HS301 FINANCIAL ECONOMICS

Financial Market (Overview)

Borrowers ← Financial Market → Lenders

Lenders → BANK → Borrowers

BANK: intermediary.
- → UNIFORM INTEREST RATE
- → ABILITY TO GET MONEY MAY DIFFER

* Govt → Central Bank → Control policy of commercial and public banks.

* Investment Banks → take money from very rich people and invest that in stock markets (exclusive in nature) eg. Goldman Sachs.

* Based on nature of Raising money

Debt Market	Equity Market (Stock Market).
Short Run: less than 1 yr. – Borrow money. / Long Run.	stocks are issued by companies to public or private entities to raise money.

Nature of origin:
- Primary market – direct sell of stocks to important people of company.
- Secondary market – stocks are issued to common public (one example).

Long Run and Short Run.

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* **Interest Rate** :-
time value of money
future value = Today's value $\left(1 + \frac{\text{Time value}}{\text{interest rate}}\right)$

* ₹100 $\xrightarrow[+1yr]{}$ ₹105 $\xrightarrow[+1yr.]{}$ ₹110.25 (5% — // —)

* **finding time period**
– future value, current value, years (time period) are given.

$$\ln(FV) = \ln(PV) + n \ln(1+r)$$

eg. PV = 50000, FV = 100000, r = 4.5% per annum

$$n = \frac{\ln(100000) - \ln(50000)}{\ln(1+r)} = 15.xx \text{ yrs}$$

PV = 1, FV = 2, r = 100% per annum.

$$n = \frac{\ln(2) - \ln(1)}{\ln\left(1 + \frac{100}{100}\right)} = 1 \text{ yr.}$$

* **Annuity (ordinary)**.
→ Fixed amt. of money to be paid at the end of time period for certain number of yrs.
eg. LOAN. EDUCATION

Annuity (due).
→ when paid in advance. eg. INSURANCE.

eg.

future value of Annuity

$$FVA = P(1+r)^{n-1} + P(1+r)^{n-2} + P(1+r)^{n-3} + \ldots$$

if $n=3$, $r=5\%$, $P=100$.

$FVA = 315.25$ INR

$P \rightarrow$ regular payment.

$n \rightarrow$ no. of yrs. to be paid.

$$FVA = P\left[\frac{(1+r)^n}{r} - \frac{1}{r}\right]$$

* MS Excel command = FV (r, n, P, PV)
(by default PV = 0)

→ total sum of all the payments in the whole time period.

* Present value of Ordinary Annuity

$$PVA = \frac{PMT}{(1+r)} + \frac{PMT}{(1+r)^2} + \frac{PMT}{(1+r)^3} + \ldots + \frac{PMT}{(1+r)^n}.$$

PMT ⇒ payment amount ~~paye~~ paid every period.
r ⇒ rate of interest.
PVA ⇒ present value of annuity

Alternative, $PVA = PMT\left[\left(\frac{1}{r}\right) - \left(\frac{1}{r(1+r)}\right)\right].$

eg. PMT = ₹100, $r = 5\%$, $n = 3$ yr.
PVA = ₹272.32

* Excel Command = PV (r, n, PMT, FV)
= PV (0.05, 3, −100, 0) (for above eg)

Present value of Annuity Due

$= PVA$ of ordinary annuity $(1+r)$

$= 272.32(1+0.05)$

$= 285.94$

Excel command = PV(r, n, PMT, FV, Type)

= PV(0.05, 3, −100, 0, 1)

* **Finding Annuity Payments, Periods, Interest Rates**

Excel for Payment = PMT (r, n, PV, FV)

Excel for finding Period = NPER (r, PMT, PV, FV)

Excel for finding Interest Rate = RATE (n, PMT, PV, FV)

* **Semiannual and other compounding periods**

$$FV = PV\left(1+\frac{r}{M}\right)^{Mn}$$

M ⇒ compounding times a year

monthly = 12
quaterly = 4
yearly = 1

if compounding occurs every moment i.e infinite times $(M \rightarrow \infty)$

1₹ ⇒ 2.718₹

if r = 100% & for n=1.

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Effective Annual Rate [EAR] :-

same result if we had compounded at a given periodic rate M times per year

$$EAR = \left(1 + \frac{r}{M}\right)^M - 1$$

r/M is periodic rate.

M is no. of periods.

$$\text{real value of money} = \frac{(1+r)}{(1+f)}$$

→ interest rate
→ inflation

(More FDI) ⇔ ($r\uparrow$ $f\downarrow$)

* Real and Nominal interest rate :-

$$\left(\frac{1+r}{1+f}\right)$$

$r \Rightarrow$ nominal interest rate
$f \Rightarrow$ rate of inflation

R be real interest rate.

$$1+R = \left(\frac{1+r}{1+f}\right)$$

$$\boxed{R = \left(\frac{r-f}{1+f}\right)}$$

* Notion of IRR :- Internal Rate of Return

An interest rate which yeild a PV = 0 for a cash flow over a period of time

$$0 = x_0 + \frac{x_1}{(1+r)} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} ; \quad r =$$

Using IRR.

$$c = \frac{1}{1+r}$$

(a) $(-1, 2)$

$$0 = -1 + \frac{2c}{\cancel{(1+0.1)^2}} \Rightarrow c = \frac{1}{2} \Rightarrow \boxed{r = 1.0}$$

(b). $(-1, 0, 3)$

$$0 = -1 + \frac{0c}{\cancel{(1+0.1)}} + \frac{3c^2}{\cancel{(1+0.1)^2}} \Rightarrow c = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{1}{1+r} \Rightarrow \boxed{r = 1.73 - 1 = 0.73}$$

Car1 (A)	Car2 (B)
20000	30000
1000/yr	2000/yr
4 years	6 years

12 year long duration ; $r = 10\%$

~~One cycle~~.

$$PV_A = 20000 + 1000 \sum_{k=1}^{3} \frac{1}{(1.1)^k}$$

$$= 22487$$

One cycle.

$$PV_B = 30000 + 2000 \sum_{k=1}^{5} \frac{1}{(1.1)^k}$$

$$= 37582$$

Three cycles

$$PV_{A_3} = PV_A \left[1 + \frac{1}{(1.1)^4} + \frac{1}{(1.1)^8}\right]$$

$$= 48336$$

$$PV_{B_2} = PV_B \left(1 + \frac{1}{(1.1)^6}\right)$$

$$= 58795$$

Random Variable

X be a RV with value X_1, X_2, X_3, X_4 ... and P_1, P_2, P_3, P_4 probabilites (respectively).

$$E(X) = X_1P_1 + X_2P_2 + X_3P_3 + X_4P_4$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= (X_1^2P_1 + X_2^2P_2 + \cdots) - [E(X)]^2$$

$$\sigma_{XY} = cov(X,Y) = E[(X-E(X))(Y-E(Y))]$$
$$cov(X,Y) = E(XY) - E(X)E(Y)$$

#

correlation coefficient $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1\sigma_2}$ (only for measuring linear association)

$\begin{array}{r} 3 \\ 36 \\ \times 6 \\ \hline 216 \end{array}$

★ $X = 2, 3, 4, 5, 6$ $\quad Y = 4, 9, 16, 25, 36$.

$E(X) = \dfrac{20}{5} = 4$ $\qquad E(Y) = \dfrac{90}{5} = 18$.

$E(X^2) = \dfrac{90}{5} = 18$ $\qquad E[Y^2] = 454.8$.

$Var(X) = (18) - (16)$
$= 2$

$Var(Y) = 454.8 - 324$
$= 130.8$.

$E(XY) = 88$.

$cov(X,Y) = 88 - (4)(18) = \underline{16}$

$\rho_{12} = \dfrac{16}{\sqrt{2}\cdot\sqrt{130.8}} = 0.9892$.

$$V(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2+Y^2+2XY] - (E[X])^2 - (E[Y])^2 - 2E(X)E(Y)$$

$$= E(X^2) - (E[X])^2 + E(Y^2) - (E[Y])^2 + 2E(XY) - 2E(X)E(Y)$$

$$= V(X) + V(Y) + 2\,\text{cov}(XY)$$

$\text{cov}(X,Y) = 0$; X & Y are independent RV.

$$S = \{ (1,1)\ (1,2)\ \ldots\ (1,6)$$
$$(2,1)\ (2,2)\ \ldots\ (2,6)$$
$$\vdots$$
$$(6,1)\ (6,2)\ \ldots\ (6,6) \}$$

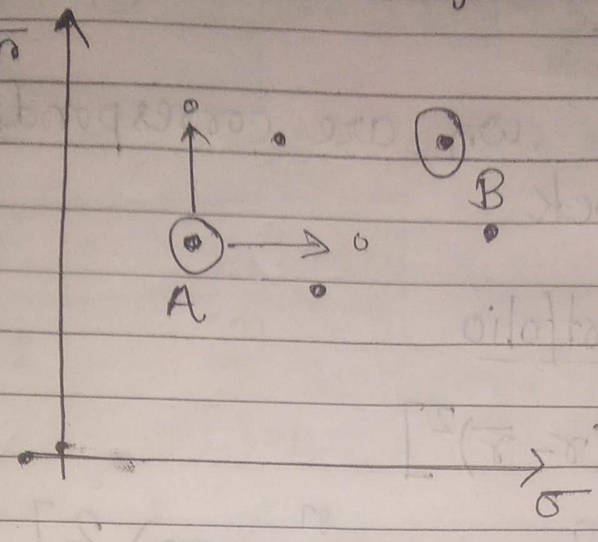
Z = Average of two roll of die.

$$Z = \frac{1}{2}(x+y) \Rightarrow \bar{Z} = \frac{1}{2}(\bar{x}+\bar{y}) = 3.5$$

$$\text{Var}(Z) = \text{Var}(x) + \text{Var}(y)$$

$$= \frac{1}{4}\sigma_X^2 + \frac{1}{4}\sigma_Y^2 = 1.46$$

* Mean- Std deviation diagram :-



$\bar{r} \Rightarrow$ return

$\sigma \Rightarrow$ risk.

for comparision between pts. A and B, depends on the amount of risk the investor would love to take as

for A	for B
$\bar{r}$ lesser	$\bar{r}$ greater
σ lesser	σ lesser.

* Expected value / mean return of a portfolio :-

$$r = \omega_1 r_1 + \omega_2 r_2 + \dots + \omega_n r_n.$$

$$E(r) = \omega_1 E(r_1) + \omega_2 E(r_2) + \dots + \omega_n . E(r_n).$$

[Growth rate for a day (i.e. Return of any particular stock)]

$$\frac{\text{today's price} - \text{yesterday's price}}{\text{yesterday's price}} \times 100.$$

$r_1, r_2, \ldots, r_n$ are avg. returns of each stock. (past)

$w_1, w_2, \ldots, w_n$ are corresponding weights of each stock

★ Variance of Portfolio

$$\sigma^2 = E[(r-\bar{r})^2]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\right) \cdot \left(\sum_{j=1}^{n} w_j (r_j - \bar{r}_j)\right)\right]$$

$$= E\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right]$$

$$= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

★ Calculate mean & variance ($w_1 = 0.25$)

$r_1 = 0.12$ $\quad \sigma_1 = 0.20$ $\quad w_1 = 0.25$

$r_2 = 0.15$ $\quad \sigma_2 = 0.18$ $\quad w_2 = 0.75$

$\sigma_{12} = 0.1$

$$r = (0.25)(0.12) + (0.75)(0.15)$$

$$= \frac{1}{4}(0.12) + \frac{3}{4}(0.15)$$

$$= 0.03 + 0.1125$$

$$\boxed{r = 0.1425}$$

$$\sigma^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} w_i w_j \sigma_{ij}$$

$$= w_1 w_1 \sigma_{11} + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_{22}$$

$$= w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2^2 \sigma_2^2$$

$$= (0.25)^2 (0.2)^2 + ((0.25)(0.75)(0.1))^2 + (0.75)^2 (0.18)^2$$

$$= 0.024475$$

$$\boxed{\sigma = 0.15644488}$$

★ Notion of diversification :-

diversification - not keeping all your eggs in the same basket.

Hedge - mitigating the risk (Not synonomous)

Don't put your whole investment into similar stocks. Instead you should invest into different stocks (i.e. with varying risk and return profiles).

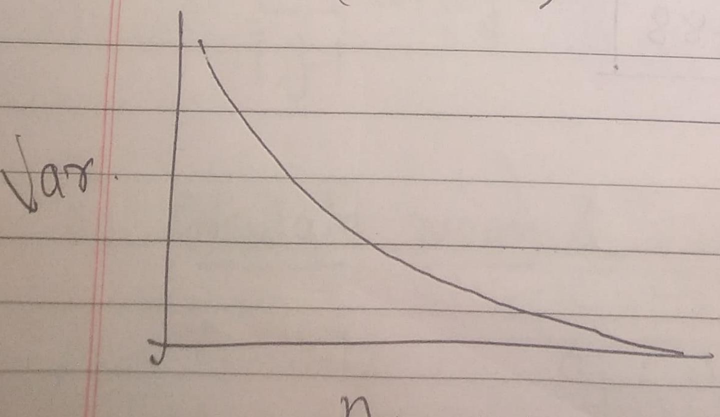
$$r = \frac{1}{n}\sum_{i=1}^{n} r_i \qquad var(r) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.$$

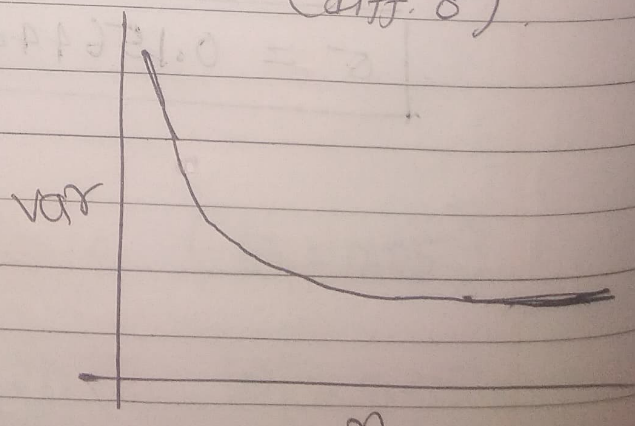
When we have n independent stocks (with equal σ), by increasing the value of n, the portfolio risk ~~σ~~ will decrease.

Ideally, if $n \to \infty$, $var(r) \to 0$.
(Make the portfolio riskless).

(a) Uncorrelated
(same σ)

(b) Correlated
(diff. σ)





eg. n stock portfolio with equal weightage and each with equal sd $= \sigma$. Covariance betn each pair of stocks $= 0.3\sigma^2$. Find variance of portfolio.

$\Rightarrow$

$$Var(r) = E\left[\sum_{i=1}^{n} \frac{1}{n}(r_i - \bar{r})\right]^2$$

$$= \frac{1}{n^2} E\left[\left[\sum_{i=1}^{n}(r_i - \bar{r})\right]\left[\sum_{j=1}^{n}(r_j - \bar{r})\right]\right]$$

$$= \frac{1}{n^2}\sum_{i,j}\sigma_{ij} = \frac{1}{n^2}\left\{\sum_{i=j}\sigma_{ij} + \sum_{i \neq j}\sigma_{ij}\right\}$$

$$= \frac{1}{n^2}\left\{n\sigma^2 + 0.3(n^2 - n)\sigma^2\right\}$$

$$= \frac{\sigma^2}{n} + 0.3\sigma^2\left(1 - \frac{1}{n}\right)$$

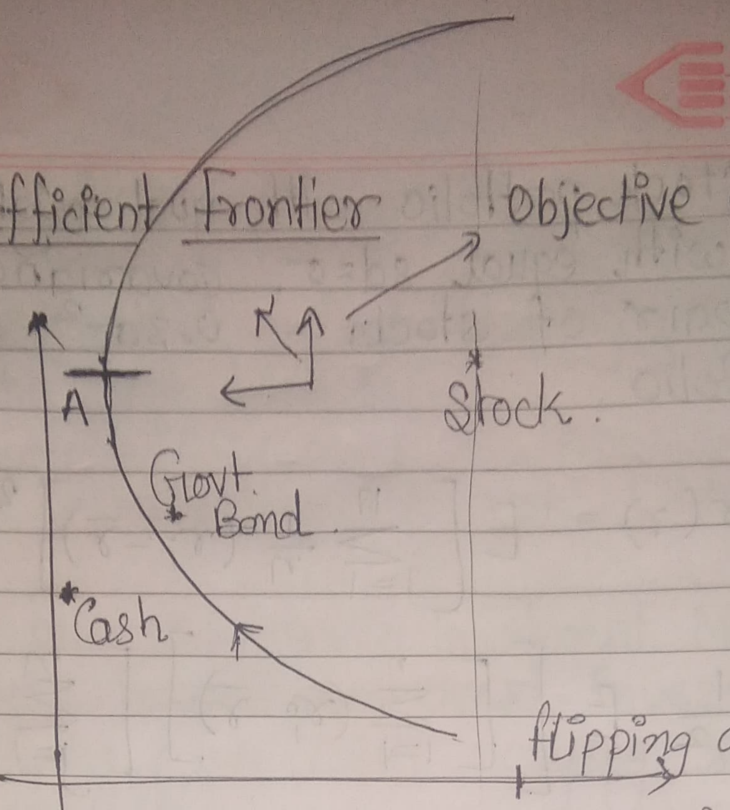
$$= \frac{0.7\sigma^2}{n} + 0.3\sigma^2$$

So, as we keep increasing n, only first term $\left(\frac{0.7\sigma^2}{n}\right)$ goes on decreasing, while we can't make $Var(r)$ less than $0.3\sigma^2$

Related to (b) graph

★ Efficient Frontier

Efficient Frontier is the graph above the point A (min. variance point)



The curve is called as Efficient frontier.

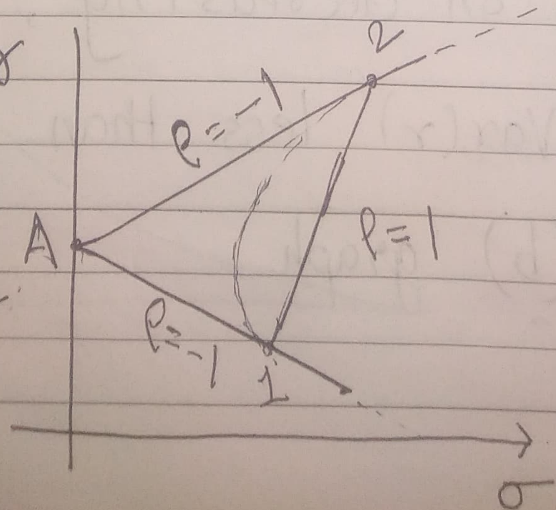
Two assets with different returns and risks (SD)

$$\bar{r}(\alpha) = (1-\alpha)\bar{r}_1 + \alpha \bar{r}_2 \qquad (0 < \alpha < 1.)$$

$$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{12} + \alpha^2\sigma_2^2}$$

$$= \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\rho\,\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$

$$\left(\text{As } \rho = \sigma_{12}/\sigma_1.\sigma_2\right) \quad \left(-1 \le \rho \le +1\right)$$



$\alpha = \dfrac{\sigma_1}{\sigma_1+\sigma_2}$

if $\rho = +1$;

$$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$

$$= \sqrt{((1-\alpha)\sigma_1 + \alpha\sigma_2)^2}$$

$$\sigma(\alpha) = (1-\alpha)\sigma_1 + \alpha\sigma_2$$

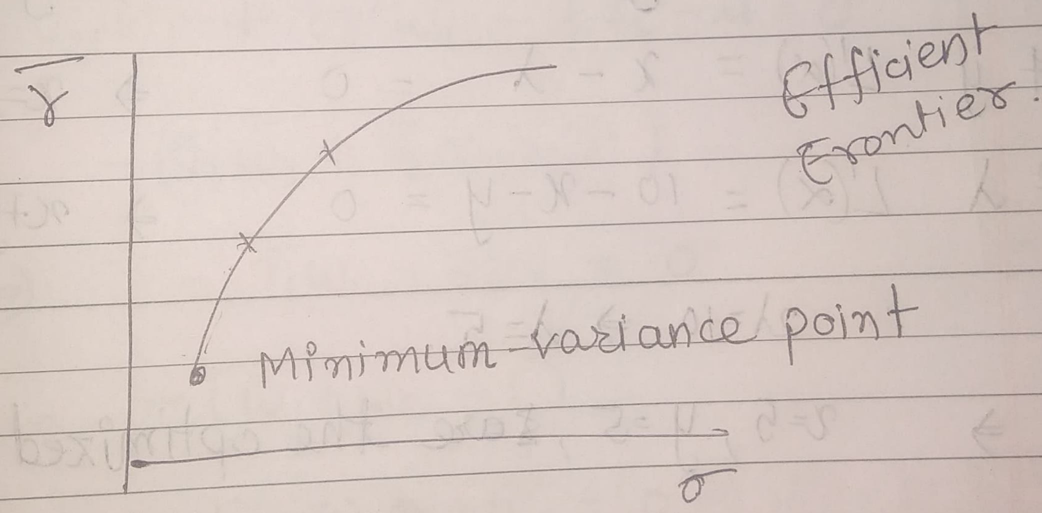
if $\rho = -1$;

$$\sigma(\alpha) = |(1-\alpha)\sigma_1 - \alpha\sigma_2|.$$

if $\sigma = 0$; $\quad 0 = (1-\alpha)\sigma_1 - \alpha\sigma_2$

$$\Rightarrow \quad 0 = \sigma_1 - \alpha(\sigma_1 + \sigma_2)$$

$$\Rightarrow \quad \boxed{\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}}$$



★ Markowitz Model

minimize $\frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$ (to be optimized) Here minized.

subject to $\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$

$\sum_{i=1}^{n} w_i = 1$ } Constraints

Constraint Optimization.

* $U = xy \,; \quad x+y = 10$.

Lagrange function. $L(\alpha) = U + \lambda\,[10 - x - y]$.

λ lagrange multiplier.

for Linear Constraint.

NOT REQ (for non-linear constraint KunTucker method).

$$L(\alpha) = xy + \lambda(10 - x - y)$$

w.r.t x $\quad L'(\alpha) = y - \lambda = 0 \quad \Rightarrow y = \lambda$

w.r.t y $\quad L'(\alpha) = x - \lambda = 0 \quad \Rightarrow x = \lambda$

w.r.t λ $\quad L'(\alpha) = 10 - x - y = 0 \quad \Rightarrow x+y = 10$.

$\therefore\ 2\lambda = 10 \Rightarrow \lambda = 5$.

$\Rightarrow$ $x = 5,\ y = 5$, are the optimized values.

Sol^n

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \omega_i \omega_j \sigma_{ij} - \lambda\left(\sum_{i=1}^{n} \omega_i \bar{r}_i - r\right) - \mu\left(\sum_{i=1}^{n} \omega_i - 1\right)$$

for 2 stocks;

$$L = \frac{1}{2}\left(\omega_1^2\sigma_1^2 + \omega_1\omega_2\sigma_{12} + \omega_2\omega_1\sigma_{2}\sigma_1 + \omega_2^2\sigma_2^2\right) - \lambda\left(\bar{r}_1\omega_1 + \bar{r}_2\omega_2 - \bar{r}\right) - \mu\left(\omega_1 + \omega_2 - 1\right)$$

$$\frac{\partial L}{\partial \omega_1} = \frac{1}{2}\left(2\sigma_1^2\omega_1 + \sigma_{12}\omega_2 + \sigma_{21}\omega_2\right) - \lambda\bar{r}_1 - \mu$$

$$\frac{\partial L}{\partial \omega_2} = \frac{1}{2}\left(\sigma_{12}\omega_1 + \sigma_{21}\omega_1 + 2\sigma_2^2\omega_2\right) - \lambda\bar{r}_2 - \mu.$$

Using the fact $\sigma_{12} = \sigma_{21}$; set to zero.

$$\Rightarrow \sigma_1^2\omega_1 + \sigma_{12}\omega_2 - \lambda\bar{r}_1 - \mu = 0.$$

$$\Rightarrow \sigma_{21}\omega_1 + \sigma_2^2\omega_2 - \lambda\bar{r}_2 - \mu = 0.$$

$$\frac{\partial L}{\partial \lambda} = -\left(\bar{r}_1\omega_1 + \bar{r}_2\omega_2 - \bar{r}\right)$$

$$\Rightarrow \bar{r} - \bar{r}_1\omega_1 - \bar{r}_2\omega_2 = 0$$

$$\frac{\partial L}{\partial \mu} = -\left(\omega_1 + \omega_2 - 1\right)$$

$$\Rightarrow 1 - \omega_1 - \omega_2 = 0$$

Financial Economics (HS 301)

20% Grp. Presentation (8-12) mse 25%

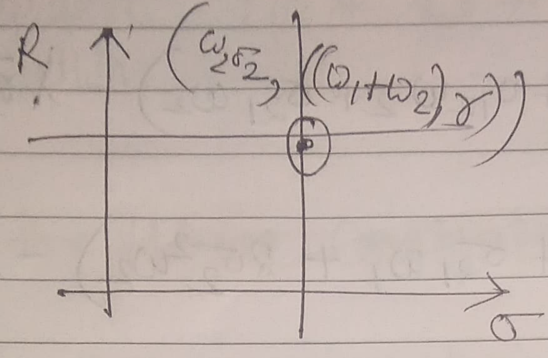
20% quizzes/assignments. ese 25%

10% viva (indually)

$r_1 \quad r_2$ $\qquad R = w_1 r_1 + w_2 r_2$

$\sigma_1 \quad \sigma_2$ $\qquad \sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 w_1 w_2$

Case (1). $r_1 = r_2 \quad \sigma_1 = 0 \quad \sigma_2 \neq 0$



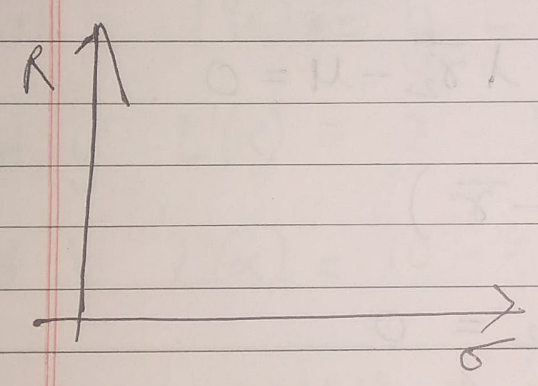
$R = (w_1 + w_2) r_1$

$\sigma^2 = w_2^2 \sigma_2^2$

$\sigma = (w_2 \sigma_2)$

a point $\qquad$ Invest only in first stock.

Case (2) $r_1 = r_2 \quad \sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \rho = 0 \quad \sigma_1 \neq \sigma_2$



$R = (w_1 + w_2) r$

$\sigma^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$

Invest amount in low risk stock.

Case (3) $r = r_1 = r_2 \quad \sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \rho = 0 \quad \sigma_1 = \sigma_2 = \sigma'$

$R = (w_1 + w_2) r \qquad \sigma^2 = (w_1^2 + w_2^2)\sigma'^2$

σ^2 is min for $w_1 = w_2 = 1/2$

Invest same amount in both the stocks to minimize risk (σ).

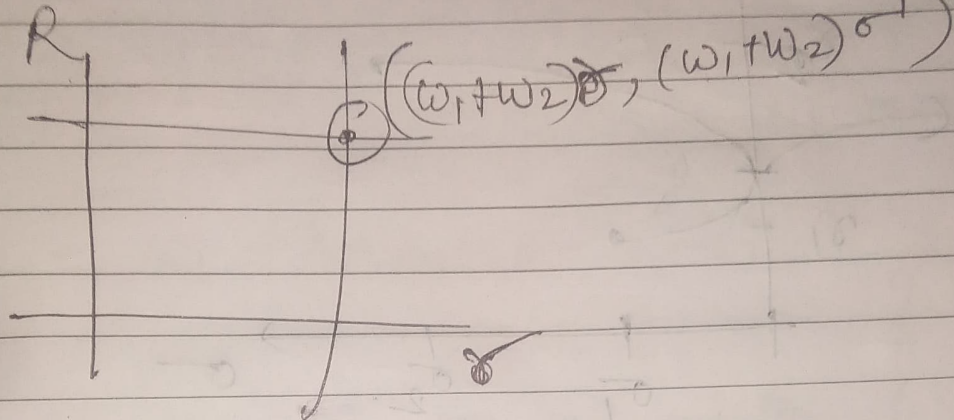
diversification

Case (4) $r_1 = r_2 = r \quad \sigma_1 \neq 0 \; \sigma_2 \neq 0 \quad \rho = 1 \quad \sigma_1 = \sigma_2 = \sigma'$

$$R = (w_1 + w_2) r \qquad \sigma^2 = (w_1\sigma_1 + w_2\sigma_2)^2$$

$$\sigma^2 = (w_1 + w_2)^2 \sigma'^2$$

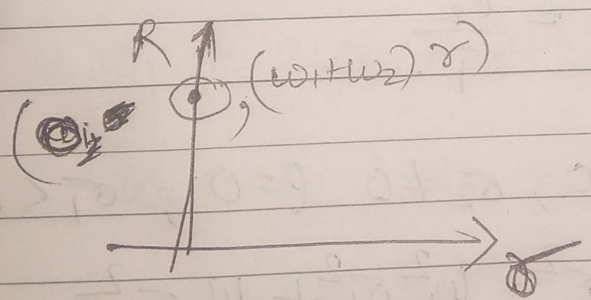
$$\sigma = (w_1 + w_2)\sigma'.$$



Case 5 $r_1 = r_2 = r \quad \sigma_1 \neq 0 \; \sigma_2 \neq 0 \quad \rho = -1 \quad \sigma_1 = \sigma_2 = \sigma'$

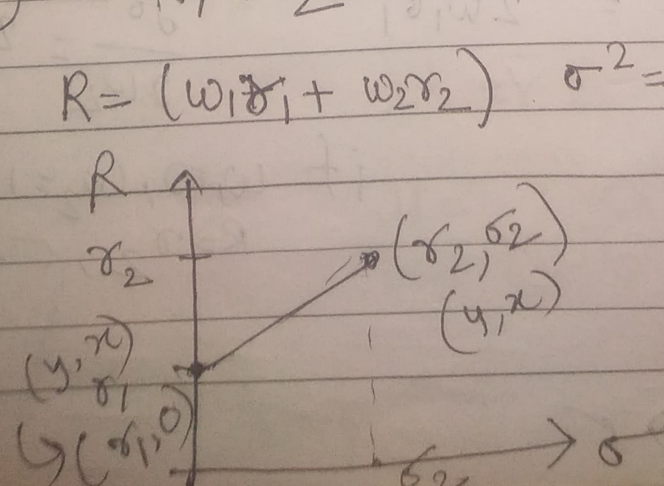
$$R = (w_1 + w_2) r \qquad \sigma^2 = (w_1\sigma_1 - w_2\sigma_2)^2$$

$$\sigma = |w_1 - w_2| \sigma'$$



Case (6) $r_1 < r_2, \; \sigma_1 = 0, \; \sigma_2 \neq 0, \; \rho = 0$

$$R = (w_1 r_1 + w_2 r_2) \qquad \sigma^2 = w_2^2 \sigma_2^2 \qquad \sigma = w_2\sigma_2$$

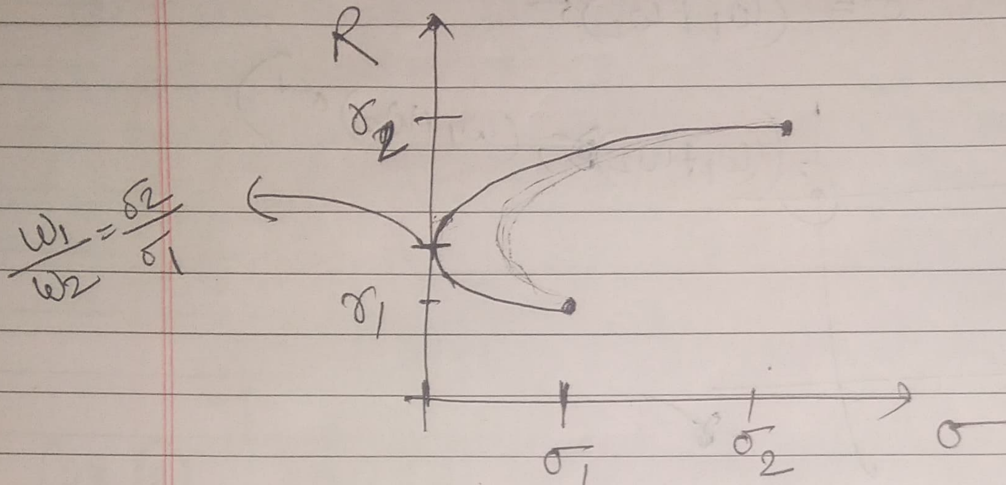


Case 7 $\quad r_1 < r_2 \quad \sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \rho = -1 \quad \sigma_1 < \sigma_2$

$$R = \omega_1 r_1 + \omega_2 r_2 \qquad \sigma^2 = \omega_1^2 \sigma_1^2 - 2\sigma_1\sigma_2\omega_1\omega_2 + \omega_2^2\sigma_2^2$$

$$\sigma^2 = (\omega_1\sigma_1 - \omega_2\sigma_2)^2$$

$$\sigma = |\omega_1\sigma_1 - \omega_2\sigma_2|$$



if $w_1 = 1,\ w_2 = 0 \quad R = r_1 \quad \sigma = \sigma_1$

$\omega_1 = 0,\ \omega_2 = 1 \quad R = r_2 \quad \sigma = \sigma_2$

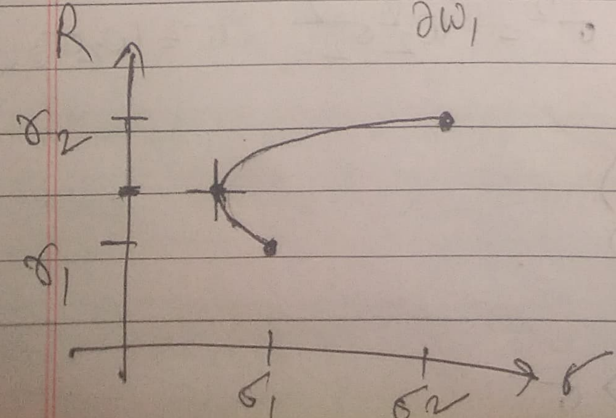
optimal $\quad \sigma = 0 \Rightarrow \quad \omega_1\sigma_1 = \omega_2\sigma_2$

$$\frac{\omega_1}{\omega_2} = \frac{\sigma_2}{\sigma_1}.$$

Case 8 $\quad r_1 < r_2 \quad \sigma_1 \neq 0,\ \sigma_2 \neq 0 \quad \rho = 0,\ \sigma_1 < \sigma_2$

$$R = \omega_1 r_1 + \omega_2 r_2 \qquad \sigma^2 = \omega_1^2\sigma_1^2 + \omega_2\sigma_2^2$$

$$\frac{\partial \sigma^2}{\partial \omega_1} = 2\omega_1\sigma_1^2 \qquad \frac{\partial \sigma^2}{\partial w_2} = 2W_2\sigma_2^2$$



if $w_1 = 0,\ w_2 = 1$

$R = r_2,\ \sigma = \sigma_2$

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* Let's assume there are three uncorrelated assets with equal variance 1 and mean return 1, 2, 3 resp. What would be the optimal allocation of money for these three assets?

$\Rightarrow$ $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$.

$$\alpha = \frac{1}{2} \sum_{i,j=1}^{3} w_i w_j \sigma_{ij} - \lambda \left[\sum w_i r_i - \bar{r} \right] - \mu \left(\sum w_i - 1 \right)$$

$$\frac{\partial \alpha}{\partial w_1} = w_1 - \lambda - \mu = 0 \quad \text{—— (1)}$$

$$\frac{\partial \alpha}{\partial w_2} = w_2 - 2\lambda - \mu = 0 \quad \text{—— (2)}$$

$$\frac{\partial \alpha}{\partial w_3} = w_3 - 3\lambda - \mu = 0 \quad \text{—— (3)}$$

$r_1 = 1$
$r_2 = 2$
$r_3 = 3$

$w_1 + 2w_2 + 3w_3 = \bar{r}$

$w_1 + w_2 + w_3 = 1$

$w_1 = \lambda + \mu$
$w_2 = 2\lambda + \mu$
$w_3 = 3\lambda + \mu$

$\Rightarrow$ $6\lambda + 3\mu = 1$ $\quad$ $6\mu + 14\lambda = \bar{r}$

$\Rightarrow$ $12\lambda + 6\mu = 2$

$\Rightarrow$ $2\lambda = \bar{r} - 2$ $\quad \Rightarrow \quad \boxed{\lambda = (\bar{r} - 2)/2}$

$\Rightarrow$ $\mu = \dfrac{1 - 6\lambda}{3} = \dfrac{\cancel{1 - (6)(+1/2)}}{3} = \dfrac{4}{3}$

$\dfrac{1 - \cancel{6}^{3}\frac{(\bar{r}-2)}{\cancel{2}}}{3}$

$\dfrac{7 - 3\bar{r}}{3}$

$$\boxed{\mu = \frac{7 - 3\bar{r}}{3}}$$

$\cancel{w_1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}}$

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$$w_1 = \lambda + \mu = \left(\frac{\bar{r}-2}{2}\right) + \left(\frac{7 - 3\bar{r}}{3}\right)$$

$$w_1 = \frac{3\bar{r} - 6 + 14 - 6\bar{r}}{6} = \frac{-3\bar{r} + 8}{6}$$

$$w_1 = \frac{-3}{6}\bar{r} + \frac{4}{3} = \boxed{\frac{4}{3} - \left(\frac{\bar{r}}{2}\right)}$$

w_1

$$w_2 = \cancel{2}\left(\frac{\bar{r}-2}{\cancel{2}}\right) + \left(\frac{7 - 3\bar{r}}{3}\right)$$

$$= \cancel{\bar{r}} - 2 + \frac{7}{3} - \cancel{\bar{r}}$$

$$\boxed{w_2 = 1/3}$$

$$w_3 = 3\lambda + \mu = 3\left(\frac{\bar{r}-2}{2}\right) + \left(\frac{7 - 3\bar{r}}{3}\right)$$

$$= \frac{9\bar{r} - 18 + 14 - \cancel{6\bar{r}}}{6} = \frac{3\bar{r} - 4}{6}$$

$$\boxed{w_3 = \left(\frac{\bar{r}}{2}\right) - \frac{2}{3}}$$

$$\sigma^2 = \sum_{i,j=1}^{3} w_i w_j \sigma_{ij}$$

$$\sigma^2 = w_1^2 + w_2^2 + w_3^2$$

On solving $\sigma = \sqrt{7/3 - 2\bar{r} + \frac{\bar{r}^2}{2}}$

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$$0 = \frac{7}{3} - 2\bar{r} + \frac{\bar{r}^2}{2}$$

$$\Rightarrow \quad 3\bar{r}^2 + 14 - 12\bar{r} = 0$$

$$\Rightarrow \quad 3\bar{r}^2 - 12\bar{r} + 14 = 0$$

$$(144) - 4(14)(3) < 0$$

$\Rightarrow$ No real value for $\bar{r}$.

(if real soln for $\bar{r}$ exists, then for optimal weights w_1, w_2, w_3 use this value)

$$\sigma^2 = \left(\frac{4}{3} - \frac{\bar{r}}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{\bar{r}}{2} - \frac{2}{3}\right)^2$$

$$= \frac{16}{9} + \frac{\bar{r}^2}{4} - 2\left(\frac{4}{3}\right)\left(\frac{\bar{r}}{2}\right) + \frac{1}{9} + \frac{\bar{r}^2}{4} + \frac{4}{9} - 2\left(\frac{\bar{r}}{2}\right)\left(\frac{2}{3}\right)$$

$$= \frac{21}{9} + \frac{2\bar{r}^2}{4} - \frac{6\bar{r}}{3}$$

$$\sigma^2 = \frac{7}{3} + \frac{\bar{r}^2}{2} - 2\bar{r}$$

$$\sigma = \sqrt{\frac{\bar{r}^2}{2} - 2\bar{r} + 7/3}$$

All possible outcomes line. Efficient frontier

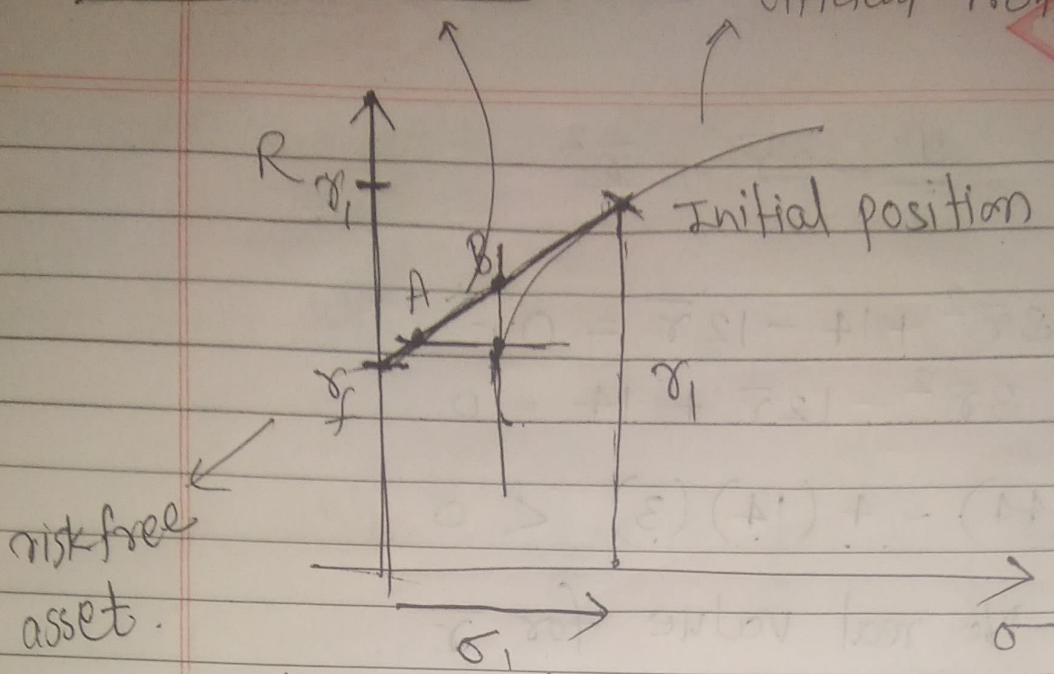


fig. inclusion of risk-free asset in a portfolio.

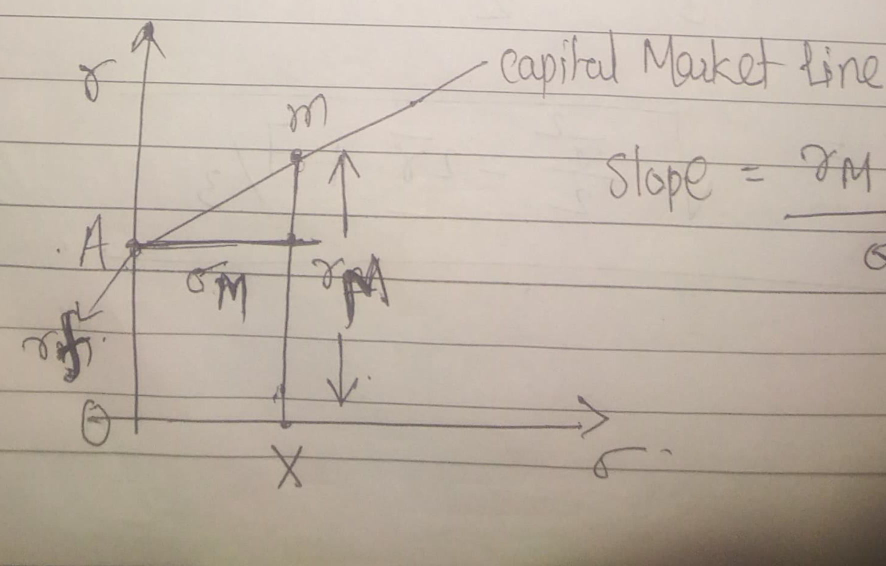
So, it depends on the risk-appetite of the individual where to stay on the straight line.

* Insider Trading

Sensex - Stock Prices (Demand and Supply).

Algorithm/frequency Trading → Short Time.

* Capital Asset price Model (CAPM) :-



$$\text{Slope} = \frac{r_M - r_f}{\sigma_M}$$

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Q. Smith is an investor. He notices that $r_f = 6\%$ and market portfolio has return $= 12\%$ and $15\% = S.D.$

$1000\$$ to $1000000\$$ will take around 118 years & He wants $1000.000\$$ in only 10 years.

* CAPM

If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset "i" satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_m - r_f)$$

where $\beta_i = \sigma_{iM}/\sigma_M^2$

$\sigma_{iM} \Rightarrow$ covariance betn ith stock and total market

Proof :- Assume 'α' is invested in asset 'i' and '$(1-\alpha)$' is invested in market M.

$0 \leq \alpha \leq 1$

portfolio's return $= \alpha\bar{r}_i + (1-\alpha)\bar{r}_M = \bar{r}_\alpha$

portfolio's risk $= [\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2]^{1/2} = \sigma_\alpha$

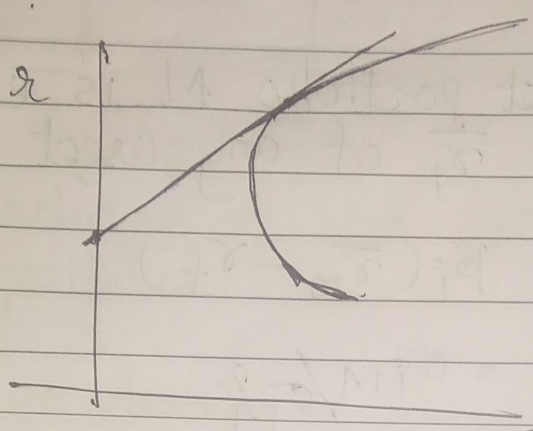
$$\frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$$

$$\frac{d\sigma_\alpha}{d\alpha} = \alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)^2\sigma_M^2$$

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} =$$

Further, $\dfrac{d\bar{r}_\alpha}{d\sigma_\alpha} = \dfrac{d\bar{r}_\alpha / d\alpha}{d\sigma_\alpha / d\alpha}$

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{\sigma_{iM} - \sigma_M^2}$$



$\alpha = 0$; all money is in the market.

Slope must be equal to slope of capital market line. Hence,

$$\frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$\bar{r}_i - \bar{r}_M = \frac{(\bar{r}_M - r_f)(\sigma_{iM} - \sigma_M^2)}{\sigma_M^2}$$

$$\bar{r}_i = \cancel{\bar{r}_M} + \frac{\bar{r}_M \sigma_{iM}}{\sigma_M^2} - \frac{\cancel{\bar{r}_M \sigma_M^2}}{\cancel{\sigma_M^2}} - \frac{r_f \sigma_{iM}}{\sigma_M^2} + r_f$$

$$\bar{r}_i = r_f + \frac{(\bar{r}_M - r_f)\,\sigma_{iM}}{\sigma_M^2}$$

$$\boxed{\bar{r}_p = r_f + \beta_i(\bar{r}_M - r_f)}$$

where, $\beta_i = \dfrac{\sigma_{iM}}{\sigma_M^2}$

β denotes risk profile of 'i' asset

$(\bar{r}_i - r_f)$ = excess return of 'i' asset.

$(\bar{r}_M - r_f)$ = excess return of market.

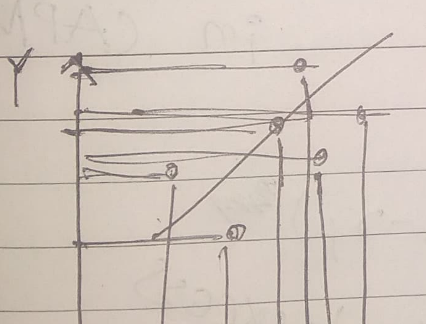
excess return of 'i' stock $\propto$ excess return of market

(β is constant)

* (M does not include i).

* <u>linear Regression</u>

$\hat{y} = \hat{\alpha} + \hat{\beta} x$.



y is dependent

x is independent

error = $y - \hat{y}$ (Actual − fitted)

$\sum(\text{error})^2 = \sum(y-\hat{y})^2$

$= \sum(y - \hat{\alpha} - \hat{\beta} x)^2$

minimize $\sum(\text{error})^2$ w.r.t. parameter (α, β).

$\dfrac{\partial \sum(\text{error})^2}{\partial \hat{\alpha}} = 0 = -2\sum(y_i - \hat{\alpha} - \hat{\beta} x_i)$

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$$\frac{\partial \sum (\text{error})^2}{\partial \hat{\beta}} = 0 = -2 \sum x_i (y_i - \hat{\alpha} - \hat{\beta} x_i) \quad (2)$$

Solving (1) & (2) $\Rightarrow$

$$\hat{\beta} = \frac{\sum x_i' y_i'}{\sum x_i'^2} \qquad x_i' = x_i - \bar{x}, \quad y_i' = y_i - \bar{y}$$

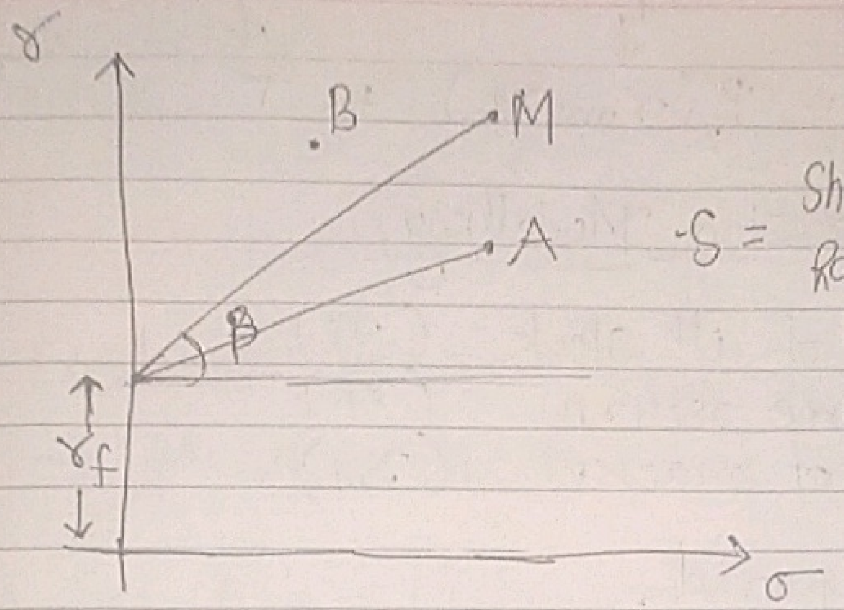
$$\hat{\alpha} = \hat{y} - \hat{\beta} \bar{x} \qquad \bar{x} = \text{Mean of } x, \quad \bar{y} = \text{Mean of } y.$$

$$\text{cov}(x,y) = \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y})$$

$$\left\{ \hat{\beta} = \frac{\text{cov}(x,y)}{\text{var}(x)} \right\} \rightarrow \text{risk factor in CAPM.}$$

in CAPM Model.

$$\underbrace{(r_i - r_f)}_{\text{ith stock's excess return}} = \beta \underbrace{(r_m - r_f)}_{\text{market excess return}}$$



$$S = \text{Sharpe Ratio} = \frac{r_A - r_f}{\sigma_A}$$

↳ Risk adjusted excess return of an individual stock /portfolio

higher Sharpe ratio is always better.

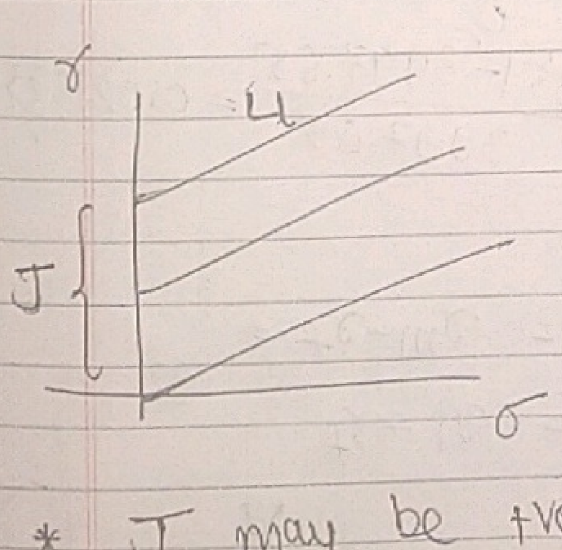
$$(r_i - r_f) = \beta\,(r_m - r_f) \longrightarrow \text{in CAPM}$$

* $(r_i - r_f) = \beta(r_m - r_f) + J$

⇓

Jensen's index.

for CAPM, $J = 0$.



$J \Rightarrow$ constant intercept term (excess return)

$\beta \Rightarrow$ some risk is associated with it.

* J may be +ve or −ve

* Higher the value of J, that is better

$$Y = \alpha + \beta_1 X$$

$$(r_i - r_f) = \beta_2 (r_m - r_f) + J$$

for Regression Modelling,

Return of i^{th} stock (r_i)
Risk-free return (r_f)
Return of market (r_m)

are required.

$$\text{Growth Rate in percentage} = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100$$

$P_t \Rightarrow$ Price today
$P_{t-1} \Rightarrow$ Yesterday's Price.

Stock A	sensex	Govt. Bond (r_f)	return A (r_i)
3997.65	13619.7	0.07	0
4085.1	13928.33	0.07	$\frac{4085.1 - 3997.65}{3997.65} = 0.021875$

* return Market (r_m) (M)

0

$$\frac{13928.33 - 13619.7}{13619.7} = 0.022661$$

$X = r_m - r_f$

$Y = r_i - r_f$

$\cancel{X}$ $\cancel{Y}$

Y	X
excess return A.	excess return M.
(return A − Govt Bond)	(return Market − Govt Bond)
−0.048124648	−0.047339442.

Excel → File → Options ⇒ Add Ins → GO
→ (Analysis Tool PAK / Solver Add IN) ✓.

↳ For Adding data analyisis.
(For Regression Modelling)

☆ CAPM as pricing formula

If you sell a stock at Q and buy at P

rate of return = $\frac{Q-P}{P}$.

From CAPM ⇒ $(r_i - r_f = \beta(r_m - r_f))$

$$\frac{Q-P}{P} = r_f + \beta(r_m - r_f).$$

$$\therefore P = \frac{Q}{1 + r_f + \beta(r_m - r_f)}$$

If we have two assets with price P_1 and P_2 and end price as → (Q_1) and (Q_2)

$$P_1 = \frac{Q_1}{1 + r_f + \beta_1(r_m - r_f)} \qquad P_2 = \frac{Q_2}{1 + r_f + \beta_2(r_m - r_f)}$$

$P_1 + P_2 = ??$

Application

X is planning to invest in mutual fund which has

(90 – 10)% composition.

Market Portfolio (return = 15%) ; Fixed Income (7% return)

One share of mutual fund = 100 INR.

Let, $\beta = 0.9$

$$(r_i - r_f) = \beta(r_M - r_f).$$

$$r_i - 0.07 = 0.9(0.15 - 0.07)$$

$$r_i = 0.07 + 0.9 \times 0.08$$
$$= 0.07 + 0.072$$
$$= 0.142$$

$r_i = 14.2\%$
in %

$$\Rightarrow \quad Q = (0.9 \times 1.15 + 0.1 \times 1.07) = 1.142 \times 100 = 11$$

$$P = \frac{Q}{1 + \beta(r_m - r_f) + r_f} = \frac{1.142 \times 100}{1 + 0.9(0.15 - 0.07) + 0.07}$$

$$= \underline{100}.$$

* Arbitrary Price Theory :- (APT)

$r_i - r_f = \beta_1 (r_M - r_f) + \beta_2 (\text{gdp growth})$

$+ \beta_3 (\text{inflation}) + \beta_4 (\text{Exchange rate movement})$

$+ \beta_5 (\text{Interest rate}) + \text{Error}$

An extension of CAPM with more explanato-ry variables to affect excess return of ith stock.

☆ Fama-French Three Factor Model :-

$(r_i - r_f) = \alpha + \beta_1 (r_M - r_f) + \beta_2 r_{SMB} + \beta_3 r_{HML}$
$+ \text{Error}.$

r_{SMB} = Historical rate of return on (small size portfolio – big size portfolio).

r_{HML} = Historical rate of return on (high B/M ratio portfolio – Low B/M ratio)

B/M ratio → Book to Market ratio.

Book Value = If all assets are sold and all debts are paid

Market Value = Total stock × Each stock price.

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In APT,

β_1 β_2 β_3 β_4 β_5

All these signs depends on how the corresponding factor like (gdpgrowth, inflation, Exchange rate movement, interest rate) affects that particular stock. They are positive if it favours the stock otherwise negative.

☆ Notion of systematic risk

From CAPM,

$$r_i = r_f + \beta_i (r_M - r_f) + \varepsilon_i \quad \text{—①}$$

Here, ε_i is random / idiosyncratic error

Assumption $\begin{cases} E(\varepsilon_i) = 0 \\ \text{and } \operatorname{cov}(\varepsilon_i, \sigma_M) = 0 \end{cases}$

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + var(\varepsilon_i) \longrightarrow \text{Why?}$$

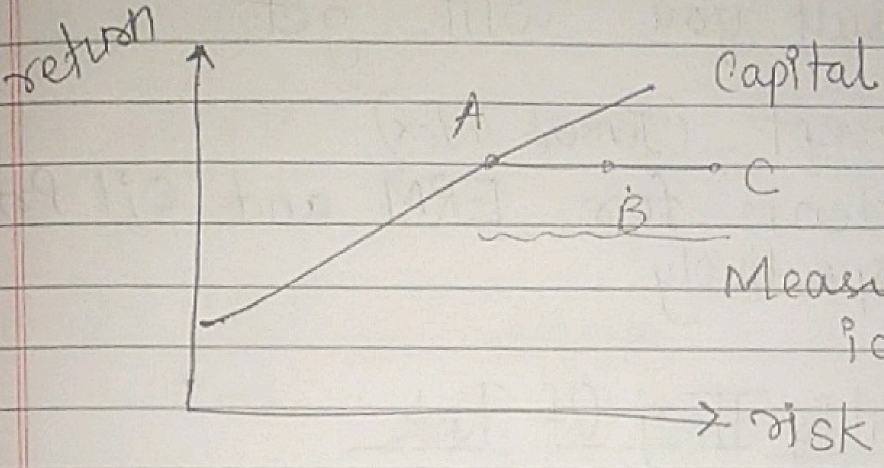
$$E(r_i) = E(r_f) + \beta_i E(r_M) - \beta_i E(r_f) + 0 \quad \text{—②}$$

① − ②

$$E\{r_i - E(r_i)\}^2 = E\left[\beta_i\{r_M - E(r_M)\}\right]^2 + var(\varepsilon_i)$$

$$\therefore \sigma_i^2 = \beta_i^2 \sigma_M^2 + var(\varepsilon_i)$$

systematic risk (can't be reduced)	Non-systematic or idiosyncratic risk (Uncorrelated with market)



Capital Market Line.

Measure of idiosyncratic risk

* Application of APT :-

given data

Stock-A	Sensex	Govt. Bond	Oil price
3997.65	13619.7	0.07	16.98667
4085.1	13928.33	0.07	18.6333
4087.9	13872.37	0.07	19.47667

ERA (excess return A) | ERM (excess return M) | Oil

② $\overset{0}{\left(\frac{4085.1 - 3997.65}{3997}\right)} - (0.07)$ $\quad \overset{0}{\left(\frac{13928.33 - 13619.7}{13619.7}\right)} - (0.07)$ $\quad$ 16.98667 186333

$\underline{r_A - r_f}$ is a function of $(r_M - r_f)$ and (Oil price).

linear

then do data analysis (regression) using excel

Here Y = ERA.

X = ERM, Oil Price.

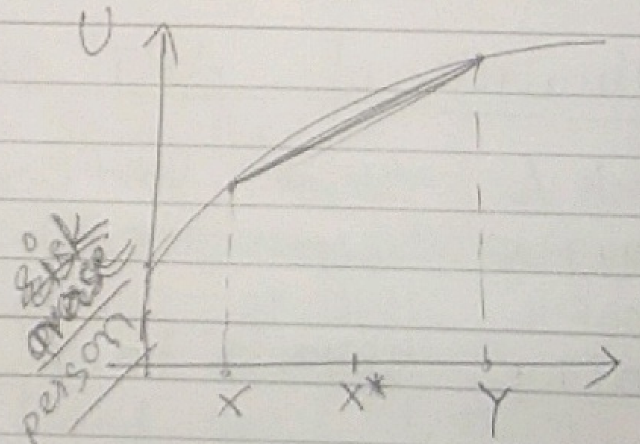
in result you will get

intercept. (Jensen's Index).

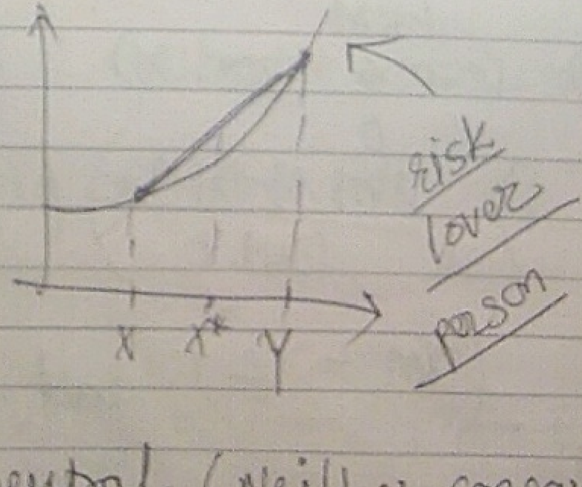
coefficients for ERM and Oil Price respectively.

* Utility Theory Of Risk

Utility function



concave w.r.t. origin



convex w.r.t. origin

straight line — risk neutral (Neither concave nor convex) eg. You have two option.

head = +10 INR Tail = 0 INR.

U = utility function = $x - 0.4x^2$

$$\frac{dU}{dx} = -0.4(2x) = -0.8x.$$

$$\frac{d^2U}{dx^2} = -0.8 < 0 \quad \text{concave}$$

if $\frac{d^2U}{dx^2} > 0$ convex.

---

eg. Treasury bill = 6 INR. (for sure).

Second option = 10 INR, 5 INR, 1 INR with 0.2, 0.4, 0.4 probability.

E(x) = $0.2 \times 10 + 0.4 \times 5 + 0.4 \times 1$

Mean $= 2 + 2 + 0.4 = \underline{4.4 \text{ INR}}$

Utility function is $U(x) = \sqrt{x}$

$$\frac{dU}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{d^2U}{dx^2} = \frac{1}{2}\frac{d}{dx}(x^{-1/2}) = \frac{1}{2}\left(\frac{-1}{2}\right)\frac{1}{x\sqrt{x}}$$

$$\frac{d^2U}{dx^2} = \frac{-1}{4x\sqrt{x}}$$

~~As x is positive~~

< 0 concave

$\therefore$ The person is risk averse

So, if Treasury bill = 4.4 INR
then also he would have gone with fixed one

★ Arrow - Pratt measure of absolute risk aversion

$$A = -\frac{U''(x)}{U'(x)}$$

U ⇒ Utility function (Y - axis)
X ⇒ Income (X - axis

A > 0 ; for risk averse
A < 0 ; for risk taker.

★ Arrow - Pratt measure of relative risk aversion

$$A' = x\,A = -\frac{x \cdot U''(x)}{U'(x)}$$

Income level also comes into consideration.
where x is the payoff of a given lottery and $U(x)$ is the utility derived from that payoff.

eg ① $U(x) = \ln(x)$

$$U'(x) = \frac{1}{x}$$

$$U''(x) = \frac{-1}{x^2} < 0 \quad \underline{\text{concave}}$$

$$\therefore \quad A = -\frac{U''(x)}{U'(x)} = -\frac{(-1/x^2)}{(1/x)} = \frac{1}{x} > 0$$

$\therefore$ risk averse.

$A' = x.A = x.\frac{1}{x} = 1$

A' is constant
no change in
risk apetite
with x.

eg ② $U(x) = x - cx^2$

$$U'(x) = 1 - 2cx$$
$$U''(x) = -2c \quad < 0 \quad \text{concave.}$$

$$\therefore \quad A = -\frac{U''(x)}{U'(x)} = -\frac{(-2c)}{1-2cx}$$

$$A = \frac{2c}{1-2cx}$$

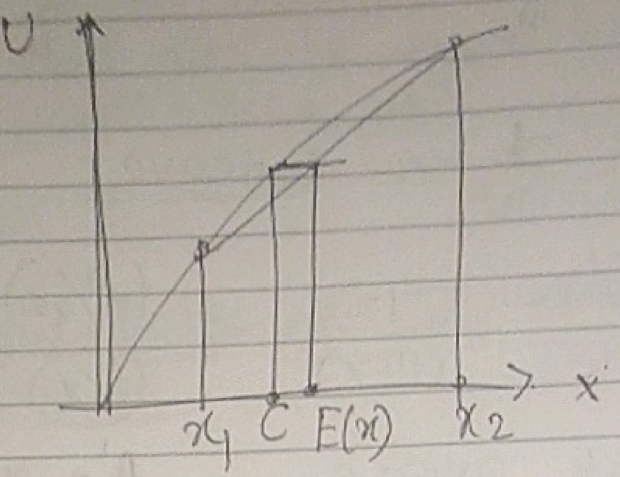
$\therefore$ risk averse.

$$A' = \frac{2cx}{1-2cx} = \frac{2cx-1+1}{1-2cx}$$

$$= \frac{1}{1-2cx} - 1$$

As $x\uparrow$ $A'\uparrow$

* Certainty Equivalence :-



C and $E(x)$ has same level of utility.

Here, C is called certainty equivalence

for risk averse ; $C < E(x)$

risk lover ; $C > E(x)$

risk neutral ; $C = E(x)$

min. amt. of money for a person when it does not matter the person participates or not.

Derivate Market

* Forward :-

Forward contract is an agreement b/w two parties to buy or sell on a pre-determined future date at a 'agreed-upon' price.

Price and data are fixed in advance.

eg In the month of August, a rice mill agrees to buy 2.35 tonnes of rice from a farmer (X) in the following february 38000 INR / tonne. This is a forward contract.

If the price goes up in Feb to 40000 INR then rice mill will gain

and if the price goes down to 36000 INR then rice mill will loose.

→ Thales' Strategy :- He used to read stars and predict rainfall / harvest.

* Future Contract :-

Futures, traded in exchange → special type of forwards'

Other types - forwards
can be traded OTC
over the counter.

There is mediator in between for Futures.

Why? transparency,

* Options → Right to buy or sell but no obligation unlike future/forward

Call option → right to buy
Put option → right to sell.

- Comes with extra cost of premium

2008 Financial Crisis

Low interest rate.
People borrowed money from bank.
(Higher (redit score) Prime borrower
(Lower — // —) Sub prime borrower (Expectation that house price will rise)
All loans were securitized

'packaged'

↓

(MBS) Morgage Backed Securities
(Several MBS).

↓

(CDO) Collatorallized Debt Obligations.

Very Risky | Moderately Risky | Poorly Risky.

Crisis errupted

↓

CDS ← Moody — (Credit Ratings)
Credit Default Swap

Options

① Strike price, ② date of maturity / expiration day.

ex. A enters into a contract with B whereby A has the right to buy 100gm gold from B for 1 lakh INR at any time before 20 October. A has to pay an option premium of 10000 for 100gm.

CALL OPTION

American option	European Option
any time before expiration date	only at the expiration date

ex. A enters into contract with B. whereby A has the option to sell 100gm gold to B at a price of 120000 INR any time before october 20. for granting this right to A, it has to buy 12000 INR as option premium to B.

PUT OPTION

Bermuda type → on fixed dates of a month
American type → on any day before maturity
European type → only on maturity day
* Asian type → Avg. of different spot prices = strick price

it is not decided earlier.

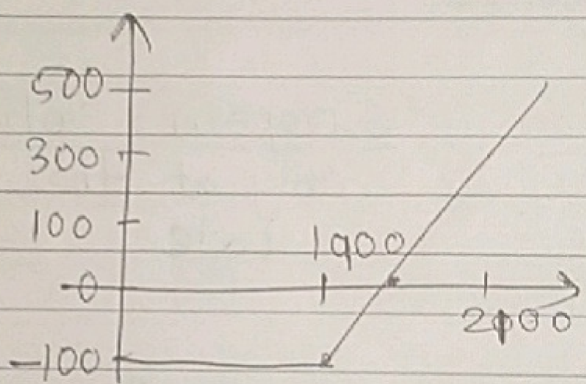
Capped type – pre determined cap price

cap price > strike price (call)

cap price < strike price (put).

Once price reaches the cap its' automatically exercised.

Pay off of option strategy



current spot price = 1900

premium for SBI call = 100

(with 1900 strike price).

Stock Price at expiry	Premium Paid	gain by exercising option	Net gain
1500	-100	0	-100
1700	-100	0	-100
1900	-100	0	-100
2100	-100	200	100

A third variety is called call-cum-put option (Teji - Mandi - Indian version) :

A enters into a contract with B, whereby he has the option to buy or sell 100 ounce of gold at \$1400 per ounce any time befor 1 Nov. For granting this option, B charges a premium of \$ 31 from A.

Option

On 1 Mar, the price of copper is \$ 9800 per ton. A expects a rise in price and buys an American option on June copper.

Maturity - 15 June
Strike price = \$ 9800.

15 May - 10100 \$/ton
15 June - 9700 \$/ton

As the option is an American one, A exercises it on 15th May, takes delivery of copper at strike price 9800 and sells it in spot market at profit (10100 - 9800) = 300 \$/ton.

In case of European option he had to exercise it on 15 June only, He will earn only (-100) \$/ton or he incurs loss

This is why American Option Premium > European Option Premium.

Relation between strike price & market price.

An option is said to be "at the money" (ATM). if current market price = strike price

An option is said to be "in the money" (ITM) if when strike price relates to the market price in such a way it is advantageous to execute the deal /exercise the option.

ITM { for CALL option strike price < current price
PUT option strike price > current price

Out of the money (OTM) is just reserve to ITM

OTM { for CALL option strike price > current price
PUT option strike price < current price

* Concept of "intrinsic value".

Premium for an option = intrinsic value + time value.

An option is said to have any intrinsic value when it is "ITM".
Intrinsic value is zero for ATM or OTM.

Time value depends on time to maturity higher the remaining time to maturity, higher is uncertainty about it, and higher is the time value.

So, intrinsic value = premium − time value
also intrinsic value = strikeprice − mkt. price.

for an ITM option :- An example,
A company's share is trading at 110 INR and 105 call has 5 intrinsic value.
premium > intrinsic value

⇒ Intrinsic Value of any call option.
= mkt price − discounted present value of the exercise price

Intrinsic Value of American Put.
= exercise price − mkt. price

Intrinsic Value of European put option.
= discounted value of strike price − mkt. price

Factors that affect option provides.
① Intrinsic Value
② Time Value
③ Volatility

frequency and magnitude of change in price

Measure (S.D. ⇒ or variance).

Coefficient of variation $= \frac{SD}{Mean} \times 100$.

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* Delta δ = $\dfrac{\text{\% Change in option price}}{\text{\% change in price of underlying asset}}$

* Gamma γ = $\dfrac{\text{Change of } \delta}{\text{change in underlying price}}$

* Rho ρ = $\dfrac{\text{Expected change in price of option}}{\text{\% change in risk free interest rate}}$

* Kappa : K = $\dfrac{\text{Change in price of option}}{\text{Change in votality}}$

Properties of a Markov process :-

Suppose that a stock price follows Markov process whose current price is 10 and its change in next 1 year can be represented as $\phi(0, 1)$

probability distribution ← ϕ, mean ← 0, std deviation ← 1

What will be p.d. of its change in next 2 years ?

The change in two years is sum of 2 Normal distⁿ (which are independent) when we add two normal distributions (independent) then resultant distⁿ is normal with mean (0) = sum of two means and variance of sum of two variances (2)

$\therefore$ SD of resultant $= \sqrt{2}$

This is why risk $\propto \sqrt{\text{time}}$
(increase less proportionality of time.

Generalized Weiner Process :-

Process of a stock price

follows a generalized weiner process i.e. It has a constant drift rate and a constant variance rate However, this type of model fails to capture a key aspect of stock price movement People look at return rather Than absolute value (think of GDP growth rate!)

If we assume the drift rate is constant then

$$\Delta S = \mu S \Delta t \quad \left(\text{where } \mu \text{ is the constant drift}\right)$$

here we believe there is no volatility term

As $\Delta t \to 0$ then

$$\frac{dS}{S} = \mu dt$$

or % change in $S = \mu dt$ (if multiplied by 100)

$$\therefore \int \frac{dS}{S} = \mu \int dt$$

$$\ln S = \mu t + c$$

or $$S_T = S_0 e^{\mu T} \quad (\text{between } t=0 \text{ and } T)$$

But in reality stock price exhibits volatility.

$\therefore \quad dS = \mu S dt + \sigma S dz$

where σ is volatility (SD) factor

$$\frac{dS}{S} = \mu dt + \sigma dz$$

{ z follows weiner process. $\Delta z = \epsilon \sqrt{\Delta t}$ }

Monte Carlo (named after city of casino).

$$\frac{dS}{S} = \mu dt + \sigma dz.$$

For a time interval Δt the change in stock price ΔS will be.

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta z$$

~~let's assume~~ $\left\{ \begin{array}{l} \mu = 0.15 \\ \sigma = 0.3. \end{array} \right.$ (~~from past data of stock we calculate~~ historical figures

Expressed in annual % form.

$$\therefore \frac{\Delta S}{S} = 0.15 \Delta t + 0.30 \epsilon \sqrt{\Delta t}$$

$\epsilon \sim N(0,1)$, let us assume a time interval of one week = 0.0192 year.

$\therefore \quad \Delta t = 0.0192$ yr.

$\therefore \Delta S = (S \times 0.15 \times 0.0192) + (S \times 0.30 \times \sqrt{0.0192} \times \epsilon)$

$\Delta S = (0.002888)S + (0.0416)S\epsilon$

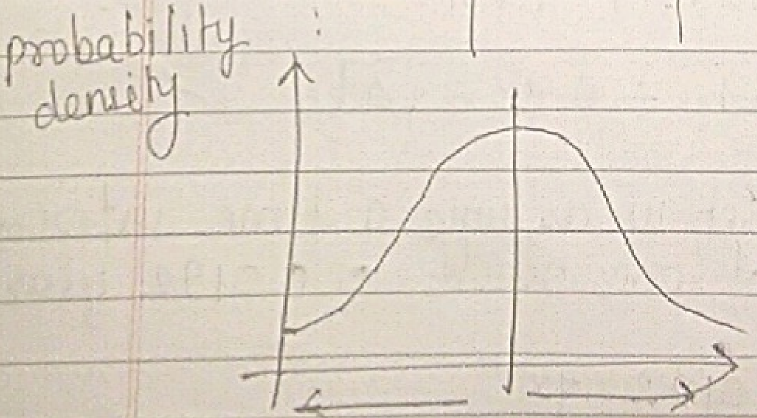
Now, ε's value can be attained / simulated from repeated sampling.

In excel we can artificially generate 'ε' values,

= NORMSINV (RAND()) → 0.52, → 0.44, → −0.85, → 1.46.

If we assume stock price = 100 ₹ the simulation of the stock price will be.

Stock price	'ε' (random sample)	ΔS (change in stock price)
100.00	0.52	2.45
102.45	0.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70.
⋮	⋮	⋮



$$N = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\pi^2}}$$

$$-\infty < x < +\infty$$

$$\int_{x=-\infty}^{+\infty} f(x)\,dx = 1 \quad \text{(Total area)}$$

When we are generating 'ε' we have to draw random number between 0 and 1 first and then inverse commulative normal $dist^n$ afterwards.

ε follows a p.d. of a standard normal $dist^n$.

∴ $\varepsilon \sim N(0,1)$ mean = 0 & variance = 1.

Itô Process (named after mathematician K itô)

This is a generalized weiner process where 'a' and 'b' parameters are not constant but they are dependent on time & underlying assets' value.

Algebraically, $dx = a(x,t)\,dt + b(x,t)\,dz$

$$\Delta x = a(x,t)\,\Delta t + b(x,t)\,\varepsilon\sqrt{\Delta t}$$

↑ (small interval of time Δt).

$$\begin{cases} a(x,t) \rightarrow \text{drift term} \\ b(x,t) \rightarrow \text{variance term} \end{cases}$$

Itô's Lemma: Stock price behaves stochastically and its changes depends on time. Hence any options price depends on time and the underlying stock. Behaviour of stochaistic process for an option was proposed by (Itô).

Accor. to Itô's lemma, $\underset{\text{option price}}{h}$, a function of $\underset{\text{stock price}}{x}$ & $\underset{\text{time}}{t}$

will follow the dynamic →

$$dh = \left(\frac{\partial h}{\partial x} a + \frac{\partial h}{dt} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} b^2\right)dt + \frac{\partial h}{\partial x} b\, dz$$

where x follows a weiner process

Derivation of Itô's Lemma :-

If x (underlying asset or stock) changes by small unit Δx then

$$\Delta h \approx \frac{dh}{dx}\Delta x \quad \text{(from ordinary calculus)}$$

A precise expression (From Taylor's expansion)

$$\Delta h = \frac{dh}{dx}\Delta x + \frac{1}{2}\frac{d^2 h}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3 h}{dx^3}\Delta x^3 + \cdots \quad \text{——} (1)$$

→

$$\Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial y}\Delta y \quad \left(\text{If } h \text{ is a fn of } x \,\&\, y\right.$$

Similar to (1) we can write

$$\Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}\Delta y^2$$

$$+ \frac{\partial^2 h}{\partial x \partial y}\Delta x \Delta y + \cdots$$

~~Exam~~ for small values of Δx & Δy,

$$\Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial y}\Delta y$$

Now

if we want to assume,

$dx = a(x,t)dt + b(x,t)\,dZ$ & y is nothing time then

$$\Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 h}{\partial t^2}\Delta t^2$$

$$+ \frac{\partial^2 h}{\partial x \partial t}\Delta x \Delta t \quad \text{—— (3)}$$

Previously, we saw.

$\Delta x = a(x,t)\Delta t + b(x,t)\Delta z$

$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$ (a,b are fn of x,t).

$$\Delta x^2 = a^2(\Delta t)^2 + b^2\varepsilon^2\Delta t + 2ab\varepsilon\,\Delta t^{3/2}$$

Ignoring higher term of Δt $(\because \Delta t \to 0)$

$$\Delta x^2 = b^2\varepsilon^2\Delta t + \text{neglible term}.$$

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\varepsilon^2 dt$$

—— (from 3)

$$Var(\varepsilon) = E[\varepsilon - E(\varepsilon)]^2$$

$$1 = E(\varepsilon^2) - [E(\varepsilon)]^2 \quad (\text{Std Nor dis}^n)$$

$$E(\varepsilon^2) = 1 \quad [\because E(\varepsilon) = 0]$$

var = const.

$\therefore E(\varepsilon^2) = 1$ is also constant. It happens when $\varepsilon^2 = 1$

So, $$dh = \frac{\partial h}{\partial x}(a\,dt + b\,dz) + \frac{\partial h}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} b^2\,dt$$

$$= \underbrace{\left(\frac{\partial h}{\partial x} a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} b^2\right)}_{\text{new drift}} dt + \underbrace{\left(\frac{\partial h}{\partial x} b\right)}_{\text{new var-factor}} dz$$

new drift   new var-factor.

<u>Application of Itô's Lemma</u> :-

we saw $S_T = S_0 e^{rT}$ (under constant drift assumption)

$$\therefore F_0 = S_0 e^{rT}$$

$r \Rightarrow$ risk free interest rate.

$F_0 \Rightarrow$ forward price at $t=0$)

$S_0 \Rightarrow$ Spot price.

$T \Rightarrow$ Time to maturity.

If we take gap/interval of time between t & T then

$$F = Se^{r(T-t)}$$

$$\frac{\partial F}{\partial S} = e^{r(T-t)} \qquad \frac{\partial^2 F}{\partial S^2} = 0$$

$$\frac{\partial F}{\partial t} = -rSe^{r(T-t)}.$$

$\therefore$ Using Ito's Lemma,

$$dF = \left[e^{r(T-t)} \mu S - rSe^{-r(T-t)} \right] dt$$

$$+ \ e^{r(T-t)} \sigma S dz$$

If we use $F = Se^{r(T-t)}$.

$$dF = (\mu - r) F dt + \sigma F dz.$$

F (Forward price) follows a geometric brownian motion with $(\mu - r), \sigma$.

Black - Scholes (Merton) Eq^n for option pricing

stock price process -

$$dS = \mu S dt + \sigma S dz \quad - (1)$$

Δz follows weiner process.

$\mu \Rightarrow$ avg. return

$\sigma \Rightarrow$ std. deviation

$S \Rightarrow$ stock prices . $t \Rightarrow$ time.

Assume f is price of a call option whose value depend on underlying stock (price) and time (t)

According to Ito's lemma,

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt$$

$$+ \frac{\partial f}{\partial S}\sigma S\, dz \quad \text{———} \quad (2)$$

Change in price of f.

lets create a portfolio where holder short one derivative
(borrow).
& long an amount $\frac{\partial f}{\partial S}$ of share.
(investment)

∴ Value of portfolio →

$$\pi = -f + \frac{\partial f}{\partial S} \times (S)$$

or $$\Delta\pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S.$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \quad \text{———} \quad (3)$$

$$\delta f = \left(\frac{\delta f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt$$

$$+ \frac{\partial f}{\partial S}\sigma S\, dz \quad \text{———} \quad (4).$$

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$$\delta f = \left(\frac{\delta f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$

$$+ \frac{\partial f}{\partial S} \sigma S \, dz \quad \text{—— (4)}$$

Putting 3 & 4 in $\Delta\Pi$ we get

$$\Delta\Pi = -\cancel{\frac{\partial f}{\partial S}\mu S\,dt} - \frac{\partial f}{\partial t}dt - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 dt$$

$$-\cancel{\frac{\partial f}{\partial S}\sigma S\,dz} + \cancel{\frac{\partial f}{\partial S}\mu S\,dt} + \cancel{\sigma S\,dz\frac{df}{\partial S}}.$$

$$\Delta\Pi = -\frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 dt$$

$$\Delta\Pi = \left(\frac{-\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t.$$

one can always go for riskless return in bond market with 'r' rate of interest.

$$\Delta\Pi = r\Pi\Delta t \quad \text{(change from bond earning.}$$

$$\therefore \left(\frac{-\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\Pi\Delta t$$

Rearranging all terms -

$$\left(\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf\right)$$

$\longrightarrow$ PDE.

The PDE can be solved under boundary condition for a CALL option.

$$\left[f = \max(S-k, 0) \text{ when } t = T \right]$$

↳ value of a CALL option.

for PUT option

$$C = S_0 N(d_1) - k e^{-rT} N(d_2)$$

$$P = k e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where
$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

$N(x)$ = Commulative normal dist. for standarized normal variable.

S_0 = Stock price at $t=0$

k = Strike price

r = risk free rate.

T = time in year

Commulative Normal distribution :

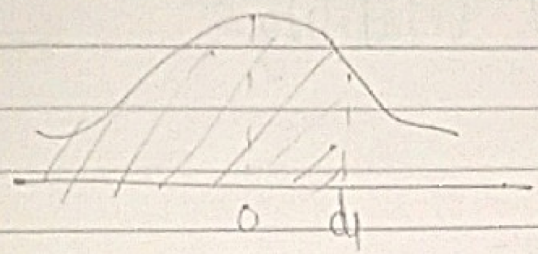
$N(d_1)$

Total Area till the value d_1 for a std normal $dist^n$

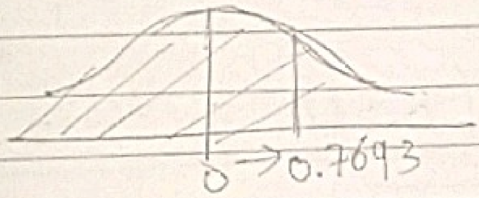
$= \phi(0,1)$

$0 \Rightarrow$ mean

$1 \Rightarrow$ var.



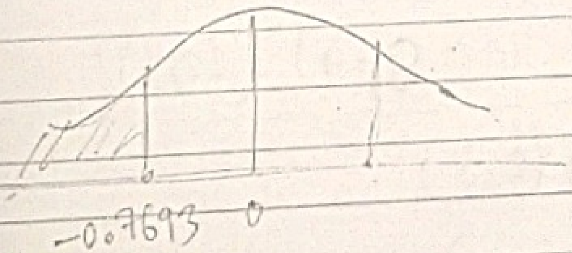
Excel command = NORMSDIST.



$N(0.7693) = 0.7791$

$N(-0.7693) = 1 - N(0.7643)$

$= 0.2209$



Symmetric Nature

x

The greek letters (revisited) :-

* $\frac{\partial c}{\partial S}$ Delta (Δ) $= \dfrac{\text{Change in option price}}{\text{Change in price of underlying asset}}$

for call option ;

$$C = SN(d_1) - ke^{-rT}N(d_2)$$

$$\frac{\partial c}{\partial S} = N(d_1) \quad \text{(partial derivative)}$$

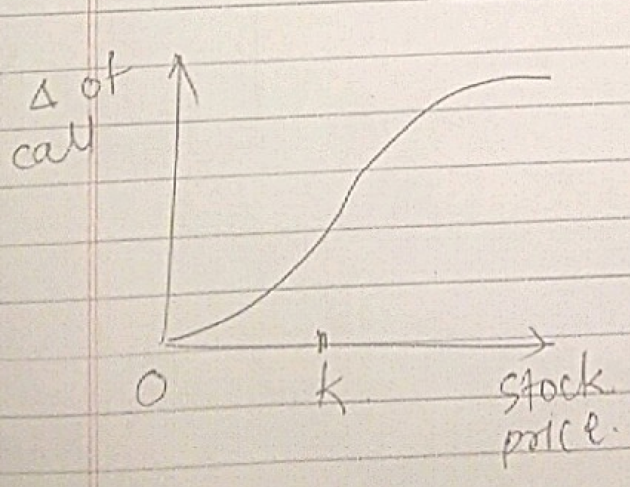
$\therefore$ $N(d_1)$ = Commulative norm. $dist^n$ (std) is the Δ of a call option.

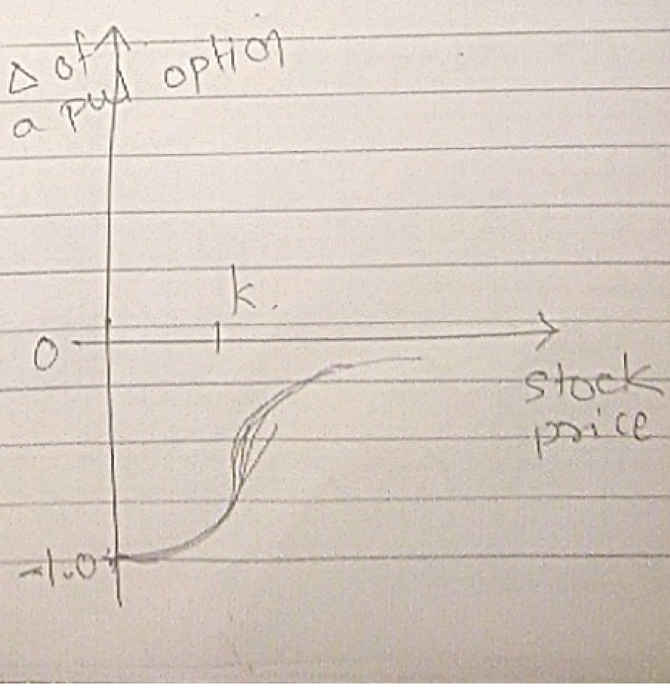
For put option :-

$$P = ke^{-rT}N(-d_2) - S\{N(-d_1)\}$$

$$\frac{\partial P}{\partial S} = -N(-d_1) = -[1 - N(d_1)]$$

Δ of a put option $= N(d_1) - 1$. (Always negative)





★ Theta (θ) rate of change of the value of a portfolio w.r.t passage of time

time decay of a portfolio

for call

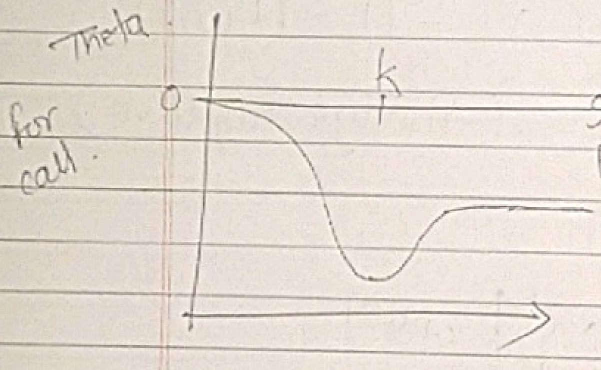
$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

$$\frac{\partial c}{\partial t} = \frac{SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

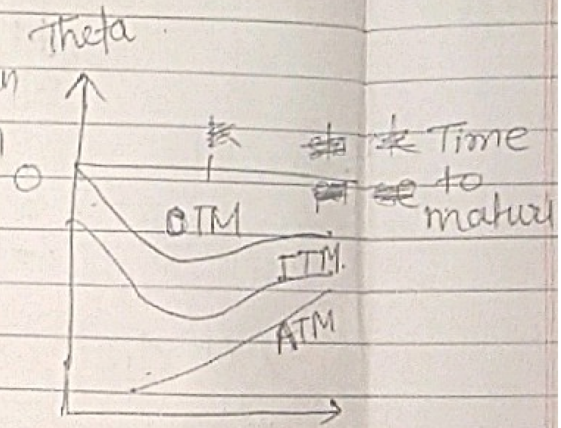
for Put

$$\frac{\partial P}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

$$N(-d_2) = 1 - N(d_2)$$



European call option

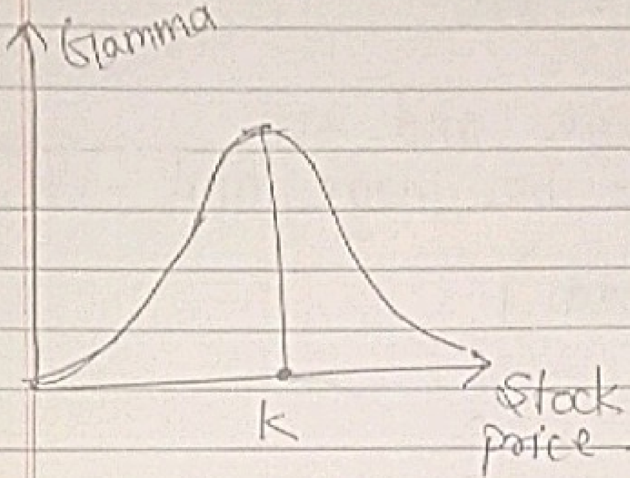


of a portfolio is

★ Gamma (Γ) double derivative of its value w.r.t stock price.

first order derivative of Δ

$$\Gamma = \frac{\partial^2 \pi}{\partial S^2}$$



Gamma of any call/put option $= \dfrac{N'(d_1)}{S_0 \sigma \sqrt{T}}$

Relation b/w Δ, θ and Γ :-

If π is the value of the portfolio of derivatives (call/put) then it must follow the B-S-M eq^n

Time to maturity

$$\frac{\partial \pi}{\partial t} + rS\frac{\partial \pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} = r\pi$$

$$\theta = \frac{\partial \pi}{\partial t} \; ; \; \Delta = \frac{\partial \pi}{\partial S} \; ; \; \Gamma = \frac{\partial^2 \pi}{\partial S^2}$$

$$\therefore \boxed{\theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\pi}$$

If we can make a portfolio Δ neutral i.e. $\Delta = 0$; then

(Portfolio's value is neutral to changes in stock price).

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

when Θ is large and +ve then Γ tends to be large and -ve.

(because $r\Pi$ is fixed)

☆ Rho (ρ) →

Partial derivative of an option wrt interest rate.

$$\frac{\partial \Pi}{\partial r} = \rho$$

ρ for call option $= kTe^{-rT}N(d_2)$

ρ for put option $= -kTe^{-rT}N(-d_2)$.

☆ vega (ν) $= \frac{\partial \Pi}{\partial \sigma}$ (value's derivative wrt σ)

high vega means high sensitivity of the portfolio to stock's volatility.

ν for call $= S\sqrt{T}N'(d_1)$.