30/7/19

MA201 - MATHEMATICS - III

Complex Analysis

* A complex number can be represented as

$$z = x + iy \quad (x, y \in \mathbb{R}) \; ; \; i = \sqrt{-1}$$

It can also be represented as an ordered pair (x, y).

It is represented geometrically on a gaussian plane or Argand plane

Addition of two complex numbers is also a complex number since the set of complex no.s ($\mathbb{C}$) is a field. Similarly for subtraction.

- $i \equiv (0, 1)$

- $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

(or)

$$(x_1, y_1) \cdot (x_2, y_2) = ((x_1x_2 - y_1y_2), (x_1y_2 + x_2y_1))$$

- $z = x + iy \Rightarrow \operatorname{Re}(z) = x, \; \operatorname{Im}(z) = y$

- $x + iy = (x, 0) + (0, 1)(y, 0)$

- $i = (0,1) \Rightarrow i^2 = (0,1) \cdot (0,1)$

$$= (0 - 1, \; 0 \cdot 1 + 1 \cdot 0)$$
$$= (-1, 0) \Rightarrow i^2 = -1$$

- z_1/z_2 exists only when $z_2 = x_2 + iy_2 \neq 0$

$z_1/z_2 = \alpha + i\beta$ (let us say), then

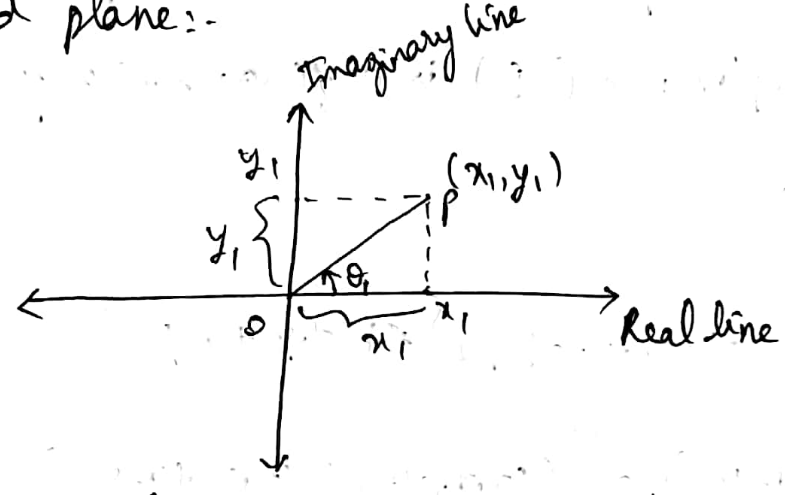
$z_1 = x_1 + iy_1 = (\alpha + i\beta)(x_2 + iy_2)$

Multiplying and equating real parts and imaginary parts on both sides gives α and β as solutions of simultaneous eqns.

$$\Rightarrow \quad \alpha = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \;; \qquad \beta = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

$x_1 + iy_1 = \alpha + i\beta$ only if $x_1 = \alpha$ and $y_1 = \beta$

* Argand plane:-



The complex number $z_1 = x_1 + iy_1$ represents a vector $\vec{OP}$

Modulus (or magnitude) of the complex no. / vector is given by

$\underset{\text{magnitude}}{r} = |z_1| = \sqrt{x_1^2 + y_1^2}$ (from pythagorus theorem)

Amplitude of the complex number,

Amp(z_1) represents angle (anticlockwise) from real line to the complex number.

$\text{Amp}(z_1) = \theta_1 = \tan^{-1}(y_1/x_1) \;;\; \tan(2n\pi + \theta_1) = y_1/x_1$

$\Rightarrow$ Amplitude is not unique for given magnitude.

$\Rightarrow$ Principal amplitude is limited as $-\pi \leq \theta \leq \pi$

* Polar representation of complex number :-

$$z_1 = r e^{i\theta}$$

$i\theta \to$ amplitude

r $\to$ magnitude

(De-Moivre's theorem :-

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where $n \in \mathbb{Z}$ (→ integer)

if $n \in \mathbb{Q}$ (rational) (p/q, $q \neq 0$) then one of the values is given by $\cos n\theta + i\sin n\theta$).

$$z_1 = re^{i\theta} = r(\cos\theta + i\sin\theta) \quad ; \quad |e^{i\theta}| = |\cos\theta + i\sin\theta|$$

r $\to$ mag.

$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

* Square root of complex no.:-

Let $z = x+iy$ be a complex number.

then $\sqrt{z} = \sqrt{x+iy}$. Let $(\alpha+i\beta)^2 = x+iy$

$\Rightarrow$ $\sqrt{z} = \alpha + i\beta$.

$\Rightarrow$ $(\alpha+i\beta)\cdot(\alpha+i\beta) = x+iy$

$\Rightarrow$ On solving, $\alpha^2 - \beta^2 = x$ — ①

$2\alpha\beta = y$ — ②

(It is biquadratic, verify all solns. at the end).

from ① and ②

Find the values of

(i) $(1+2i)^3$ (ii) $\frac{5}{-3+4i}$ (iii) $(1+i)^n+(1-i)^n$

$z=x+iy$, then find

(i) z^4 (ii) $\frac{1}{z}$ (iii) $\frac{z-1}{z+1}$ (iv) $\frac{1}{z^2}$

* Multiplication and division in polar form

$z_1 = x_1+iy_1 = r_1 e^{i\theta_1}$

$z_2 = x_2+iy_2 = r_2 e^{i\theta_2}$

$\Rightarrow z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

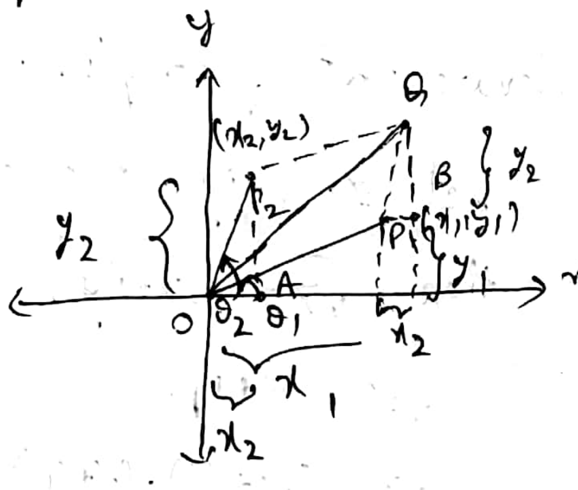
$\Rightarrow z_1/z_2 = r_1/r_2 \, e^{i(\theta_1-\theta_2)}$

* Graphical representation of operations :-

- $z_1+z_2 = \vec{OP_1}+\vec{OP_2}$

$= \vec{OP_1}+\vec{P_1Q}$

$= \vec{OQ}$ (from parallelogram law)



$\Delta^{le} OP_2A \cong \Delta^{le} OP_1B$ (RHS)

$\Rightarrow \vec{OP_1} = x_1+iy_1$; $\vec{P_1Q} = x_2+iy_2$

$\Rightarrow \vec{OP_1}+\vec{P_1Q} = (x_1+x_2)+i(y_1+y_2)$

and $\vec{OQ} = (x_1+x_2)+i(y_1+y_2)$ (larger diagonal)

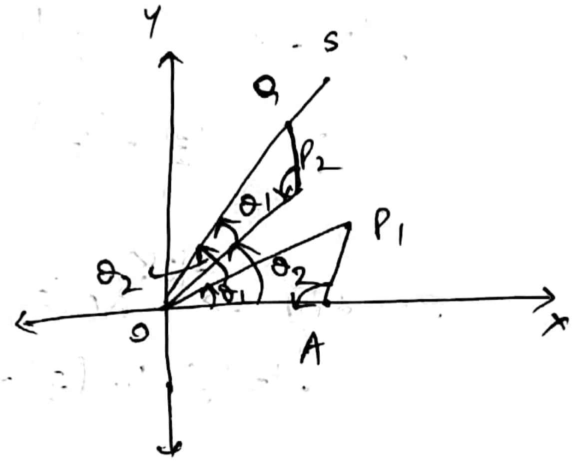
- $z_1 - z_2 = \vec{OP_1} - \vec{OP_2}$

$= \vec{OP_1} + \vec{P_2O}$ reverse vector

$\Rightarrow$ following similar process ($\cong$) as before,

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$ (shorter diagonal)

- $z_1 \cdot z_2 = \vec{OP_1} \cdot \vec{OP_2}$



Let a point A on the real line Ox is of unit modulus

Also, construction is made such that

$\angle OAP_1 = \angle OP_2Q$

$\Delta^{le}\, OAP_1 \cong \Delta^{le}\, OP_2Q$ (AAA) $\Rightarrow \frac{OP_1}{OA} = \frac{OQ}{OP_2}$

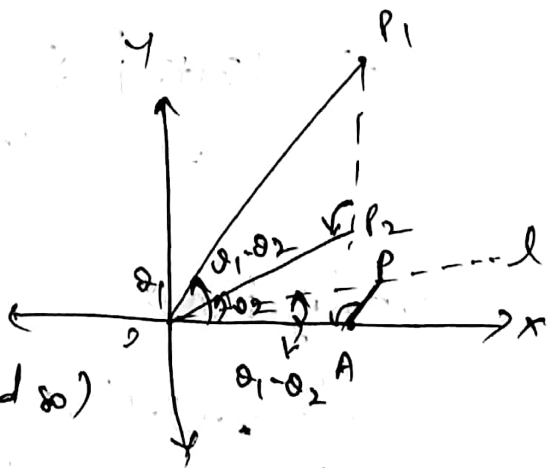
$\Rightarrow OP_1 \cdot OP_2 = OA \cdot OQ$; since $|OA| = 1 \Rightarrow OP_1 \cdot OP_2 = OQ$

$\Rightarrow OQ = |z_1| \cdot |z_2|$

Hence, it is clear that, $\text{amp}(z_1 z_2) = \theta_1 + \theta_2$

$|z_1 \cdot z_2| = r_1 r_2$

- $z_1 / z_2 = \vec{OP_1} / \vec{OP_2}$



$|OA| = 1$

line l is plotted such that

$\angle AOl = \theta_1 - \theta_2$.

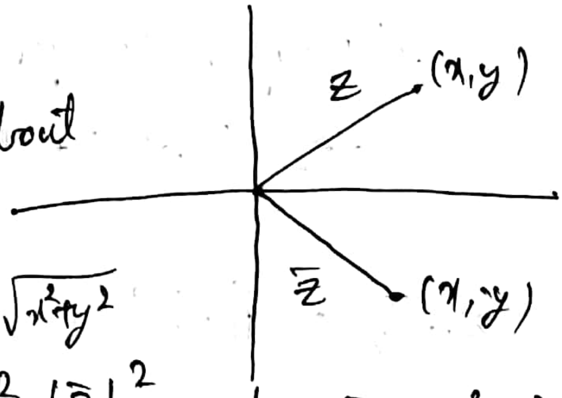
Then, also, $\angle OAP = \angle OP_2P_1$ (constructed so)

$\Rightarrow \Delta^{le}\, OAP \cong \Delta^{le}\, OP_2P_1$ (AAA) $\Rightarrow \frac{OP}{OA} = \frac{OP_1}{OP_2} = |z_1/z_2|$

$\Rightarrow \text{amp}(OP) = \theta_1 - \theta_2$, $OP = |z_1/z_2| = r_1/r_2$

* Conjugate of a complex no. :-

If $z = x + iy$, then conjugate of z is denoted by $\bar{z}$ and defined as $\bar{z} = x - iy$.



⇒ conjugate is reflection (image) about real axis.

$|z| = \sqrt{x^2+y^2}$ $\quad |\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2}$

$\Rightarrow \underline{|z| = |\bar{z}|} \Rightarrow \underline{|z|^2 = |\bar{z}|^2}$ and $\underline{z \cdot \bar{z} = x^2 + y^2 = r^2}$

$z = x + iy \Rightarrow -|z| \leq Re(z) = x \leq |z|$

$-|z| \leq Im(z) = y \leq |z|$

$\Rightarrow \quad |z_1| + |z_2| \geq |z_1 + z_2|$ (Δ^{les} inequality)

Arithmetically proving,

$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$\Rightarrow |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$

$\Rightarrow |z_1| + |z_2| \geq |z_1 + z_2|$ (taking +ve sqrt)

Generalization ⇒

$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 \leq \sum_{i=1}^{n} |z_i|^2 \cdot \sum_{i=1}^{n} |w_i|^2$$

* Parametric representation of complex no.:-

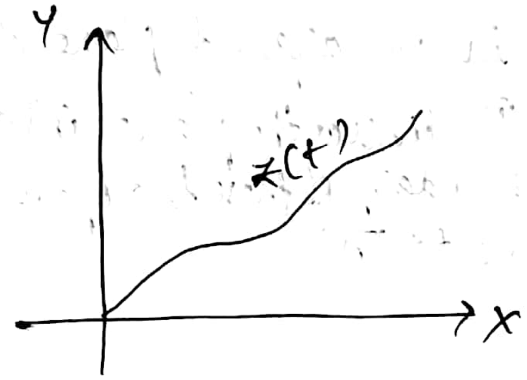
$$z = x + iy \;;\; x, y \in \mathbb{R}$$

$y = f(x) \rightarrow$ can be expressed.

Now, if we decide a parameter $a \leq \underline{t} \leq b$

$\Rightarrow \quad x = x(t), \; y = y(t) \Rightarrow \quad z(t) = x(t) + iy(t)$

By taking individual values of t_i and plotting on graph, we get the curve $z(t)$



Curve in Argand plane:

A curve Γ(gamma) in argand plane is a continuous complex valued function, $z(t) = x(t) + iy(t)$ depends upon the real parameter t, $a \leq t \leq b$ (a, b also $\in \mathbb{R}$).

A curve $z(t) = x(t) + iy(t)$ is said to be closed if $z(a) = z(b)$, where $t \in [a, b] \in \mathbb{R}$.

$z(t) = t^2 \quad , -1 \leq t \leq 1$

$z(-1) = 1 \;;\; z(1) = 1 \Rightarrow z(t)$ is closed.

since $z(t)$ is a polynomial function, it is naturally continuous.

$z(t) = \cos t + i \sin t \quad 0 \leq t \leq 2\pi$

$z(0) = z(2\pi) = 1 \Rightarrow$ closed.

$\cos t$ is continuous, $\sin t$ is continuous $\Rightarrow z(t)$ is also continuous.

Simple curve in argand plane :-

A curve Γ $z(t) = x(t) + iy(t)$ is said to be a simple curve on the argand plane if

$$\boxed{t_1 < t_2 \Rightarrow z(t_1) \neq z(t_2)}$$

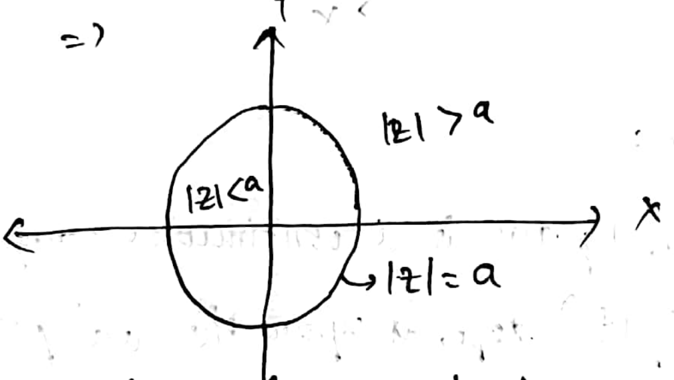
and if $z(a) = z(b) \Rightarrow$ simple closed curve.

A simple curve is also known as Jordan arc.

- Every Jordan curve in an argand plane defined on the finite and bounded domain D divides the plane into two open domains and the curve is the common boundary of these two open domains.

$z(t) = \cos t + i\sin t \quad 0 \le t \le 2\pi$

$\Rightarrow$



$z(t) = a(\cos t + i\sin t) \Rightarrow |z| = a \Rightarrow \underbrace{|z| < a \text{ and } |z| > a}$

these are the two open domains.

* Complex valued function :-

A rule or set of rules defined on a bounded (and closed) domain D (which is a set of $\mathbb{C}$) is said to be a complex valued function if for each $z \in D$, we get one or more than one values of z under that rule.

(In case of a real function, we get only one value of the function under a rule, but not so for a complex value function).

If corresponding to z, we have only a single value, then it is said to be a single valued function.

If we get more than one values of z, then it is a multivalued function.

eg:- $\sqrt{z}$, $2n\pi i +\log z$ are all multivalued.

* Limit of a complex valued function:-

let $w = f(z) = u+iv$ $\Rightarrow z \in \underset{\sim}{D}$ and w also $\in$ say $\underset{\sim}{E}$

$z = x+iy$

D: subset of complex domain

E: also subset of $\mathbb{C}$ domain.

can also be writen as $u(x,y)+iv(x,y)$ as ultimately, u and v depend on x and y only $\Rightarrow$ u and v are real valued functions of x and y.

Limit:- Let $w = f(z)$ be a single complex valued function defined on a finite and bounded domain D. We say $f(z)$ tends to a limit l as z tends to a point $a \in D$, if there exists an smallest number $\delta > 0$ for given $\epsilon > 0$ (however small) such that

$$|f(z) - l| < \epsilon, \text{ for all } z \in D \text{ satisfying } |z-a| < \delta$$

- Deleted neighbourhood:-

ie, $|f(z) - l| < \epsilon$, whenever $|z-a| < \delta$

ie, $z \xrightarrow{\lim} a \Rightarrow f(z) = l$ (finite)

ie, $z \in \delta(a)$, then $f(z)$ must $\in \epsilon(l)$

$\Rightarrow$ Irrespective of the infinite no. of ways of approaching a point a, the limiting value will be same.

* Continuity :-

As $z \longrightarrow a$,

$$\lim_{z \to a} f(z) = f(a)$$

$\Rightarrow$ for $\epsilon > 0$, $\exists\ \delta > 0$ such that $|f(z) - f(a)| < \epsilon$ when $|z - a| < \delta$

- If $w = f(z) = u(x,y) + i\,v(x,y)$ is continuous at $z = a \in D$, then $u(x,y)$ and $v(x,y)$ are also continuous at that point.

$$|f(z) - f(a)| < \epsilon, \quad |z - a| < \delta$$

$\Rightarrow$ $|u(x,y) + i\,v(x,y) - (a_1 + i a_2)| < \epsilon$

$\Rightarrow$ $|u(x,y) - a_1 + i(v(x,y) - a_2)|$, using Δles inequality,

* Differentiability :-

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$ exists whatever the path of approaching the point 'a'.

* Uniform continuity :-

$z_1, z_2 \in D$

such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$

Let $w = f(z)$ be single valued function defined on $D (\subseteq \mathbb{C})$.
Then derivative of $f(z)$ is defined as

$$\frac{df(z)}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided the limit exists and has the same value for the different ways in which δz approaches 0.

eg :-

$f_1(z) = z$

$f_2(z) = |z|$ and $\frac{df_4(z)}{dz} = 2z$

$f_3(z) = \bar{z}$

$f_4(z) = z^2$ and $\frac{d \sin z}{dz} = \cos z$ and so on,

$f_5(z) = |z|^2$

but we should also prove by using first principle (using complex valued functions)

$$\Rightarrow \lim_{a \to 0} \frac{f(z) - f(a)}{z - a}$$

(If function is differentiable on a domain, then it is differentiable at every point in the domain).

* Analytic function :-

A function $w = f(z)$ (single valued) is said to be analytic at a point 'a' $\in D$ if $f(z)$ is differentiable at each point of some neighbourhood of the point a.

Every analytic func. is differentiable, but every differentiable func. is not analytic because a func. that is not analytic may not be having the same limit when we consider a different neighbourhood.

* Necessary and sufficient condition for differentiability (analyticity):-

Let $w = f(z) = u + iv$ be a single valued function defined on $D (\in \mathbb{C}^*)$. The necessary and sufficient condition for $w = f(z)$ to be differentiable (or analytic) are that,

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous

and (ii) $\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Cauchy - Riemann equations}}$ → Necessary condition

To show analyticity, both criteria have to be proved.

Proof of C-R equation:-

Let $w = f(z)$; $w = u + iv$, $z = x + iy$.

Let $z + \delta z$ be a neighbouring point of z such that changes in x and y, say δx and δy giving us δu and δv which are the changes in u and v respectively.

Let $w = f(z)$ is differentiable at $z \in D$. This means

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists and has unique value irrespective of path of $\delta z \to 0$.

Now, we assume $\delta z \to 0$ by the path parallel to real line

$\Rightarrow$ $Im(\delta z) = 0 \Rightarrow \delta y = 0$ and $\delta z = \delta x$.

$$\Rightarrow \frac{df(z)}{dz} = \lim_{\delta x \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta x \to 0} \frac{(u+\delta u)+i(v+\delta v) - u + iv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + \frac{i\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now, we assume $\delta z \to 0$ by path $\|$ to Im line $\Rightarrow$ $\delta x = 0$, $\delta z = i\delta y$

$$\Rightarrow \frac{df(z)}{dz} = \underbrace{\lim_{\delta y \to 0}}_{\text{since } i \text{ is nonzero value}} \frac{(u+\delta u)+i(v+\delta v) - u + iv}{i\delta y} = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y} \right)$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

since limits have to be same $\Rightarrow$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(C-R eqn.)}$$

- Taylors theorem :-

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{1}{n!} f^n(a+\theta h)$$

$$0 < \theta < 1$$

- Taylor's series :-

- Taylor's theorem extension to 2-variable functions:-

If n^{th} order continuous partial derivatives of the function exists, then,

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + R_n \cdots$$

- Proof for sufficient condition for $w = f(z)$ to be differentiable (or analytic)

Let $w = f(z) = u(x,y) + iv(x,y)$ be a single valued function defined on D such that $u(x,y), v(x,y)$ possess continuous partial derivatives of first order and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $z \in D$ be arbitrary and $z + \delta z$ be a neighbouring point of z.

Then

$$w + \delta w = f(z + \delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

Using Taylors theorem,

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\delta x \frac{\partial u}{\partial x} + \delta y \frac{\partial u}{\partial y}\right) + \cdots$$

$$+ i\left\{ v(x,y) + \left(\delta x \frac{\partial v}{\partial x} + \delta y \frac{\partial v}{\partial y}\right) + \cdots \right\}$$

δx and δy are choosen so small that we can omit terms containing powers 2 and higher of δx and δy.

$$\Rightarrow f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) i\delta y$$

(using C-R equation)

$$\Rightarrow \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u+iv)$$

$$\Rightarrow \boxed{\frac{df(z)}{dz} = \frac{\partial}{\partial x} f(z)}$$

Also can be written as,

$$\frac{df(z)}{dz} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) = -i^2\frac{\partial v}{\partial y} - i\left(\frac{\partial u}{\partial y}\right)$$

$$= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial}{\partial y}(u+iv) = \boxed{-i\frac{\partial}{\partial y} f(z)}$$

Remarks:- $w = f(z) = u(x,y) + iv(x,y)$

is analytic, then C-R equations must be satisfied,

let $u(x,y) = c_1$ and $v(x,y) = c_2$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \Rightarrow m_1 = \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$m_2 = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$; $m_1 m_2 = -1$ (condition of orthogonality)

$$\therefore \boxed{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=0} \rightarrow \text{orthogonality condition.}$$

We know from C-R eqns. that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \quad \text{multiplying both,}$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=0, \text{ we get orthogonality condn}$$

$\Rightarrow$

- $f(z)=u+iv$ is an analytic function then $u(x,y)$ and $v(x,y)$ are said to be conjugate of eachother.
- conjugate function of analytic function are orthogonal to each other.

* Polar form of C-R equations:-

$z=x+iy \equiv z=re^{i\theta}$

$x=r\cos\theta \quad y=r\sin\theta \quad r=\sqrt{x^2+y^2} \quad \theta=\tan^{-1}(y/x)$

$w=f(z)=f(re^{i\theta})$

$$\Rightarrow \quad \frac{\partial w}{\partial r}=f'(re^{i\theta})\,e^{i\theta}$$

$$\frac{\partial w}{\partial \theta}=\left(\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}\right)=f'(re^{i\theta}).re^{i\theta}.i$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}} \rightarrow \text{polar form of C-R eqns.}$$

Differentiating again w.r.t r and θ,

$$\frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} \quad \text{and} \quad \frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r}$$

we have another useful relation,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$= \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial \theta \partial r}\right)$$

$$= \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} = \underline{\underline{0}}$$

Thus proved.

* Harmonic function:-

A function of two variables x and y is said to be a harmonic function if partial derivatives of first and second order exist and satisfy the Laplace equation.

Let function be $z = u(x,y)$ $\left\{ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial^2 u}{\partial x^2} \quad \frac{\partial^2 u}{\partial y^2} \quad \frac{\partial^2 u}{\partial x \partial y} \quad \frac{\partial^2 u}{\partial y \partial x} \right.$

Laplace eqn. in two dimension:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in three dimension:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (and $u = f(x,y,z)$)

- Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function over the domain.
Prove that $u(x,y)$ and $v(x,y)$ are harmonic functions.

Proof:- Therefore C-R eqns. hold =>

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since function is analytic, both u and v are continuously differentiable (definition)

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding both =>

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = \underline{\underline{0}}$$

(since continuously differentiable and also u and v are uniformly continuous)

=> u is Harmonic function

Similar proof for v.

=> u and v are thus also said to be harmonic conjugate of each other.

If $f(z) = z^2$. Find $\frac{d}{dz} f(z)$

$z = x+iy \Rightarrow z^2 = x^2 - y^2 + 2ixy$. We know $\frac{df}{dz} = \frac{\partial}{\partial x} f(z)$

$= 2x + 2iy = 2(x+iy) = \underline{\underline{2z}}$.

If $f(z) = \text{Im}(z) = iy$. Determine the analyticity of $f(z)$.

$$f(z) = \underset{u(x,y)}{0} + i\underset{v(x,y)}{y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 1$$

Since C-R eqns. are not satisfying, $f(z)$ is not analytic.

Check analyticity of $f(z) = |z|^4$

$$f(z) = |z|^4 = (x^2+y^2)^2 = \underbrace{x^4+y^4+2x^2y^2}_{u} + \underbrace{0i}_{v}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 4x^3 + 4y^2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 4y^3 + 4x^2y \qquad \frac{\partial v}{\partial y} = 0$$

Since C-R eqns. are not satisfying, $f(z)$ is not analytic.

Show that $f(z) = e^z$ is analytic everywhere.

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

$$= \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\qquad \| \qquad\qquad\qquad \| \qquad \Rightarrow \text{Analytic}$$

$$\frac{\partial v}{\partial y} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

- Theorem :- If $f'(z) = 0$ in a domain D, then $f(z)$ is constant in D.

If $w = \log(z)$, find $\frac{dw}{dz}$ and determine where w is non analytic.

$$\Rightarrow w = \log(z) = \log(x+iy) = \log(r) + \log e^{i\theta}$$

$$= \log(\sqrt{x^2+y^2}) + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$\Rightarrow \frac{dw}{dz} = \frac{dw}{dx} =$$

Since w is not defined at $z=0 \Rightarrow$ it is not differentiable at $z=0 \Rightarrow$ Non-analytic at $z=0$

* $z = x+iy$, $\bar{z} = x-iy$

$\Rightarrow x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ $\Rightarrow$, we can arrive at another differential relation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$ ($u \to$ harmonic function)

Proof:- $f(z) = u+iv$ is analytic, then $u(x,y)$ and $v(x,y)$ are harmonic functions $\Rightarrow$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial u}{\partial y} \cdot \left(\frac{-1}{2i}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{0,\text{ since it satisfies laplace eqn.}}\Big)$$

$$= \underline{\underline{0}}.$$ Thus proved

* Methods of finding one harmonic function from other:-

I:- If $f(z) = u+iv$ is an analytic function, where both $u(x,y)$ and $v(x,y)$ are conjugate functions, given one of these, say $u(x,y)$, to find the other,

Since $v(x,y)$ is a real valued functions of real variables x and y,

$$\therefore dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy,$$

Using C-R equations,

$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \quad \text{—} \ (1)$$ ⇒ the RHS of (1) is of the form,

$M\,dx + N\,dy$, where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

For this diff. eqn. to be exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ⇒

$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$. Since $u(x,y)$ is a harmonic function, it has to satisfy laplace equation. ⇒

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$ ⇒ it is exact.

Thus $-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ satisfies exact differential. ⇒

$dv = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ can be integrated to find $v(x,y)$.

If $f(z) = u(x,y) + iv(x,y)$, an analytic function is given, then $u(x,y)$ can be obtained.

The constant of integration can be obtained by using C-R equation $\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$

Note :- $f(z) = u + iv$

$if(z) = iu + i^2 v = iu - v = -v + iu$

⇒ if v is the harmonic conjugate of u, then u is the harmonic conjugate of $-v$ also.

* Milne - Thomson Method (construction)

Step 1:- Since $f(z) = u(x,y) + iv(x,y) = u\left[\frac{z+\bar{z}}{2}, \frac{1}{2i}(z-\bar{z})\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$ — ①

i.e, we may regard ① as a formal identity in two independent variables $z, \bar{z}$.

Step 2:- On putting $z = \bar{z}$, we get $f(z) = u(z,0) + iv(z,0)$

$$\Rightarrow \frac{df(z)}{dz} = f'(z) = \frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \text{By C-R eqns.,}$$

$$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$\Rightarrow f'(z) = u_x(z,0) - iu_y(z,0)$, can be writen as,

$$= \phi_1(x,y) - i\phi_2(x,y).$$

Step 3:- Integrating it, we will get,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$$

Similarly, if $v(x,y)$ is given, then we have $f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + C'$

where $\psi_1(z,0) = v_y(z,0)$ and $\psi_2(z,0) = v_x(z,0)$

\# Show that $u(x,y) = e^{-x}[x\sin y - y\cos y]$ is harmonic and find its harmonic conjugate $v(x,y)$ such that $f(z) = u + iv$ is analytic.

$$\Rightarrow \frac{\partial u}{\partial x} =$$

If $u(x,y) = e^x(x\cos y - y\sin y)$, find the analytic function $f(z) = u + iv$

we have
$$u(x,y) = e^x(x\cos y - y\sin y)$$
$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$
$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) \; ; \; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ $u(x,y)$ is a harmonic function.

To calculate $v(x,y)$, we have $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$

$$\Rightarrow dv = e^x(x\sin y + \sin y + y\cos y)\,dx + e^x(x\cos y - y\sin y + \cos y)\,dy$$

$$v = \int_{y\text{ const}} e^x(x\sin y + \sin y + y\cos y)\,dx + \int \left[\begin{array}{l}\text{those terms}\\ \text{which do not}\\ \text{contain } x\end{array}\right] dy + C$$

$$v = e^x(x\sin y + y\cos y) + c$$

$$\Rightarrow f(z) = u(x,y) + iv(x,y) = e^x[x\cos y - y\sin y + ix\sin y + iy\cos y] + ci$$
$$= e^x(x+iy)(\cos y + i\sin y) + ci$$
$$= e^x z e^{iy} + Ci = ze^{x+iy} + Ci$$

C here is $\equiv g(y)$ because we kept y constant while integrating first integral.

$\Rightarrow$ calculate $\frac{\partial v}{\partial y}$, use C-R eqn. to determine C.

* Theorem:- If $f'(z)=0$ in a domain D, then $f(z)$ is constant in D.

Proof:-

$$f'(z) = \frac{d f(z)}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$ (or can even be expressed in dy using C-R eqn.)

$$= 0$$

$\Rightarrow$ $\frac{\partial u}{\partial x}$ (or) $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial v}{\partial x}$ (or) $\frac{\partial v}{\partial y} = 0$ $\Rightarrow$ $u(x,y)$ and $v(x,y)$ are constants

$f(z) = u + iv \equiv$ constant.

* Theorem:- Let $|f(z)|$ be constant in a region where $f(z)$ is analytic. Then $f(z)$ is constant.

Proof:- Let $f(z)$ be analytic in the domain D and $|f(z)| \equiv$ constant

$\Rightarrow$ $|f(z)| = |u+iv| = c$ (say)

then $u^2 + v^2 = c^2$, partially differentiating this,

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$; using C-R equations,

$u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial y} = 0$ and $u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial x} = 0$

$\Rightarrow$ $(u^2+v^2)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$. similarly $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$, $\frac{\partial u}{\partial y} = 0$

$\Rightarrow$ u and v are constant functions $\Rightarrow$ $z = u+iv$ is also constant.

\# If $w = f(z) = u+iv$ and
$u - v = e^x(\cos y - \sin y)$, find w in terms of z

We have $u - v = e^x(\cos y - \sin y)$

$$\Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \quad -①$$

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^{x}(-\sin y - \cos y)$ — ②

$\Rightarrow -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^{x}(\sin y + \cos y)$ (Using C-R equations)

$\Rightarrow \frac{\partial u}{\partial x} = e^{x}\cos y = \phi_1(x,y)$ and $\frac{\partial v}{\partial x} = e^{x}\sin y = \phi_2(x,y)$

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \phi_1(x,y) + i\phi_2(x,y)$

$= e^{x}\cos y + ie^{x}\sin y \Rightarrow \int e^{x}\cos y + ie^{x}\sin y \, dz + c = f(z)$

$\Rightarrow f(z) = \int e^{x+iy}\,dz + c = \int e^{z}\,dz + c = \underline{e^{z} + c}$.

\# ⊗ Prove that the function $f(z)$ defined by $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ $(z \neq 0)$

$= 0 \quad (z = 0)$

is continuous and the C-R equations are satisfied at the origin, yet $f'(0)$ does not exist.

$\Rightarrow f(z) = \underbrace{\frac{x^3 - y^3}{x^2+y^2}}_{u} + i\underbrace{\left(\frac{x^3+y^3}{x^2+y^2}\right)}_{v}$

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{x^3-y^3}{x^2+y^2} + i\lim\left(\frac{x^3+y^3}{x^2+y^2}\right)$

$= \lim_{x\to 0}\lim_{y\to 0} \frac{x^3-y^3}{x^2+y^2} + i\lim_{x\to 0}\lim_{y\to 0}\left(\quad\right) = \underline{\underline{0}}$

$\lim_{y\to 0}\lim_{x\to 0}\left(\quad\right) + i\lim_{y\to 0}\lim_{x\to 0}\left(\quad\right) = \underline{\underline{0}}$.

Using $y = mx$, we can again see, $\lim = \underline{\underline{0}}$.

For a more general solution,

$$x = r\cos\theta, \quad y = r\sin\theta \Rightarrow f(z) = f(r,\theta) = \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(1)} + i\frac{r^3(\cos^3\theta + \sin^3\theta)}{(1)}$$

$$= r(\cos^3\theta - \sin^3\theta) + i\, r^3(\cos^3\theta + \sin^3\theta)$$

$\Rightarrow \lim \Rightarrow \lim_{r \to 0} f(r,\theta)$ (θ takes any value such that various points are generated)

$$\Rightarrow \lim_{r \to 0} f(r,\theta) = \underline{\underline{0}}.$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{\delta x \to 0} \frac{u(0+\delta x, 0) - u(0,0)}{\delta x}$$ (~~...~~ checking)

$$= \frac{\delta x}{\delta x} = \underline{\underline{1}}.$$

$$\left(\frac{\delta u}{\partial y}\right)_{(0,0)} = \lim_{\delta y \to 0} -\frac{\delta y}{\delta y} = \underline{\underline{-1}}$$

$$\left(\frac{\delta v}{\partial x}\right)_{(0,0)} = \underline{\underline{1}} \quad \text{and} \quad \left(\frac{\delta v}{\partial y}\right)_{(0,0)} = \underline{\underline{1}} \Rightarrow \text{C-R eqns are satisfied.}$$

$$f'(0) = \lim_{z \to 0} \frac{f(\delta z) - f(0)}{\delta z}$$ z here can be taken to represent δz.

along real line $$= \lim_{x \to 0} \lim_{y \to 0} \frac{\delta z + i\delta z}{\delta z} = 1+i$$

along imaginary axis $$= \lim_{y \to 0} \lim_{x \to 0} = \frac{-\delta z + i\delta z}{\delta z} = i-1$$

Since values are not equal $\Rightarrow$ function is not differentiable at $(0,0)$

Remark :- Even though function is continuous at $(0,0)$ and also satisfying the C-R equations, yet the function is not differentiable at $(0,0)$.

\# If $f(z) = \dfrac{x^3 y(y-ix)}{x^6+y^2}$, $(z \neq 0)$ Prove that,

$= 0 \quad (z=0)$

$\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$

along any radius vector but not as $z \to 0$ in any manner.

→ In polar form, $f(r,\theta) = \dfrac{r^3\cos^3\theta \cdot r\sin\theta(r\sin\theta - ir\cos\theta)}{r^6\cos^6\theta + r^2\sin^2\theta}$

$\Rightarrow f(r,\theta) = \dfrac{r^{\cancel{5}3}(\sin\theta - i\cos\theta)\sin\theta\cos\theta}{\cancel{r^2}(r^4\cos^6\theta+\sin^2\theta)}$

$\Rightarrow f(r,\theta) = \dfrac{r^3\sin^2\theta\cos\theta}{r^4\cos^6\theta+\sin^2\theta} - i\dfrac{r^3\cos^2\theta\sin\theta}{r^4\cos^6\theta+\sin^2\theta}$

$\Rightarrow \lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z} = \lim\limits_{r\to 0}(\quad) = \underline{\underline{0}}$

$\lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z} = \lim\limits_{z\to 0}\dfrac{x^3y(y-ix)}{(x^6+y^2)(x+iy)} = 0$ (Along $y = mx$)

$\lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z}$ along $y = x^3$ $= \underline{\underline{\dfrac{-i}{2}}}$

Since we have two diff. values (0 and $-i/2$), it does not have a limit.

Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}, \quad z \neq 0, \quad f(0) = 0$$ when in a region including the origin.

(Here even though C-R equations satisfy, function is not differentiable at $z=0$).

* Integration of complex valued functions:-

- Rectifiable curves:-

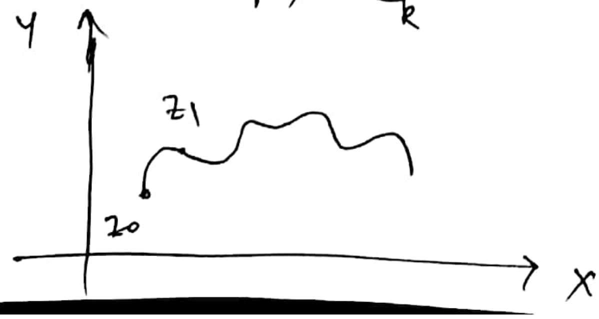
Let L be a continuous curve with equation $z(t) = x(t) + i\,y(t)$, $\alpha \leq t \leq \beta$

and suppose we divide the interval $[\alpha, \beta]$ into n subintervals, $[t_{k-1}, t_k]$ by introducing $(n-1)$ intermediate points $t_1, t_2, \dots t_{n-1}$ satisfying the inequalities

$$\alpha = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = \beta$$

The set $P = \{t_0, t_1, \dots, t_n\}$ is called a partition of the interval $[\alpha, \beta]$ and the largest of the numbers $t_1 - t_0, t_2 - t_1, \dots t_n - t_{n-1}$, ie, the maximum length of the subintervals is called the norm of the partition, denoted by $\|P\|$.

Let $z_0, z_1, z_2 \dots z_n$ be the points on the curve corresponding to the values of $t_0, t_1, \dots t_n$ ie, $z(t_k) = z_k$



Clearly, the length of the polygon curve inscribed in the curve L obtained by joining the points $z_0, z_1; z_1, z_2; \ldots$ etc. successively by straight lines segments (as in figure) is given by $\sum_{k=1}^{n} |z_k - z_{k-1}|$

The curve L is said to be rectifiable if supremum of the sum

$\sum_{k=1}^{n} |z_k - z_{k-1}| = l < \infty$ where the supremum (or least upper bound) is taken over all possible partitions given by ①.

- Contour :-

Let $z = z(t) = x(t) + iy(t)$ — ①

where t runs through the interval $\alpha \leq t \leq \beta$ and $x(t)$ and $y(t)$ are continuous functions of t, represents a continuous arc L in the complex plane.

If eq^n ① is satisfied by more than one values of t in the given range, then the point z or (x, y) is a multiple point of the arc.

A continuous arc without multiple points is called Jordon arc.

If for a point z on a Jordan arc, z as expressed in equation ① is one valued function, $x(t)$ and $y(t)$ are continuous and $x'(t)$ and $y'(t)$ are continuous in the range $\alpha \leq t \leq \beta$, then the arc is called a regular arc of the Jordan curve.

A Jordan curve consisting of continuous chain of a finite number of regular arc is called a contour.

- Complex integration:-

If $f(z)$ is a continuous function of the complex variable $z = x+iy$ defined at all points of a curve C having end points A and B.

We want integral of $f(z)$ over C.

$(z = x(t) + iy(t))$

Divide C into n parts at the points

$$A = A_0(z_0),\ A_1(z_1),\ A_2(z_2),\ \ldots A_i(z_i), \ldots A_n(z_n) = B$$

let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $A_{i-1}A_i$, then the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches to zero, is called, the line integral of the function $f(z)$ taken along the curve C

ie, $\int_C f(z)\,dz$

$$\Rightarrow \int_C f(z)\,dz = \int_C (u(x,y) + iv(x,y))(dx + idy)$$

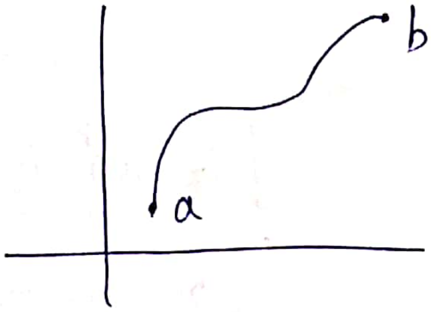
$$= \int_C (u(x,y)\,dx - v(x,y)\,dy) + i\int_C u(x,y)\,dy + v(x,y)\,dx$$

Using the definition of integral as the limit of sum, evaluate the following integrals:

(i) $\int_C dz$ (ii) $\int_C |dz|$ (iii) $\int_C z\,dz$ (iv) $\int_C |z|\,dz$

where C is any rectifiable curve (arc) joining the points a and b.

Let C be



$z = x + iy$

(i) $\int 1 \cdot dz = \lim_{n\to\infty} \sum_{r=1}^{n} 1 \cdot (z_r - z_{r-1}) = \lim_{n\to\infty} \sum_{r=1}^{n} z_r - z_{r-1}$

$= z_n - z_{n-1} + z_{n-1} - z_{n-2} + z_{n-2} + \dots - z_0 \quad \lim_{n\to\infty} z_n - z_0 = \underline{\underline{b - a}}$.

(ii) $\int |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} 1 \cdot |z_r - z_{r-1}| = \lim_{n\to\infty} |z_n - z_{n-1}| + |z_{n-1} - z_{n-2}| + |z_{n-2} - z_{n-3}| \dots + |z_1 - z_0|$

$=$ arclength $(z_n z_{n-1})$ + arc length $(z_{n-1} z_{n-2})$ + ...

$=$ $\underline{\text{arc length } (C)}$.

(iii) $\int_C \underset{f(z)}{z}\, dz = \lim_{n\to\infty} \sum_{r=1}^{n} \xi_r (z_r - z_{r-1})$ — ξ_r: any point on arc $z_r z_{r-1}$

Taking first $\xi_r = z_r$ and then $\xi_r = z_{r-1}$

$\int_C = \lim_{n\to\infty} \sum_{r=1}^{n} z_r (z_r - z_{r-1})$ — (2) and $\int_C = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1} (z_r - z_{r-1})$ — (3)

(2)+(3) $\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{n} z_r (z_r - z_{r-1}) + \sum_{r=1}^{n} z_{r-1} (z_r - z_{r-1}) = 2 \times \int$

$$\Rightarrow 2\times\int = \lim_{n\to\infty}\sum_{r=1}^{n}(z_r+z_{r-1})(z_r-z_{r-1})$$

$$\Rightarrow 2\times\int = \lim_{n\to\infty}\sum_{r=1}^{n} z_r^2-(z_{r-1})^2 = \lim_{n\to\infty} z_n^2 - z_{n-1}^2 + z_{n-1}^2 - z_{n-2}^2 + \dots - z_0^2$$

$$\Rightarrow 2\times\int = b^2-a^2 \Rightarrow \int_C = \frac{b^2-a^2}{2}$$

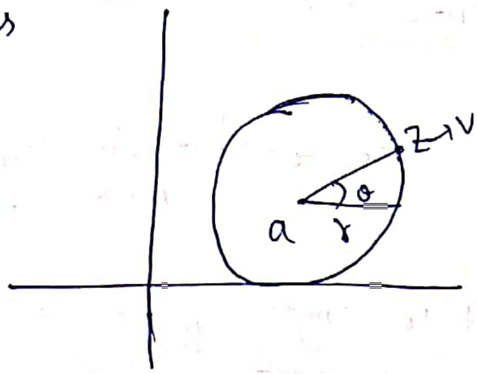
Prove that

(i) $\int_C \frac{1}{z-a}\,dz = 2\pi i$ (ii) $\int_C (z-a)^n\,dz = 0$

[n is any integer other than $n=-1$]

where C is the circle centred at a with radius r
$(\Rightarrow |z-a|=r)$

we can take C as



$\Rightarrow$ The parametric equation is, $|z-a|=r$

$$\Rightarrow z-a = re^{i\theta}$$

$$\Rightarrow z = a+re^{i\theta} \Rightarrow dz = ire^{i\theta}\,d\theta$$

$$\Rightarrow \int \frac{1}{z-a}\,dz = \int_0^{2\pi}\frac{1}{\cancel{re^{i\theta}}}\,i\cancel{re^{i\theta}}\,d\theta = i\,\theta\Big|_0^{2\pi} = 2\pi i$$

(iii) $\int_C (z-a)^n\,dz = \int_0^{2\pi} (re^{i\theta})^n\, i r e^{i\theta}\,d\theta$

$= \int_0^{2\pi} r^{n+1} e^{ni\theta}\, i\, e^{i\theta}\,d\theta = i r^{n+1} \int_0^{2\pi} e^{(n+1)i\theta}\,d\theta$

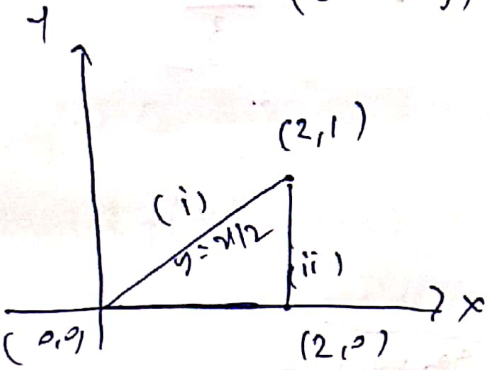
$= \cancel{i}\, r^{n+1} \left. \frac{e^{(n+1)i\theta}}{(n+1)\cancel{i}} \right|_0^{2\pi} = \frac{r^{n+1}}{n+1}\left[e^{(n+1)i(2\pi)} - e^{0}\right]$

$= \frac{r^{n+1}}{n+1}\left[\underbrace{\cos(n+1)(2\pi)}_{1} + i\underbrace{\sin(n+1)(2\pi)}_{0} - 1\right]$ $\quad \underline{(n \neq -1)}$

$= \underline{0}$.

Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along (i) the line $y = x/2$ (ii) the real axis to 2 and then vertically to $2+i$

$(z = x+iy)$



$\Rightarrow$ (i) $2+i$, limits in z

$\int_0^{1} (2-i)^2\, y^2 (2+i)\,dy$

$= (2-i)^2(2+i)\int_0^1 y^2\,dy = (2-i)^2(2+i)\,\frac{1}{3}$

$= \frac{(4+i^2-4i)(2+i)}{3}$

$= \frac{(3-4i)(2+i)}{3} = \frac{(6-8i+3i+4)}{3}$

$= \underline{\frac{10}{3} - \frac{5}{3}i}$

$z = x+iy$
$\bar{z} = x-iy$

(ii) $I = \int_{\text{along } x} (\bar{z})^2\,dz + \int_{\text{along vertical}} (\bar{z})^2\,dz = \int_0^2 x^2\,dx + \int (2-iy)^2 \cdot i\,dy$

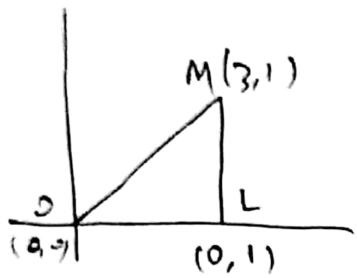
We can also check for analyticity of this function $f(z) = (\bar{z})^2$

$z = x+iy \Rightarrow f(z) = (x-iy)^2 = \underbrace{x^2-y^2}_{u} - i\underbrace{2xy}_{v}$

$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = -2x \qquad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = -2y$

$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \Rightarrow$ Not analytic.

* $\int_0^{3+i} z^2 dz$



(i) Along OM $y = x/3$
(ii) Along OL − LM

$\Rightarrow$ (i) $z = x+iy \rightarrow$ Along OM, $(3+i)\int (3y+iy)^2 dy$

$z = (3+i)y$

$dz = (3+i)dy$

$(3+i)^3 \int_0^{3+i} y^2 dy = \frac{(3+i)^3 \cdot (3+i)^3}{3}$

* Simply connected and Multiply connected regions/domain:-

A region is said to be connected region if any two points of the region can be connected by a curve which lies entirely within the region.
Eg. Circle, triangle, parallelogram, etc.

A connected region is said to be simply connected if every closed curve in the region can be shrunk to a point without passing out of the region, otherwise it is said to be multiply connected region.

* **Cauchy's fundamental theorem for Integration:-**

If $f(z)$ is analytic with a continuous derivative in a simply connected domain G and C is a closed contour lying in G, then

$$\int_C f(z)\,dz = 0.$$

Green's theorem:-

Let C be a positively oriented, piecewise smooth (analytic), simple closed curve in a plane and let D be the region bounded by C. If L and M are functions of x and y defined on an open region containing D and have continuous partial derivatives over the curve C in region D, then,

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

Proof of Cauchy's fundamental theorem:-

Let $f(z) = u(x,y) + iv(x,y)$ —(1) be an analytic function with a continuous derivative in a simply connected domain G,

$$\therefore,\ f'(z) = u_x + iv_x$$

By using C-R equations, we have,

$$f'(z) = v_y - iu_y$$ These are true for all points in the given domain.

Also, $z = x+iy \Rightarrow dz = dx + i\,dy$

$$\Rightarrow \int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$\Rightarrow$ By Green's theorem, we have,

$$\int_C f(z)dz = \iint_G \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\Rightarrow$ Using C-R equations,

$$= \iint_G \underbrace{\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)}_{0} dx\,dy + i \iint_G \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)}_{=0} dx\,dy = \underline{\underline{0}}.$$

Thus proved.

* Absolute value of a complex integral:-

If $f(z)$ is continuous on a closed contour C of length l and $|f(z)| \leq M$ ($\Rightarrow |f(z)|$ is bounded), for every point z of C, then

$$\left|\int_C f(z)dz\right| \leq Ml$$

ie, $\left|\int_C f(z)\,dz\right| \leq |f(z)| \cdot \underbrace{|dz|}_{\text{arc length}}$

Proof:- By definition of integration as limit of sum,

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

where ξ_r is any point on the curve (arc) $z_r z_{r-1}$.

Now, $\left|\int_C f(z)\,dz\right| = \left|\lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})\right| \leq \lim_{n\to\infty} \sum |f(\xi_r)|\,|z_r - z_{r-1}|$

$$\leq \lim_{n\to\infty} M \sum |z_r - z_{r-1}|$$

$\Rightarrow \left|\int f(z)dz\right| \leq M \underbrace{\lim_{n\to\infty}\sum|z_r - z_{r-1}|}_{\text{arc length of curve as seen before}} \leq |f(z)|\,|dz|$

* Some properties of integration:

① $\int_L (f(z) + \phi(z))\,dz = \int_L f(z)\,dz + \int_L \phi(z)\,dz$

② $\int_L f(z)\,dz = -\int_{-L} f(z)\,dz$

③ $\int_{L_1+L_2} f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz$ (here end point of L_1 must be starting point of L_2)

* Inverse point with respect to a circle:-

General equation of circle in real plane:-

$$x^2+y^2+2gx+2fy+c=0$$

$$C \equiv (-g,-f) \quad r = \sqrt{g^2+f^2-c}$$

In the complex plane, $|z-z_0| = r$ represents a circle.

The equation,

$$a z\bar{z} + \alpha z + \bar{\alpha}\bar{z} + c = 0 \quad ①$$

represents a real circle or a pair of straight lines.

provided $\alpha\bar{\alpha} > ac$ — ②

from $|z-z_0|^2 = r^2$, $(z-z_0)(\bar{z}-\bar{z}_0) = r^2$

where a, c are real constants, α is complex constant and z is the complex variable.

Let $\alpha = a_1 + ia_2$, $\bar{\alpha} = a_1 - ia_2$

$z = x+iy$, $\bar{z} = x - iy$

Then from ①, we have,

$$a(x+iy)(x-iy) + (a_1+ia_2)(x+iy) + (a_1-ia_2)(x-iy) + c = 0$$

$$\Rightarrow a(x^2+y^2) + 2a_1 x - 2a_2 y + c = 0 \text{ — circle equation in real plane}$$

$\Rightarrow$ Centre $= \left(-\frac{a_1}{a}, \frac{a_2}{a}\right)$ and $r = \sqrt{\frac{a_1^2}{a^2} + \frac{a_2^2}{a^2} - \frac{c}{a}}$

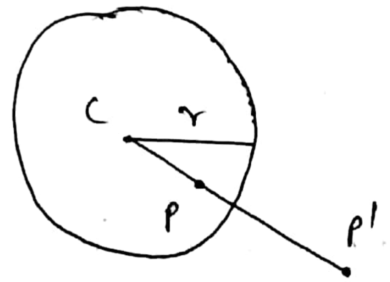
$\Rightarrow r = \sqrt{\frac{a_1^2 + a_2^2 - ac}{a^2}}$

Transporting this to an asgand plane, the centre becomes

$$-\frac{a_1}{a} + \frac{ia_2}{a} = -\frac{a_1 - ia_2}{a} = -\frac{\bar{\alpha}}{a}$$

and $r = \sqrt{\frac{\alpha\bar{\alpha} - ac}{a^2}}$ (from here, we get the condition that $\alpha\bar{\alpha} > ac$ for a real circle or pair of st. lines)

Inverse point w.r.t circle :-



$\Rightarrow$ P and P' are said to be inverse points of each other w.r.t the circle if

$CP \cdot CP' = r^2$

(C, P, P' have to be collinear)

(If point lies inside, then inverse is outside and vice versa)

- Inverse point also satisfies the equation of the circle, if the original point lies on the circle

$|z| = r \Rightarrow |z|^2 = r^2 \Rightarrow z\bar{z} = r^2$

$\Rightarrow z'\bar{z} = r^2$ (assuming z' is the inverse point)

$\Rightarrow z' = \frac{r^2}{\bar{z}}$

$\hookrightarrow$ for a circle centred at the origin.

We know for a general circle,

$$\left| z + \frac{\bar{\alpha}}{a} \right|^2 = r^2$$

$$\Rightarrow \left| z' + \frac{\bar{\alpha}}{a} \right| \left| z + \frac{\bar{\alpha}}{a} \right| = r^2 .$$

$\Rightarrow$ c, z, z' are collinear

$$\Rightarrow \arg\left(z + \frac{\bar{\alpha}}{a}\right) = \arg\left(z' + \frac{\bar{\alpha}}{a}\right)$$

Also, $\arg\left(z + \frac{\bar{\alpha}}{a}\right) = -\arg\left(\bar{z} + \frac{\alpha}{a}\right)$

$$\Rightarrow \arg\left(z' + \frac{\bar{\alpha}}{a}\right) = -\arg\left(\bar{z} + \frac{\alpha}{a}\right)$$

$$\Rightarrow \arg\left[\left(z' + \frac{\bar{\alpha}}{a}\right) \cdot \left(\bar{z} + \frac{\alpha}{a}\right)\right] = 0$$

Since $\arg = 0 \Rightarrow$ $\left(z' + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right) = r^2$

$\Rightarrow$ (Replacement of z by z' in the circle equation does not affect it).

* Cauchy's integral formula:-

Let $f(z)$ be an analytic function in a simply connected domain G bounded by a rectifiable Jordan curve C and is continuous on C, then,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz$$

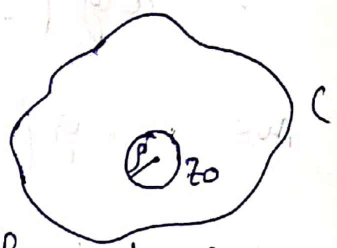
where z_0 is a point within C.

If $f(z)$ is analytic within and on closed contour C and z_0 is any point within C, then,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz.$$

$f(z)$ be an analytic within and on a closed contour C and z_0 is any point inside C. We describe a circle C_1 defined by the equation

$$|z - z_0| = \rho$$



(Distance of the point z_0 from the contour
= minimum radial distance of z_0 w.r.t the contour)

if d is the min. radial distance of z_0

$$\Rightarrow \quad |z - z_0| = \rho < d$$

We are taking a very small circle enclosing z_0 as,

Cauchy fundamental theorem states that $f(z_0)$ is analytic everywhere within and on C, but we can't say that

$\frac{f(z)}{z - z_0}$ will also be analytic all over C.

($\int_C f(z)$ analytic all over C $= 0$)

$$\Rightarrow \quad \int \frac{f(z)}{z - z_0} \, dz = 0 \quad \text{only outside the small circle}$$

and within the contour C.