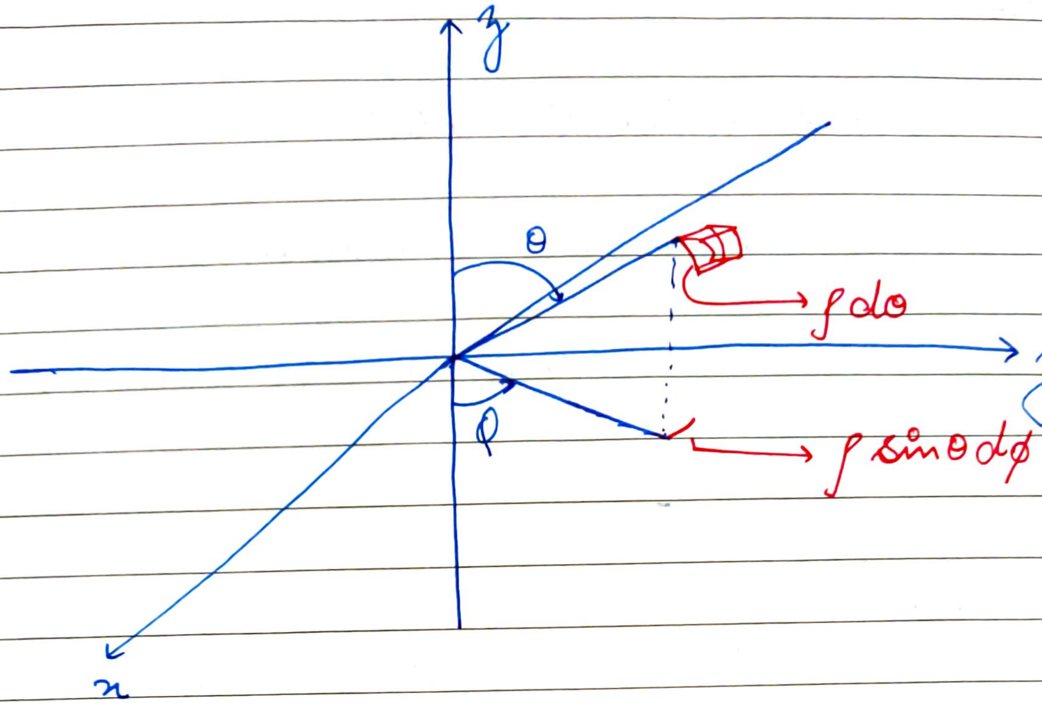


$$\boxed{\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\theta \end{aligned}}$$

θ = zenith angle
φ = azimuthal angle

$y, z, x \in [-\infty, \infty]$

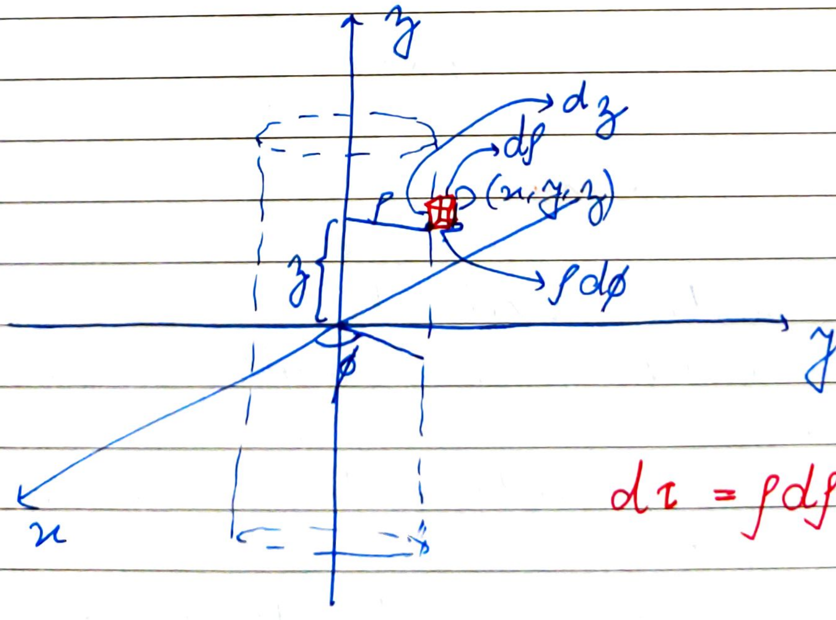
$$\left.\begin{aligned} \rho &\in [0, \infty] \\ \theta &\in [0, \pi] \\ \phi &\in [0, 2\pi) \end{aligned}\right\} \text{most standard form}$$



$$d\vec{A}_\rho = \hat{\rho}\,\rho^2 \sin\theta\, d\theta\, d\phi$$

$$d\tau = \rho^2 \sin\theta\, d\theta\, d\phi\, d\rho$$

b) Cylindrical Polar Co-ordinates



$$d\tau = \rho\, d\rho\, d\phi\, dz$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \& \quad z = z$$

$\cancel{R = d\rho\cos\phi}$

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RECIPE:

a) Sph. Polar Coordinates

$$\boxed{\vec{R} = x\hat{i} + y\hat{j} + z\hat{k} = \rho\sin\theta\cos\phi\,\hat{i} + \rho\sin\theta\sin\phi\,\hat{j} + \rho\cos\theta\,\hat{k}}$$

now

$$h_\rho = \left|\frac{\partial\vec{R}}{\partial\rho}\right| = |\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}|$$

$$= 1$$

$\Rightarrow h_\rho = 1$

now

$$h_\theta = \left|\frac{\partial\vec{R}}{\partial\theta}\right| = |\rho\cos\theta\cos\phi\,\hat{i} + \rho\cos\theta\sin\phi - \rho\sin\theta\hat{k}|$$

$$= \rho$$

$\Rightarrow h_\theta = \rho$

now

$$h_\phi = \left|\frac{\partial\vec{R}}{\partial\phi}\right| = |-\rho\sin\theta\sin\phi\,\hat{i} + \rho\sin\theta\cos\phi\,\hat{j} + 0\hat{k}|$$

$$= \rho\sin\theta$$

$\Rightarrow h_\phi = \rho\sin\theta$

now note that $|d\vec{R}|$ can be obtained by multiplying h_ρ by $d\rho$, h_θ by $d\theta$ & h_ϕ by $d\phi$

$\therefore$ we can obtain $d\tau$ by multiplying all 3 of them

ie $d\tau = (d\rho)\rho(d\theta)\rho\sin\theta\,d\phi$

$\therefore$ we can avoid imagining & do the math of it to calculate volume element

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b) Cylendrical Polar coordinates

$$\boxed{\vec{R} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}}$$

now $\left|\frac{\partial \vec{R}}{\partial \rho}\right| =$

$h_\rho = |\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\hat{k}|$

$= 1$

$\Rightarrow h_\rho = 1$

$h_\phi = \left|\frac{\partial \vec{R}}{\partial \phi}\right| = |-\rho\sin\phi + \rho\cos\phi|$

$= \rho$

$\Rightarrow h_\phi = \rho$

$h_z = \left|\frac{\partial \vec{R}}{\partial z}\right| = |1\hat{k}|$

$\Rightarrow h_z = 1$

$\therefore d\tau = \rho\, d\phi\, d\rho\, dz$

Hence, we can calculate volume element

NOTE: To prove ~~it~~ that a force is conservative, we write

$$\nabla \times \vec{F} = \begin{vmatrix} h_{u_1}\hat{u}_1 & h_{u_2}\hat{u}_2 & h_{u_3}\hat{u}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_{u_1}F_{u_1} & h_{u_2}F_{u_2} & h_{u_3}F_{u_3} \end{vmatrix}$$

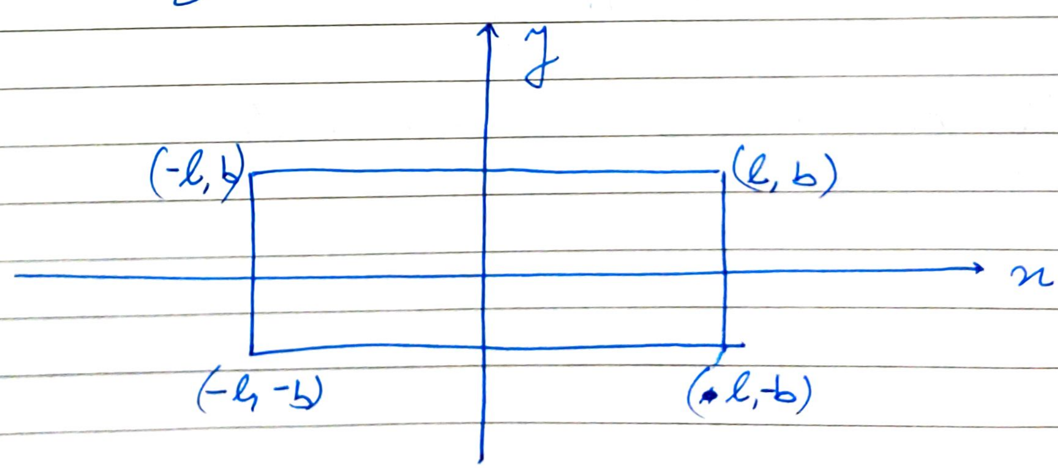
$$h_{u_i} = \left|\frac{\partial \vec{R}}{\partial u_i}\right|$$

$\cancel{\vec{F} = (k\rho\cos\theta\,\hat{j} + k\rho\sin\theta\,\hat{\phi}}$

At any point at any instant unit vectors will be $\perp$.

CONSTRAINTS →

Let us consider a billiard table of length $2l$ & breadth $2b$ and a billiard ball of radius r



$\therefore$ we have a constraint on billiard ball.

$$-l+r \leq x \leq l-r$$

$$\& \quad -b+r \leq y \leq b-r$$

$$\& \quad z = r$$

→ CLASSIFICATION OF CONSTRAINTS →

1. ON TIME DEPENDENCE

a) Scleronomic : constraints are time independent

b) Rheonomic : constraints are time dependent

2. ON VELOCITY DEPENDENCE

a) Holonomic : constraint eq^n indep of velocity

b) Non Holonomic : constraint eq^n dep on velocity

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3. ON ENERGY LOSS DEPENDENCE (deformation)

a) DISSIPATIVE : eg- constraints on a soft ball

b) NON-DISSIPATIVE : eg - constraints on a rigid ball

4. ON EQUATIONS

a) UNILATERAL : when we have inequalities in the equation

b) BILATERAL : when we have equalities in the equation

Tutorial - I

1 Que:

a) $\nu = R^a \rho^b S^c$

Dimensions of R, ρ & S

$R = [L]$
$\rho = [ML^{-3}]$
$S = [MT^{-2}]$

We know
$\nu = [T^{-1}]$
$\therefore$ clearly $c = 1/2$

$\Rightarrow b = -1/2$

$a = -3/2$

$\Rightarrow \nu = R^{-3/2} \rho^{-1/2} S^{1/2}$
If R is doubled then

$\nu = \nu_0 (2)^{-3/2}$ where ν_0 is old freq.

b) $h = [ML^2T^{-1}]$

$c = [LT^{-1}]$

$G = [M^{-1}L^3T^{-2}]$

$M^P = h^a c^b G^d$ $\quad a + d = 0 \Rightarrow 2a = 1$
$\therefore$ we can say $\quad a = 1/2$
$a - d = 1$; $2a + b + 3d = 0$; $-a - b - 2d = 0$

$d = -1/2 \qquad b = 1/2$

c) $-a-b-2d = 1$

$2a+b+3d=0$

$a = d$

$\Rightarrow -3a-b=1 \quad \& \quad 5a+b=0$

$\Rightarrow 2a = 1$

$\Rightarrow a = 1/2 \qquad b = -5/2 \qquad d = 1/2$

Q 2) a) $V = IR$

$\Rightarrow 1 = 10^{-6} R$

$\Rightarrow 10^{6} = R$

$\Rightarrow C = 10^{-6}\, S$

b) Applying Ohm's law in analogy

Heat current = Conductance × Temp-Difference

$\Rightarrow \frac{5\times10^{-3}}{1} = C \times 100$

$\Rightarrow C = 5\times10^{-5}\ J/sK$

c) $\frac{dV}{dt} = -F(P_2 - P_1)$

$F = \frac{\pi r^4}{8\eta l} \qquad r = 10^{-2}$

$l = 1 \qquad \eta = 1.5\times10^{-2}$

$\Rightarrow F = 2.618\times10^{-7}$

3 Que

a) ~~dissipative~~, ~~Rheonomic~~ Bilateral, non-dissipative (conservative), Scleronomic, Holonomic

b) dissipative, ~~xxx~~ holonomic, Rheonomic, bilateral
since $|\bar{r}_i - \bar{r}_k| = f(t)$ ie dist b/w 2 particles

c)

4 Que

a) $(2\times\cancel{2})-1 = 3$

b) $3-2=1$

c)

H/W

1 Que:

Ans: $M_p = \hbar^{1/2} c^{1/2} G^{-1/2}$

$h = 6.626 \times 10^{-34}$ $c = 3\times 10^{8}$ $G = 6.67\times 10^{-11}$

$$M_p = \frac{2.574 \times 10^{-17} \times 1.732 \times 10^{4}}{\sqrt{2\pi} \times 8.167 \times 10^{-6}}$$

$= 0.545 \times 10^{-7}/\sqrt{2\pi}$

$\Rightarrow M_p = \frac{5.45 \times 10^{-7}}{\sqrt{2\pi}}$ kg $= 2.18 \times 10^{-7}$ kg

$$T_p = \hbar^{1/2} \quad c^{-5/2} \quad d^{1/2}$$

$$= \frac{2.574 \times 10^{-17}}{2.5} \times \frac{8.167 \times 10^{-6}}{c^{5/2}}$$

$$=$$

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NOTE: CONSTRAINTS

RIGID BODY	DEFORMABLE BODY
$\lvert\vec{r}_i - \vec{r}_j\rvert = c_{ij}$ $\forall i,j$	$\lvert\vec{r}_i - \vec{r}_j\rvert = c_{ij}(t)$ or $= c_{ij}(t, i_k)$

might be any one depending on the eq.

NEWTON'S LAWS OF MOTION

First Law

A body tends to stay in its state of rest or of uniform motion in a straight line until and unless acted about by an external unbalanced force in an inertial frame of reference.

Second Law

$$\frac{d\vec{p}}{dt} = \vec{F}$$

~~(2D version of~~

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→ 2-D version of Newton's Second Law →

NOTE: Pitch (θ_P) Roll (θ_R) Yaw (θ_Y)

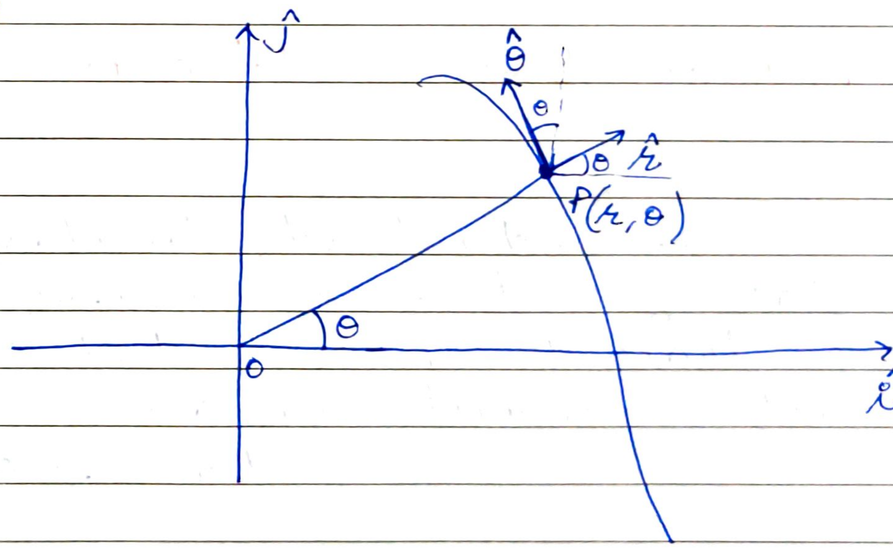
(Elevation (θ_E) Banking (θ_B) Heading (θ_H))

Like in an aeroplane

We can use this sometimes ~~to~~ as a co-ordinate system

× —— ×

Now let us assume a particle moving in 2D as



$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$$

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NOTE: $\dot{\{\}} = \frac{d\{\}}{dt}$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\dot{\theta} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}$$

$$\Rightarrow \underbrace{\ddot{\vec{r}}}_{} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

equivalent to acceleration

$$\Rightarrow \mu\ddot{\vec{r}} = \mu\left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}\right]$$

In case of a central force

$$F(\vec{r}) = f(\vec{r})\hat{r} \quad \& \quad (f'(\vec{r})\hat{\theta} = 0)$$
(fprime)

$$\Rightarrow \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

$$\forall \vec{r} \neq 0, \quad \mu[r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}] = 0$$

$$\Rightarrow \frac{d}{dt}\underbrace{[\mu r^2\dot{\theta}]}_{L} = 0$$

$\Rightarrow$ L = constant of motion (conserved)

now equation of motion EoM is

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(\vec{r})$$

now to solve such differential equation, we will use substitution

$u = 1/r$

$\Rightarrow r = 1/u$

$\Rightarrow \dot{r} = -\frac{1}{u^2} \frac{du}{d\theta} \dot{\theta}$

$\Rightarrow \dot{r} = -\frac{L}{\mu} \frac{du}{d\theta}$ $\qquad [\because L = \mu r^2 \dot{\theta}]$

$\Rightarrow \ddot{r} = -\frac{L}{\mu} \frac{d^2u}{d\theta^2} \dot{\theta}$

$\Rightarrow \ddot{r} = -\frac{L^2u^2}{\mu^2} \frac{d^2u}{d\theta^2}$

Also

$r\dot{\theta}^2 = \frac{1}{u} \frac{L^2u^4}{\mu^2} = \frac{L^2u^3}{\mu^2}$

$\therefore$ From EoM now

$$-\mu \left[\frac{L^2u^2}{\mu^2} \frac{d^2u}{d\theta^2} + \frac{L^2u^3}{\mu^2}\right] = f(\tfrac{1}{u})$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2} f(\tfrac{1}{u})$$

conjecture: $f(\frac{1}{u}) = -ku^2$ $\quad$ (ie $f(r) = -\frac{k}{r^2}$)

If it is valid, then, $\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{\mu k}{L^2}$

$\Rightarrow \frac{d^2\xi}{d\theta^2} + \xi = 0$ $\qquad \Big| \quad \xi = u - \frac{\mu k}{L^2}$

$\Rightarrow \xi = A\cos(\theta - \theta_0)$

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$\Rightarrow \frac{1}{r} = \frac{\mu k}{L^2} + A\cos(\theta-\theta_0)$

$\Rightarrow \boxed{\frac{l}{r} = 1 + \varepsilon\cos(\theta-\theta_0)}$; where $l = \frac{L^2}{\mu k}$ & $\varepsilon = \frac{L^2 A}{\mu k}$

$\longrightarrow$ (i)

now above equation is basically polar form of conic sections where
l is latus rectum and ε is eccentricity

$\therefore$ we can determine orbit shape from ε and size from l

NOTE: The conjecture assumes inverse square law and there isn't a single known orbit that doesn't satisfy it.

now,

$\vec{\dot{r}} = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

$\boxed{T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}}$

total KE

$\boxed{V = \frac{-k}{r}}$

total PE

$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}$

We actually put $\frac{dr}{dt} = 0$ to obtain the maximum & minimum values of r

$\Rightarrow \frac{1}{2}\mu\dot{r}^2 = -\frac{L^2}{2\mu}\left(\frac{1}{r}\right)^2 + k\left(\frac{1}{r}\right) + E$

$=0$ for const. rad

$$\Rightarrow \frac{1}{r} = \frac{-k \pm \sqrt{k^2 + 2EL^2/\mu}}{\frac{-L^2}{\mu}} \qquad (ii)$$

$$= \frac{\mu k}{L^2} \mp \frac{\mu k}{L^2}\sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

From (ii)

$$\Rightarrow \left(\frac{1}{r_<} - \frac{1}{r_>}\right) = \frac{2\mu k}{L^2}\sqrt{1+\frac{2EL^2}{\mu k^2}} \qquad (iii)$$

Similarly from (i) earlier, we have

$$\left(\frac{1}{r_<} - \frac{1}{r_>}\right) = \frac{2\varepsilon}{l} = 2A \quad (iv)$$

$\therefore$ equating (iii) & (iv), we have

$$\boxed{\varepsilon = \sqrt{1 + \frac{EL^2}{\mu k^2}}}$$

$\therefore$ we have a relation b/w energy and eccentricity & using ε, we can find A as

$$\boxed{A = \frac{\varepsilon}{l}}$$

Tutorial 2

2. $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0-0) = 0$$

$$= 0$$

$\therefore$ Conservative

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-1-1)$$

$$= -2\hat{k}$$

$\therefore$ non conservative.

4. $r = \frac{b}{2}\left(e^{\omega t} + e^{-\omega t}\right)$ (where $\omega = \Omega$)

~~w.r.t. (the central axis~~
~~w.r.t. ground)~~

$\dot{r} = \frac{b\omega}{2}\left(e^{\omega t} - e^{-\omega t}\right)$

w.r.t. ground

$$\text{speed} = \frac{b\omega}{2}\sqrt{2\left(e^{2\omega t} + e^{-2\omega t}\right)}$$

$$\text{acceleration} = \frac{b\omega^2}{\sqrt{2}}\left(e^{\omega t} + e^{-\omega t}\right)$$

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3.

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

On comparing

$\ddot{r} = r\dot{\theta}^2$ & $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 3\dot{r}\dot{\theta}$

$\Rightarrow \ddot{r} = r(cr)^2$ $\Rightarrow r\ddot{\theta} = \dot{r}\dot{\theta}$

$\Rightarrow \ddot{r} = c^2 r^3$ $\Rightarrow \dfrac{r\ddot{\theta} - \dot{r}\dot{\theta}}{r^2} = 0$

$\Rightarrow \int \dot{r}\ddot{r} = \int \dot{r} r^3 c^2$

$\Rightarrow \dfrac{d}{dt}\left(\dfrac{\dot{\theta}}{r}\right) = 0$

$\Rightarrow \dfrac{\dot{r}^2}{2} = \dfrac{c^2 r^4}{4} + B$ $\Rightarrow \dfrac{\dot{\theta}}{r} = c$

$\Rightarrow \dot{\theta} = cr$

$\Rightarrow \boxed{\dot{r} = \pm\sqrt{Ar^4 + B}}$

Rough

$\dot{r} = r\dot{\theta}^2$ & $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2\dot{r}\dot{\theta}$

$\boxed{\dot{r} = rk}$ $\Rightarrow r\ddot{\theta} = 0$

$\ddot{\theta} = 0$

$\dot{r}\ddot{r} = rk\dot{r}$

$r = \text{const}$

$\dot{\theta} = \text{const}$

$\dot{r} = 0$

$\dfrac{\dot{r}^2}{\cancel{2}} = \dfrac{r^2 k}{\cancel{2}} + C$

$r = 0$

$\ddot{r} = 0$

$\boxed{\dot{r} = \pm\sqrt{r^2 k + C}}$

5. $\vec{\dot{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\Rightarrow \vec{\dot{r}} = \dot{r}\hat{r} + r\frac{u}{a}\hat{\theta}$

Speed of lion $= U$ (const.)

Integrals of Motion (Central Force)

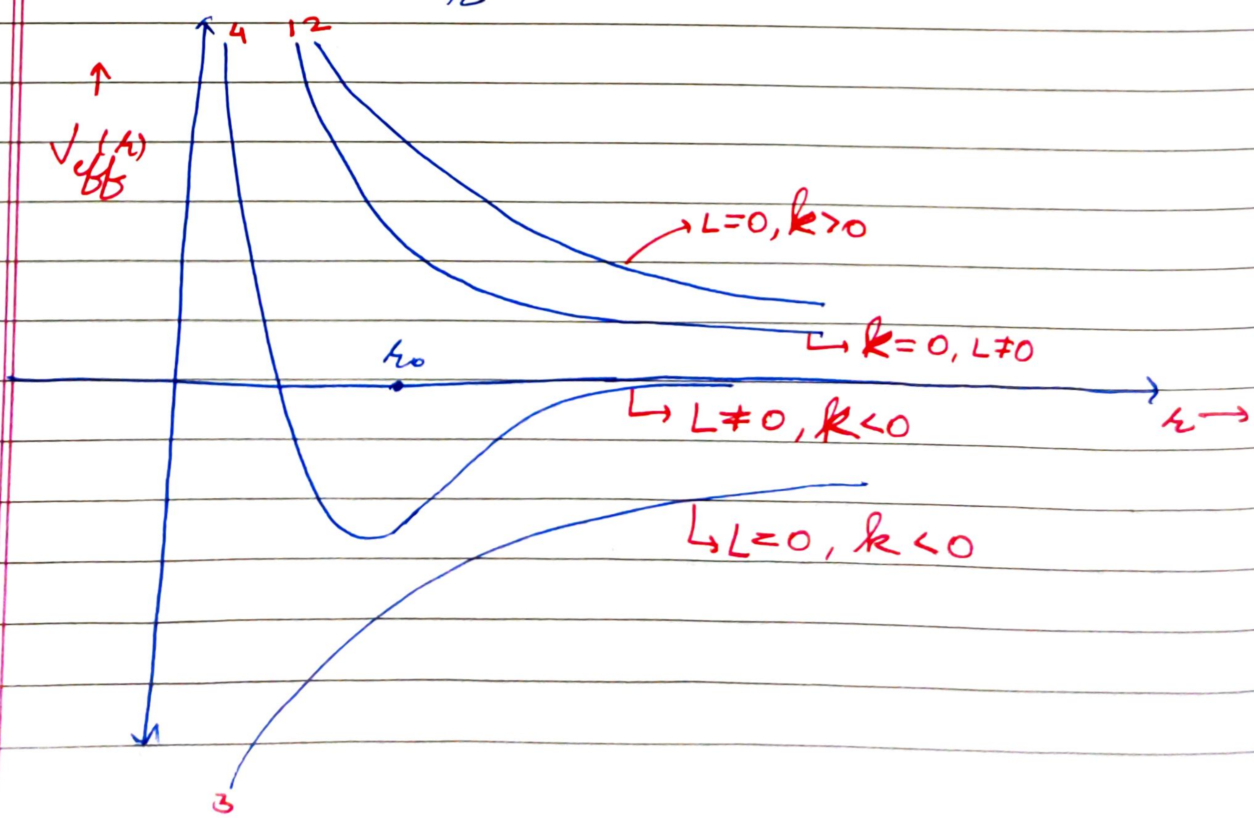
$$\mu \ddot{r} - \underbrace{\mu r \dot{\theta}^2}_{\frac{L^2}{\mu r^3}} = -\frac{dV(r)}{dr}$$

$$\mu \ddot{r} = -\frac{d}{dr}\underbrace{\left(V(r) + \frac{L^2}{2\mu r^2}\right)}_{V_{eff}(r)}$$

$$\Rightarrow \boxed{\mu \ddot{r} = -\frac{d}{dr}\underbrace{\left(V(r) + \frac{L^2}{2\mu r^2}\right)}_{V_{eff}(r)}} \quad -(i)$$

now we plot graphs for diff values of $V(r)$ & L

Let $V(r) = \frac{k}{r}$



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imp NOTE: Above in curves 1, 2, 3, 4 note very carefully that in curves 1, 2 & 3, we never get bound orbits since there is a direction where potential is decreasing. The system can easily tend to always go to a lower potential. But however, in 4th curve we have min. V at r_0 and this is where we will get at stable equilibrium point & thus a bounded orbit.

Now in eq. (i) multiplying $\frac{dr}{dt}$ both sides

$$\mu \dot{r} \ddot{r} = -\frac{dr}{dt}\frac{d}{dr}\left(V(r) + \frac{L^2}{2\mu r^2}\right)$$

multiplying dt on both sides

$$\int \mu \dot{r}\, d\dot{r} = -\int d\left(V(r) + \frac{L^2}{2\mu r^2}\right)$$

$$\Rightarrow \frac{1}{2}\mu(\dot{r})^2 = -\left(V(r) + \frac{L^2}{2\mu r^2}\right) + \text{term going to zero}$$

$$\Rightarrow \frac{1}{2}\mu(\dot{r})^2 + V(r) + \frac{L^2}{2\mu r^2} = \text{term going to zero}$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{L^2}{2\mu r^2}\right)}_{E} = 0$$

conservation law

$$\dot{r}^2 = \frac{2}{\mu}\left[E - V(r) - \frac{L^2}{2\mu r^2}\right] \quad -(ii)$$

$$\Rightarrow \boxed{\int_0^t dt = \int_{r_0}^{r} \frac{dr}{\left(\frac{2}{\mu}\right)^{1/2}\left[E - V(r) - \frac{L^2}{2\mu r^2}\right]^{1/2}}}$$

I^{st} integral of motion

* This integral will get us r.

now

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\mu\dot{\theta}}{\mu\dot{r}}dr \quad (iii)$$

now also in (ii)

$$\mu^2\dot{r}^2 = 2\mu\left[E - V(r) - \frac{L^2}{2\mu r^2}\right]$$

$$\Rightarrow \mu\dot{r} = \sqrt{2\mu\left[E - V(r) - \frac{L^2}{2\mu r^2}\right]}$$

$\therefore$ in (iii) we get

$$\boxed{\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{L/r^2}{\left[2\mu\left(E - V(r) - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}}}$$

II^{nd} integral of motion

NOTE: These 2 integrals of motion might help in very complicated motions such as missile guidance systems.

HW Q: $2m\dot{r}\dot{\theta}\hat{\theta} = \vec{F}$

Ans: $\ddot{r} = r\dot{\theta}^2$

Case 1:

$\ddot{r} = 0$ (not possible)

$\Rightarrow \dot{r} = \text{const}$

$\Rightarrow \dot{r} = 0$ [since $r=0$]

Case 2:

$\ddot{r} = rk$ $(k>0)$

$\dot{r}\ddot{r} = \dot{r}rk$

$\frac{(\dot{r})^2}{2} = \frac{kr^2}{2} + \frac{c}{2}$

$\Rightarrow \dot{r} = \pm\sqrt{kr^2 + C}$

Case 3:

similar to case 1.

$\therefore \dot{r} = 0$ (not possible)

$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2\dot{r}\dot{\theta}$

$\Rightarrow r\ddot{\theta} = 0$

Case 1:

$r=0$; $\ddot{\theta} \neq 0$

Case 2:

$r \neq 0$; $\ddot{\theta} = 0$

$\Rightarrow \dot{\theta} = \text{const}$

Case 3:

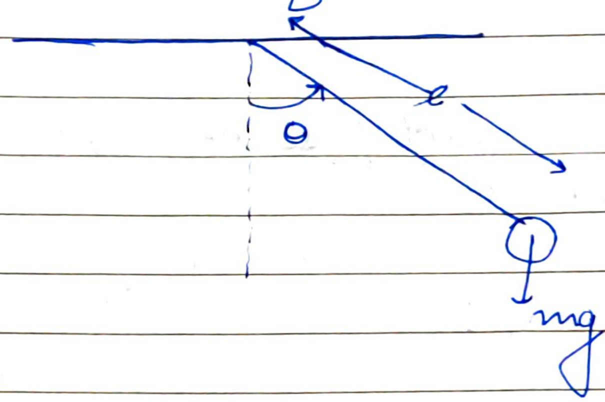
$r = 0$; $\ddot{\theta} = 0$

Note: If $r=0$, then $\ddot{\theta}$ is automatically zero.

$\therefore$ Case 1 is not possible

APPROXIMATIONS / (APPROXIMATE) MODELS

- Consider situation of a simple pendulum



$l\ddot{\theta} = -g\sin\theta$

For small θ we have

$\theta \simeq \sin\theta$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

This becomes an SHM equation.

- Example – An object thrown vertically up under the influence of gravity + drag force proportional to velocity. (it could also have been a more complicated function of v.)

$z=0$ $\uparrow$ at $t=0$, $v=u$ & $z=0$

$$\therefore m\frac{dv}{dt} = -mg - m\kappa v$$

"kappa"

Case I : $K = 0$

$$\frac{dv}{dt} = -g$$

$$\Rightarrow \int dv = -\int g\,dt$$

$$\Rightarrow v = -gt + c_1$$

at $t = 0$, $v = u$

$$\Rightarrow (v = u - gt) \quad \text{(i)}$$

At $t = t_{max}$

$v = 0$

$$\Rightarrow u = gt$$

$$\Rightarrow t_{max} = u/g \quad \Rightarrow \boxed{t_{max} = u/g}$$

From (i)

$$\frac{dz}{dt} = u - gt$$

$$\Rightarrow z = ut - \frac{gt^2}{2} + c_2$$

At $t = 0$, $z = 0$ $\therefore c_2 = 0$

$$\Rightarrow z = ut - \frac{1}{2}gt^2$$

$$\Rightarrow z_{max} = u\left(\frac{u}{g}\right) - \frac{g}{2}\left(\frac{u}{g}\right)^2$$

$$\Rightarrow \boxed{z_{max} = \frac{u^2}{2g}}$$

Case 2: $K \neq 0$

$$\frac{dv}{dt} = -(g + Kv)$$

$$\Rightarrow \int \frac{dv}{g+Kv} = \int dt$$

$$\Rightarrow \frac{1}{K} \ln(g+Kv) = -t + D_1$$

At $t=0$, $v=u$

$$\Rightarrow \frac{1}{K} \ln(g+Ku) = D_1$$

$$\Rightarrow \frac{1}{K}\left(\ln(g+Kv) - \ln(g+Ku)\right) = -t$$

$$\Rightarrow t = \frac{1}{K} \ln\left(\frac{g+Ku}{g+Kv}\right)$$

$$\Rightarrow \boxed{t = \frac{1}{K} \ln\left(\frac{g+Ku}{g+Kv}\right)}$$

At $t = t_{max}$, $v=0$

$$\Rightarrow t_{max} = \frac{1}{K} \ln\left(1 + \frac{Ku}{g}\right)$$

$$\Rightarrow \boxed{t_{max} = \frac{1}{K} \ln\left(1 + \frac{Ku}{g}\right)}$$

now

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

$$\Rightarrow \frac{v\,dv}{dz} = -(g + Kv)$$

$$\Rightarrow \frac{v\,dv}{g+Kv} = -dz$$

$$\Rightarrow \frac{1}{K}\left[1 - \frac{g}{g+Kv}\right] dv = -dz$$

$$\Rightarrow \frac{1}{K}\left[v - \frac{g}{K}\ln(g+Kv)\right] = -z + D_2$$

At $z=0$, $v=u$

$$\Rightarrow \frac{u}{K} - \frac{g}{K^2}\ln(g+Ku) = D_2$$

$$\Rightarrow \frac{u-v}{K} - \frac{g}{K^2}\ln\left(\frac{g+Ku}{g+Kv}\right) = z$$

At $v=0$, $z = z_{max}$

$$\Rightarrow z_{max} = \frac{u}{K} - \frac{g}{K^2}\ln\left(1 + \frac{Ku}{g}\right)$$

$$\Rightarrow \boxed{z_{max} = \frac{u}{K} - \frac{g}{K^2}\ln\left(1 + \frac{Ku}{g}\right)}$$

compiling all of this into a table

	$K=0$	$K\neq 0$
t_{max}	u/g	$\frac{1}{K}\ln\left(1+\frac{Ku}{g}\right)$
z_{max}	$\frac{u^2}{2g}$	$\frac{u}{K} - \frac{g}{K^2}\ln\left(1+\frac{Ku}{g}\right)$

For t_{max}, to check some kind of ordered behaviour (leading order behaviour)

$$\frac{1}{K}\ln\left(1+\frac{Ku}{g}\right) = \frac{1}{K}\left[\frac{Ku}{g} - \frac{K^2u^2}{2g^2} + \frac{K^3u^3}{3g^3} - + \dots\right]$$

$$= \frac{u}{g}\left[1 - \frac{Ku}{2g} + \frac{K^2u^2}{3g^2} - + \dots\right]$$

$\therefore$ leading order behaviour term for $K\neq 0$ t_{max}

$$\boxed{t_{max} = \frac{u}{g}\left[1 - \frac{Ku}{2g} + \frac{K^2u^2}{3g^2} - + \dots\right]}$$

imp NOTE: We use leading order behaviour to get the expression in $K_v \neq 0$ as a multiple of original expression in $K_v = 0$. eg- $\frac{u}{g}$ & $\frac{u}{g}\left[1 - \frac{K_v u}{2g} + \frac{K_v^2 u^3}{3g^2} - + \ldots\right]$

For z_{max}, to check leading order behaviour

$$\frac{u}{K_v} - \frac{g}{K_v^2}\ln\left(1 + \frac{K_v u}{g}\right) = \frac{u}{K_v} - \frac{g}{K_v^2}\left[\frac{K_v u}{g} - \frac{K_v^2 u^2}{2g^2} + - \ldots\right]$$

$$= \frac{u}{K_v} - \frac{u}{K_v} + \frac{u^2}{2g} - \frac{K_v u^3}{3g^2} + - \ldots$$

$$= \frac{u^2}{2g}\left[1 - \frac{2}{3}\frac{K_v u}{g} + \ldots\right]$$

$$\therefore \boxed{z_{max} = \frac{u^2}{2g}\left[1 - \frac{2}{3}\frac{K_v u}{g} + - \ldots\right]}$$

~~PERTURB~~

→ PERTURBATION THEORY (CLASSICAL) ⟶

- For a simple pendulum, we get eq^n

$$\ddot{\theta} + \omega_0^2 \theta = 0 \qquad \left[\text{whr. } \omega_0 \text{ (ang freq)} = \frac{g}{l}\right]$$

Amplitude $\theta = \theta_0 \overset{\sin}{\cos}(\omega_0 t + \phi)$ (i)

- Here we used to take assumption $\sin\theta = \theta$ & then we obtained above eqn. But can we do a better job?

- Lets keep an additional term for $\sin\theta$ ie $\sin\theta = \theta - \frac{\theta^3}{6}$

- ∴ we get

$$\boxed{\ddot{\theta} + \omega_0^2\theta - \omega_0^2\frac{\theta^3}{6} = 0}$$

↳ Anharmonic oscillator

RECALL: $\sin 3\omega t = 3\sin\omega t - 4\sin^3\omega t$

$$\boxed{\sin^3\omega t = (3/4)\sin\omega t - \frac{1}{4}\sin 3\omega t}$$

This eq^n is of anharmonic oscillator since anharmonic terms will be added.

- We now make "Ansatz" solution (intelligent guess)

From (i)
what we do here is that since the angle is increased, it now has a different angular freq = ω

$$\boxed{\theta(t) = \theta_0 \sin\omega t + \varepsilon\theta_0 \sin 3\omega t} \quad (i)$$

$\varepsilon \ll 1$ & $\theta_0 < 1$

To eq^n $\theta = \theta_0 \sin\omega t$, we add the above written term to solve the eq^n. In the diff. eq^n, we have term 'θ^3' ie we have term $\sin^3\theta$ ∴ we add this correction term in accordance with this

The expression for θ becomes kind of similar to that of $\sin^3\omega t$.

NOTE:

θ_0 (degree)	$\frac{T_{Anharmonic}}{2\pi\sqrt{\frac{L}{g}}}$
0	1.0000
5	1.0005
10	1.0019
15	1.0043
20	1.0077
30	1.0174

In (iii)+(iv)+(v) when we add, we get

$$\ddot{\theta} + \omega_0^2\theta - \frac{\omega_0^2\theta^3}{6} = \ldots\ldots\ldots$$

~~This must be = 0 ∴ we put coefficients~~ of terms = 0

NOTE: $O(\varepsilon)$ is called order of ε. In our calculations further on, we will neglect higher orders of ε ie $O(\varepsilon^2), O(\varepsilon^3)$ etc.

Now

$$\theta = \theta_0 \sin \omega t + \varepsilon \theta_0 \sin 3\omega t \quad (ii)$$

$$\Rightarrow \theta^3 = \theta_0^3\left[\frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t\right] + 3\varepsilon\theta_0^3 \sin^2\omega t \sin 3\omega t + O(\varepsilon^2)$$

$$\Rightarrow -\frac{\omega_0^2}{6}\theta^3 = -\frac{\omega_0^2}{6}\theta_0^3\left[\frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t\right] - \frac{\varepsilon\omega_0^2\theta_0^3}{2}\sin^2\omega t \sin 3\omega t \quad (iii) \quad + O(\varepsilon^2)$$

neglected

Multiplying (ii) $\times \omega_0^2$

$$\omega_0^2\theta = \omega_0^2\theta_0 \sin\omega t + \varepsilon\omega_0^2\theta_0 \sin 3\omega t \quad (iv)$$

Double diff. (ii)

$$\ddot{\theta} = -\omega^2\theta_0 \sin\omega t - \varepsilon 9\omega^2\theta_0 \sin 3\omega t \quad (v)$$

Now from (iii) (iv) (v)

In RHS we collect coefficients of $\sin\omega t$ upto $O(\varepsilon)$ ie we add coeff. of $\sin\omega t$ in these eqn

$$-\omega^2\theta_0 + \omega_0^2\theta_0 - \frac{1}{8}\omega_0^2\theta_0^3 = 0$$

we put their sum = 0

$$\omega^2 = \omega_0^2\left(1 - \frac{\theta_0^2}{8}\right)$$

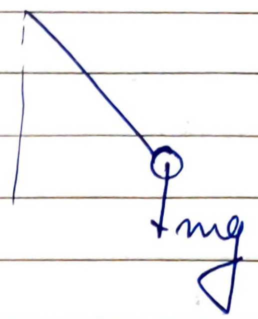
$$\omega = \omega_0\sqrt{1 - \frac{\theta_0^2}{8}}$$

Similarly, we do the same for coefficients of $\sin 3\omega t$

$\therefore$ we get $\varepsilon = \theta_0^2/192$

TUTORIAL 3

2.



$$-mg\sin\theta = mv\frac{dv}{dx}$$

$$dx = l\,d\theta$$

$$-\int_{\theta_0}^{\theta} mgl\sin\theta\,d\theta = \int_0^{v} mv\cdot dv$$

$$\Rightarrow v = \pm\sqrt{2gl(\cos\theta - \cos\theta_0)}$$

$$\cancel{T/t} \quad \int dt = \int \frac{dx}{v}$$

$$\Rightarrow T = 4\int_0^{\theta_0} \frac{l\cdot d\theta}{v} = 4\int_0^{\theta_0} \frac{l\,d\theta}{\sqrt{2gl(\cos\theta - \cos\theta_0)}}$$

Given

$$\left.\begin{aligned} &\cos\phi = 1 - 2\sin^2\phi/2 \\ &\& \ \sin x = \frac{\sin\theta/2}{\sin\theta_0/2} \end{aligned}\right] \text{use them to integrate}$$

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$$\Rightarrow T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow T = 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sin\theta_0/2 \ \sqrt{1-\sin^2 x}}$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-\sin^2\left(\frac{\theta_0}{2}\right)\sin^2 x}}$$

$$\simeq 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left[1 + \frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2 x + \ldots\right] dx$$

$$\simeq 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left[1 + \left(\frac{\theta_0}{2}\right)^2 \frac{1}{2}\sin^2 x + \ldots\right] dx$$

$\Rightarrow T$

4. $$m\frac{dv}{dt} = mg - mkv^2$$

$$v_T^2 = \frac{F}{k}$$

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_T^2}\right)$$

$$\Rightarrow t = \frac{v_T}{2g} \ln\left(\frac{v_T + v}{v_T - v}\right)$$

$$\Rightarrow \quad e^{2gt/v_T} = \frac{v_T + v}{v_T - v}$$

$$\Rightarrow \quad \frac{e^{2gt/v_T} - 1}{e^{2gt} + 1} = \frac{\cancel{2}v}{v_T}$$

$$\Rightarrow \quad v = v_T \tanh\left(\frac{gt}{v_T}\right)$$

$$\Rightarrow \quad \int_0^H \frac{dx}{\cancel{dt}} = v_T \int_0^t \tanh\left(\frac{gt}{v_T}\right) dt$$

$$\Rightarrow \quad H = \frac{v_T^2}{g} \ln\left[\cosh\left(\frac{gt}{v_T}\right)\right]_0^t$$

5. $\vec{F} = m\vec{a}$

a) $\ddot{x} = -\alpha \dot{x}$

$\ddot{y} = -g - \alpha \dot{y}$

$x(t) = \left(\frac{v_0 \cos\theta}{\alpha}\right)\left[1 - e^{-\alpha t}\right]$

$y(t) = \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(1 - e^{-\alpha t}\right) - \frac{gt}{\alpha}$

b) $m\alpha v_0 = mg$

$\dot{y} = (v_0 \sin\theta + v_0)\, e^{-\alpha t} - v_0$

At highest point $\dot{y} = 0$, $t = t_p$

$e^{-\alpha t_p} = \frac{1}{1 + \sin\theta}$

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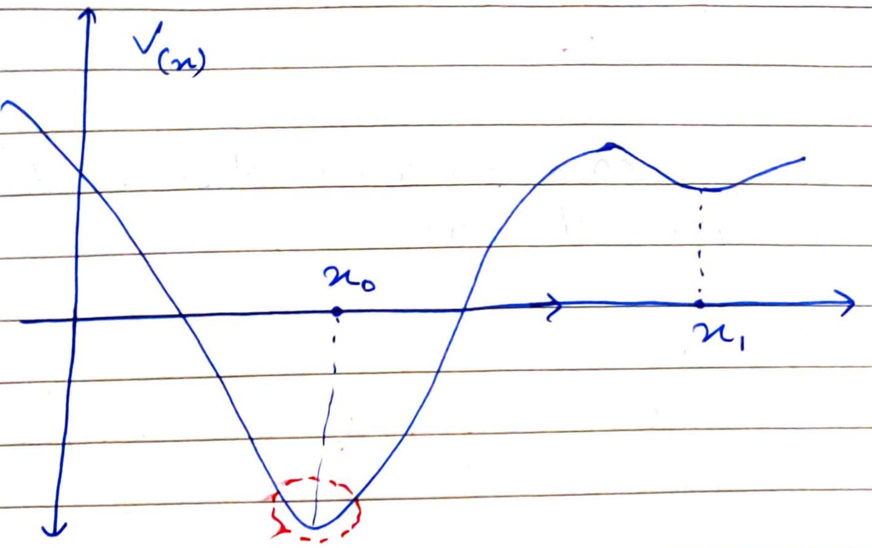
TAYLOR SERIES

consider



We are trying to define a function which is valid in encircled neighbourhood

Look at the note on next page

$$\boxed{f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k}$$ (in neighbourhood of $x=x_0$)

$\Rightarrow f(x) = a_0(x-x_0)^0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + \dots\dots$

We observe

$$\boxed{f(x_0) = a_0}$$

now

$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots\dots$

$$\boxed{f'(x_0) = a_1}$$

now

$f''(x) = 2a_2 + 3!\, a_3(x-x_0) + \dots\dots$

$\Rightarrow \boxed{f''(x_0) = 2a_2}$

Similarly

$\Rightarrow \boxed{f'''(x_0) = 3!\, a_3}$

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$$\Rightarrow \boxed{a_k = \frac{1}{k!} f^{(k)}(x_0)}$$

$\therefore$ we can finally define the function required as

$$\boxed{f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k} \qquad (i)$$

with $\boxed{a_k = \frac{1}{k!} f^{(k)}(x_0)}$

NOTE: Any differentiable function which we may draw can be expressed as a linear combination of various polynomial functions infinitely.

eg- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm - - - -$

ie $\sin x$ can be expressed as a linear combination of polynomials.

This is what we are trying to do here for the given neighbourhood.

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Now

$V'(x_0) = 0$

$V''(x_0) > 0$

$\therefore$ From (i) defining Taylor series

$$V(x) = V(x_0) + \underbrace{(x-x_0)\,V'(x_0)}_{=0} + V''(x_0)\,\frac{(x-x_0)^2}{2}$$

$$+ V'''(x_0)\,\frac{(x-x_0)^3}{3!} \;\; \text{------} \rightarrow \text{very very small}$$

$\therefore$ we have final expression

$$\boxed{V(x) = V(x_0) + V''(x_0)\,\frac{(x-x_0)^2}{2}}$$

Note that $V''(x_0)\,\frac{(x-x_0)^2}{2}$ is similar to potential for spring $= \frac{1}{2}kx^2$ & $V(x_0)$ is reference potential

$\therefore$ we can say, $V''(x_0)$ is spring constant

$$\therefore \quad \omega = \sqrt{\frac{V''(x_0)}{m}}$$

Examples →

a) $V(x) = \dfrac{A}{x^3} - \dfrac{B}{x^2}$; $A = u_0 a_0^3$, $B = u_0 a_0^2$

Find ω near minima. given μ = reduced mass

Ans. $V'(x) = 0 = -\dfrac{3A}{x_0^4} + \dfrac{2B}{x_0^3}$

$\Rightarrow \dfrac{3A}{x_0} = +2B$

$\Rightarrow x_0 = +\dfrac{3A}{2B}$

$\Rightarrow x_0 = \dfrac{3}{2} a_0$

$V''(x) = +\dfrac{12A}{x^5} - \dfrac{6B}{x^4}$

$V''(x_0) = \dfrac{12A}{x_0^5} - \dfrac{6B}{x_0^4}$

$\Rightarrow V''(x_0) = \dfrac{6 u_0 a_0^2}{x_0^4}\left(\dfrac{2a_0}{x_0} - 1\right)$

$\Rightarrow V''(x_0) = \dfrac{16 \times 6\, u_0\, \cancel{a_0^2}}{81\, \cancel{a_0^4}\, a_0^2}\left(\dfrac{4\cancel{a_0}}{3\cancel{a_0}} - 1\right)$

$\Rightarrow V''(x_0) = \dfrac{32}{27}\dfrac{u_0}{a_0^2}\left(\dfrac{1}{3}\right) = \dfrac{32}{81}\dfrac{u_0}{a_0^2}$

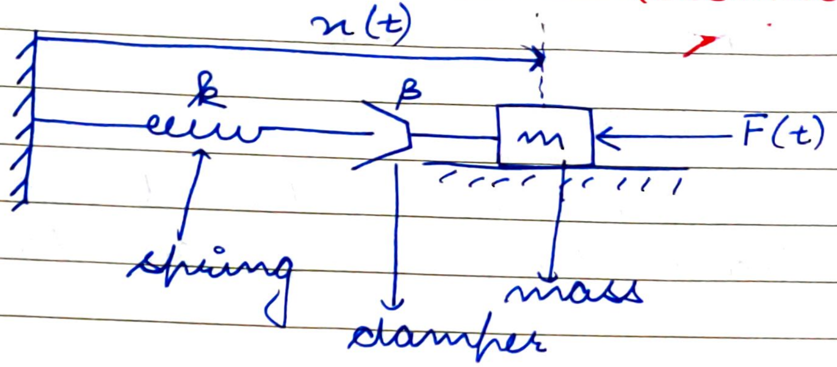
$$\omega = \sqrt{\dfrac{32\, u_0}{81 a_0^2 \times \mu}}$$

Lennard - Jones potential

$$V(r_{ij}) = u_0 \left[\frac{r_m^{12}}{r_{ij}^{12}} - \frac{1.2 \, 2 \, r_m^6}{r_{ij}^6} \right] \quad ; \quad r_{ij} = |\vec{r_i} - \vec{r_j}|$$

Find ω ; given μ = reduced mass of system in potential $V(r_{ij})$ & r_m = constant

* → FORCED DAMPED OSCILLATIONS →
(OSCILLATORS)



Writing the equation

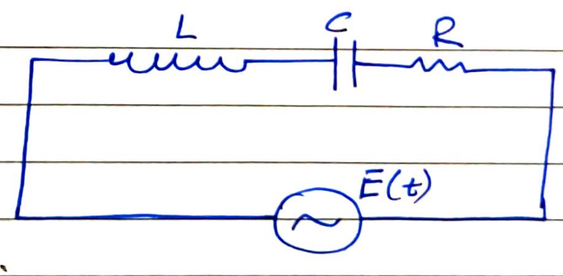
$\cancel{m\ddot{x} + \beta \frac{dx}{dt}}$

$$m\ddot{x} = -\beta\frac{dx}{dt} - kx + F(t)$$

$$\Rightarrow \boxed{\ddot{x} = -2\Gamma\frac{dx}{dt} - \omega_0^2 x + F_0(t)} \qquad (i)$$

where $2\Gamma = \frac{\beta}{m}$; $\omega_0^2 = \frac{k}{m}$; $F_0(t) = \frac{F(t)}{m}$
gamma

Now remember LCR circuit



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t) \qquad (ii)$$

Now equation (ii) is comparable to (i)

$$F_0(t) = \frac{E(t)}{L} \ ; \quad 2\Gamma = \frac{R}{L} \ ; \quad \omega_0^2 = \frac{1}{LC}$$

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NOTE: SHORT PRIMER ON HOMOGENOUS LINEAR SECOND ORDER DIFFERENTIAL EQUATION

HOMOGENEOUS $\Longleftrightarrow$ $F_0(t) = 0$

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = 0$$

LINEAR $\Rightarrow$ If $x_1(t)$ and $x_2(t)$ are solutions then $C_1 x_1(t) + C_2 x_2(t)$ is also a solution
C_1, C_2 are constants.

Using ANSATZ (intelligent guess)

$x = e^{\lambda t}$ $\longrightarrow$ "Irreducible"

putting this in eqn

$$(\lambda^2 + 2\Gamma\lambda + \omega_0^2)\, e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 + 2\Gamma\lambda + \omega_0^2 = 0$$

$$\Rightarrow \lambda = \frac{-2\Gamma \pm \sqrt{4\Gamma^2 - 4\omega_0^2}}{2}$$

$$\Rightarrow \lambda = -\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2}$$

$\therefore$ we have 2 values of λ $\therefore$ the solution will be linear combination of $x(t)$ for 2 lambda

$$\therefore \boxed{x(t) = c_1 e^{\left(-\Gamma + \sqrt{\Gamma^2 - \omega_0^2}\right)t} + c_2 e^{\left(-\Gamma - \sqrt{\Gamma^2 - \omega_0^2}\right)t}}$$

This is called complimentary function

Coming back to original eq^n (i)

- Case I : $F(t)=0$; $\omega_0 > \Gamma$ (underdamped) ; $\Omega_D^2 = \omega_0^2 - \Gamma^2$

$$x(t) = e^{-\Gamma t}\left(\underbrace{c_1}_{\frac{C-iD}{2}} e^{i\Omega_D t} + \underbrace{c_2}_{\frac{C+iD}{2}} e^{-i\Omega_D t}\right)$$

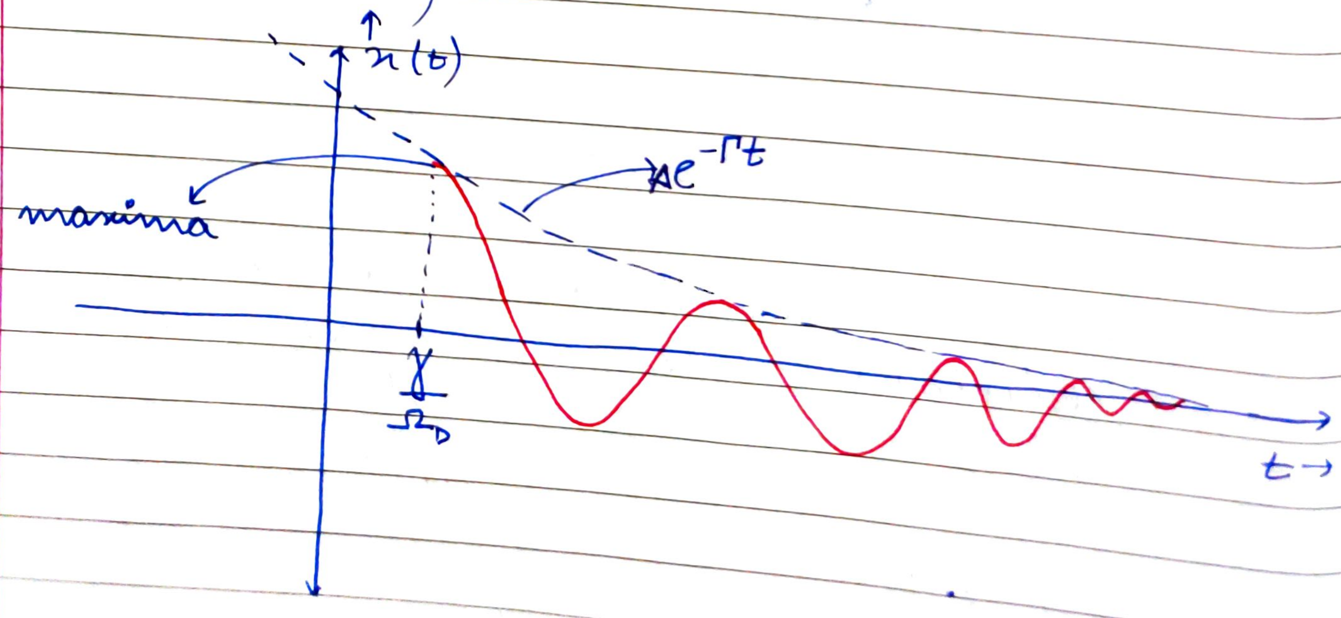
$$\Rightarrow \boxed{x(t) = e^{-\Gamma t}\left(C\cos\Omega_D t + D\sin\Omega_D t\right)}$$

$$\Rightarrow \boxed{x(t) = Ae^{-\Gamma t}\cos\left(\Omega_D t - \gamma\right)} \quad (iii)$$

where $C = A\cos\gamma$ & $D = A\sin\gamma$

$$\Rightarrow \boxed{\gamma = \tan^{-1}\left(\frac{D}{C}\right)}$$

now we plot (iii)

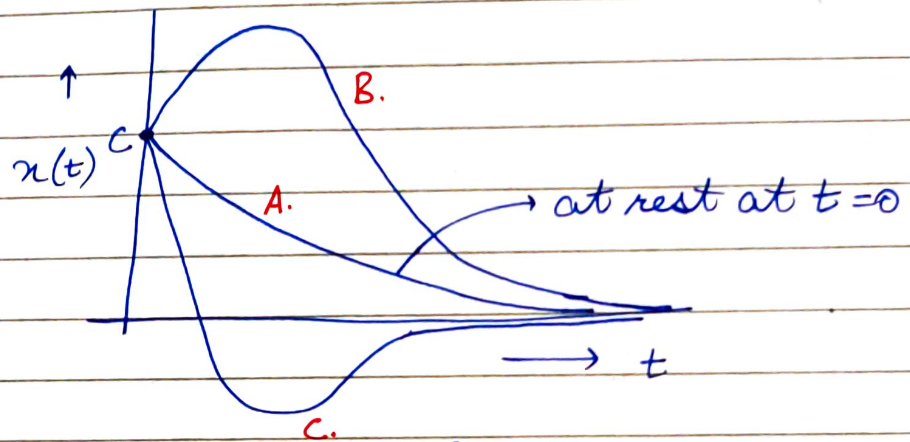


- Case II: $\Gamma > \omega_0$ (overdamped) ; $F_0(t) = 0$

$$\lambda = -\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2}$$

$$= -\Gamma \pm \delta_0 \quad ; \quad \delta_0 = \sqrt{\Gamma^2 - \omega_0^2} > 0$$

$$\boxed{x(t) = e^{-\Gamma t}\left(Ae^{\delta_0 t} + Be^{-\delta_0 t}\right)}$$



NOTE: This case is when damping is too high. In this case there will be no oscillations practically (theoretically there should be some) ie its unobservable. eg- if a spring mass system is kept in a very viscous fluid, say, honey.

(A) ~~x(t)=0~~

There are 3 cases in this case

A. $x(t) = c$; $\dot{x}(t) = 0$
B. $x(t) = c$; $\dot{x}(t) > 0$
C. $x(t) = c$; $\dot{x}(t) < 0$

(Look at graphs above)

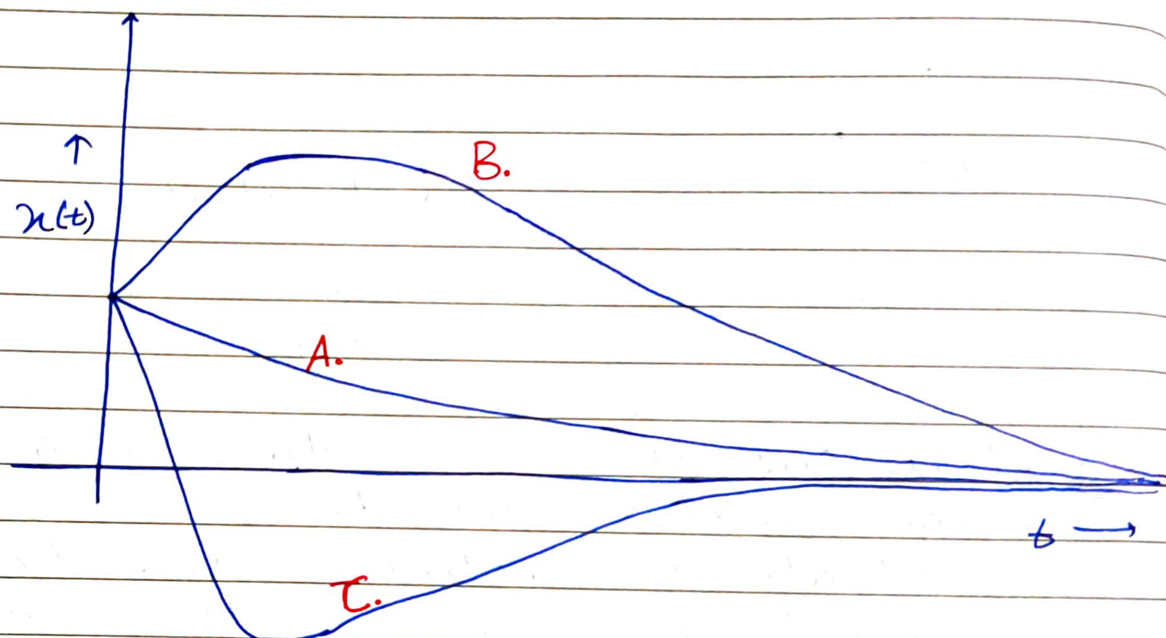
NOTE: At max, there will only be one overshooting of natural length whatsoever, practically.

- **Case III**: Critical damping $\Gamma = \omega_0$

$$\lambda = -\Gamma$$

$$\cancel{x(t) = C e^{-\Gamma t}}$$

Again we have previously mentioned 3 cases: A., B. & C.



The solution for this ~~equation~~ case is

$$\boxed{x(t) = e^{-\Gamma t}(A + Bt)}$$

NOTE: The solution has extra 'Bt' term otherwise it isn't valid.

imp

NOTE: The graph for critical damping is stretched in length as compared to over-damping inspite of similar shapes ie damping is faster in over damping

• Case IV : Forced (driven) damped oscillation
$F_0(t) \neq 0$

PI = Particular Integral ; CF = Complimentary function

$$\boxed{x(t) = \underbrace{x_{CF}(t)}_{\text{solution for homogenous part}} + x_{PI}(t)}$$

now we write original eqn

$$\boxed{\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = f\cos(\omega_f t)}$$

→ we choose $F_0(t)$ ~~F0t~~ $f\cos(\omega_f t)$ since the force provided is periodic in nature

we convert this to complex form ~~part~~

$$\ddot{z} + 2\Gamma\dot{z} + \omega_0^2 z = f e^{i\omega_f t} \quad ; \quad x_{PI}(t) = Re(z(t))$$

Let $z(t) = a e^{i(\omega_f t - \theta)}$

$a\cos(\omega_f t - \theta)$

$$\Rightarrow \left[(\omega_0^2 - \omega_f^2) + 2i\Gamma\omega_f\right] a e^{i(\omega_f t - \theta)} = f e^{i\omega_f t}$$

$\therefore$ ~~α~~ $\therefore \; a = \dfrac{f e^{i\theta}}{(\omega_0^2 - \omega_f^2) + 2i\Gamma\omega_f}$

$$\Rightarrow \boxed{|a| = \frac{f}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4\Gamma^2\omega_f^2}}} \; ; \; \boxed{\theta = \tan^{-1}\left(\frac{2\Gamma\omega_f}{\omega_0^2 - \omega_f^2}\right)}$$

At $\omega_0 = \omega_f$ phase of tan changes from $\pi/2$ to $-\pi/2$ ie value of tan shifts from ∞ to $-\infty$

Q→ $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 10\cos t$

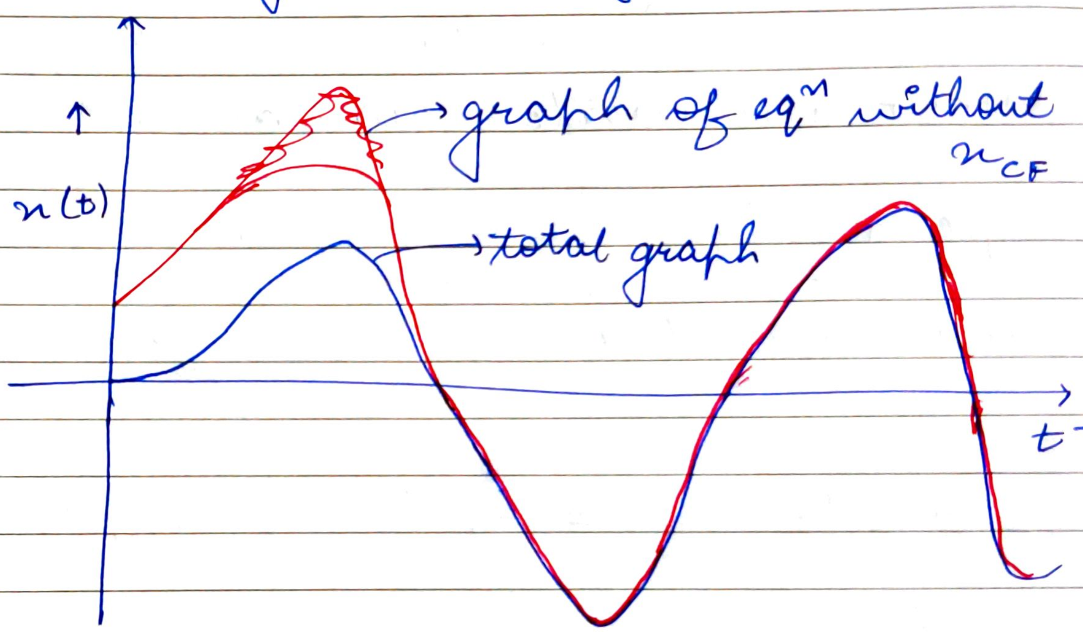
at $t=0$

~~Solve~~

$\dot{x}(t)$ & $x(t) = 0$

Ans: $x(t) = 5e^{-t} + 4e^{-2t} + \cos t + 3\sin t$

we actually see its graph



NOTE: ∴ At larger ~~time~~ time, the x_{CF} part doesn't matter and net equation matches with $x_{P.I}$ part.

classmate

Date ______

Page ______

TUTORIAL 4

1. $TE = KE + PE$

$\Rightarrow$ ~~E~~ ~~TE~~ $U = \frac{1}{2}m(\dot{x})^2 + P$

$F = -kx - \beta\dot{x}$

$\int_x^0 F\,dx = +k\int_x^0 x\,dx + \beta\int_x^0 \dot{x}\,dx \rightarrow 0$

since for calculating PE, the object is moved slow i.e. $v = 0$

$\int_0^x F\,dx = k\int_0^x x\,dx$

$\Rightarrow P = \frac{kx^2}{2}$

$\Rightarrow P = \frac{1}{2}kx^2$

$\Rightarrow \frac{dP}{dt} = kx\dot{x}$

$\Rightarrow \frac{dP}{dt} = kx\dot{x}$

$\frac{dU}{dt} = m\dot{x}\ddot{x} + kx\dot{x}$

$\frac{dU}{dt} = -\beta(\dot{x})^2$

$\Rightarrow \frac{dU}{dt} = -2m\Gamma(\dot{x})^2$

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Date
Page

2. $F = -kx - \beta \dot{x}$

Damping force $= -\beta \dot{x}$

$$\text{Work done} = -\beta \int_{x_0}^{0} \dot{x}\, dx$$

$$\Rightarrow -\beta \int_{x_0}^{0} \dot{x}\, dx = m \int \ddot{x}\, dx + k \int_{x_0}^{0} x\, dx$$

$$\Rightarrow -\beta \int_{x_0}^{0} \dot{x}\, dx = m \int_{v_0}^{0} v\, dv + k \int_{x_0}^{0} x\, dx$$

$$= \frac{mv^2}{2}\Big|_{v_0}^{0} + \frac{kx^2}{2}\Big|_{x_0}^{0}$$

$$\Rightarrow W = -\frac{mv_0^2}{2} - \frac{1}{2} k x_0^2$$

3. a) Overdamped

$$x = e^{-\Gamma t}\left(A e^{\delta_0 t} + B e^{-\delta_0 t}\right)$$

$$\Rightarrow 0 = e^{-\Gamma t}\left(A e^{\delta_0 t} + B e^{-\delta_0 t}\right)$$

$$\Rightarrow A e^{2\delta_0 t} + B = 0$$

$$\Rightarrow e^{2\delta_0 t} = \frac{-B}{A}$$

$$\Rightarrow 2\delta_0 t = \ln(-B/A)$$

$$\Rightarrow t = \frac{1}{2\delta_0} \ln(-B/A)$$

Q.5

∴ only one solution

similarly

$$x = e^{-\Gamma t}(A+Bt) = 0$$

$$\Rightarrow A+Bt = 0$$

$$\Rightarrow t = -A/B$$

b) $\boxed{A+Bt>0}$

$$\dot{x} = -\Gamma e^{-\Gamma t}(A+Bt) + Be^{-\Gamma t}$$

$$\dot{x} = e^{-\Gamma t}\left(B - \Gamma(A+Bt)\right)$$

$$\Rightarrow \dot{x} = \left(Be^{-\Gamma t} - \Gamma x\right)$$

At $x=0$, at max velocity case, v must be $=0$

$$\Rightarrow \dot{x} = \left(Be^{-\Gamma t} - \Gamma 0\right) = 0$$

$$\Rightarrow \boxed{B=0}$$

∴ velocity function is

$$v = -\Gamma x$$

∴ At x_0, max velocity to prevent crossing the origin

$$v = -\Gamma x_0$$

* → COUPLED PENDULUM →

- EIGEN VALUES AND VECTORS

Suppose we have a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then

λ_i are eigen values and v_i are eigen vectors if

$$\boxed{M v_i = \lambda_i v_i}$$

NOTE: v_i is basically a column matrix.

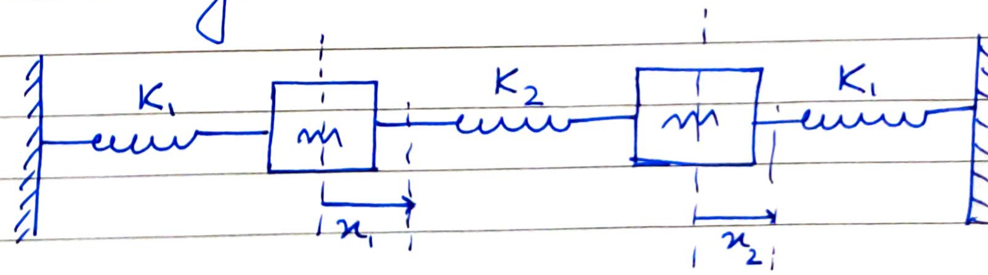
eg-

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{v_i} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{v_i}$$

$\therefore$ For M and eigen vector v_i, eigen value is 1.

- Consider system

+ve direction →



At time t, let displacements be $= x_1$ & x_2

NOTE: remember, in $\ddot{x} = -\frac{k}{m}x$

$\frac{k}{m} = \omega^2 = (\text{ang freq})^2$

$$\therefore \quad m\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$$

&

$$m\ddot{x}_2 = -k_1 x_2 - k_2(x_2 - x_1)$$

$$\therefore \quad m\ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2 \qquad (i)$$

$$m\ddot{x}_2 = -(k_1+k_2)x_2 + k_2 x_1 \qquad (ii)$$

On adding (i) & (ii), surprisingly the eq^n decouples easily in this case

$$(\ddot{x_1 + x_2}) = -\frac{k_1}{m}(x_1 + x_2)$$

On subtracting (ii) from (i)

$$(\ddot{x_1 - x_2}) = -\frac{(k_1 + 2k_2)}{m}(x_1 - x_2)$$

$\therefore$ we can say that $(x_1 + x_2)$ and $(x_1 - x_2)$ are oscillating with angular frequencies

$$\sqrt{\frac{k_1}{m}} \quad \& \quad \sqrt{\frac{k_1 + 2k_2}{m}}$$

$\therefore$ $(x_1 + x_2)$ & $(x_1 - x_2)$ are normal mode coordinates.

Now we can write eq^n (i) & (ii) in matrix form as

PTO

NOTE: '$\leftrightarrow$' on top of symbol denotes a tensor: a higher-dimensional quantity (here matrix)

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_1+K_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Writing this in matrix form

$$\boxed{\overleftrightarrow{M}\ddot{\vec{X}} = -\overleftrightarrow{K}\vec{X}}$$

$$\Rightarrow \boxed{\ddot{\vec{X}} = -\overleftrightarrow{M}^{-1}\overleftrightarrow{K}\vec{X}} \qquad \text{(iii)}$$

Taking analogy of a typical spring mass system (eg- $\ddot{x} = -\omega^2 x$)

$$\ddot{\vec{X}} = -\omega^2 \overleftrightarrow{\mathbb{1}}\vec{X} \qquad \text{(iv)} \qquad (\overleftrightarrow{\mathbb{1}} = \text{Identity matrix})$$

Comparing (iii) & (iv)

$$\boxed{\overleftrightarrow{M}^{-1}\overleftrightarrow{K} = \omega^2 \overleftrightarrow{\mathbb{1}}} \qquad \text{(v)}$$

$$\det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2\overleftrightarrow{\mathbb{1}}\right| = 0$$

$\Downarrow$ (Eigenvalue eqn)

Equation for obtaining ω^2

PTO

To determine ω, we use $\boxed{\det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \mathbb{1}\right| = 0}$

NOTE: The actual condition is

$$\overleftrightarrow{M}^{-1}\overleftrightarrow{K}\vec{X} = \omega^2 \mathbb{1}\vec{X}$$

we can't directly remove $\vec{X}$

$\therefore$ we can't actually write

$$\overleftrightarrow{M}^{-1}\overleftrightarrow{K}\vec{X} = \omega^2 \mathbb{1}$$

but for non-trivial solutions of $\vec{X}$, we can however write

$$\det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \mathbb{1}\right| = 0$$

$\therefore$ use $\det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \mathbb{1}\right| = 0$

$$\overleftrightarrow{M}^{-1}\overleftrightarrow{K} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \begin{pmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_1+K_2 \end{pmatrix}$$

$$\Rightarrow \quad = \begin{pmatrix} \frac{K_1+K_2}{m} & -\frac{K_2}{m} \\ -\frac{K_2}{m} & \frac{K_1+K_2}{m} \end{pmatrix}$$

$$\Rightarrow \det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \mathbb{1}\right| = \left(\frac{K_1+K_2}{m} - \omega^2\right)^2 - \left(\frac{K_2}{m}\right)^2 = 0$$

PTO

$$\Rightarrow \left(\frac{K_1+K_2}{m}-\omega^2\right)^2-\left(\frac{k_2}{m}\right)^2=0$$

$$\Rightarrow \frac{K_1+K_2}{m}-\omega^2 = \pm\frac{k_2}{m}$$

We obtain two solutions

$$\boxed{\omega_A^2=\frac{k_1}{m} \quad ; \quad \omega_B^2=\frac{k_1+2k_2}{m}}$$

(EIGEN VALUES FOR $\overleftrightarrow{M}^{-1}\mathbb{K}$)

Now we must also have an $\vec{X}_A$ & $\vec{X}_B$ for which ω_A & ω_B are respectively applicable.

For this

We take Ansatz

$$\vec{X}=\vec{X}_0\, e^{i\omega t}$$

Case A:

$$\omega^2=\omega_A^2=\frac{K_1}{m} \quad ; \quad \vec{X}_{0A}=\begin{pmatrix} c \\ d \end{pmatrix}$$

Let $\begin{pmatrix} c \\ d \end{pmatrix}$ be an eigen vector

$$\Rightarrow \begin{pmatrix} \frac{K_1+K_2}{m} & \frac{-K_2}{m} \\ \frac{-K_2}{m} & \frac{K_1+K_2}{m} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}=\frac{K_1}{m}\begin{pmatrix} c \\ d \end{pmatrix}$$

NOTE: $\vec{X}_{OA}$ actually denotes ratio of 2 masses and their relative direction of motion.

$$\Rightarrow \left(\frac{K_1+K_2}{m}\right)c - \frac{K_2}{m}d = \frac{K_1}{m}c$$

$$\therefore \Rightarrow c = d$$

$$\therefore \vec{X}_{OA} = \begin{pmatrix} c \\ c \end{pmatrix}$$

$$\vec{X}_{OA}^{+} = \begin{pmatrix} c^* & c^* \end{pmatrix} \longrightarrow$$ transpose of $\vec{X}_{OA}$ and with conjugate elements

$$\vec{X}_{OA}^{+} X_{OA} = 1$$

$$\Rightarrow 2|c|^2 = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{2}}$$

this is a^2+b^2

pronounced as (ket - X_p)

$$\therefore \boxed{\vec{X}_{OA} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} \to a \\ \to b \end{matrix} \equiv |X_{OA}\rangle = |X_p\rangle}$$

pendulum like motion

$$\boxed{\begin{aligned} \langle X_p| &= \left(|X_p\rangle\right)^{+} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \end{aligned}}$$

This actually means if mass 1 moves +1 unit then mass 2 moves +1 unit & also that the 2 masses are same

PTO

eg- $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ would mean $m_2 = 3m$, & that both move in opposite direction ie breathing mode (look at next page)

Case B:

$$\omega^2 = \omega_B^2 = \frac{K_1 + 2K_2}{m}$$

Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be eigen vector

$$\Rightarrow \begin{pmatrix} \frac{K_1+K_2}{m} & -\frac{k_2}{m} \\ -\frac{K_2}{m} & \frac{K_1+K_2}{m} \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \frac{K_1+2K_2}{m}$$

we get on comparing

$$\left(\frac{K_1+K_2}{m}\right) e - \frac{K_2}{m} f = \frac{K_1+2K_2}{m} e$$

$$\Rightarrow e = -f$$

$\therefore$ Normalised

$$\boxed{\vec{X}_{0B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv |X_{0B}\rangle \equiv |X_b\rangle}$$

breathing like

$$\boxed{\langle X_b| = (|X_b\rangle)^{+} = \frac{1}{\sqrt{2}} (1 \ \ -1)}$$

warning

NOTE: Pendulum mode :- when 2 masses move in same direction.

Breathing mode :- when the 2 masses move in opposite direction.

compiling the solutions

$\omega_A^2 = \frac{K_1}{m} \equiv \omega_p^2$ | $\omega_B^2 = \frac{K_1 + 2K_2}{m} \equiv \omega_b^2$

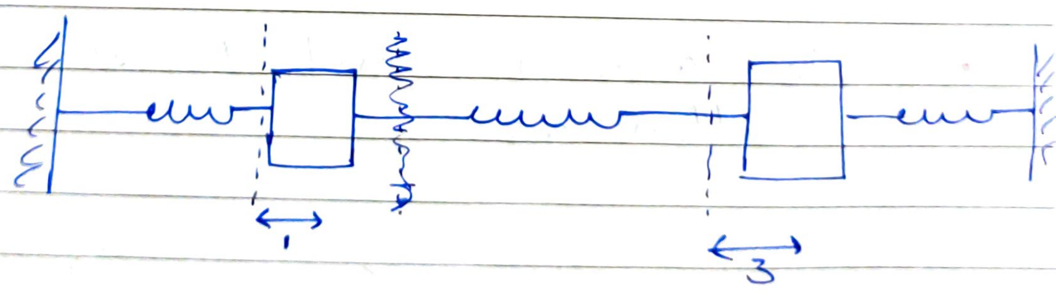
$|X_{A0}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \equiv |X_p\rangle$ | $|X_{B0}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |X_b\rangle$

we can write $|X\rangle$ $\therefore$ as:-

$$\boxed{|X\rangle = C_p e^{i\omega_p t} |X_p\rangle + C_b e^{i\omega_b t} |X_b\rangle}$$

NOTE: Now observe carefully. In above equation we have a pendulum mode part & a breathing part. $\therefore$ any position of the 2 masses can be defined.

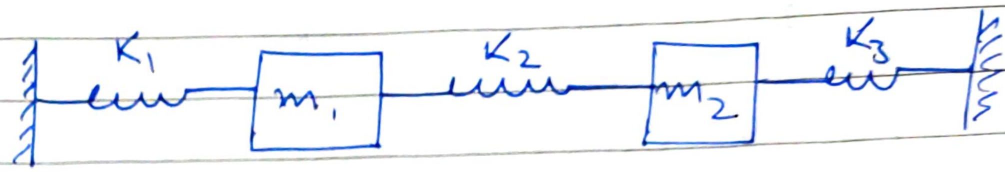
Consider



Now above case can be described as

pendulum part = 2
& breathing part = 1

- now consider the situation



NOTE: For 2 different masses, we would have,

$$\frac{1}{\sqrt{m_1^2+m_2^2}}\begin{pmatrix}m_1\\ m_2\end{pmatrix}$$ for pendulum mode

and $$\frac{1}{\sqrt{m_1^2+m_2^2}}\begin{pmatrix}m_1\\ -m_2\end{pmatrix}$$ for breathing mode

NOTE: For 3 masses our $\vec{X}_{OA}$ would be of form

$$\begin{pmatrix}+\\ +\\ +\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}+\\ +\\ -\end{pmatrix}$$ & all combinations possible & would give us all the modes

- We have a completeness relation written as

$$\boxed{\sum_i \underbrace{|x_i\rangle\langle x_i|}_{\text{projection operator}} = \overleftrightarrow{\mathbb{1}}}$$

valid for coupled pendulum like motion

We can check this in our calculated case

$$|X_p\rangle\langle X_p| + |X_b\rangle\langle X_b|$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{1}$$

$\therefore$ it is valid.

NOTE:

$$\langle X_p|X_b\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\times 0 = 0$$

indicates dot product.

$$\langle X_b|X_b\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \mathbb{1} = \langle X_p|X_p\rangle$$

$\therefore$ We can imagine $|X_p\rangle$ & $|X_b\rangle$ as vectors. They are $\perp$ & their dot product is $= 0$

Q→ $|X_b\rangle = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$; Find $|X_b\rangle$

Ans: We have two conditions

1.) dot product must be zero

2.) $\sum_i |X_i\rangle\langle X_i| = \overleftrightarrow{\mathbb{1}}$

$\therefore$

$\langle X_p|X_b\rangle = 0$

$\Rightarrow \frac{1}{4}\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{4}(\sqrt{3}a+b)$

$\therefore \sqrt{3}a+b=0$

& $\frac{1}{4}\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}a & b\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}\begin{pmatrix}\sqrt{3} & 1\end{pmatrix} = \overleftrightarrow{\mathbb{1}}$

$\Rightarrow \frac{1}{4}\begin{pmatrix}a^2 & ab\\ab & b^2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}3 & \sqrt{3}\\\sqrt{3} & 1\end{pmatrix} = \overleftrightarrow{\mathbb{1}}$

$\Rightarrow \frac{1}{4}\begin{pmatrix}3+a^2 & ab+\sqrt{3}\\ab+\sqrt{3} & b^2+1\end{pmatrix} = \mathbb{1}$

$\Rightarrow a = \pm 1$ & $b = \mp\sqrt{3}$

$\therefore$ since cases would actually represent same physical situation

$\therefore$ take $a=1$ & $b=-\sqrt{3}$

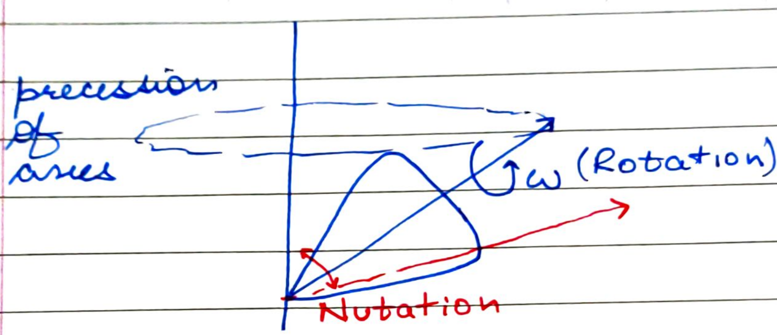
$\therefore |X_b\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

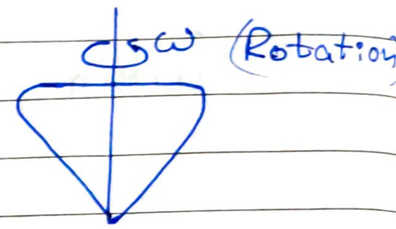
NOTE: The fact that dot product of $|X_p\rangle$ and $|X_b\rangle$ is zero signifies that the two vectors are independent of each other. eg- like $\hat{i}$ & $\hat{j}$ are independent ie anything along $\hat{i}$ can't be represented along $\hat{j}$

Chapter
RIGID BODY MOTION

→ Primer : Rotational frames

- Consider the case of लट्टू (spinning top)





In this case we would have three ω's for
a) Rotation
b) precession
c) nutation

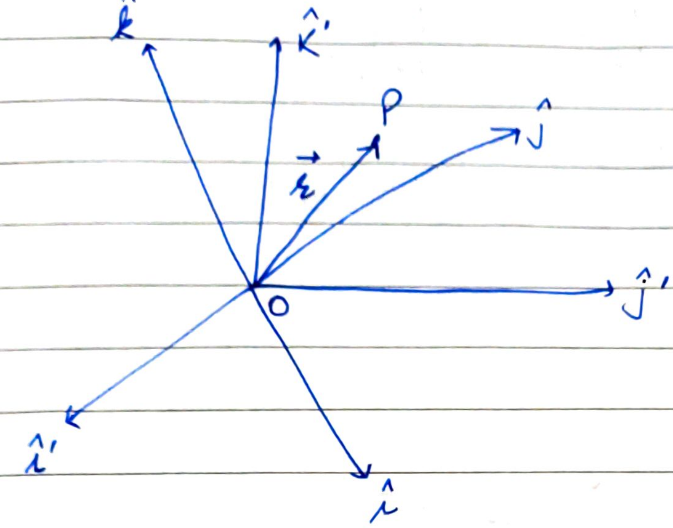
- We will develop equations to solve these 3 ω's in this chapter : OUR MOTIVE

→ • Firstly, we will

Obtain relationship between fixed frame and rotational frame of reference

GOALS
• EULER EQUATIONS
• PSEUDO FORCES

For this we consider 2 frames



$\hat{i}', \hat{j}', \hat{k}'$ - fixed frame of reference
(primed frame)

$\hat{i}, \hat{j}, \hat{k}$ - rotating frame of reference
(fixed w.r.t. the body)
(unprimed frame)

$\cancel{\hat{r}}$ $\quad \vec{r} = x'\hat{i}' + y'\hat{j}' + z\hat{k}'$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

now

$$x' = \vec{r}\cdot\hat{i}' = \underbrace{x\,\hat{i}\cdot\hat{i}' + y\,\hat{j}\cdot\hat{i}' + z\,\hat{k}\cdot\hat{i}'}_{\text{direction cosines}}$$

$$y' = \vec{r}\cdot\hat{j}'$$

$$z = \vec{r}\cdot\hat{k}'$$

NOTE: we will use notations

$$\left(\frac{d}{dt}\right)' \equiv \left(\frac{d}{dt}\right)_{fin}$$

$$\left(\frac{d}{dt}\right) \equiv \left(\frac{d}{dt}\right)_{rot}.$$

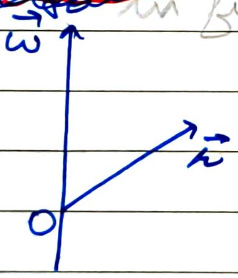
---

now

$$\boxed{\left(\frac{d}{dt}\right)_{fin}\vec{r} = \left(\frac{d\vec{r}}{dt}\right)_{fin} = \overbrace{\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}}^{(dr/dt)_{rot}} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}} \quad -(i)$$

we get these 3 extra terms since in fin frame of reference $\hat{i}, \hat{j}, \hat{k}$ are not fixed

Also



$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \Rightarrow \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \omega \times (\hat{i}x + \hat{j}y + z\hat{k})$$

$$\Rightarrow \boxed{\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}\ ;\ \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}\ ;\ \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}} \quad (ii)$$

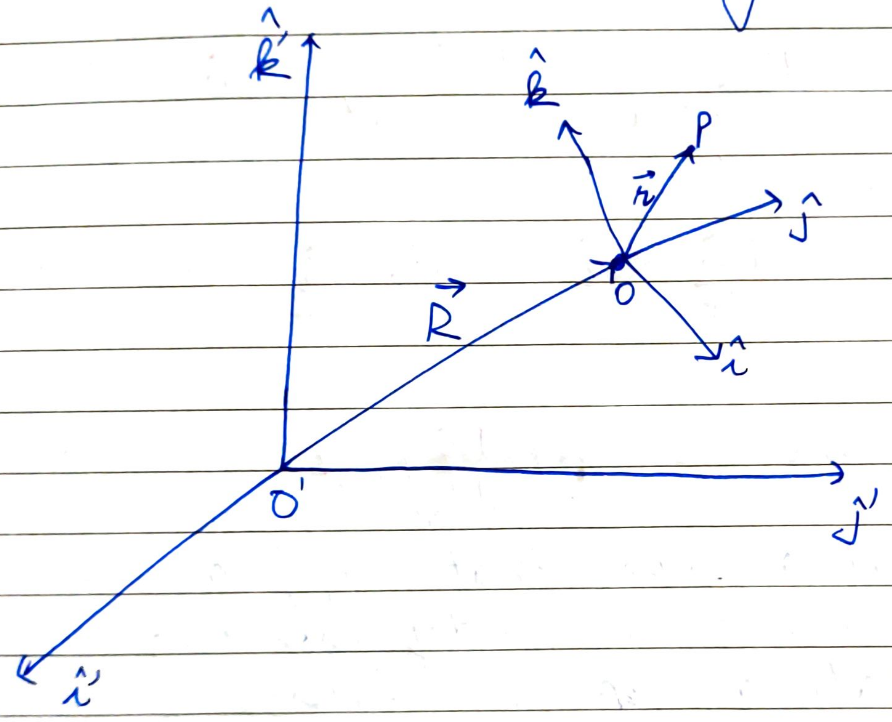
using (ii) in (i)

$$\Rightarrow \left(\frac{d\vec{r}}{dt}\right)_{fin} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

$$\Rightarrow \boxed{\left(\frac{d(\,)}{dt}\right)_{fin} = \left(\frac{d(\,)}{dt}\right)_{rot} + \vec{\omega} \times (\,)}$$

NOTE: The above equation is valid for all vectors considered. i.e ( ) the brackets can have any vector.

- now we will obtain a relation if the primed and unprimed frame don't have coinciding origins.



For the new point P we can write

$$\vec{r}\,' = \vec{R} + \vec{r}$$

$$\Rightarrow \left(\frac{d\vec{r}\,'}{dt}\right)_{fin} = \left(\frac{d\vec{R}}{dt}\right)_{fin} + \left(\frac{d\vec{r}}{dt}\right)_{fin}$$

$$\Rightarrow \left(\frac{d\vec{R}}{dt}\right)_{fin} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

Similarly,

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{fin} = \left(\frac{d^2R}{dt^2}\right)_{fin} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot}$$

$$+ \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot}$$

$$+ \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$\boxed{\left(\frac{d^2\vec{r}}{dt^2}\right)_{fin} = \left(\frac{d^2R}{dt^2}\right)_{fin} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega} \times \hat{r}}$$

NOTE: The term $2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot}$ is basically an extra force that appears when we observe a body moving in a rotating frame from outside.

If we look at, say, rivers from space, we will observe they have a force acting on them called the coriolis force. arising out of the acceleration term $2\vec{\omega} \times \left(\frac{dr}{dt}\right)_{rot}$

using subscripts f & r for fixed & rotating frames respectively.

$\cancel{\dot{r}_{ff} = (\dot{\vec{R}}_f + \dot{r}_{r}) + (\vec{\omega} \times \vec{r})}$

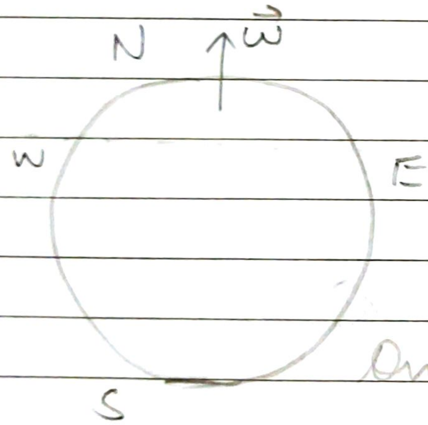
$$\dot{r}_f = \dot{\vec{R}}_f + \dot{\vec{r}}_r + \vec{\omega} \times \vec{r}$$

CENTRIPETAL ACCELRATION

$$\boxed{\ddot{\vec{r}}_f = \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times \vec{\omega} \times \vec{r} + \dot{\vec{\omega}} \times \vec{r}}$$

angular acc of particle due to acc. of rotatory frame

CORIOLIS FORCE ACCELERATION

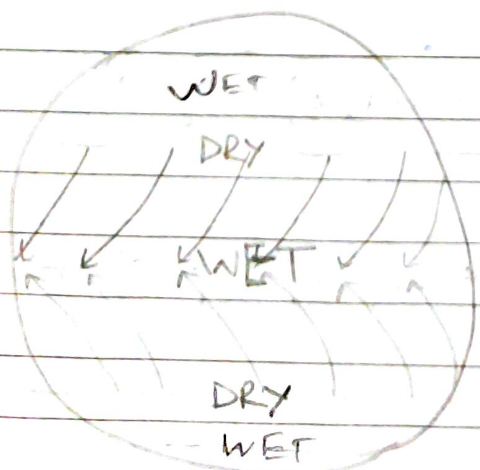


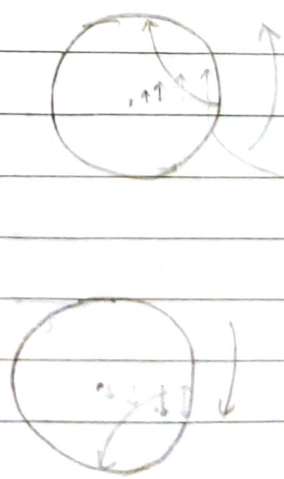
In space we see coriolis acceleration as $2\vec{\omega} \times \dot{\vec{r}}_r$

On earth, we will see this relatively as $-2\vec{\omega} \times \dot{\vec{r}}_r$

On earth we observe it as $\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot}$

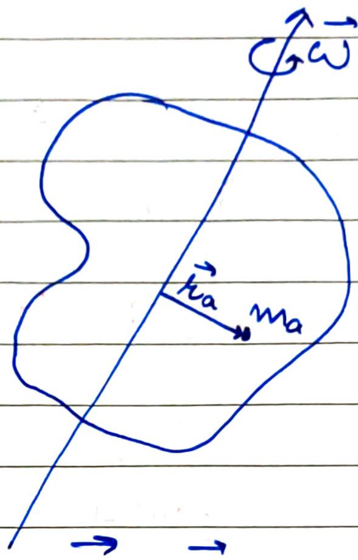
TURNTABLE





an object thrown from centre

consider a mass element on the object



$$v_a = \vec{\omega} \times \vec{r_a}$$

$$\vec{l_a} = \vec{r_a} \times m_a \vec{v_a}$$

$$\Rightarrow \vec{l_a} = m_a \, \vec{r_a} \times (\vec{\omega} \times \vec{r_a})$$

$$\Rightarrow \vec{L} = \sum_a m_a \vec{r_a} \times (\vec{\omega} \times \vec{r_a})$$

$$= \sum_a \left\{ m_a r_a^2 \vec{\omega} - m_a \vec{r_a} (\vec{r_a} \cdot \vec{\omega}) \right\}$$

$$= \sum_a \left\{ (m_a r_a^2) \vec{\omega} - (m_a (\vec{r_a} \cdot \vec{\omega})) \vec{r_a} \right\}$$

Finding angular momentum along x, y & z

$$L_x = \sum_a \left\{ m_a(r_a^2 - x_a^2)\omega_x - m_a x_a y_a \omega_y - m_a x_a z_a \omega_z \right.$$

$$L_y = \sum_a \left\{ m_a(r_a^2 - y_a^2)\omega_y - m_a y_a z_a \omega_z - m_a y_a x_a \omega_x \right.$$

$$L_z = \sum_a \left\{ -m_a z_a x_a \omega_x - m_a z_a y_a \omega_y + m_a(r_a^2 - z_a^2)\omega_z \right.$$

Hence we can write imp

This term basically represents how mass is distributed across the body

$\therefore$

$$\boxed{\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\overleftrightarrow{I}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}};$$

Here we are using tensors to represent the above equations

$$I_{xx} = \sum_a m_a(r_a^2 - x_a^2)$$

$$I_{yy} = \sum_a m_a(r_a^2 - y_a^2)$$

$$I_{zz} = \sum_a m_a(r_a^2 - z_a^2)$$

$$I_{xy} = I_{yx} = -\sum_a m_a x_a y_a$$

$$I_{yz} = I_{zy} = -\sum_a m_a y_a z_a$$

$$I_{xz} = I_{zx} = -\sum_a m_a x_a z_a$$

$\therefore$ we can write

$$L_i = \sum_j I_{ij}\,\omega_j$$

eg -

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

imp **NOTE:** I_{xx}, I_{yy} & I_{zz} are moment of inertia and I_{xy}, I_{yz}, I_{zx} are products of inertia.

PTO

now writing the kinetic energy

$$T = \frac{1}{2} \sum_a m_a |v_a|^2$$

$$= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r_a}) \cdot (\vec{\omega} \times \vec{r_a})$$

$$= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot [\vec{r_a} \times (\vec{\omega} \times \vec{r_a})]$$

[using $(a \times b) \cdot c = a \cdot (b \times c)$]

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

$$= \frac{1}{2} \sum_i \omega_i L_i$$

$$\Rightarrow \boxed{T = \sum_{i,j} I_{ij}\, \omega_i \omega_j}$$

NOTE: In rigid body dynamics, we have to take care of the dimensions of the body as well.

NOTE: Products of inertia are I_{xy}, I_{yz}, I_{xz} & moments of inertia are I_{xx}, I_{yy}, I_{zz}

→ PRINCIPAL AXES

- These are a set of three mutually perpendicular axis about which the products of inertia of a rigid body vanishes.
- Generally these axes are the ones which pass through the COM & ~~a~~ have symmetry about them.
- using principal axes makes calculatio[n] easy.

→ TORQUE →

- From class XI we know

$$\vec{\tau}_{ext} = \left(\frac{d\vec{L}}{dt}\right)_{fix}$$

$$\therefore \quad \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{fix} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega}\times\vec{L}$$

$$= \frac{d}{dt}(\overleftrightarrow{I}\cdot\vec{\omega})_{rot\ (\text{or body})} + \vec{\omega}\times\vec{L}$$

$$\Rightarrow \boxed{\tau = \overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\vec{L}}$$

$\left(\because \overleftrightarrow{I}\right.$ was fro[zen] as the body is defined ie $\overleftrightarrow{I}$ is fixed [...]

2

→ Look at lecture 12 in handouts (except variable mass)

About principal axes,

$I_{xx} = I_1$; $I_{yy} = I_2$; $I_{zz} = I_3$

$\omega_1 = \omega_x$; $\omega_2 = \omega_y$; $\omega_3 = \omega_z$

Also since its about principal axes, $\overleftrightarrow{I}$ becomes

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

∴ From earlier

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

∴ Calculating τ_1, τ_2, τ_3

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{1} & \hat{2} & \hat{3} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} \quad \therefore (\vec{\omega} \times \vec{L})_1 = (I_3 - I_2)\omega_2\omega_3$$

$$\therefore \boxed{\begin{aligned} \tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}} \Longrightarrow \text{EULER EQUATION}$$

REST OF THE TOPICS TO BE DISCUSSED LATER

TUTORIAL 5

1. $F_c = 2m\omega v$

$a = \frac{F}{m} = 2\omega g t$

$\Rightarrow \int a\,dt = \omega g t^2$

$\Rightarrow v = \omega g t^2 + c \qquad (c=0)$

$\Rightarrow v = \omega g t^2$

$\therefore F_c = 2m\omega(\omega g t^2)$

$\Rightarrow F_c = 4m\omega^2\left(\frac{1}{2} g t^2\right)$

$\Rightarrow F_c = 4m\omega^2 d$

correction

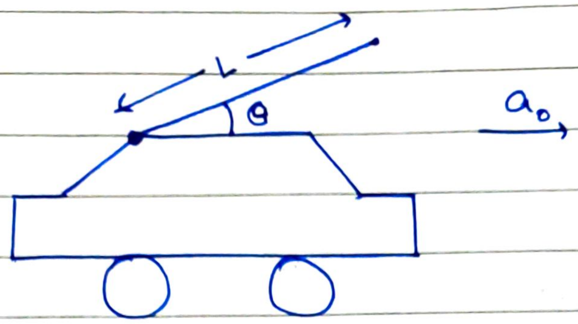
$= \omega^2 d - 4\omega^2 d$

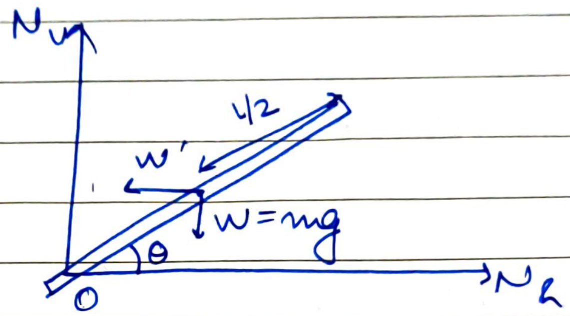
$= -3\omega^2 d$

Q2.

Ans:

a)





$$N_v - W = 0$$

$$N - w' = 0$$

$$w' = +Ma_0$$

Torque about O

$$\tau_0 = \frac{L}{2}\cos\theta\,(W) - \frac{L}{2}\sin\theta\,(W')$$

At Equn $\tau_0 = 0$

$$\Rightarrow \cancel{\frac{L}{2}}\cos\theta\, W = \cancel{\frac{L}{2}}\sin\theta\,(Ma_0)$$

$$\Rightarrow g\cos\theta = a_0\sin\theta$$

$$\Rightarrow a_0 = g\cot\theta$$

$\Rightarrow \tan\theta = \frac{g}{a_0}$

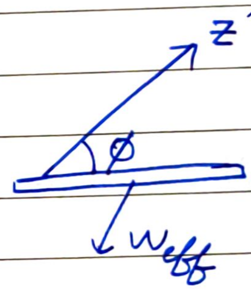
b) new coordinate system z' is along the point of pivot ie along eq^m axis.

For small displacement, torque

$$\tau = \frac{L}{2}\sin\phi\,(Mg_{eff})$$

ϕ small $\therefore$

$$\tau = \frac{L\phi}{2} Mg_{eff}$$



Now

$$W_{eff} = Mg_{eff}$$

$$g_{eff} = \sqrt{g^2+a_0^2}$$

equation of motion

$$I\frac{d^2\phi}{dt^2} = \frac{L}{2}(Mg_{eff})\phi$$

$$\Rightarrow \frac{ML^2}{3}\frac{d^2\phi}{dt^2} - \frac{L}{2}\phi\,Mg_{eff} = 0$$

$$\Rightarrow \frac{d^2\phi}{dt^2} - \left(\frac{6g_{eff}}{L}\right)\phi = 0$$

Solution

$$\phi = \phi_0 e^{\pm\gamma t}, \quad \gamma = \sqrt{\frac{6g_{eff}}{L}}$$

4. a) $x=2$; $y=0$; $z=3$

$r=\sqrt{13}$

$I_{xx} = m(13-4)$

$\Rightarrow I_{xx} = 9m$

$I_{xy}=0$

$I_{yy} = 13m$

$I_{yz}=0$

$I_{zz} = 4m$

$I_{xz}=-6m$

$$\therefore \quad \overleftrightarrow{I} = \begin{pmatrix} 9m & 0 & -6m \\ 0 & 13m & 0 \\ -6m & 0 & 4m \end{pmatrix}$$

b) $x = 2\cos\beta \quad y = 2\sin\beta$

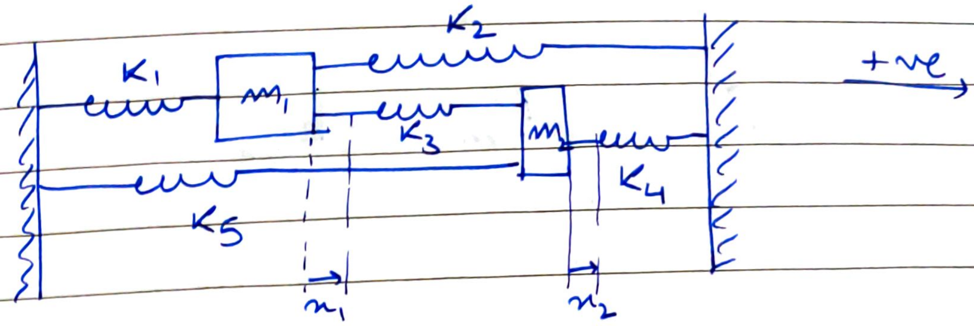
$z=3$

$\cos\beta = 1-\frac{\beta^2}{2}$ (upto 1st order)

$\sin\beta = \beta$ (" " ")

PRE - MID SEM CLASS

Q→



Ans: a) $m_1\ddot{x}_1 = -k_1x_1 - k_2x_1 - k_3(x_1 - x_2)$

$\Rightarrow m_1\ddot{x}_1 = -(k_1 + k_2 + k_3)x_1 + k_3x_2$

$m_2\ddot{x}_2 = -k_3(x_2 - x_1) - k_5x_2 - k_4x_2$

$\Rightarrow m_2\ddot{x}_2 = -(k_3 + k_4 + k_5)x_2 + k_3x_1$

b) $$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2+k_3 & -k_3 \\ -k_3 & k_3+k_4+k_5 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix}$$

$$\therefore \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

c) $\overleftrightarrow{M}^{-1}\overleftrightarrow{K} = \begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$

$\overleftrightarrow{M}^{-1}\cdot\overleftrightarrow{K} = \begin{pmatrix} 2k/m_1 & -k/m_1 \\ -k/m_2 & 2k/m_2 \end{pmatrix}$

$\det\left(\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \overleftrightarrow{\mathbb{1}}\right) = 0$

$\Rightarrow \begin{vmatrix} \frac{2k}{m_1} - \omega^2 & -k/m_1 \\ -k/m_2 & \frac{2k}{m_2} - \omega^2 \end{vmatrix} = 0$

$\Rightarrow \left(\frac{2k}{m_1} - \omega^2\right)\left(\frac{2k}{m_2} - \omega^2\right) - \frac{k^2}{m_1 m_2} = 0$

$\Rightarrow \omega^4 - 2k\omega^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{3k^2}{m_1 m_2} = 0$

$\Rightarrow \omega^2 = \dfrac{2k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm \sqrt{4k^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{12k^2}{m_1 m_2}}}{2}$

$$\Rightarrow \omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\sqrt{\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2}\right)}$$

$m_1 = m_2$

$$\Rightarrow \omega^2 = \frac{3k}{m} \text{ or } \frac{k}{m}$$

d) $$\vec{M}^{-1}\vec{K} = \begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix}$$

For $\omega^2 = 3k/m$, let $|e_i\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\cancel{k/m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\cancel{\frac{k}{m}}\begin{pmatrix} a \\ b \end{pmatrix}$$

$2a - b = 3a$

$\Rightarrow \underline{a = -b}$

$\therefore$ eigen vector $= \begin{pmatrix} a \\ -a \end{pmatrix} = |e_1\rangle$

$\therefore$ Normalised $|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\omega^2 = k/m$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\Rightarrow 2e - f = e$$

$$\Rightarrow e = f$$

$$\therefore \begin{pmatrix} e \\ e \end{pmatrix} = |e_2\rangle$$

$$\therefore \text{normalised } |e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|e_1\rangle\langle e_1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\left(\because \langle e_1| = \left(|e_1\rangle\right)^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\right)$$

$$|e_2\rangle\langle e_2| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\therefore \sum_{i=1}^{2} |e_i\rangle\langle e_i| = \overleftrightarrow{I}$$

$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$

Ansatz $\Rightarrow$ $x = \alpha e^{\lambda t}$

$$a\lambda^2 + b\lambda + c = 0$$

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \begin{cases} \lambda_1 \\ \lambda_2 \end{cases}$$

Complimentary function $x_{CF}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

RULE

when $\lambda_1 = \lambda_2 = \lambda$; $x_{CF}(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

TUTORIAL 5

3. Along north and south directions

$$-\Omega_E \times v_{m/s} = 0$$

(Ω_E is ω for earth)

$$\Rightarrow \frac{\Delta g}{g} = 0$$

Along east

$$-\Omega_E \times v_{East}$$

$\therefore$ $\frac{\Delta g}{g}$ is -ve

1 mile/hour = 1.46 ft/sec

$$\left|\frac{\Delta g}{g}\right| = -\frac{2\Omega v_{East}}{g}$$

Along west

$$-\Omega \times v_w =$$

$\frac{\Delta g}{g}$ is +ve

$$\therefore \frac{\Delta g}{g} = -\frac{2\Omega v_E}{g}$$

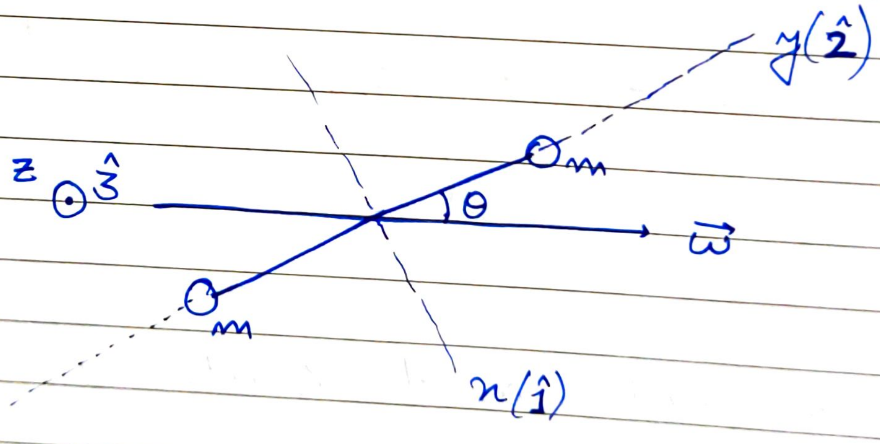
imp

\# Time scale was large in critical damping

imp

\#

Q→



Ans

$$I_1 = I_{xx} = 2ma^2$$

$$I_2 = I_{yy} = 0$$

$$I_3 = I_{zz} = 2ma^2$$

Since we have $I_{xy}=0$; $I_{yz}=0$; $I_{xz}=0$

$\therefore$ the axis is principle axis

$$\therefore \overleftrightarrow{I} = \begin{pmatrix} 2ma^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix}$$

We can conclude

$$\vec{\omega} = \begin{pmatrix} \omega \sin\theta \\ \omega \cos\theta \\ 0 \end{pmatrix}$$

$$\vec{L} = \overleftrightarrow{I}\vec{\omega}$$

$$\Rightarrow \vec{L} = \begin{pmatrix} 2ma^2\omega\sin\theta \\ 0 \\ 0 \end{pmatrix} \equiv 2ma^2\omega\sin\theta\,\hat{1}$$

now using EULER EQN's

$$\tau_1 = 0$$
$$\tau_2 = 0$$
$$\tau_3 = -2ma^2\omega^2\sin\theta\cos\theta$$

We have assumed that $\vec{\omega}$ and θ are constant & we get a result that for this to be satisfied, a torque of

$-2ma^2\omega^2\sin\theta\cos\theta\,\hat{3}$ is required.

SECTION - 2

QUANTUM MECHANICS

Chapter - 1
The Ideas and Experiments

- Quantum mechanics is applicable in certain cases where newtonian mechanical calculations fail accuracy.

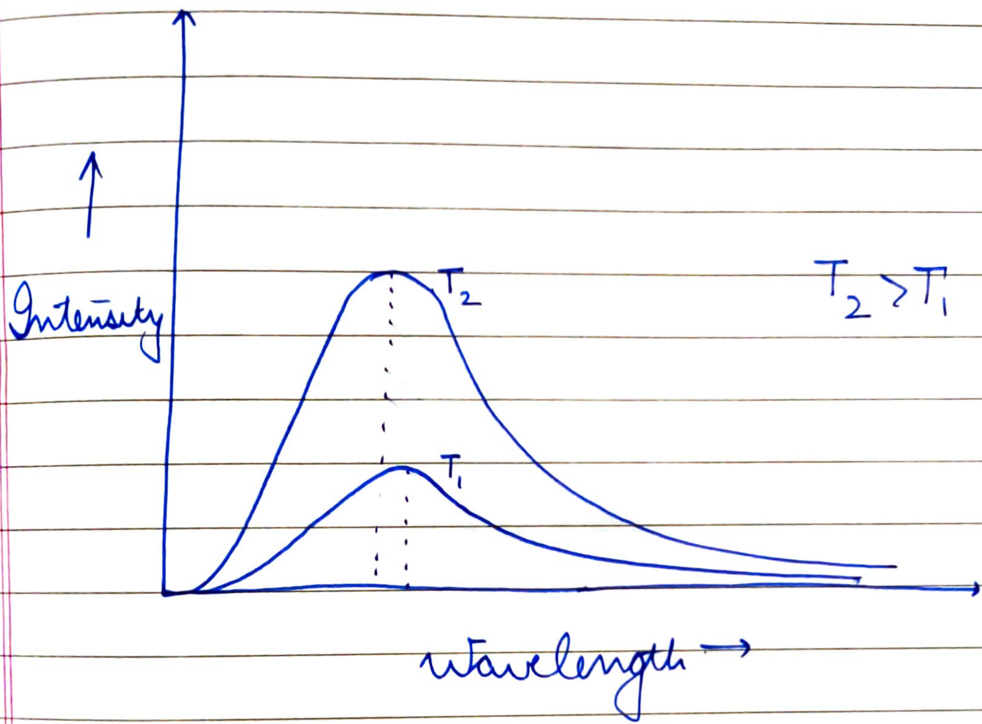
- It is not only limited to small objects but also on very large objects:

a) It is applicable where the momentum of a body is high ie the deBroglie wavelength $\lambda = \frac{h}{mv}$ becomes very small and insignificant with respect to size of the body.

b) It is also applicable on bodies with large large densities say 10^{10} kg/m³ like stars and black holes.

1. Black Body Radiation →

- At different temperatures, each and every body radiates a specific spectrum of wavelengths
- every wavelength is radiated as such but in different intensities.
-



classmate

Date

Page

HAMILTONIAN OPERATORS

- We can write Lagrangian formation equation as

$$L = T - V$$

$$\Rightarrow L = \sum_{i=1}^{n} \frac{1}{2} m \dot{q}_i^2 - V(q_i, t)$$

(For a body with n degrees of freedom)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \quad \forall\ i \qquad + 2n \text{ initial conditions.}$$

- To get rid of this above second order differential equation form, we define

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \& \quad \boxed{H(q_i, p_i) = \sum_i \dot{q}_i p_i - L}$$

LEGENDRE TRANSFORMATION

This 'H' is called Hamiltonian Operator

If we use this 'H', we can transform second order differential equations to first order equations. On this sort of process, we will get

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

And hence we have equations reduced to first order differential equations.

• eg- <u>SHO in 1d</u>

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \quad ; \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\& \quad H = \dot{x} p_x - \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\Rightarrow H = m\dot{x}^2 - \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

now H is a function of q_i and p_i only as per definition. $\therefore$ we eliminate $\dot{x}$

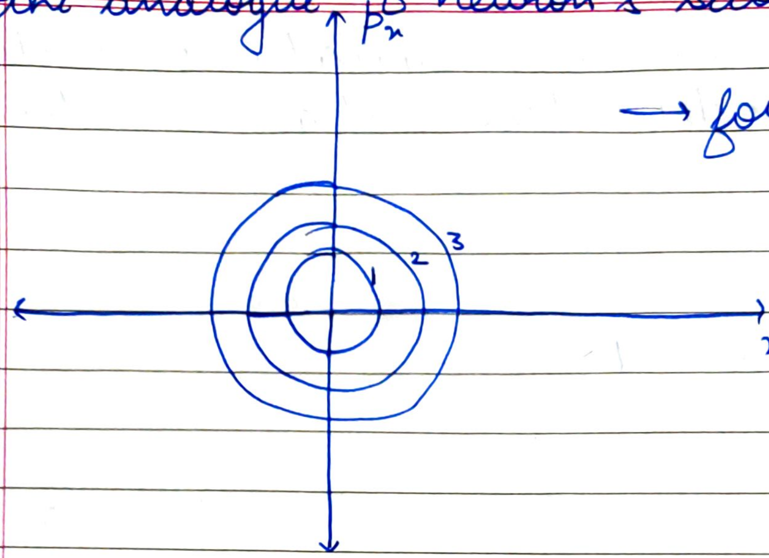
$$\Rightarrow \quad H = \frac{p_x^2}{2m} + \frac{1}{2} k x^2 \qquad (i)$$

$$\therefore \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

<u>NOTE:</u> In a way, 'H' represents the total energy of the system (think upon it! 😛). From equation (i) carefully see that if we plot graph b/w p_x and x, we would either get an ellipse or a circle (depending on k and m) for a given value of 'H'.

NOTE: In classical mechanics newton's second law is used to predict the conditions of a system (velocity etc) when some initial conditions are given. In quantum mechanics, ~~the analogue of Newton's second law is Schrödinger eqn.~~



$\rightarrow$ for $\rightarrow \frac{1}{m} = k$

$H_1 < H_2 < H_3$

NOTE: p_i vs q_i graphs are very useful in motion analysis.

~~WORKING ON CANONICALLY CONJUGATE VARIABLES~~

* → SCHRÖDINGER EQUATION →

- From earlier discussion on Heisenberg principle, we know canonically conjugate variables satisfy.

$$\boxed{\Delta v_1 \cdot \Delta v_2 \geqslant \frac{1}{2} \left| \langle [v_1, v_2] \rangle \right|}$$

v_1 and v_2 are canonically conjugate variables.

$\langle \ldots \rangle$ are "expectation value" & $\langle \;=\; \rangle$ represents average over a probability distribution function.

$$\boxed{\left| \Psi_{(x,t)} \right|^2 = \rho(x)} \text{ and } |\Psi(x,t)|^2 = \Psi^*(x,t)\, \Psi(x,t)$$

PROBABILITY ~~densi~~ FUNCTION

NOTE: In a wave the amplitude square represents intensity of wave. Similarly here, the amplitude squared represents sort of an intensity of the particle, ie its probability density.

- Consider a general solution for $\psi_{(x,t)}$

$$\cancel{\psi_{(x,t)} = Ae^{\frac{i}{\hbar}}}$$

$$\psi_{(x,t)} = A e^{\frac{i}{\hbar}\left(\frac{2\pi}{\lambda}\hbar x - \hbar\omega t\right)}$$

where $\hbar = \frac{h}{2\pi}$

now we take analogies,

- from photoelectric effect, we write

$$E = h\nu = \hbar\omega$$

- and also from de Broglie hypothesis.

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

$\therefore$ now we can write

$$\boxed{\psi_{(x,t)} = A e^{i/\hbar\,(p_x x - Et)}}$$

now,

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar} p_x \psi$$

$$\Rightarrow p_x\psi = -i\hbar\frac{\partial}{\partial x}\psi$$

Also

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} E \psi$$

$$\Rightarrow E\psi = i\hbar \frac{\partial}{\partial t}\psi$$

Operator Relations

why did we assume p_x as an operator?

$$p_x \longrightarrow -i\hbar \frac{\partial}{\partial x}$$

$$E \longrightarrow i\hbar \frac{\partial}{\partial t}$$

now if we write Hamiltonian Operator

$$H = \frac{p_x^2}{2m} + V(x) = E$$

1-D Equation

$$\Rightarrow \boxed{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar \frac{\partial}{\partial t}\left(\psi(x,t)\right)}$$

From this 1-D equation, we can generalise a 3-D equation as

3-D Equation

$$\boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)\right]\psi(x,y,z,t) = i\hbar \frac{\partial}{\partial t}\psi(x,y,z,t)}$$

NOTE: $[A, B] = AB - BA$

$\hbar [x, p_x] \psi = \hbar \left[-i x \frac{\partial \psi}{\partial x} + i \frac{\partial (x\psi)}{\partial x} \right] = i \hbar \psi$

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- COMMUTATOR

- $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is a commutator

- If written as $[\hat{A}, \hat{B}]$, here $\hat{A}$ and $\hat{B}$ are operators eg - $\hat{x}, \hat{p}_x, \hat{L}$.

- Properties →

a) $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$

b) $[\alpha \hat{A}, \hat{B}] = \alpha [\hat{A}, \hat{B}]$ where $\alpha \in \mathbb{C}$ (complex plane)

c) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

d) $[\hat{A}, \alpha] = 0$ where $\alpha = \text{const} \in \mathbb{C}$ (complex plane)

- example-

i) Obtain $[\hat{x}, \hat{p}_x]$

$\Rightarrow [\hat{x}, \hat{p}_x] f = [\hat{x}, -i\hbar \frac{\partial}{\partial x}] f$ {f is an element on which operation is being performed}

$= -i\hbar [\hat{x}, \frac{\partial}{\partial x}] f$

$= -i\hbar \left\{ x \frac{\partial f}{\partial x} - \frac{\partial}{\partial x}(xf) \right\}$

$= -i\hbar \left\{ x \frac{\partial f}{\partial x} - f - x\frac{\partial f}{\partial x} \right\}$

$= i\hbar f \quad \forall f \in \mathbb{C}$

$\therefore [x, p_x] = i\hbar$

NOTE: you can write $[\hat{x}, \hat{p}_x]$ as $[x, p_x]$; its understood

ii) $[\hat{x}, \hat{p}_x^2]$

[illegible] using Prop(c)

$$= p_x [x, p_x] + [x, p_x] p_x$$

$\because [x, p_x] = i\hbar$

$$\therefore [x, p_x^2] = 2i\hbar p_x$$

NOTE: We had $\Delta v_1 \cdot \Delta v_2 \geq \frac{1}{2} \left| \langle [v_1, v_2] \rangle \right|$

Taking $v_1 = x$ & $v_2 = p_x$

$$\therefore \Delta x \cdot \Delta p_x \geq \frac{1}{2} \left| \langle i\hbar \rangle \right|$$

$$\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$$

$$\therefore \Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$$

$\therefore$ we can obtain Heisenberg's uncertainty equation.

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imp

- **Remarks / consequences / Interpretation of Schrödinger equations**

- $$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t)$$

a) $\psi(x,t) \in \mathbb{C}$ $\forall$ $V(x) \in \mathbb{R}$

This can be seen by assuming ψ real. In that case, RHS is complex but LHS is real $\therefore$ it is not possible.

b) If $\psi_1(x,t)$ & $\psi_2(x,t)$ are solutions of the equation, then $\alpha\psi_1(x,t) + \beta\psi_2(x,t)$ is also a solution ; $\alpha, \beta \in \mathbb{C}$

(Linearity property)

c) If $\psi(x, t_0)$ is known for all x at time t_0 then $\psi(x,t)$ can be determined $\forall x$ at all times t.

d) $P(x,t) \equiv \rho(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\,\psi(x,t)$ is defined (copenhagen interpretation) as the probability density function, s.t. $P(x,t)dx$ is the probability of finding the "QM object" in $[x, x+dx]$ at time t.

e) Sum of all probabilities is 1. i.e.

$$\int_{-\infty}^{\infty} P(x,t)\, dx = \underbrace{\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1}_{\text{normalization}}$$

Remark: If $\psi'(x,t)$ is not normalized

& $\int_{-\infty}^{\infty} |\psi'(x,t)|^2 dx = N$; then

$\psi(x,t) = \frac{1}{\sqrt{N}} \psi'(x,t)$ is normalized

f) $\psi(x,t)$ is continuous and bounded;

$\frac{\partial \psi(x,t)}{\partial(x)}$ is bounded.

$\psi(x,t) \longrightarrow 0$ as $x \rightarrow \pm\infty$

g) [Exercise] $\frac{d}{dt}\left[\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx\right] = 0$

NOTE: For a given valid wave function $\psi_0(x,t)$, for required wave func. we know

Normalization is its requirement

$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$ ∴ we find $\int_{-\infty}^{\infty} |\psi_0(x,t)|^2 dx = a$

and scale down $\psi_0(x,t)$ by $\sqrt{a}$ to get $\psi(x,t) = \frac{1}{\sqrt{a}} \psi_0(x,t)$

TUTORIAL 6

1. a) $\frac{h}{p} = \lambda$

$\Rightarrow \frac{h}{\sqrt{2mKE}} = \frac{h}{\sqrt{2 \times 1.6 \times 10^{-27} \times 100 \times 10^{6} \times 1.6 \times 10^{-19}}}$

$= \frac{h}{1.6 \times 10^{-23} \times 10^{4} \sqrt{2}} = \frac{6.6 \times 10^{-34}}{1.6 \times 10^{-19} \times \sqrt{2}} = \frac{6.6 \times 10^{-15}}{1.6\sqrt{2}}$

$= 2.917 \times 10^{-15}$

b) $\frac{h}{0.1 \times 1000} = 6.6 \times 10^{-36}$

2. $\cancel{11.239 \times 1.6 \times 10^{-19}} =$

$\frac{hc}{80nm} =$

3 a) $[x, p_x] f$

$$= \left[x, -i\hbar \frac{\partial}{\partial x}\right] f$$

$$= -i\hbar \left[x, \frac{\partial}{\partial x}\right] f$$

$$= -i\hbar \left[x\frac{\partial}{\partial x} - \frac{\partial}{\partial x} x\right] f$$

$$= -i\hbar \left[\cancel{x\frac{\partial f}{\partial x}} - \cancel{x\frac{\partial f}{\partial x}} - f\right]$$

$$= i\hbar f$$

$\therefore [x, p_x] = i\hbar$

b) $[x^2, p_x] f$

$$= \left[x[x, p_x] + [x, p_x] x\right]$$

$$= \left[2i\hbar x\right]$$

$$= 2i\hbar x$$

c) $[x, p_x^2]$

$$= p_x[x, p_x] + [x, p_x]p_x$$

$$= 2i\hbar p_x$$

d) $[x^2, p_x^2]$

$$= x[x, p_x^2] + [x, p_x^2]x$$

$$= \hat{x}\, 2i\hbar \hat{p}_x + 2i\hbar \hat{p}_x \hat{x} = 2i\hbar(\hat{x}\hat{p}_x + \hat{p}_x \hat{x})$$

now, $[\hat{x}, \hat{p}_x] = i\hbar = xp_x - p_x x \Rightarrow xp_x = i\hbar + p_x x$

$\therefore [x^2, p_x^2] = 2i\hbar(i\hbar + 2p_x x)$

4. $$\int_{-\infty}^{\infty} \left| \sin^2\left(\frac{\pi x}{a}\right) e^{2i/\hbar\, E_1 t} \right| dx$$

$$= \int_{-a}^{a} \sin^2\left(\frac{\pi x}{a}\right) dx = \int_{-a}^{a} \left(\frac{1 - \cos\left(\frac{2\pi x}{a}\right)}{2}\right) dx$$

$$= a \pm \int_{-a}^{a} \cos\left(\frac{2\pi x}{a}\right) = a$$

$$= a$$

$\therefore$ Normalized $\psi = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) e^{i/\hbar (E_1 \ldots}$

NOTE: Schrödinger eqⁿ is basically Hamiltonian operator on $\Psi(x,t)$

$$\boxed{\hat{H}\,\Psi(x,t) = \hat{E}\,\Psi(x,t)}$$

ie Schrödinger eqn is different for different Ψ

SOLVING THE SCHRÖDINGER EQUATION →

- We have Schrödinger Equation as

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \qquad \text{(i)}$$

- In solving multivariable functions, it may be easier if we split it variable-wise

- Ansatz: $\boxed{\Psi(x,t) = \psi(x)\,\phi(t)}$ (ii)

(variable separation trick)

Substituting (ii) in (i) and dividing throughout by $\Psi(x,t)$ $(= \psi(x)\,\phi(t))$

$$\Rightarrow \frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2}{\partial x^2}\psi(x) + V(x) = i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = E \quad \text{(constant)} \qquad \text{— (iii)}$$

The LHS is a function of 'x' and RHS is a function of 't'. This is only possible if LHS = RHS = constant.

NOTE: It was later realised that this constant is nothing but total energy of system.

NOTE: $\dfrac{\hat{E}\,\Psi(x,t)}{\Psi(x,t)} = E$ (const.)

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- Now equation (iii) can be written by splitting into two equations.

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)} \quad \text{(iv-a)}$$

↳ (Schrödinger time independent equa.)

$$\boxed{i\hbar \frac{\partial \phi(t)}{\partial t} = E\phi(t)} \quad \text{(iv-b)}$$

NOTE: $\left(\dfrac{\partial^2}{\partial x^2} = \dfrac{d^2}{dx^2}\right)$ here

↳ (Schrödinger position indep. eqn)

- Solving iv-b

$$\int \frac{\partial \phi(t)}{\phi(t)} = -\int \frac{iE}{\hbar}\,dt$$

$$\Rightarrow \ln \phi(t) = -\frac{iEt}{\hbar} + c'$$

$$\Rightarrow \phi(t) = c\,e^{-iEt/\hbar} \quad \text{(v)}$$

$$\therefore \quad \Psi(x,t) = A\psi(x)\,e^{-iEt/\hbar} \quad \text{(vi)}$$

- We will come back to solving the time indep. part after some examples.

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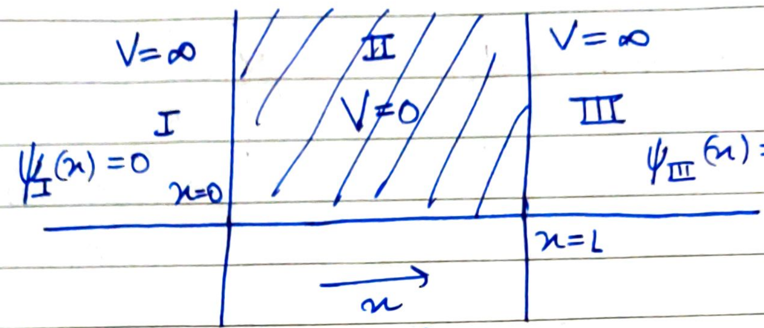
• EXAMPLES

Ⓐ Infinite Potential well.



Its a model where V=0 b/w 2 x coordinates and ∞ outside it.

A particle never prefers to get into a region of high potential (since a particle on itself moves from a region of high to low poten.)

$\therefore \quad \psi(x)_I = \psi_{III}(x) = 0$

Also we can see it using eq (iv-a) that if $V(x) = \infty$ $\psi(x)$ must be zero whatsoever

In region II, $\psi_{II}(x)$ satisfies from (iv(a))

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}(x)}{dx^2} = E\,\psi_{II}(x)$$

$$\frac{d^2\psi_{II}(x)}{dx^2} + K^2\psi_{II}(x) = 0 \quad ; \quad K^2 = \frac{2mE}{\hbar^2}$$

We can observe that this equation is the equation like that of SHM

$\therefore$

$$\boxed{\psi_{II}(x) = A\sin kx + B\cos kx}$$

Since $\psi(x)$ is continuous (we know) $\therefore$ it must be continuous at $x=0$ and $x=L$

$\therefore$ ~~$\psi(x)$~~ at $x=0$ $\quad \psi_{II}(0) = \psi_{I}(0) = 0 \Rightarrow B=0$

at $x=L$, $\psi_{II}(L) = \psi_{III}(L) = 0 \Rightarrow A\sin kL = 0$

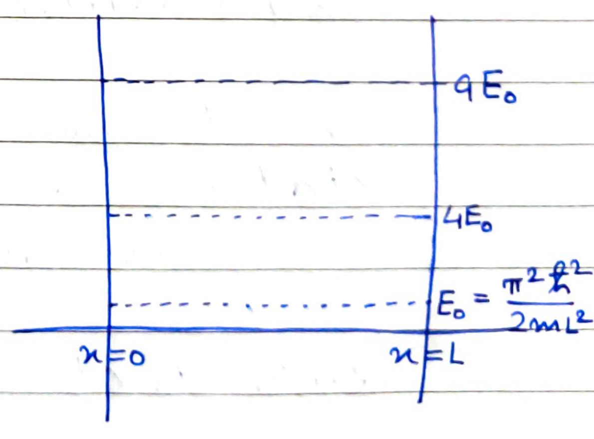
we could have $A=0$ as a trivial solution but its useless 😑, $\therefore$ for non-trivial solutions of $A\sin kL = 0$

$\Rightarrow kL = n\pi \quad , \quad n \in \mathbb{N} \ (1, 2, 3, 4, \ldots)$

(Quantization; n is quantum number)

$$\Rightarrow \sqrt{\frac{2mE}{\hbar^2}}\, L = n\pi$$

$$\Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$



$$\Rightarrow \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

$\therefore$ Any object in this potential well will be in a specified energy state. say, its at $9E_0$, then on coming to E_0, it will release photons of energy $8E_0$. Similarly, for going from E_0 to $9E_0$, object will absorb an energy of $8E_0$ to do it.

- Energy "eigen states" (wave function) and "eigen values" (energy levels) for infinite potential well.

For

$n=1$, $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$, $\psi_1(x) = A_1 \sin\dfrac{\pi x}{L}$

$n=2$, $E_2 = \dfrac{4\pi^2\hbar^2}{2mL^2}$, $\psi_2(x) = A_2 \sin\left(\dfrac{2\pi x}{L}\right)$

$n=3$, $E_3 = \dfrac{9\pi^2\hbar^2}{2mL^2}$, $\psi_3(x) = A_3 \sin\left(\dfrac{3\pi x}{L}\right)$

⋮

Now we can see we have E defined as $E = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$; $n \in \mathbb{N}$; This signifies that any particle, if present, can only be present in these specific energy states. And for every such state E_n, we will have a distinct $\psi_n(x)$ such that we have a distinct $\psi_n(x,t)$.

∴ For the particle $\psi(x,t)$ will ultimately be a summation of $\psi_n(x,t)$ from $n=1$ to $n=\infty$.

$$\therefore \psi(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\, e^{-i\left(\frac{n^2\pi^2\hbar}{2mL^2}\right)t}$$

Here c_n is taken such that ψ is normalised

NOTE: The ~~3~~ different ψ we obtained, all were obtained as valid solutions to Schrödinger equation. ∴ all of them when multiplied by time part, describe the particle ∴ All of their addition completely describes the particle.

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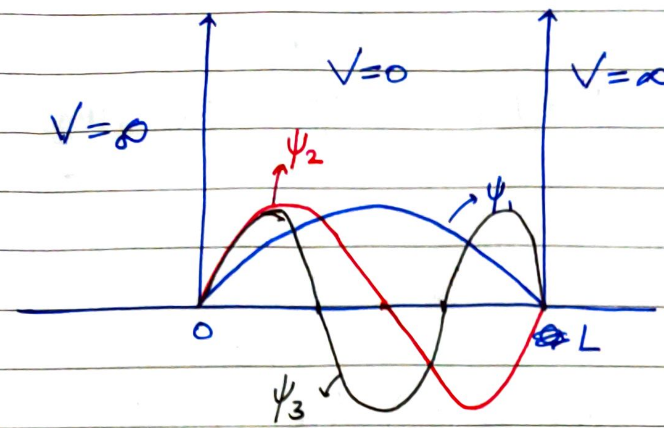


Here we will plot graphs of ψ_1, ψ_2, ψ_3

• normalization

Say we consider

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \psi_n(x)\, dx$$

$$= \boxed{\int_0^a \psi_m^*(x)\, \psi_n(x)\, dx = \delta_{m,n} = \begin{cases} 0 & ; m \neq n \\ 1 & ; m = n \end{cases}}$$

This expression denotes orthonormality of the ψ wave functions & All of them are like orthogonal vectors.

Consider

$$\Psi_1(x,t) = A_1 \psi_1(x)\, e^{-iE_1 t/\hbar}$$

$$\Psi_1^*(x,t) = A_1 \psi_1(x)\, e^{iE_1 t/\hbar}$$

NOTE: In normalization in $\Psi^*\Psi$, the time dependent part gets cancelled out anyway ∴ ∴ normalization can be carried out without time dependent part.

$$\boxed{\int_{-\infty}^{\infty} \Psi_1^* \Psi_1\, dx = 1 = \int_{-\infty}^{\infty} |\psi(x)|^2\, dx}$$

This is our basic principal. The probabilities must always add up to 1!

writing

$$\int_0^a \psi_1^*(x)\,\psi_1(x)\,dx = A_1^2\int_0^a \sin^2\left(\frac{\pi x}{L}\right) = 1$$

$$\Rightarrow A_1^2\int_0^a\left(\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right)dx = 1$$

$$\Rightarrow A_1^2 = \frac{2}{a}$$

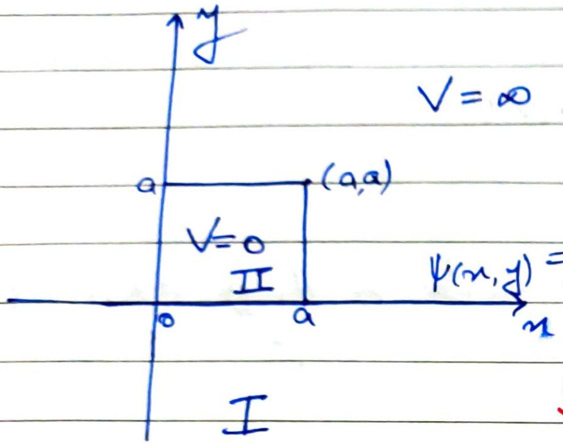
$$\Rightarrow A_1 = \sqrt{\frac{2}{a}}$$

Similarly, we find $A_1 = A_2 = A_3 \dots = \sqrt{\frac{2}{a}}$

NOTE: we previously observed how $\psi_n(x)$ are all orthonormal. This denotes that all of them are independent of each other (like orthogonal vectors). This denotes that they are independent states in which a quantum mechanical object is capable of existing indefinitely until and unless energy is supplied to it or it releases photons. (This is because of independence of every individual state.)

(B) Generalization to higher dimensions

2D case



$$V(x,y) = \begin{cases} 0 & ; \text{ if } x \in [0,a] \ \& \ y \in [0,a] \\ \infty & \text{otherwise} \end{cases}$$

In region I: $\psi(x,y) = 0$

In region II: using eq^n (iv-a) from before

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) + \cancel{V(x,y)}^{\,0}\,\psi(x,y) = E\,\psi(x,y)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) + K^2\,\psi(x,y) = 0 \ ; \quad K^2 = \frac{2mE}{\hbar^2}$$

Using variable separable form

$$\psi(x,y) = \psi(x)\,\psi(y)$$

$$\Rightarrow \psi(y)\frac{\partial^2\psi(x)}{\partial x^2} + \psi(x)\frac{\partial^2\psi(y)}{\partial y^2} + K^2\psi(x)\,\psi(y) = 0$$

Assume $K^2 = \tilde{K}_x^2 + \tilde{K}_y^2$ & dividing by $\psi(x,y)$

$$\Rightarrow \frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{\psi(y)}\frac{\partial^2\psi(y)}{\partial y^2} + (\tilde{K}_x^2 + \tilde{K}_y^2) = 0$$

$$\Rightarrow \frac{1}{\psi(x)} \frac{\partial^2 \psi(x)}{\partial x^2} + \tilde{K}_x^2 = -\left[\frac{1}{\psi(y)} \frac{\partial^2 \psi(y)}{\partial y^2} + \tilde{K}_y^2\right] = C$$

Let $\tilde{k}_x^2 - c = k_x^2$ & $\tilde{K}_y^2 - c = K_y^2$

∴ we get 2 equations

$$\frac{\partial^2 \psi(x)}{\partial x^2} + K_x^2 \psi(x) = 0 \quad \& \quad \frac{\partial^2 \psi(y)}{\partial y^2} + K_y^2 \psi(y) = 0$$

where ∴ $\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right)$; $k_x = \frac{n_x \pi}{a}$; $n_x \in N$

$\psi(y) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi y}{a}\right)$; $k_y = \frac{n_y \pi}{a}$; $n_y \in N$

$$\boxed{E = \frac{(n_x^2 + n_y^2)\pi^2 \hbar^2}{2mL^2}}$$

∴ We can see, we have two ψ functions one dependent on x and other on y. A particle's ψ is completely determined by both.

In this case we have energy states dependent on coordinates (n_x, n_y)

This is similar to degeneracy in orbits

degenerate energy levels (ie probability is equally divided in both)

(3,3)

(2,3) (3,2)

(2,2)

(1,2) (2,1)

(1,1)

Energy levels

NOTE: The different quantized values of energy are called eigen values of energy.

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CONTINUITY EQUATION →

- We consider

J → probability current $(=?)$, ρ → probability density $(= \psi^*(x,t)\,\psi(x,t))$

we relate them by writing $\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$ for 1D

- We know, Schrödinger for 1D

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

∴ we can write

$$\Rightarrow \quad -i\hbar \frac{\partial \psi^*(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

- now

$$\frac{\partial \rho}{\partial t} = \frac{\partial |\psi|^2}{\partial t} = \frac{\partial (\psi^*\psi)}{\partial t} = \left(\frac{\partial \psi^*}{\partial t}\right)\psi + \psi^*\frac{\partial \psi}{\partial t}$$

$$= \left(-\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V(x)\psi^*\right)\psi + \psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V(x)\psi\right)$$

$$= -\left[\frac{i\hbar}{2m}\left\{\left(\frac{\partial^2 \psi^*}{\partial x^2}\psi\right) - \left(\psi^*\frac{\partial^2 \psi}{\partial x^2}\right)\right\}\right]$$

$$= -\frac{\partial J}{\partial x}; \quad J = \frac{i\hbar}{2m}\left[\left(\frac{\partial \psi^*}{\partial x}\right)\psi - \psi^*\left(\frac{\partial \psi}{\partial x}\right)\right] \equiv J_x$$

$\therefore$ we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad \text{in 1-D}$$

In 3-D

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial J}{\partial x}\hat{i} + \frac{\partial J}{\partial y}\hat{j} + \frac{\partial J}{\partial z}\hat{k}}_{\nabla J} = 0$$

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla J = 0}$$

NOTE: $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{(not a vector!)}$$

- Schrödinger equation in 3-D

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi$$

To solve for H-atom, we take ∇^2 in spherical polar coordinates and $V = -\frac{Z}{r}$

NOTE: Finding ∇^2 in spherical polar coordinates

$$\nabla^2 = \sum_{i=1}^{3} \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \frac{h_1 h_2 h_3}{h_i^2} \frac{\partial}{\partial u_i}$$

now remember from first few lectures, for sperical polar coordinates,

$$h_1 = h_r = 1 \;;\; h_2 = h_\theta = r \;;\; h_3 = h_\phi = r\sin\theta$$

$$\therefore \nabla^2 = \left(\frac{1}{r^2\sin\theta} \frac{\partial}{\partial r} r^2\sin\theta \frac{\partial}{\partial r}\right) + \left(\frac{1}{r^2\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta}\right)$$

$$+ \left(\frac{1}{r^2\sin\theta} \frac{\partial}{\partial \phi} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}\right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta} \frac{\partial^2}{\partial \phi^2}$$

→ **SCHRÖDINGER EQUATION FOR HYDROGEN ATOM** →

- we have $V = -\frac{Z}{r} e^2 K$ for H-atom.
- We know Schrödinger Time Independent equation as

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

- To solve the equation, we use variable separable technique to write

$$\boxed{\psi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)}$$

- Now we substitute $\psi(r,\theta,\varphi)$ and V in equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\varphi) + V(r)\,\psi(r,\theta,\varphi) = E\,\psi(r,\theta,\varphi)$$

ie

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\varphi) - \frac{Zke^2}{r}\psi(r,\theta,\varphi) = E\,\psi(r,\theta,\varphi)}$$

→ TRANSFORMING REPRESENTAIONS

- We can have

Co-ordinate representation ⟷ momentum representation

ie $\psi(x) \longleftrightarrow \phi(p_x)$

- $|\psi|^2 dx$ = probability of finding the object b/w x and $x+dx$ at time t

- $|\phi(p_x)|^2 dp_x$ = probability of finding object with momentum b/w p_x and p_x+dp_x at time t.

Fourier transformation $$\boxed{\phi(p_x) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{-ip_x x}{\hbar}} \psi(x)\, dx}$$

Inverse Fourier Transformation $$\boxed{\psi(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{ip_x x}{\hbar}} \phi(p_x)\, dp_x}$$

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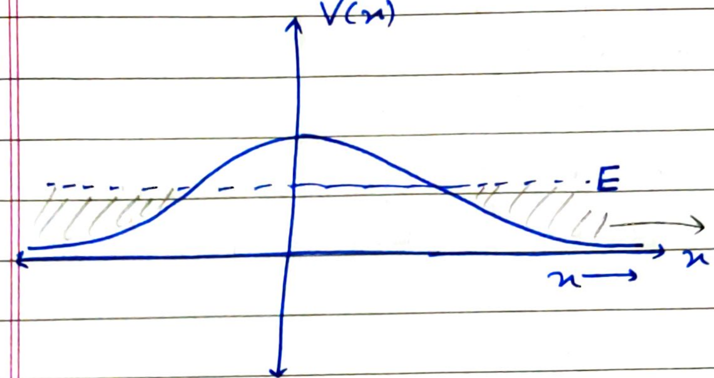
→ BOUND STATES AND SCATTERING STATES

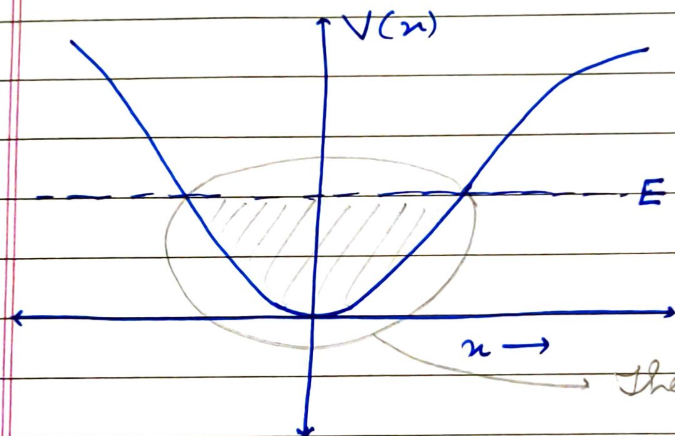
Consider total energy of object = E

a)



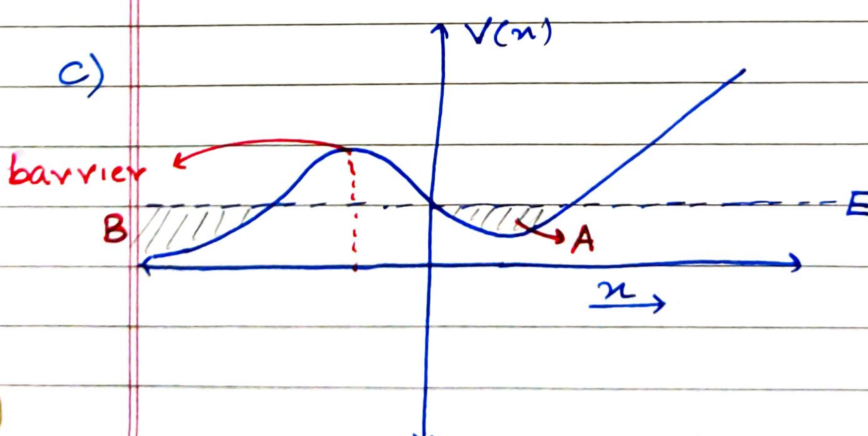
The object goes towards $x \to \infty$ indefinitely with $V(x) \to 0$ ∴ it is a scattering state clearly.

b)



The object is bound to this region. ∴ it is a bound state clearly

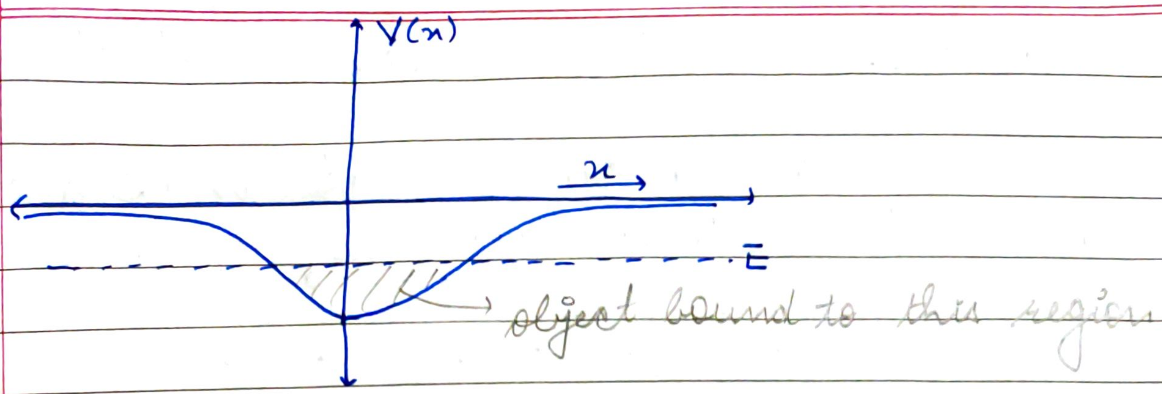
c)



- In classical mechanics, object when bounded to A, can't cross over the barrier & thus B region is forbidden.

- However quantum mechanically, there will be some probability of object being in B. (ie transmission across barrier is sort of possible.)

d)



- ∴ from above examples, we conclude

<u>a)</u> <u>Bound state in quantum mechanics</u>

if $\boxed{E < V(x) \text{ for } x \to \infty \text{ and } x \to -\infty}$

<u>b)</u> <u>Scattering state in quantum mechanics</u>

if $\boxed{E > V(x) \text{ for either } x \to \infty \text{ or } x \to -\infty}$

→ ATTRACTIVE DELTA FUNCTION POTENTIAL

- ~~Attractive~~ delta function potential is defined as

$$\boxed{\delta(x) = \begin{cases} \infty & \text{at } x = 0 \\ 0 & \text{otherwise} \end{cases}}$$

$\boxed{V(x) = -\alpha\,\delta(x)}$ ⟿ Attractive delta function potential

- We use this sort of a potential to form the

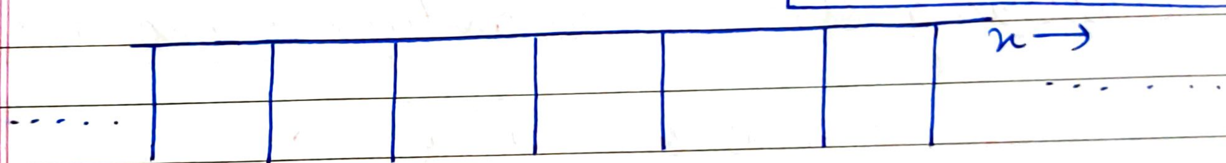
theory of bands in semiconductors

- consider an arrangement in a semiconductor as (representing nuclei) (+ve ly charged)

. . . .

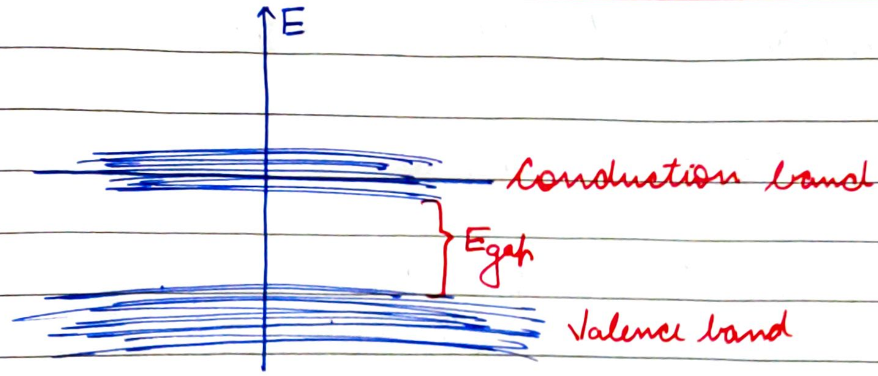
the potential near these nuclei is very very highly attractive as such and we can replace it with the following model

- We have a potential $\boxed{V(x) = -\alpha \sum_{n=-\infty}^{\infty} \delta(x_n - na)}$



and the above structure represents such an arrangement as shown above. The V(x) expression satisfies it. In the arrangement, vertical lengths denote V(x) at that x. This representation is called Dirac Comb potential.

- It was quantum mechanically solved by Krönig-Penney and thus its called Krönig - Penney model. They obtained energy band spectrum as follows

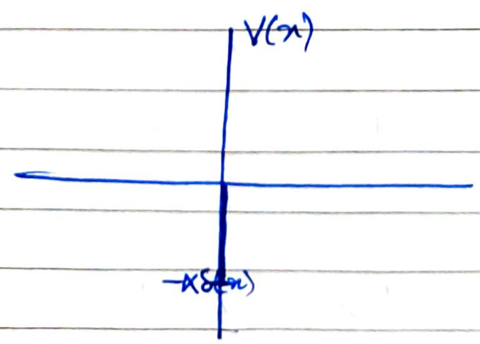


As from this diagram, it was deduced that we have 2 bands of energy. The valence (lower) band and conduction (upper) band

- **Proper Definition and Properties of Delta function Potential**

- $$\delta(x) = \begin{cases} \underset{\text{scalable}}{\infty} & \text{at } x=0 \\ 0 & \text{otherwise} \end{cases}$$

- $$V(x) = -\alpha\,\delta(x)$$



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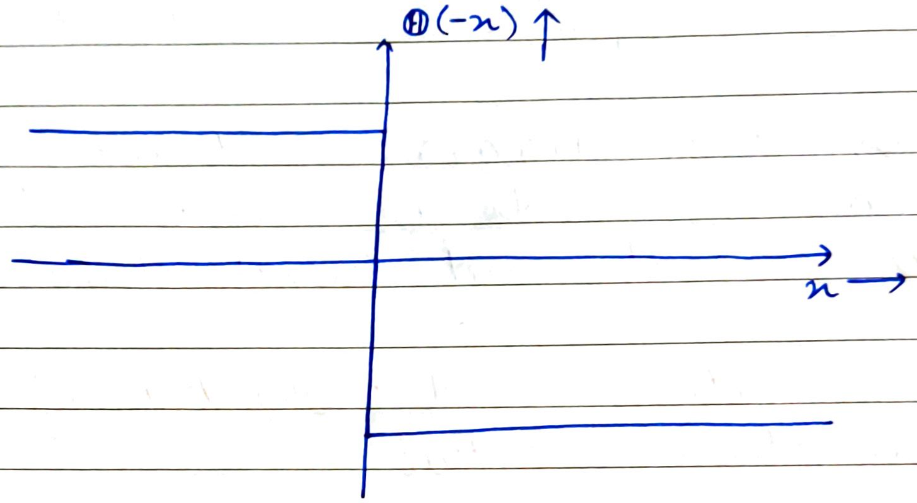
- We can express $\delta(x)$ as

$$-\delta(x) = \frac{d\Theta(-x)}{dx}$$

where $\Theta(-x)$ is like



- Some properties shown by $\delta(x)$

a) $$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

b) $$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)$$

c) $$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, dx = f(a)$$

d) $$\int \delta(x-a)\, f(x)\, dx = f(a)$$ (with limits containing a in interval)

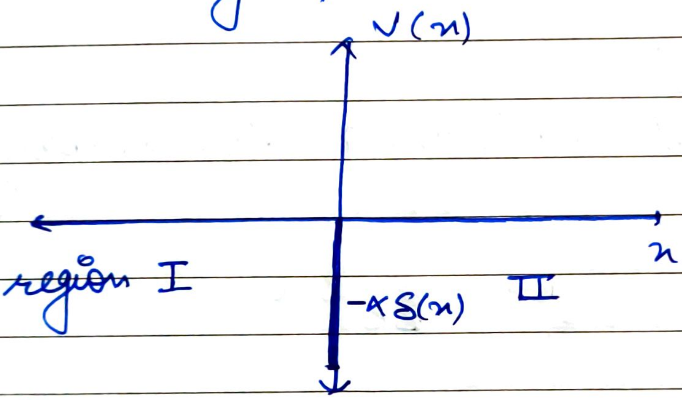
eg $$\int_{-3a}^{3a} \delta(x-a)\, f(x)\, dx = f(a)$$

eg- $\int_{-5}^{-1} \delta(x-4)\, x^2 dx = 0$

but $\int_{-5}^{-1} \delta(x+4)\, x^2 dx = 16$

- now we will discuss attractive δfunction potential

- we have a graph like



FOR BOUND STATES

- considering bound states ie $E<0$, (since $V(\infty)$ & $V(-\infty) = 0$)

In region I

$V(x) = 0$

so, time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I(x)}{dx^2} = E\psi_I(x)$$

$$\Rightarrow \frac{d^2\psi_I(x)}{dx^2} - K^2\psi_I(x) = 0 \quad \text{where } K^2 = -\frac{2mE}{\hbar^2}$$

(since $E<0$)

$\therefore$

$$\psi_I(x) = Ae^{Kx} + Be^{-Kx}$$

But here $x<0$
so $Be^{-Kx} \longrightarrow \infty$ for low values of x
but we know that $\psi(x)$ is bounded

$\therefore \quad B=0$

$$\therefore \boxed{\psi_I(x) = Ae^{Kx}}$$

- For region II

$V(x)=0$ and solving similarly, we get

$$\psi_{II}(x) = Ce^{Kx} + De^{-Kx} \quad ; \quad \left(K = \frac{-2mE}{\hbar^2}\right)$$

since $x>0$, $Ce^{Kx} \longrightarrow \infty$ as $x\to\infty$
& since $\psi(x)$ is bounded, therefore

$$C=0$$

$$\therefore \boxed{\psi_{II}(x) = De^{-Kx}}$$

- Now we know that $\psi(x)$ is continuous whatsoever.

$\therefore$ at $x=0$ it must be continuous.

$\therefore \quad Ae^{Kx} = De^{-Kx}$ at $x=0$

$\therefore A = D$

$\therefore$ we can write for the entire region

$$\boxed{\psi(x) = Ae^{-K|x|}}$$ everywhere.

- now we will normalize it:

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$\Rightarrow |A|^2 \left[\int_{-\infty}^{0} e^{2Kx} dx + \int_{0}^{\infty} e^{-2Kx} dx \right] = 1$$

$$\Rightarrow \frac{2|A|^2}{2K} = 1$$

$$\Rightarrow A = \sqrt{K}$$

$$\Rightarrow \boxed{\psi(x) = \sqrt{K}\, e^{-K|x|}}$$ everywhere

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- Integrate the Schrödinger eq^n once w.r.t. x in the range $-\epsilon$ to $+\epsilon$ (such that $\epsilon \to 0$)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\lim_{\epsilon\to 0}\left\{-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\Big|_{+\epsilon} - \frac{d\psi}{dx}\Big|_{-\epsilon}\right] - \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx\right\}$$

$$= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} E\psi(x)\,dx$$

Look at RHS

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} E\psi(x)\,dx = 0$$

($\because$ E & $\psi(x)$ both are finite)

On LHS

$$\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx = \psi(0)$$ (property remember)

v.v. imp.

$$\therefore \Longrightarrow \boxed{\Delta\left(\frac{d\psi}{dx}\right)\Big|_{x=0} = -\frac{2m\alpha}{\hbar^2}\psi(0)} \quad (i)$$

now

$$\psi(x) = \sqrt{K}\,e^{-K|x|} = \begin{cases}\sqrt{K}\,e^{-Kx} & ; x \geq 0\\ \sqrt{K}\,e^{Kx} & ; x \leq 0\end{cases}$$

$\therefore$ from (2), $\Delta\left(\frac{d\psi}{dx}\right) = -2K\sqrt{K}$ at $x=0$

$$\Rightarrow -2K\sqrt{K} = \frac{-2m\alpha\sqrt{K}}{\hbar^2}$$

$$\Rightarrow K = \frac{m\alpha}{\hbar^2}$$

$$\therefore K^2 = \frac{m^2\alpha^2}{\hbar^4} = -\frac{2mE}{\hbar^2}$$

$$\therefore \boxed{E = -\frac{m\alpha^2}{2\hbar^2}} , \quad \boxed{\psi(x) = \sqrt{K}\, e^{-K|x|}}$$

So, wave function is unique and so a single delta func. potential will have one bound state. (ie only one energy state, which is bound)

$\therefore$ Analysis of δ f^n potential for bound state of energy is complete.

TUTORIAL

3.

$$\psi(x,t) = A e^{-a[(mx^2/\hbar)+it]}$$

a) $$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$\Rightarrow A^2 = \frac{2am}{\pi\hbar}$$

$$\Rightarrow A = \sqrt{\frac{2am}{\pi\hbar}}$$

b) We know Schrödinger ~~time indep~~ equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

From $\psi(x,t)$, find $\frac{\partial \psi}{\partial x}$, $\frac{\partial^2 \psi}{\partial x^2}$, $\frac{\partial \psi}{\partial t}$

ans $$\boxed{V(x) = (2a^2 m) x^2}$$

2. $\rho(x) = A e^{-\lambda(x-a)^2}$

a) $$\int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2} dx = 1$$

$$\Rightarrow A\sqrt{\frac{\pi}{\lambda}} = 1$$

$$\Rightarrow A = \sqrt{\frac{\lambda}{\pi}}$$

b) $$\langle x \rangle = \int_{-\infty}^{\infty} x \rho(x)\, dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2} dx$$

$$= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (y+a)\, e^{-\lambda y^2} dy$$

0 (since function is odd)

$$= \sqrt{\frac{\lambda}{\pi}} \left[\cancel{\int_{-\infty}^{\infty} y\, e^{-\lambda y^2} dy} + \int_{-\infty}^{\infty} a e^{-\lambda y^2} dy \right]$$

$$= \sqrt{\frac{\lambda}{\pi}}\, a \sqrt{\frac{\pi}{\lambda}} = a$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \cancel{x} x^2 \rho(x)\, dx$$

integral $= 0$ ($\because$ odd func)

$$= \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{\infty} (u^2 + \cancel{2au} + a^2) e^{-\lambda u^2} du \right]$$

now

$$\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du$$

use Integration by parts or write as

$$-\frac{\partial}{\partial \lambda} \left[\int_{-\infty}^{\infty} e^{-\lambda u^2} du \right]$$

$$= -\frac{\partial}{\partial \lambda} \sqrt{\frac{\pi}{\lambda}} = \frac{\sqrt{\pi}}{2} \lambda^{-3/2}$$

$$= \sqrt{\frac{\lambda}{\pi}} \left[\frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + 0 + a^2 \sqrt{\frac{\pi}{\lambda}} \right]$$

$$= \left(\frac{1}{2\lambda} + a^2 \right)$$

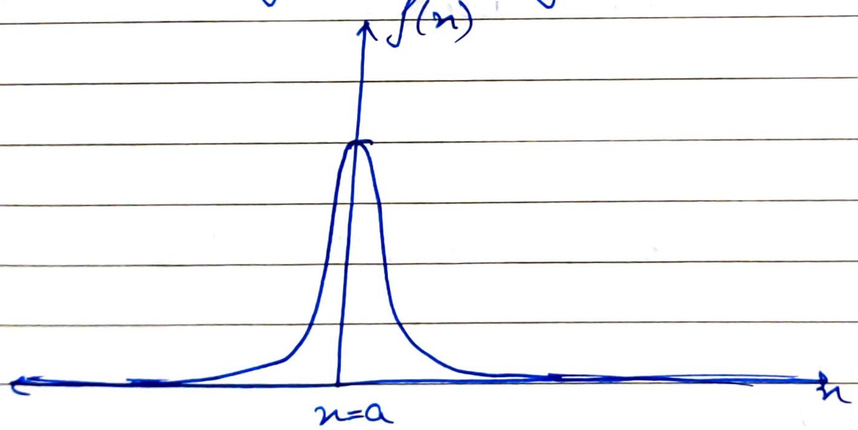
$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= a^2 + \frac{1}{2\lambda} - a^2$$

$$= \frac{1}{2\lambda}$$

$$\therefore \Delta x = \frac{1}{\sqrt{2\lambda}}$$

c) If we plot $\rho(x)$, we get



1. $\psi(x) = 2\alpha\sqrt{\alpha}\, e^{-(\alpha)(x)} \; ; \; x > 0$

$= 0$ elsewhere

a) $\rho(x) = |\psi(x)|^2$

$$= 4\alpha^3 x^2 e^{-2\alpha x} \; ; \; x > 0$$

$= 0$ elsewhere

At max $\rho(x)$

$$\frac{d}{dx} |\psi(x)|^2 = 0$$

$$\Rightarrow x = 0 \;,\; 1/\alpha$$

$$\frac{d^2}{dx^2} \rho(x) < 0$$

at $x = 1/\alpha$

$\therefore$ maxima at $x = 1/\alpha$

b) $$\langle x \rangle = \int_{-\infty}^{\infty} x \rho(x)\, dx = \int_{-\infty}^{\infty} x |\psi(x)|^2 \, dx$$

$$= \int_{0}^{\infty} x \left(4\alpha^3 x^2 e^{-2\alpha x}\right) dx$$

Let $y = 2\alpha x$

$$= \frac{1}{4\alpha} \int_{0}^{\infty} y^{4-1} e^{-y} \, dy$$

now we have a function called gamma function as

$$\Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x} \, dx$$

$$= (n-1)!$$

$\therefore \Gamma(1) = 1 \quad ; \quad \Gamma(n+1) = n\,\Gamma(n)$

$\therefore$ from our equation

$$\langle x \rangle = \frac{1}{4\alpha}\,\Gamma(4) = \frac{3!}{4\alpha} = \frac{3}{2\alpha}$$

$$\langle x^2 \rangle = \frac{1}{8\alpha^2}\int_0^\infty y^{5-1} e^{-y}\,dy$$

$$= \frac{4!}{8\alpha^2} = \frac{3}{\alpha^2}$$

c) Probability of finding b/w $x=0$ and $x = 1/\alpha$

$$= \int_0^{1/\alpha} (4\alpha^3)\, x^2 e^{-2\alpha x}\,dx$$

$$= \frac{1}{2}\int_0^2 y^2 e^{-y}\,dy$$

$$= 0.3233 \text{ (approx.)}$$

d) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty e^{-ip_x x/\hbar} \psi(x)\, dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty x\, e^{-(\alpha + ip_x/\hbar)x}\, dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\partial}{\partial\alpha} \left[\int_0^\infty e^{-(\alpha + ip_x/\hbar)x}\, dx \right]$

$=$

TUTORIAL 8

1. a) $\psi(x) = \dfrac{A}{x^2+a^2}$

and we know

Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = \underbrace{E}_{0}\psi(x) = 0$$

Double diff. ψ & substitute to get

$$V(x) = \frac{\hbar^2}{m}\frac{(3x^2-a^2)}{(x^2+a^2)^2}$$

b) $V = \alpha^2 x^2$ and $\psi(x) = \exp\left(-\sqrt{\dfrac{ma^2}{2\hbar^2}}\,x^2\right)$

Schrödinger eq^n

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

We double differentiate ψ and substitute values to get

$$E = \sqrt{\frac{\alpha^2\hbar^2}{2m}}$$

2. For

$0 \leq x \leq L$

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\langle p_x \rangle = \int \psi^*(x) \hat{p}_x \psi \, dx$$

$$= \frac{2}{L}(-i\hbar)\left(\frac{n\pi}{L}\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= 0$$

Similarly for $\langle p_x^2 \rangle = \left(\frac{n\pi\hbar^2}{L}\right)^2$

5. $$\frac{d}{dt}\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx$$ for any ψ_1 and ψ_2

($\because$ integration is over dx and diff. is w.r.t dt, $\therefore$ we can write)

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\psi_1^* \psi_2) \, dx$$

$$= \int_{-\infty}^{\infty} \left[\left(\frac{\partial \psi_1^*}{\partial t}\right)\psi_2 + \psi_1^*\left(\frac{\partial \psi_2}{\partial t}\right) dx\right] dx \quad \text{(i)}$$

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now Schrödinger eqn

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V\psi \quad \text{(ii)}$$

complex conjugate of (ii) gives

$$\frac{\partial \psi^*}{\partial t} = \frac{-i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V\psi^* \quad \text{(iii)}$$

using (ii) and (iii) in eqn (i)

$$= \frac{-i\hbar}{2m} \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 \psi_1^*}{\partial x^2}\right)\psi_2 - \psi_1^* \left(\frac{\partial^2 \psi_2}{\partial x^2}\right)\right] dx$$

using integration by parts

$$= \frac{i\hbar}{2m} \left[\left(\frac{\partial \psi_1^*}{\partial x}\right)\psi_2 \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{d\psi_1^*}{dx}\right)\left(\frac{\partial \psi_2}{\partial x}\right)dx - \psi_1^*\left(\frac{\partial \psi_2}{\partial x}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial \psi_1^*}{\partial x}\right)\left(\frac{\partial \psi_2}{\partial x}\right)dx\right]$$

1st and 3rd terms will be zero because ψ almost varies negligibly at $x \to \infty, -\infty$

4th and 2nd terms cancel each other and we get

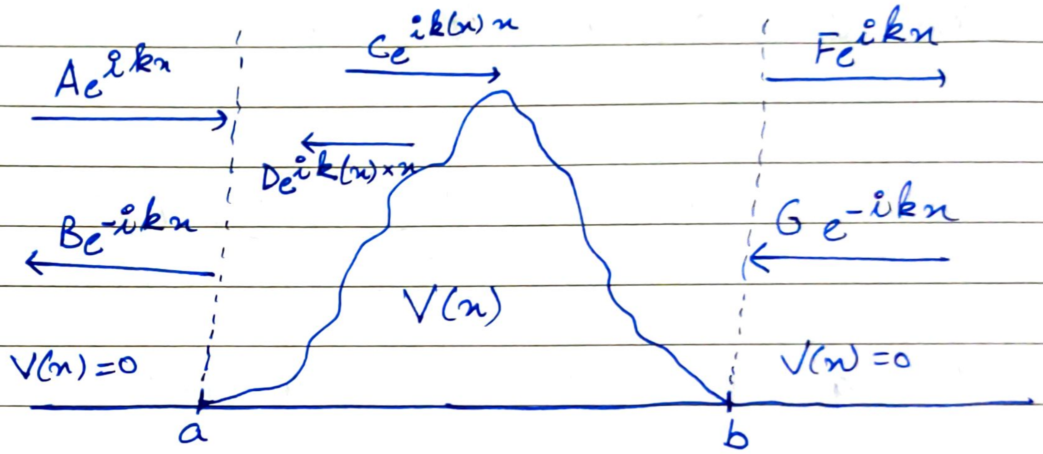
$$= 0$$

Attractive δ-f^n Potential (Positive energy states / scattering states)

~~Attractive δ-f^n potential~~

Writing the Transfer Matrix and Scattering Matrix

- Consider a situation where we represent potential $V(x)$ as



$k(x)$ denotes k as a function of x

- In above diagram, we assume

  Ae^{ikx} to be a wave coming in from left side and Ge^{-ikx} is a wave coming from right side.

- Fe^{ikx} would be a summation of the wave that crosses the potential barrier & coming from A and a wave from G which is reflected back from potential barrier.

- Be^{-ikx} is a similar summation.

- What happens between the coordinates (a, b) is very complicated and is not yet properly understood.

- To avoid calculations between a and b, we can compare the waves on both sides of the potential barrier in matrix form

$$\boxed{\begin{pmatrix} F \\ G \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\text{Transfer matrix}} \begin{pmatrix} A \\ B \end{pmatrix}}$$

This is how we relate F and G with A and B

and $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is called transfer matrix

$$\boxed{\begin{pmatrix} B \\ F \end{pmatrix} = \underbrace{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}}_{\text{Scattering matrix}} \begin{pmatrix} A \\ G \end{pmatrix}}$$

This is how we relate incoming and outgoing waves.

- The above matrix representation is useful when we have multiple potential barriers

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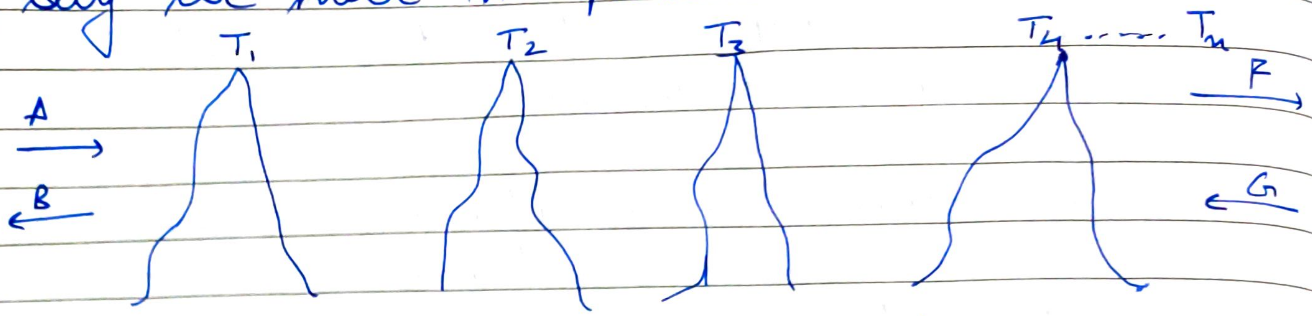
Say we have n potential barriers

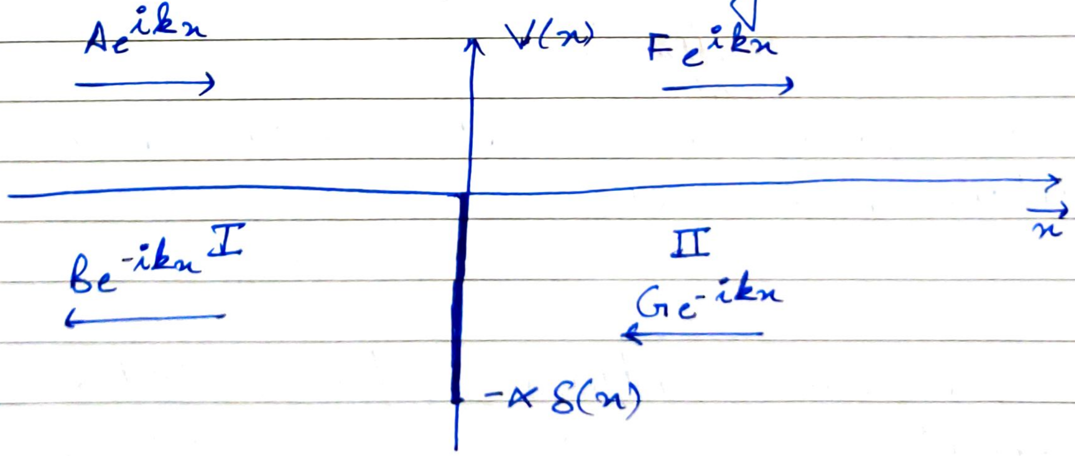


We can write the transfer matrix eq^n for it as

$$\boxed{\begin{pmatrix}F\\G\end{pmatrix} = T_n T_{n-1} \cdots\cdots T_2 T_1 \begin{pmatrix}A\\B\end{pmatrix}}$$

- Now coming to our original motive to analyze attractive δ f^n potential for +ve energy states / scattering states.

We have a barrier diagram as



$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad (\because E > 0)$$

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$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$ → solution to Schrödinger eqⁿ formed for this case

$\psi_{II}(x) = Fe^{ikx} + Ge^{-ikx}$ i.e. $\frac{d^2\psi}{dx^2} + k^2\psi = 0$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

Continuity of wavefunction at $x=0$

$\Rightarrow \quad A+B = F+G \quad -(i)$

Discontinuity of $\frac{d\psi}{dx}$ at $x=0$

we know

$$\Delta\left(\frac{d\psi}{dx}\right)\Bigg|_{x=0} = -\frac{2m\alpha}{\hbar^2}\psi(0) \quad (ii)$$

$$\frac{d\psi_I}{dx} = ikAe^{ikx} - ikBe^{-ikx}$$

$$\frac{d\psi_{II}}{dx} = ikF(e^{ikx}) - ikGe^{-ikx}$$

$\therefore$ from (ii)

$$\Rightarrow ik\left(F-G-A+B\right) = -\frac{2m\alpha}{\hbar^2}(F+G)$$

$$ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(F+G) \quad (iii)$$

~~Note~~

- We will try and find the relations between F, G, A and B and find the scattering and transfer matrices.

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Let $\frac{m\alpha}{\hbar^2 k} = \Gamma$

Add (i) and (iii)

$$B = i\Gamma F + (1+i\Gamma)G \quad \text{-(iv)}$$

Now (i) - (ii)

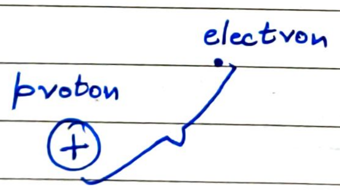
$$\Rightarrow -F + G + 2A = (1-2i\Gamma)F + (1-2i\Gamma)G$$

$$A = (1-i\Gamma)F - i\Gamma G \quad (iv)$$

⋮

In such a manner we can find T and S matrices easily by comparison of equations.

THE HYDROGEN ATOM ("Quantum mechanical Kepler problem")

- 

$$\mu = \frac{M_p m_e}{M_p + m_e}$$

μ is very close to 1 but slightly less than 1.

- $V = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$

We need to formulate & study Schrödinger eq^n for $V = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$

- Schrödinger eqⁿ in 3D cartesian coordinates.

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\hat{}\,\psi(x,y,z)+V(x,y,z)\,\psi(x,y,z)$$

$$= E\,\psi(x,y,z)$$

We need to convert this into spherical polar coordinates since it is ~~easier~~ very difficult to solve this in cartesian form

- We have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) \equiv \nabla^2 \equiv \hat{O}(r,\theta,\phi)$$

an ~~operator~~ in curvilinear coord. systems

We will try & find $\hat{O}$

- Recall,

$$dV = r^2 \sin\theta\, d\theta\, d\phi\, dr = h_1 du_1\, h_2 du_2\, h_3 du_3$$

Cartesian: $(u_1, u_2, u_3) = (x, y, z)$

$$\& \; h_1 = h_2 = h_3 = 1$$

Spherical polar : $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$

$$\& \; h_1 = h_r = 1;\; h_2 = h_\theta = r;\; h_3 = h_\phi = r\sin\theta$$

Now

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\boxed{\Rightarrow \nabla \equiv \frac{\hat{u}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial}{\partial u_3}}$$ (in general)

$$\nabla^2 \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_1 h_2 h_3}{h_1^2}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_2 h_3}{h_2^2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2 h_3}{h_3^2}\frac{\partial}{\partial u_3}\right)\right]$$

- Implementing this in Schrödinger eq^n using sph. polar coordinates.

$$\left[-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} - \frac{Z}{r}\right]\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi) \quad -(i)$$

Using method of separation of variables

a) $\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$ use this

b) Substitute in (i)

c) Divide throughout by $R(r)\,\Theta(\theta)\,\Phi(\phi)$

d) Keep all r dep. terms to LHS and shift the rest to RHS

e) Equate both sides equal to a constant.

$\Rightarrow \cancel{\left(\frac{1}{R(r)} \left(\frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) \right) + \frac{2\mu Z}{\hbar^2 r} \right) \right)}$

$\Rightarrow$ Applying the steps & multiplying by $-\frac{2\mu}{\hbar^2}$

$$\frac{1}{R(r)} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) + \frac{1}{\Theta(\theta)} \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta(\theta)}{\partial \theta}\right)$$

$$+ \frac{1}{\Phi(\varphi)} \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi(\varphi)}{\partial \phi^2} + \frac{2Z\mu}{\hbar^2} = -\frac{2\mu}{\hbar^2} E \qquad (ii)$$

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WAVE PARTICLE DUALITY →

- wave like: light , sound

Particle equivalent: photons , phonons

- 1st quantisation: particles $\xrightarrow{\text{Schrödinger Eq}^n}$ Ψ ("wave-like")

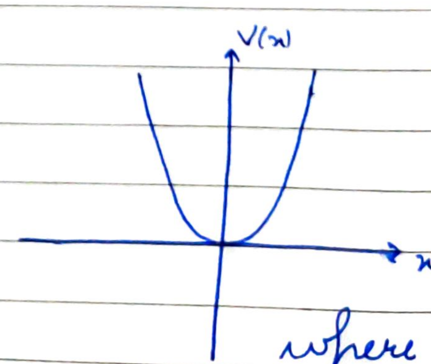
2nd quantisation: Waves → ? → particles

We will try and find what will help us out in this conversion.

NOTE: PHOTON - quanta of vibrating $\vec{E}$ and $\vec{B}$

- We consider the case of a HARMONIC OSCILLATOR in 1-D

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$$



Here note that p_x and x are both operators

where $\omega_0 = \sqrt{\frac{k}{m}}$

- We define 2 dimensionless operators

$$P = \frac{1}{p_0} p_x$$

$$Q = \frac{1}{x_0} x$$

(where function of p_0 and x_0 is to make P and Q dimensionless)

- To find dimensions of p_0 and x_0,

$$p_0 = m^{\alpha} \omega_0^{\beta} \hbar^{\gamma}$$

$$= [M]^{\alpha} [T^{-1}]^{\beta} [ML^2T^{-1}]^{\gamma}$$

$$\Rightarrow MLT^{-1} = M^{\alpha+\gamma} T^{-\beta-\gamma} L^{2\gamma}$$

$$\therefore \alpha = 1/2 ; \beta = 1/2 \ \& \ \gamma = 1/2$$

$$\therefore \boxed{p_0 = \sqrt{m\omega_0 \hbar}}$$

$$x_0 = m^a \omega_0^b \hbar^c = [M]^a [T^{-1}]^b [ML^2T^{-1}]^c$$

$$\Rightarrow L = M^{a+c} L^{2c} T^{-b-c}$$

$$\therefore a = -1/2 ; \ b = -1/2 ; \ c = 1/2$$

$$\therefore \boxed{x_0 = \sqrt{\frac{\hbar}{m\omega_0}}}$$

- Now we write H in terms of P and Q

$$\therefore \boxed{H = \frac{\hbar\omega_0}{2} (P^2 + Q^2)}$$

now we know $[x, p_x] = i\hbar$

$$\therefore [Q, P] = \frac{1}{x_0 p_0} [x, p_x] = i$$

Now

$$H = \frac{\hbar\omega_0}{2}(P^2+Q^2)$$

$$= \frac{\hbar\omega_0}{2}\left(P^2+Q^2 + i\{QP - iPQ - i[Q,P]\right)$$

$$\Rightarrow H = \hbar\omega_0\left(\left(\frac{Q-iP}{\sqrt{2}}\right)\left(\frac{Q+iP}{\sqrt{2}}\right) - \frac{i}{2}[Q,P]\right)$$

$$\Rightarrow H = \hbar\omega_0\left[a^+a + 1/2\right]; \text{ where } \quad a^+ = \frac{1}{\sqrt{2}}(Q-iP)$$

$$a = \frac{1}{\sqrt{2}}(Q+iP)$$

$$\Rightarrow \boxed{H = \hbar\omega_0\left[a^+a + 1/2\right]}$$

$$Q = \sqrt{\frac{m\omega_0}{\hbar}}\,x$$

$$P = \frac{1}{\sqrt{m\omega_0\hbar}}\,p_x$$

- This solving was done by Sakurai, a Japanese Scientist

$a^+ \equiv$ "Brahma" Creation operator

$a \equiv$ "Shiva" Destruction operator

$\boxed{N = a^+a} \equiv$ "Vishnu" Number operator

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By definition, we write

an operator

N such that

$$N\underbrace{|n\rangle}_{\text{a state with } n\text{-particles}} = \overbrace{n}^{\text{number}}|n\rangle$$

state (ket-n)

This operator operates on $|n\rangle$

- now we find

$$[a^+, a] = \frac{1}{2}\,[Q - iP,\; Q + iP]$$

$$= \frac{1}{2}\left\{\cancel{[Q,Q]}^{\,0} + i\,[Q,P] - i\,[P,Q] - i^2\,\cancel{[P,P]}^{\,0}\right\}$$

$$= -1$$

$$\therefore \boxed{[a^+, a] = -1} \;;\; \boxed{[a, a^+] = 1}$$

and

$$[N, a^+] = [a^+a, a^+] = a^+[a, a^+] + [a^+, a^+]a = a^+$$

and $[N, a] = [a^+a, a] = a^+[a,a] + [a^+, a]a = -a$

$$\therefore \boxed{[N, a^+] = a^+} \;;\; \boxed{[N, a] = -a}$$

- Also we can write

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \sqrt{\frac{m\omega_0}{2\hbar}}\, x + \frac{i}{\sqrt{2m\omega_0\hbar}}\, p_x$$

$$a^+ = \frac{1}{\sqrt{2}}(Q - iP) = \sqrt{\frac{m\omega_0}{2\hbar}}\, x - \frac{i}{\sqrt{2m\omega_0\hbar}}\, p_x$$

$$\therefore \quad x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a + a^+)$$

$$p_x = i\sqrt{\frac{m\omega_0\hbar}{2}}\,(a - a^+)$$

- Look at relations on previous page and now we write an equation

$$[\hat{N}, \hat{a}^+]\,|n\rangle = \{\hat{a}^+|n\rangle\} \qquad \{\because [\hat{N}, \hat{a}^+] = a^+\}$$

$$\Rightarrow \hat{N}\{\hat{a}^+|n\rangle\} - \hat{a}^+\{\underbrace{\hat{N}|n\rangle}_{=n|n\rangle}\} = \{\hat{a}^+|n\rangle\}$$

$$\Rightarrow \hat{N}\{\hat{a}^+|n\rangle\} = (n+1)\{\hat{a}^+|n\rangle\}$$

$\therefore$ ket state $\hat{a}^+|n\rangle$ has $n+1$ states and thus we can try and write it in form $\rightarrow$ where c is a coeff that needs to be calculated.

$$\Rightarrow a^+|n\rangle = c\,|n+1\rangle \qquad -(i)$$

$$\Rightarrow (a^+|n\rangle)^+ = (c|n+1\rangle)^+$$

$$\Rightarrow \langle n|\hat{a} = \langle n+1|c^* \quad \text{(ii)}$$

From (i) and (ii)

$$\Rightarrow \langle n|\hat{a}\hat{a}^+|n\rangle = c^*c\,\langle n+1|n+1\rangle$$

$$\Rightarrow \langle n|\hat{a}^+\hat{a}+1|n\rangle = |c|^2\underbrace{\langle n+1|n+1\rangle}_{=1}$$

$$\Rightarrow (n+1)\langle n|n\rangle = |c|^2\langle n+1|n+1\rangle$$

$$\Rightarrow (n+1) = |c|^2$$

$$\Rightarrow c = \sqrt{n+1}$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\therefore \boxed{a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$\therefore$ a^+ is called creation operator since it operates in ket-state of n states and creates $n+1$ states.

- Now we again write an eq^n.

$$[\hat{N},\hat{a}] = -\hat{a}$$

Operating on $|n\rangle$

$$\Rightarrow \hat{N}\hat{a}|n\rangle - \hat{a}\hat{N}|n\rangle = -\hat{a}|n\rangle$$

$$\Rightarrow \hat{N}\{\hat{a}|n\rangle\} = (n-1)\{\hat{a}|n\rangle\}$$

$\therefore$ ket state $\hat{a}|n\rangle$ has $n-1$ states and thus we can try and write it in form ← where d is a coeff. that needs to be calculated.

$$\therefore \Rightarrow \hat{a}|n\rangle = d\,|n-1\rangle \quad \text{(iii)}$$

Take + of eqⁿ (iii)
(adjoint)

$$\langle n|\hat{a}^{+} = \langle n-1|\alpha^{*} \qquad \text{-(iv)}$$

Take dot product on both sides (eqⁿ iii & iv)

$$\Rightarrow \langle n|\hat{a}^{+}\hat{a}|n\rangle = \langle n-1|\alpha^{*}\alpha|n-1\rangle$$

$$\Rightarrow n\underbrace{\langle n|n\rangle}_{=1} = |\alpha|^2 \underbrace{\langle n-1|n-1\rangle}_{=1}$$

$$\therefore \Rightarrow |\alpha|^2 = n$$

$$\Rightarrow \alpha = \sqrt{n}$$

$$\therefore \boxed{a|n\rangle = (n)\,|n-1\rangle}$$

∴ a is destruction operator.

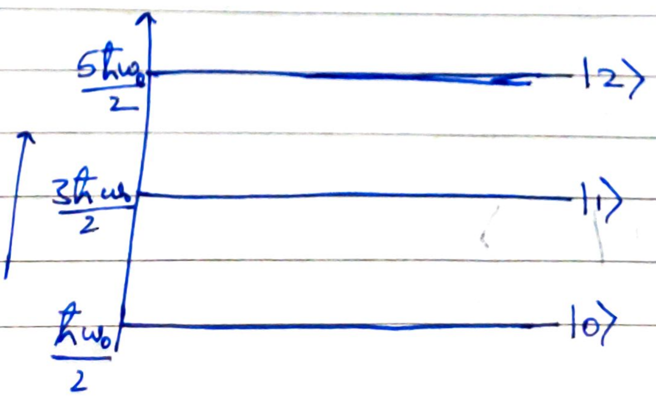
- consider Hamiltonian

$$\hat{H} = \left(a^{+}a + \frac{1}{2}\right)\hbar\omega_0$$

$$\Rightarrow \hat{H}|n\rangle = \left(n+\tfrac{1}{2}\right)\hbar\omega_0|n\rangle$$

~~For a system with ... states~~, we can write

$$E = \left(n+\tfrac{1}{2}\right)\hbar\omega_0$$



(ground state with no particles)

NOTE: $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$ & $a|0\rangle = 0$

since there is no lower state.

$\langle n|n\rangle =$

Now consider.

$$\langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle 0|(\hat{a} + \hat{a}^+)|0\rangle = 0$$

$\hat{a}|0\rangle = 0$

$\langle 0|\hat{a}^+|0\rangle = \langle 0|1\rangle = 0$

$$\langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega_0} \underbrace{\langle 0|aa + a^+a + aa^+ + a^+a^+|0\rangle}_{=1}$$

$$= \frac{\hbar}{2m\omega_0}$$

$$\langle 0|\hat{p}_x|0\rangle = -i\sqrt{\frac{m\omega_0\hbar}{2}} \langle 0|\hat{a} - \hat{a}^+|0\rangle = 0$$

$$\langle 0|\hat{p}_x^2|0\rangle = -\frac{m\omega_0\hbar}{2} \langle 0|\hat{a}\hat{a} + \hat{a}^+\hat{a}^+ - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}|0\rangle$$

$$= \frac{m\omega_0\hbar}{2}$$

$\therefore$

$$(\langle x^2\rangle - \langle x\rangle^2)(\langle p_x^2\rangle - \langle p_x\rangle^2) = \frac{\hbar}{2m\omega_0} \frac{m\omega_0\hbar}{2}$$

$$\Rightarrow \boxed{\Delta x \, \Delta p_x = \frac{\hbar}{2}}$$

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TUTORIAL 9

1. $\psi_1 = Ae^{ikx} + Be^{-ikx}$

$\psi_2 = Fe^{ikx} + Ge^{-ikx}$

cont. at $x=0$

$$A+B=F+G \qquad (i)$$

Also we know

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$\Rightarrow ik(F-A-G+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

Let $\frac{\cancel{2}m\alpha}{\hbar^2 k} = \Gamma$

$$\Rightarrow i(F+B-A-G) = -2\Gamma(A+B) \qquad (ii)$$

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

$\therefore B = S_{11}A + S_{12}G \quad ; \quad F = S_{21}A + S_{22}G$

We will try and find similar relations from (i) & (ii)

Putting F from (i) in (ii)

$$i(\cancel{A}+2B-G-\cancel{A}-G) = -2\Gamma(A+B)$$

$$\Rightarrow (i+\Gamma)2B = 2Gi - 2\Gamma A$$

$$\Rightarrow B = \frac{Gi}{i+\Gamma} - \frac{\Gamma A}{i+\Gamma}$$

$$\Rightarrow B = \frac{-G}{+i\Gamma-1} - \frac{i\Gamma A}{-1+i\Gamma}$$

$$\Rightarrow B = \frac{G}{1-i\Gamma} - \frac{i\Gamma A}{i\Gamma-1}$$

Also we can write from symmetry

$$F = \frac{A}{1-i\Gamma} - \frac{i\Gamma G}{i\Gamma-1}$$

$$\therefore S_{11} = \frac{i\Gamma}{1-i\Gamma} = S_{22} \quad ; \quad S_{12} = \frac{1}{1-i\Gamma} = S_{21}$$

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DISCUSSION

- Important que

1. Find degeneracy of 5^{th} excited state in a 3D potential well

Ans: Energy states:

ground ~~state~~ state : $(1,1,1)$
1^{st} ~~2nd~~ state : $(1,1,2)$, $(2,1,1)$, $(1,2,1)$
2^{nd} ~~3rd~~ state : $(1,2,2)$, $(2,1,2)$, $(2,2,$
3^{rd} ~~4th~~ state : $(1,1,3)$, $(1,3,1)$, $(3,1,$
4^{th} ~~5th~~ state : $(2,2,2)$
5^{th} state : $(1,2,3)$, $(3,2,1)$, $(3,1,2)$, $(2,$

$\therefore$ degeneracy of 5^{th} excited state ~~is~~ is

- TOPICS COVERED

→ Why QM?
- Black body radiation and Planck's hypothesis
- Photoelectric effect

→ Heisenberg uncertainty principle
- commutators
- Lagrangian, Hamiltonian

→ Schrödinger equation

- $H\psi = E\psi$
- TI Schrödinger eq^n (method of separation of variables)

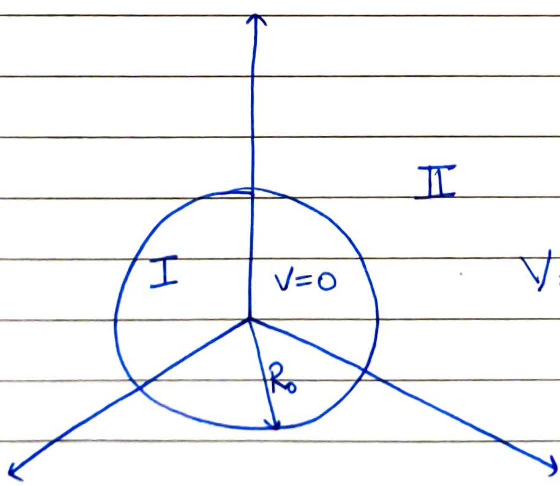
- Continuity Eq^n. $J, |\Psi|^2$
- $\phi(p_x)$

→ Applications

- Infinite potential well (2D & 3D generalisation)
- Bound & Scattering States
- δ-f^n potential
- H-atom
- Harmonic oscillator $\begin{cases} 2^{nd} \text{ quantized operation} \\ |n\rangle \;\&\; E_n \end{cases}$

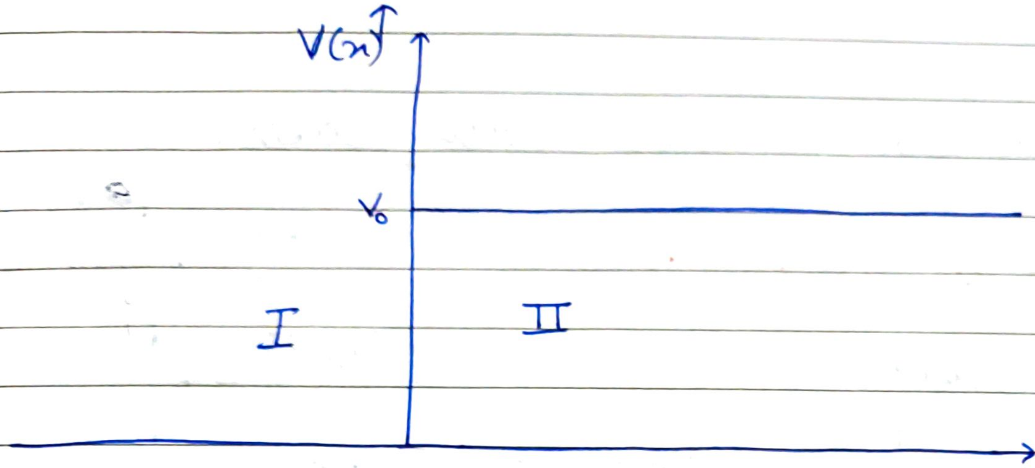
Q→ Solve for a spherical potential well of radius R_0

Ans.



Using sperical polar co-ordinates.

SCHRÖDINGER EQUATION FOR STEP POTENTIAL → AND ITS SOLUTION



- The type of potential we see, is a sort of θ function (remember) and we can write

$$\boxed{V(x) = V_0\,(\theta(x))}$$

- writing Schrödinger eqn. (time-independent)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\Rightarrow \boxed{\psi''(x) + \frac{2m(E-V(x))}{\hbar^2}\psi(x) = 0} \quad (i)$$

Imp

- For finite discontinuity in $V(x)$ at $x=a$, we have $\psi(x)$ and $\psi'(x)$ which are continuous at $x=a$

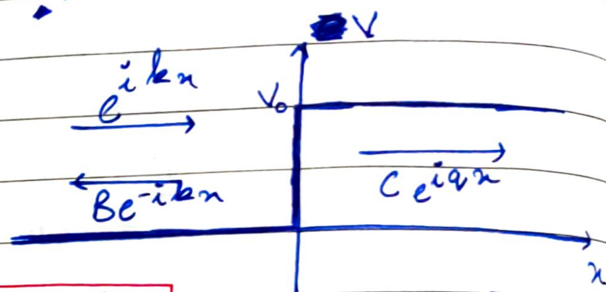
In above eqn, $E-V$ term is finite discontinuous & $\therefore$ $\psi''(x)$ must be finite discontinuous.
If we integrate above eqn wrt x, we will get continuous term, where ~~with~~ $V(x)$ is $\therefore$ $\psi'(x)$ will be continuous

- In region I $V(x) = 0$, $\therefore$ we have

$$\psi_I = Ae^{ikx} + Be^{-ikx} \quad ; \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

- Now we proceed with cases:

- **Case I:** $E > V_0$



- We write

$$\boxed{\psi_I = e^{ikx} + Be^{-ikx}} \quad ; \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

Here we take $A = 1$ forcefully to know the fraction being reflected back

and

$$\boxed{\psi_{II} = Ce^{iqx}} \quad ; \quad q = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

Here we take $D = 0$ since we assume incoming particle from left side only.

- Since ψ is continuous at $x = 0$;

$$\therefore \boxed{1 + B = C} \quad -(i)$$

Also ψ' is also continuous at $x = 0$;

$$\therefore \boxed{ik(1-B) = iqC} \quad (ii)$$

- From eq^n (i), & (ii) :

$$1 + B = C$$
$$1 - B = \frac{q}{k} C$$

$$\Rightarrow \boxed{C = \frac{2k}{k+q}}$$

$$\therefore \Rightarrow \boxed{B = \frac{k-q}{k+q}}$$

$$\therefore \boxed{C = \frac{2\sqrt{E}}{\sqrt{E} + \sqrt{E-V_0}}} \quad ; \quad \boxed{B = \frac{\sqrt{E} - \sqrt{E-V_0}}{\sqrt{E} + \sqrt{E+V_0}}}$$

now

- Simplification of B and C using binomial expansion

NOTE: Be very careful when you [illegible] else you could get horrendous results.

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- now we know

$$J(x) = \frac{i\hbar}{2m}\left[\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right]$$

We will obtain $J(x)$ in region I and II to find the reflection and transmission coefficient.

$$\boxed{r = \frac{J_{reflected}(x)}{J_{incident}(x)}}$$

$$\boxed{t = \frac{J_{transmitted}(x)}{J_{incident}(x)}}$$

- ∴ now for region I.

$$J_I(x) = \frac{i\hbar}{2m}\left[\frac{\partial\psi_I^*}{\partial x}\psi_I - \psi_I^*\frac{\partial\psi_I}{\partial x}\right]$$

$$= \frac{i\hbar}{2m}\Big[-ik\,(e^{-ikx} + B^* e^{ikx})(e^{ikx} + Be^{-ik}$$

$$- ik\,(e^{ikx} + B^* e^{ikx})(e^{ikx} - Be^{-ik}$$

$$= \frac{i\hbar}{2m}(-ik)2(1-|B|^2)$$

$\cancel{J_I(x)}$

$$\Rightarrow \boxed{J_I(x) = \frac{\hbar k}{m}\left(1-|B|^2\right) = J_{inc}(x) - J_{ref}(x)}$$

Here, carefully observe that

$$\boxed{J_{inc}(x) = \frac{\hbar k}{m}} \quad ; \quad \boxed{J_{ref}(x) = |B|^2 \frac{\hbar k}{m}}$$

- now for region II

$$J_{II}(x) = \cancel{\hbar q}\ \frac{i\hbar}{2m}\left[\frac{\partial \psi_{II}^*}{\partial(x)} \psi_{II} - \psi_{II}^* \frac{\partial \psi_{II}}{\partial x}\right]$$

$$\cancel{= \frac{i\hbar}{2m}\left[-iqC^* e^{-iqx} - iqC^* e^{-iqx} Ce^{iqx}\right]}$$

$$= \frac{i\hbar}{2m}\left[-iq C^* e^{-iqx} Ce^{iqx} - iq C^* e^{-iqx} Ce^{iqx}\right]$$

$$= \frac{i\hbar}{2m} 2\left[-iq |C|^2\right]$$

$$\Rightarrow \boxed{J_{II}(x) = \frac{\hbar q}{m}|B|^2 = J_{trans}(x)}$$

- now we will find reflection and transmission coefficient

PTO

$$r = \frac{J_{refl.}(x)}{J_{inc.}(x)} = \frac{\hbar k |B|^2}{\hbar k}$$

$$\Rightarrow r = |B|^2$$

$$\Rightarrow \boxed{r = \frac{(k-q)^2}{(k+q)^2}}$$

and

$$t = \frac{J_{trans}(x)}{J_{inc}(x)}$$

$$= \frac{\hbar q \, |C|^2}{\hbar k}$$

$$= \frac{q}{k} |C|^2$$

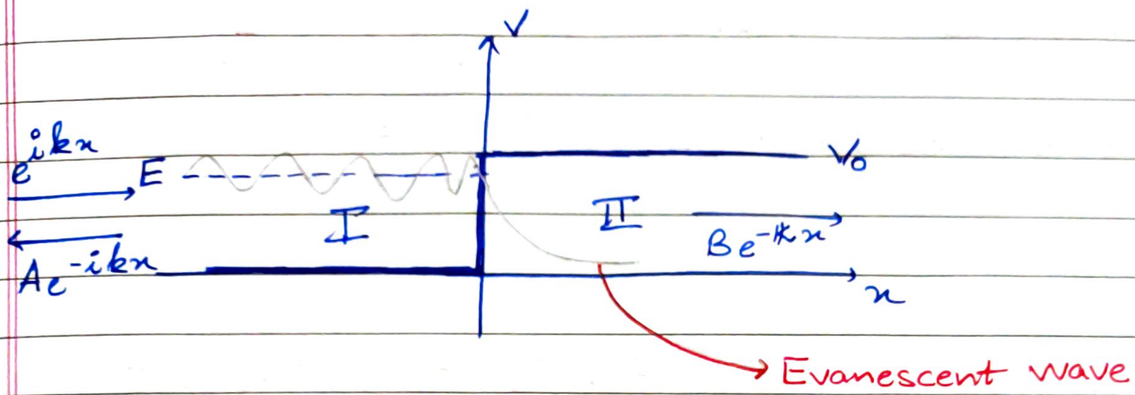
$$= \frac{4 q k^2}{k (k+q)^2}$$

$$\Rightarrow \boxed{t = \frac{4kq}{(k+q)^2}}$$

$\therefore$ for $E > V_0$

$$\boxed{r = \frac{(k-q)^2}{(k+q)^2}} \; ; \; \boxed{t = \frac{4kq}{(k+q)^2}}$$

• **Case II :** $E < V_0$



When we use Schrödinger eqⁿ in region I and region II, we will get ψ as

$$\boxed{\psi_I(x) = e^{ikx} + Ae^{-ikx}} \quad ; \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

&

$$\boxed{\psi_{II}(x) = Be^{-Kx}} \quad ; \quad K = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

note here that it is not complex

• now we write (using continuity of ψ and ψ' like before, in previous case)

$$\boxed{A = \frac{k - iK}{k + iK}} \quad \& \quad \boxed{B = \frac{2k}{k + iK}}$$

• now we proceed to write the current density in 2 regions

$$J_I(x) = \frac{\hbar k}{m}\left[1 - |A|^2\right]$$

derived similarly as in previous case

and

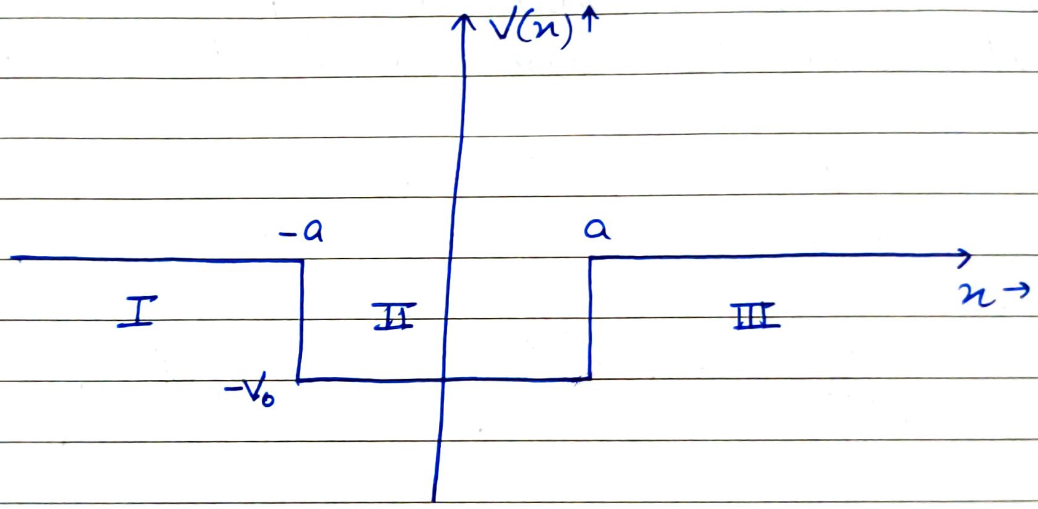
$$J_{II}(x) = 0$$

- now calculating 'r' & 't'

Reflection coefficient $\boxed{r = 1}$ $\left(\because |A|^2 = 1 \text{ ie } AA^* = 1\right)$

Transmission coefficient $\boxed{t = 0}$

★ → **BOUND STATE IN A FINITE POTENTIAL WELL**



$$V(x) = \begin{cases} -V_0 & \text{for } x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

- Since we are considering bound states

$$-V_0 \leq E \leq 0$$

- Also note that

$$V(x) = V(-x)$$

- We have an operator called Parity Operator P such that

$$P\psi(x) = \psi(-x) = \text{either } \underbrace{+\psi(x)}_{\substack{\text{even parity} \\ \text{(symmetric)}}} \text{ or } \underbrace{-\psi(x)}_{\substack{\text{odd parity} \\ \text{(anti-symmetric)}}}$$

- now we will write ~~the~~ Hamiltonian on parity of ψ

we know
$$\hat{H}\psi(x) = \underbrace{E}_{\text{where } E = \text{constant.}}\psi(x)$$

now

$$\hat{H}\hat{P}\psi(x) = \pm\hat{H}\psi(x) = \pm E\psi(x)$$

$$\hat{P}\hat{H}\psi(x) = E\hat{P}\psi(x) = \pm E\psi(x)$$

$$\therefore \Rightarrow [\hat{H}\hat{P} - \hat{P}\hat{H}]\psi(x) = 0$$

$$\Rightarrow [\hat{H}, \hat{P}] = 0$$

$\therefore$ If $\hat{H}$ commutes with an operator $\hat{O}_1$, and $\hat{H}\psi(x) = E\psi(x)$ then $\psi(x)$ is a simultaneous eigen function of $\hat{O}_1$.

- Now, we will write Schrödinger time indep. eq^n for region I, II and III

- For region I and III

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} - K^2\psi = 0 \qquad \text{where, } K^2 = \frac{-2mE}{\hbar^2}$$

$\therefore$ we have solution of form $\{e^{Kx}, e^{-Kx}\}$

- For region II

$$\frac{d^2\psi}{dx^2} + q^2\psi = 0 \qquad \text{where, } q = \sqrt{\frac{2m(V_0+E)}{\hbar^2}}$$

$\therefore$ we have sol^n of form $\{\sin qx, \cos qx\}$

- We could have an even parity ($\psi(x) =$ sol^n or odd parity sol^n ($\psi(x) = -\psi(-x)$)

PTO

NOTE: We define a quantity $\boxed{\alpha = \sqrt{\frac{2mV_0}{\hbar}}\, a}$

For even parity, the condition
$\psi(x) = \psi(-x)$
must be valid

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EVEN PARITY SOLUTION

since $\cos qx$ is an even func.

- $\psi = \begin{cases} A\cos qx & x \in [-a, a] \\ Be^{-Kx} & x \geq a \\ Be^{Kx} & x \leq a \end{cases}$

General soln.

- Now we check continuity of ψ at $x=a$ ~~...~~

$A\cos qa = Be^{-Ka}$ (i)

- Continuity of ψ' at $x=a$

$-qA\sin qa = -KBe^{-Ka}$ (ii)

- Taking ratio of (i) & (ii)

$$\boxed{\tan qa = \frac{Ka}{qa}}$$

$\psi(-x)$

- Also Ka

$$Ka = \sqrt{\alpha^2 - (qa)^2}$$

ODD PARITY SOLUTION

- $\psi = \begin{cases} A\sin qx & x \in [-a, a] \\ Be^{-Kx} & x \geq a \\ -Be^{-Kx} & x \leq -a \end{cases}$

gen. soln

At $x=a$

- Continuity of $x=a$

$A\sin qa = Be^{-Ka}$ (i)

- Continuity of ψ' at $x=a$

$qA\cos qa = -KBe^{-Ka}$ (ii)

- Taking ratio of (i) & (ii)

$$\tan qa = -\frac{qa}{Ka}$$

- Also $Ka = \sqrt{\alpha^2 - (qa)^2}$

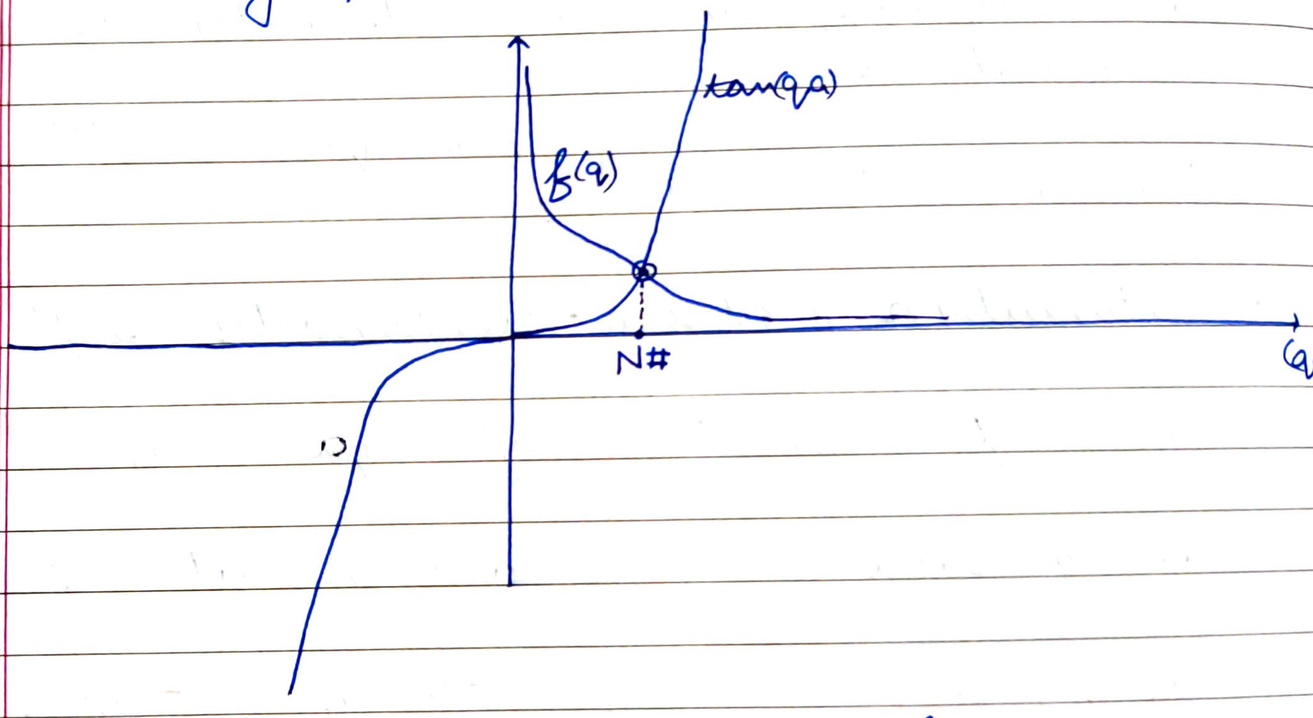
- Continuing for even parity solⁿ

∴ we have

$$\boxed{\tan(qa) = \underbrace{\frac{\sqrt{\alpha^2-(qa)^2}}{qa}}_{f(q)}}$$

Transcedental equation

now if we plot both LHS and RHS on a graph



∴ For such a condition, where we have a bound state in finite potential well, N# is one of the solutions

ie $qa = N\#$

$$\Rightarrow \sqrt{\frac{2m(E+V_0)}{\hbar^2}}\, a = N\#$$

$$\Rightarrow \boxed{E = -V_0 + \frac{N_\#^2\hbar^2}{2ma^2}}$$

- The graph of $f(q)$ ~~extends~~ extends further to ∞ and there might be many intersection points. All intersection points are different values of N# and when put in previous formula, this would give us different energy states for the quantum mechanical object in the given ~~the~~ finite potential well.

- Continuing for odd parity solⁿ

$\therefore$ we have

$$\tan(qa) = \frac{-qa}{\sqrt{\alpha^2 - (qa)^2}}$$