Chapter-2

Infinite Sequences & Series

* SEQUENCES →

- A list of ~~real~~ nos is called a sequence. like $a_1, a_2, a_3, a_4, \ldots a_n$. These are terms of the sequence.

- literal meaning is 'in order'.

- IMPORTANCE & USES

a) Used to represent a differentiable function $f(x)$ as an infinite sum of powers of x.

b) Representing a function as an infinite sum of sine and cosine functions (Fourier series

c) How to evaluate, differentiate, and integra class of functions much more general than polynomials.

d) Bisection method to find roots of non-linear equations uses sequences.

* INF

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→ INFINITE SEQUENCE

- An Infinite sequence of nos. is a function whose domain is set of infinite nos.

- Sequences can be represented as notations on a graph or on a real line as well.

- A converging sequence is the one which tends to obtain a specific limit value as n gets larger.

To prove if a ~~series~~ ~~as~~ sequence is convergent or not, we prove it in following way

- **Test for convergent series**

Suppose limit is L

We take a small no. $\varepsilon > 0$

Given $\varepsilon > 0 \ \exists \ N$
such that $\forall$
$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

$$\therefore \boxed{|a_n - L| < \varepsilon} \quad \text{for } \forall n \geq N$$

THEOREM 1 →

Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then

Sum rule: $\boxed{\lim_{n\to\infty} (a_n + b_n) = A + B}$

Difference rule: $\boxed{\lim_{n\to\infty} (a_n - b_n) = A - B}$

Product rule: $\boxed{\lim_{n\to\infty} (a_n \cdot b_n) = AB}$

Const. Multiple Rule: $\boxed{\lim_{n\to\infty} k b_n = kB}$

THEOREM 2 : (Sandwich Theorem) →

- Let $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ be sequences of real nos.

If $a_n \le b_n \le c_n$ holds for all 'n' beyond some index N and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$ also.

NOTE: An immediate consequence of Theorem 2 is that, if $|b_n| \le c_n$ and $c_n \to 0$ then $b_n \to 0$ because $\boxed{-c_n \le b_n \le c_n}$

↳ using theorem 2 on this

- eg-

THEOREM 3: (Continuous func. Theorem for Sequences)

- Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$

- eg – Sequence $\left(-\frac{1}{n}\right)$ & $f(n) = 2^n$

  ~~both~~ $f(a_n)$ converges to $f(0) = 1$

THEOREM 4: (Using L' Hopital)

- Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\boxed{\begin{aligned} &\lim_{x\to\infty} f(x) = L \\ \Rightarrow \ &\lim_{n\to\infty} a_n = L \end{aligned}}$$

- eg – $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

$$\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = e^2$$

NOTE: The following 6 sequences converge to limits listed below

- $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
- $\lim_{n \to \infty} n^{1/n} = 1$
- $\lim_{n \to \infty} x^n = 0 \quad (|x| < 1)$
- $\lim_{n \to \infty} x^{1/n} = 1 \quad (x > 0)$
- $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
- $\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

→ **THEOREMS:** (Non-decreasing sequence theorem)

A non-decreasing sequence converges if and only if it is bounded above; it converges to its least upper bound.

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CONVERGENT SEQUENCE a_n

A sequence $\{a_n\}$ of real nos. is said to converge to the real number L provided that "for each $\varepsilon > 0$ there exists a number N such that $\forall n \geq N(\varepsilon)$ implies $|a_n - L| < \varepsilon$."

~~PRO~~

- UNIQUE LIMIT

Convergent sequence has unique limit.

PROOF. Let seq. $\{a_n\}$ $\to L_1$, $\to L_2$ have 2 limits L_1 & L_2

Given $\varepsilon > 0$ $\exists$ N_1, N_2

such that $n > N_1 \Rightarrow |a_n - L_1| < \varepsilon/2$

$n > N_2 \Rightarrow |a_n - L_2| < \varepsilon/2$

$$|L_1 - L_2| = |a_n - L_2 - (a_n - L_1)| \leq |a_n - L_2| + |a_n - L_1|$$

$$< \frac{\varepsilon}{2} + \varepsilon/2 < \varepsilon$$

$\therefore$ It is very very small (smaller than 0)

& hence $L_1 = L_2$ since ε can be as small as possible.

- BOUNDED NESS

~~Lo~~ Convergent sequences are bounded.

PROOF. $\{a_n\} \longrightarrow L$

given $\varepsilon > 0$ $\exists N$

$$\forall n > N \Rightarrow |a_n - L| < \varepsilon$$

$$L - \varepsilon < a_n < L + \varepsilon$$

$$-|L| - \varepsilon < a_n < |L| + \varepsilon$$

now

$\forall n > N \Rightarrow |a_n| < |L| + \varepsilon$

let $\varepsilon = 1$

$\forall n > N \Rightarrow |a_n| < |L| + 1$

$$|a_{N+1}| < |L| + 1, \; |a_{N+2}| < |L| + 1 \; \ldots$$

$M = \max\{|a_1|, |a_2|, |a_3|, \ldots, |a_N|, |L| + 1\}$

$\therefore \boxed{|a_n| \leq M \quad \forall n \in \mathbb{N}}$

Hence proved.

PTO

- EXISTENCE OF MONOTONIC SUBSEQUENCE

- Every sequence has a monotonic subsequence.

- A subsequence is like a part of the sequence obtained on applying a particular rule.

  eg:- $a_{n_k} = \frac{1}{2k}$ is a subsequence of $a_n = \frac{1}{n}$

- BOLZANO-WEIERSTRASS THEOREM

- Every bounded sequence has a convergent subsequence.

QUESTIONS →

1Q: Let $p \in \mathbb{N}$, $a > 0$ and $a_1 \geq 0$. Define sequence $\{a_n\}$ by setting

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$$a_{n+1} = \frac{a_n(p-1) + \frac{a}{a_n^{p-1}}}{p}, \quad n \in \mathbb{N}$$

$$\lim_{n \to \infty} a_n$$

Ans. $a_{n+1} = \dfrac{\overbrace{a_n + a_n + a_n + \cdots + a_n}^{p \text{ times}} + \dfrac{a}{a_n^{p-1}}}{p}$

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$\Rightarrow a_{n+1} \geq \left(a_n^{p-1} \times \dfrac{a}{a_n^{p-1}}\right)^{1/p}$

$\Rightarrow a_{n+1} \geq a^{1/p} \geq 0 \ \forall n$ (i)

$a_{n+1} - a_n = \dfrac{(p-1)a_n + \dfrac{a}{a_n^{p-1}} - pa_n}{p}$

$\Rightarrow a_{n+1} - a_n = \dfrac{\overbrace{a - a_n^p}^{\leq 0}}{\underbrace{p\, a_n^{p-1}}_{>0}} \leq 0$ from (i)

∴ Sequence is non-increasing & bounded below, hence convergent.

→ Since $a_n > 0$ whatsoever but is non-increasing ∴ must be bounded.

Now

$\lim_{n\to\infty} a_n = L$

∴ In original eqⁿ we have

$\lim_{n\to\infty} a_{n+1} = \dfrac{1}{p}\left[\lim_{n\to\infty} a_n (p-1) + \lim_{n\to\infty} \dfrac{a}{a_n^{p-1}}\right]$

$\Rightarrow L = \dfrac{1}{p}\left[L(p-1) + \dfrac{a}{L^{p-1}}\right]$

$\Rightarrow \cancel{p}L^p = \dfrac{\cancel{p}}{\cancel{p}}\left[L^p(\cancel{p}-1) + a\right]$

$\Rightarrow L^p = a$

$\Rightarrow L = a^{1/p}$

Date
Page

Q2. Let $a_1 > b_1 \geq 0$. Define sequence $\{a_n\}$ & $\{b_n\}$ by setting

$$a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n} \quad (i), \qquad b_{n+1} = \frac{a_n + b_n}{2} \quad (ii), \quad n \in \mathbb{N}$$

Show that both $\{a_n\}$ and $\{b_n\}$ are convergent & hence show that both have same limit

Ans: First show by induction all terms are +ve

$$a_{n+1} - b_{n+1} = \frac{2a_n^2 + 2b_n^2 - a_n^2 - b_n^2 - 2a_nb_n}{2(a_n+b_n)}$$

$$= \frac{a_n^2 + b_n^2 - 2a_nb_n}{2(a_n+b_n)}$$

$$a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{2(a_n+b_n)} \geq 0$$

$$\Rightarrow a_n \geq b_n \quad \forall n$$

now for eq (i) given replace 'b_n' in b_n^2 by a_n

$$\Rightarrow a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n} \leq \frac{a_n^2 + b_na_n}{a_n + b_n} = a_n$$

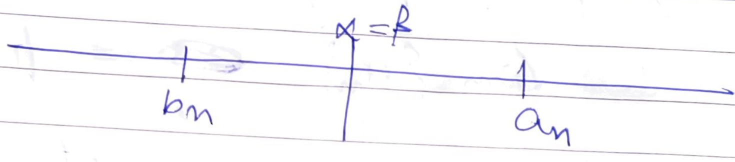
$$\Rightarrow a_{n+1} \leq a_n \quad (iii)$$

Similarly

$$b_{n+1} \geq \frac{b_n + b_n}{2}$$

$$\Rightarrow b_{n+1} \geq b_n \quad (iv)$$

Now since $a_n \geq b_n$ & looking at (iii) & (iv) we can say that b_n must have an upper bound & a_n must have a lower bound. $\therefore$ both are convergent



$$\lim_{n \to \infty} b_n = \beta$$

$$\lim_{n \to \infty} a_n = \alpha$$

$$\lim_{n \to \infty} b_{n+1} = \frac{1}{2}\left(\lim b_n + \lim a_n\right)$$

$$\lim_{n \to \infty} b_{n+1} = \frac{1}{2}\left(\lim_{n \to \infty} b_n + \lim a_n\right)$$

$$\beta = \frac{1}{2}\beta + \frac{1}{2}\alpha$$

$$\Rightarrow \cancel{\alpha} \; \alpha = \beta$$

Hence, proved.

classmate
Date
Page

TUTORIAL 2

1. $\lim u_n = u$
So, for $\varepsilon > 0$, $\exists\, n > N$ such that
$|u_n - u| < \varepsilon \quad \forall n \geq N$

Now, $\big||u_n| - |u|\big| \leq |u_n - u| < \varepsilon$
$\Rightarrow \big||u_n| - |u|\big| < \varepsilon$
$\therefore \lim \{u_n\} = |u|$

2. Let $\lim u_n = u$ & let $u < 0$
choose $\varepsilon > 0$ such that $u + \varepsilon < 0$ (i)
$\lim u_n = u$ $\exists\, k_1 \in \mathbb{N}$ s.t.
$|u_n - u| < \varepsilon \quad \forall n \geq K$
$u - \varepsilon < u_n < u + \varepsilon < 0$ (from i)
Let $k = \max\{K, m\}$
Then by hypothesis,
$u_n \geq 0 \ \forall (n \geq k)$
from $u_n < 0 \ \forall (n \geq k)$
$\Rightarrow \Leftarrow$

15.

$x_n = \sqrt{n}$

$\lim \left| \sqrt{n+1} - \sqrt{n} \right|$

$= \lim \left(\sqrt{n+1} - \sqrt{n} \right) \times \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right)$

$= \lim \left(\frac{1}{\sqrt{n+1} + \sqrt{n}} \right)$

$= 0$

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5. (i) $\log\left(\frac{n}{1+n}\right)$

$$-\varepsilon < \log\left(\frac{n}{1+n}\right) < \varepsilon$$

$$e^{-\varepsilon} < 1 - \frac{1}{1+n} < e^{\varepsilon}$$

ie $e^{-\varepsilon} - 1 < x_n < e^{\varepsilon} - 1$, $x_n = \frac{-1}{1+n}$

$\varepsilon > 0$, $e^{-\varepsilon} - 1 < 0$ & $e^{\varepsilon} - 1 > 0$

So $\exists$ m_1, m_2

$$0 < \frac{1}{m_1} < e^{\varepsilon} - 1 \; \&$$

$$0 < \frac{1}{m_2} < 1 - e^{-\varepsilon}$$

Take ~~/~~ $m = \max\left\{\frac{1}{m_1}, \frac{1}{m_2}\right\}$

Now we know $\lim \frac{-1}{1+n} = 0$

$$|x_n| < \frac{1}{m} \quad \forall n \geq k$$

$$-\frac{1}{m} < x_n < \frac{1}{m}$$

$$e^{-\varepsilon} - 1 < \frac{-1}{n+1} < e^{\varepsilon} - 1 \quad \left(\because e^{-\varepsilon} - 1 \leq \frac{-1}{m} \; \& \; e^{\varepsilon} - 1 \geq \frac{1}{m}\right)$$

$$\Rightarrow -\varepsilon < \log\left(\frac{n}{n+1}\right) < \varepsilon$$

$$\forall n \geq k$$

3.(i) $\lim (u_n + v_n) = u + v$

given $\lim u_n = u$

For $\frac{\varepsilon}{2} > 0$ $\exists$ $K_1 \in \mathbb{N}$ s.t

$|u_n - u| < \varepsilon/2$ $\forall n \geq K_1$

Again $\lim v_n = v$

For $\frac{\varepsilon}{2} > 0$, $\exists$ $k_2 \in \mathbb{N}$ s.t

$|v_n - v| < \varepsilon/2$ $\forall n \geq K_2$

now,

$$|(u_n + v_n) - (u+v)| = |(u_n - u) + (v_n - v)|$$

$$\leq |u_n - u| + |v_n - v|$$

$$< \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$$

$\forall n \geq \max\{K_1, K_2\}$

$\therefore \lim\limits_{n \to \infty} (u_n + v_n) = u + v$

3 (ii) Do yourself, its easy, you dumbo

(iii) $\cancel{\lim (u_n v_n) = uv}$

$\cancel{|u_n - u| < \varepsilon^{1/2}}$

$\cancel{\& \; |v_n - v| < \varepsilon^{1/2}}$

$\cancel{|u_n v_n - L| < \varepsilon}$

$\cancel{\Rightarrow |u_n v_n - L| < |u_n v_n - v u_n - u v_n + uv|}$

iii) $\lim u_n v_n = uv$

As u_n is convergent so it is bounded $\therefore |u_n| \leq M \;\forall n \in \mathbb{N}$

Now, $|u_n v_n - uv| = |u_n v_n - u_n v + u_n v - uv|$

$= |u_n(v_n - v) + v(u_n - u)|$

$\leq M|v_n - v| + |v||u_n - u|$

$\leq M\dfrac{\varepsilon}{2M} + |v|\dfrac{\varepsilon}{2|v|}$

$= \varepsilon$

iv)

iv) $\lim \frac{u_n}{v_n} = \frac{u}{v}$

First we will prove $\lim \frac{1}{v_n} = \frac{1}{v}$

Let $\alpha = \frac{|v|}{2}$, then $\alpha > 0$ & since $\lim v_n = v$

So, $\exists k_1 \in \mathbb{N}$ s.t. $|v_n - v| < \alpha \ \forall n \geq k_1$

we have, $||v_n| - |v|| \leq |v_n - v| < \alpha \ \forall n \geq k_1$

$|v| - \alpha < |v_n| < \alpha + |v| \ \forall n \geq k_1$

$|v_n| > |v| - \frac{|v|}{2} = \frac{|v|}{2}$ (i)

Now, $\left|\frac{1}{v_n} - \frac{1}{v}\right| = \left|\frac{v_n - v}{v_n v}\right| < \frac{2}{|v|^2}|v_n - v|$ (ii)

Since, $\lim v_n = v$ $\exists k_2 \in \mathbb{N}$ from (i)

s.t. $|v_n - v| < \frac{|v|^2 \varepsilon}{2} \ \forall n \geq k_2$

$\Rightarrow \frac{|v_n - v|}{\frac{|v|^2}{2}} < \varepsilon$

Let $k = \max\{k_1, k_2\}$ then

$\left|\frac{1}{v_n} - \frac{1}{v}\right| < \frac{|v_n - v|}{\frac{|v|^2}{2}} < \varepsilon \ \forall n \geq k$

$\therefore \lim \frac{1}{v_n} = \frac{1}{v}$

Now using (iii)

6) $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

$x_2 - x_1 = 1 > 0$

Adding

$x_3 - x_2 = \frac{1}{2}(x_2 + x_1) - x_2$

$= -\frac{1}{2}(x_2 - x_1)$

$x_4 - x_3 = \frac{1}{2}(x_3 + x_2) - x_3$

$x_4 - x_3 = \left(-\frac{1}{2}\right)^2 (x_2 - x_1)$

$\vdots$

$x_n - x_{n-1} = \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$

$\therefore x_n - x_1 = \left[1 + \left(-\frac{1}{2}\right) + \dots + \left(-\frac{1}{2}\right)^{n-2}\right]$

$= \dfrac{1 - \left(-\frac{1}{2}\right)^{n-2}}{1 + \frac{1}{2}} = \lim(x_n - x_1) = 2/3$ 8

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4. $0 < \alpha < 1$, To prove $\lim\limits_{n \to \infty} \alpha^n = 0$

given : $0 < \alpha < 1 \Rightarrow \frac{1}{\alpha} > 1$

Lim $\frac{1}{\alpha} = 1 + p$ for some $p > 0$

$\Rightarrow \alpha = \frac{1}{1+p} \Rightarrow |\alpha|^n = \frac{1}{(1+p)^n}$

we have, $(1+p)^n > 1 + np \ \forall\ n \in N$

(using binomial)

$$|x^n - 0| = \frac{1}{(1+p)^n} \leq \frac{1}{1+np} \quad \forall n \in N$$

choose $\varepsilon > 0$,

$$0 < \frac{1}{\varepsilon} < 1 + pn \Rightarrow \frac{1}{\varepsilon} - 1 < pn$$

$$\Rightarrow \frac{1-\varepsilon}{\varepsilon p} < n$$

By Archimedian property $\exists N \in N$ s.t.

$$N > \frac{1-\varepsilon}{\varepsilon p}$$

Then $\forall n \geq N$, $|x^n - 0| < \varepsilon$

$\therefore \lim x^n = 0$

8. Let $\{f(n)\}$ is mono decreasing seq. & m g.l.b

$$m \leq f(n) \;\forall n$$

for $\varepsilon > 0$, $\exists k \in N$ s.t. $m + \varepsilon > f(k)$

Cauchy Sequences

- Sequences whose terms are very very close to each other are called Cauchy sequences.

- Given $\varepsilon > 0$ $\exists$ N s.t.
  arbitrarily small
  $\forall\, m, n > N \quad |a_m - a_n| < \varepsilon$
  then $\{a_n\}$ is called Cauchy sequence.

Theorem 1:

- Convergent sequences are Cauchy sequences (Try to prove).

Theorem 2:

- Cauchy sequences are bounded. (Can be proved using definition of Cauchy seq. in point 2)

Theorem 3:

- Cauchy sequences of real nos. is convergent

Proof: Cauch seq. is bounded $q - x_n$
Also it has a monotonic sub-sequence (x_{n_k}) which is convergent & since its convergent, $\therefore$ x_n must be convergent. $\therefore$ Cauchy seq of real nos. is convergent

HW Q→ $x_1 = 1,\ x_2 = 2,\quad x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n \geq 2$.

Prove by induction that $1 \leq x_n \leq 2$.

Also show that (x_n) is not a monotone sequence.

\# Consider, in the above que.

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}}.$$

Taking $m > n$,

$$|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \dots + |x_{m-1} - x_m|$$

$$\Rightarrow \quad \leq \frac{1}{2^{n-1}} + \frac{1}{2^n} + \dots + \frac{1}{2^{m-2}}$$

$$\Rightarrow \quad \leq \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{m-n-1}}\right)$$

In worst case scenario, $m - n \longrightarrow \infty$

$$\Rightarrow \quad \leq \frac{1}{2^{n-2}} < \varepsilon$$ (when n is very very large)

Now

$x_3 = 1 + \frac{1}{2},\quad x_5 = 1 + \frac{1}{2} + \frac{1}{2^3},\quad x_7 = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5}$

$$x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots + \frac{1}{2^{2n-1}}$$

$$\Rightarrow x_{2n+1} = 1 + \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2^2}\right)^n\right)}{(1 - 1/4)}$$

$$\lim_{n\to\infty} x_{2n+1} = 1 + 2/3$$

$$= 5/3 \qquad = \lim_{n\to\infty} x_n$$

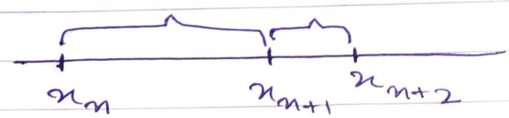
Contractive Sequences →

- A sequence is said to be a contractive sequence if it follows the following conditions

(x_n) if $\exists$ c such that $0 < c < 1$

and $\boxed{|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|}$

Then x_n is contractive.



→ Theorem 1:

- ~~For~~ every contractive sequence is Cauchy Sequence.

Proof

- $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n| \leq c^2|x_n - x_{n-1}| \cdots \leq c^n|x_2 - x_1|$

$$\Rightarrow |x_{n+2} - x_{n+1}| \leq c^n |x_2 - x_1| \qquad (i)$$

Now

triangle ineq

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots$$

In each term, we convert it to form of $|x_2 - x_1|$ using (i)

$$\Rightarrow |x_m - x_n| \leq (c^{m-2} + c^{m-3} + \cdots + c^{n-1})|x_2 - x_1|$$

$$\le c^{n-1}\left(1+c+c^2+\cdots+c^{m-n-1}\right)|x_2-x_1|$$

$$\le c^{n-1}\left(\frac{1-c^{m-n}}{1-c}\right)|x_2-x_1| < \frac{c^{n-1}}{1-c}|x_2-x_1|$$

As $n\to\infty$ ie n is large ie $n>N$

$m>n>N$

given $\varepsilon>0$ $\exists N$

$m>n>N$

$\Rightarrow |x_m-x_n|\to 0$ since RHS $\to 0$

$\therefore (x_n)$ is CAUCHY

→ FIBBONACCI FUNCTION →

- It is defined as $\boxed{x_n=\dfrac{f_n}{f_{n+1}}}$

where $f_1=f_2=1$ &

$$f_{n+2}=f_n+f_{n+1}$$

$$x_{n+1}=\frac{f_{n+1}}{f_{n+2}}=\frac{f_{n+1}}{f_n+f_{n+1}}=\frac{1}{1+x_n}$$

$$\boxed{x_{n+1}=\frac{1}{1+x_n}}$$

- Now we prove Fibonacci func. is contractive

$$|x_{n+2}-x_{n+1}|=\left|\frac{1}{1+x_{n+1}}-\frac{1}{1+x_n}\right|$$

Date
Page

$$\Rightarrow |x_{n+2} - x_{n+1}| = \frac{|x_n - x_{n+1}|}{(1+x_n)(1+x_{n+1})} \quad (i)$$

Now $\frac{1}{2} \leq x_n \leq 1$ → prove it (H.W)

$$\frac{1}{2} \leq \frac{1}{1+x_n} \leq \frac{2}{3}$$

∴ In (i), we can say

$$|x_{n+2} - x_{n+1}| = \frac{|x_n - x_{n+1}|}{(1+x_n)(1+x_{n+1})} \leq \frac{2}{3} \times \frac{2}{3} |x_n - x_{n+1}|$$

∴ x_n is contractive

- $\lim x_{n+1} = \lim \frac{1}{1+x_n}$

$$\Rightarrow x = \frac{1}{1+x}$$

$x^2 + x - 1 = 0$, ∴ $x = 0.618034$

Q→

Ans:

(i)

$x_n - x_{n+1}|$

Q. Find root of $x^3 - 7x + 2$.

Ans: $f(x) = x^3 - 7x + 2$

At $x = 0$, $f(x) = 2$

At $x = 1$ $f(x) = -4$

$\therefore$ we must have a root b/w 0 & 1

Now we assume sequence (relating to $f(x)$)

$$x_{n+1} = \frac{1}{7}(x_n^3 + 2)$$ $\because (0 \le x_n \le 1)$

$$|x_{n+2} - x_{n+1}| = \frac{1}{7}|x_{n+1}^3 - x_n^3|$$
$$= \frac{1}{7}|x_{n+1}^2 + x_{n+1}x_n + x_n^2||x_n - x_{n+1}|$$
$$\le \frac{3}{7}|x_{n+1} - x_n|$$

$$\Rightarrow |x_{n+2} - x_{n+1}| \le \frac{3}{7}|x_{n+1} - x_n|$$

Hence $\{x_n\}$ is contradiction

Now in sequence, we put

$n=1$, $x_2 = \frac{1}{7}(x_1^3 + 2) = \frac{1}{7}((0.5)^3 + 2)$

Initial guess

Similarly we carryout iterations for $n=2$, $n=3$ ..... uptil say $n=6$

Here we get

$$\boxed{x_6 = 0.28916857 1}$$

As n increases we get more and more accurate value of the root. Say if we are told to calculate uptill 4 decimal places, then we will carry out those many iterations only after which the 4 decimal places become constant.