Decidable Languages

Recall that:

A language L is **Turing-Acceptable** if there is a Turing machine M that accepts L

Also known as: *Turing-Recognizable*
or
Recursively-enumerable
languages

For any string w :

$w \in L \Rightarrow M$ halts in an accept state

$w \notin L \Rightarrow M$ halts in a non-accept state

or loops forever

Definition:

A language L is **decidable**
if there is a Turing machine (decider) M
which accepts L
and halts on every input string

Also known as *recursive languages*

For any string w :

$w \in L \Rightarrow M$ halts in an accept state

$w \notin L \Rightarrow M$ halts in a non-accept state

Every decidable language is Turing-Acceptable

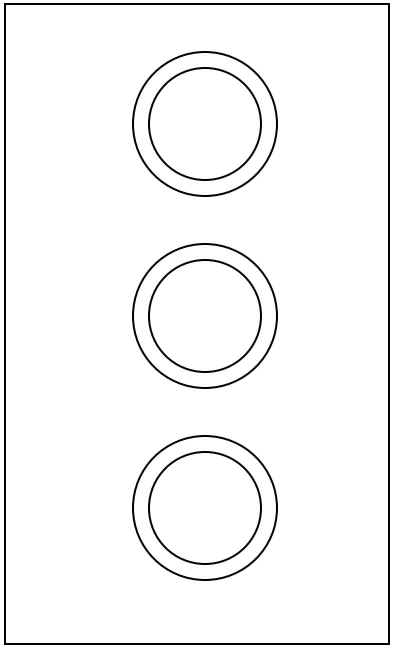
Sometimes, it is convenient to have Turing machines with single accept and reject states

These are the only halting states

That result to possible halting configurations

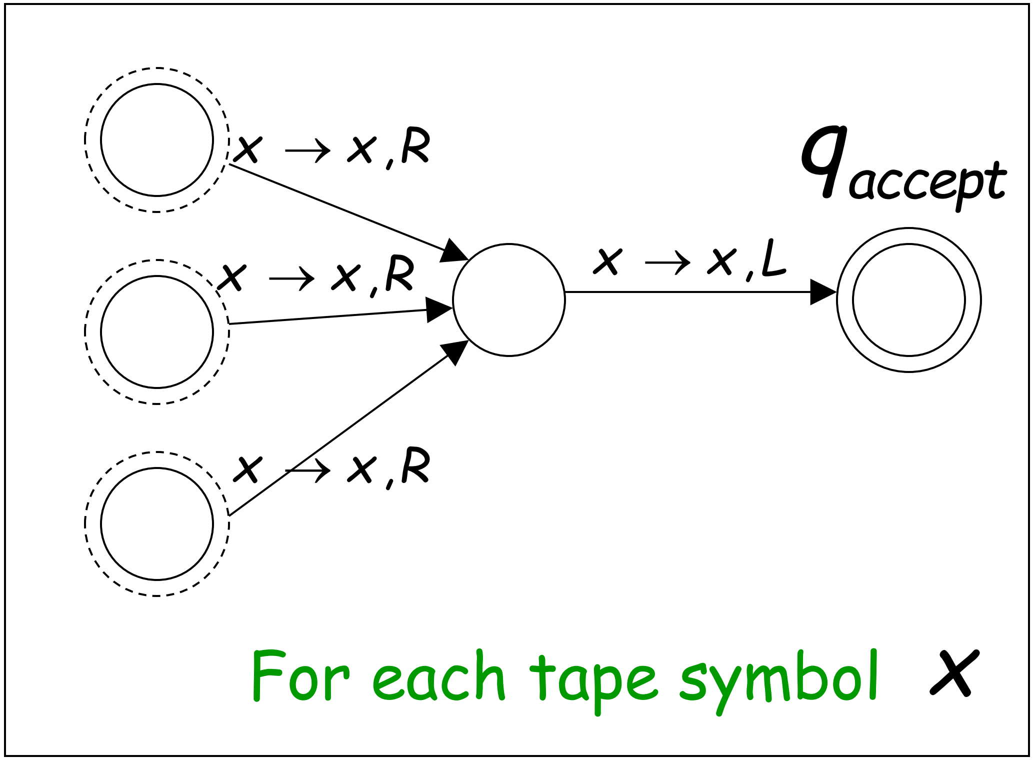
We can convert any Turing machine to have single accept and reject states

Old machine



Multiple accept states

New machine

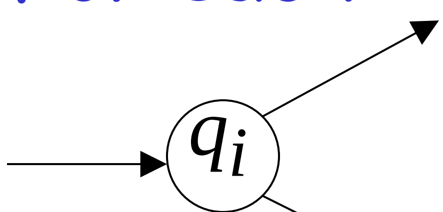


One accept state

Do the following for each possible halting state:

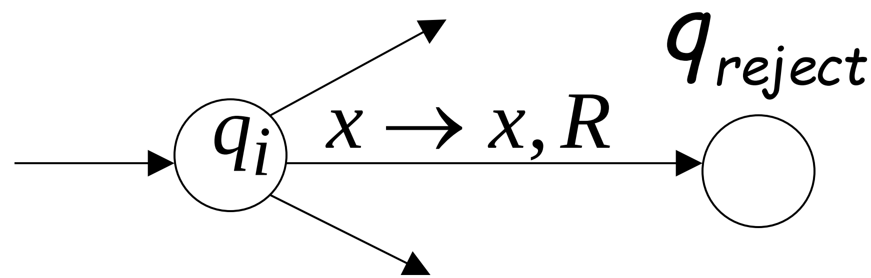
Old machine

For each



Multiple reject states

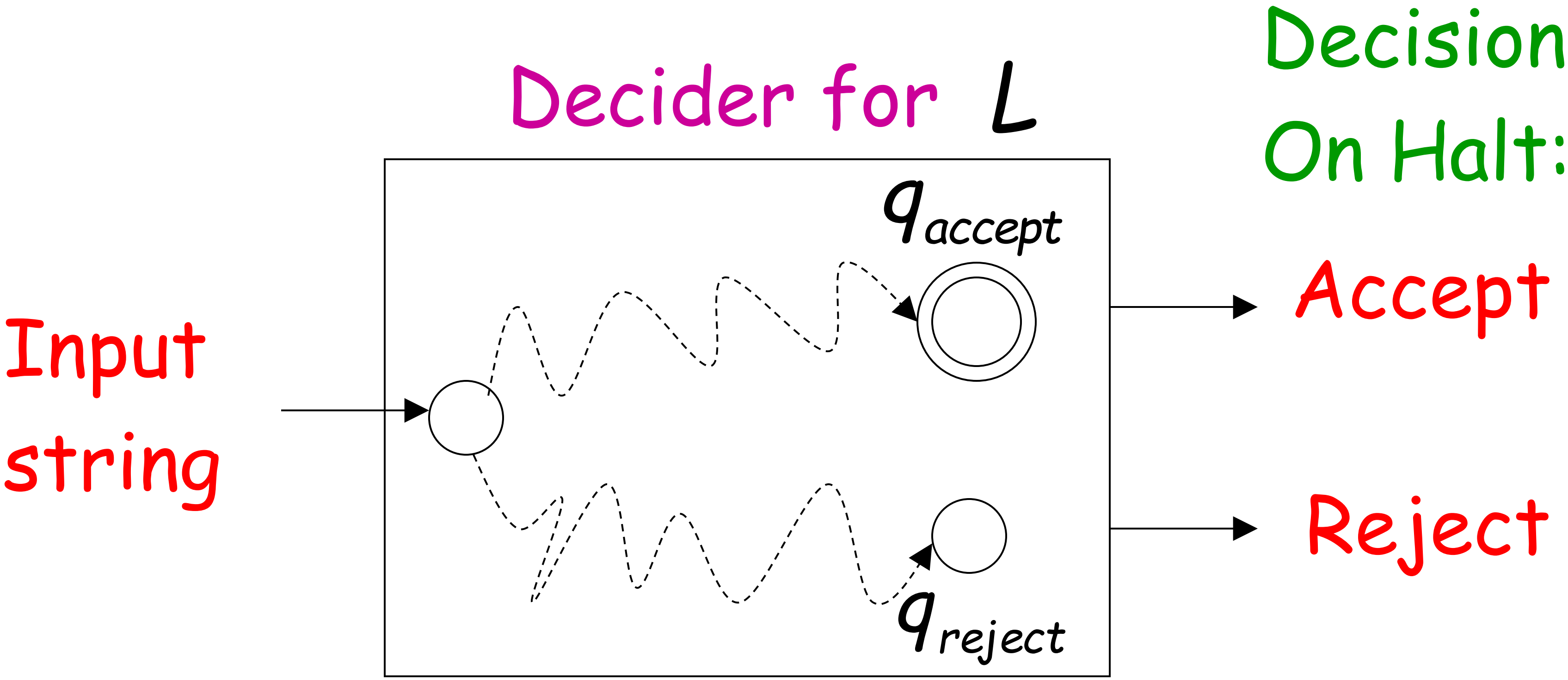
New machine



For all tape symbols x not used for read in the other transitions of q_i

One reject state

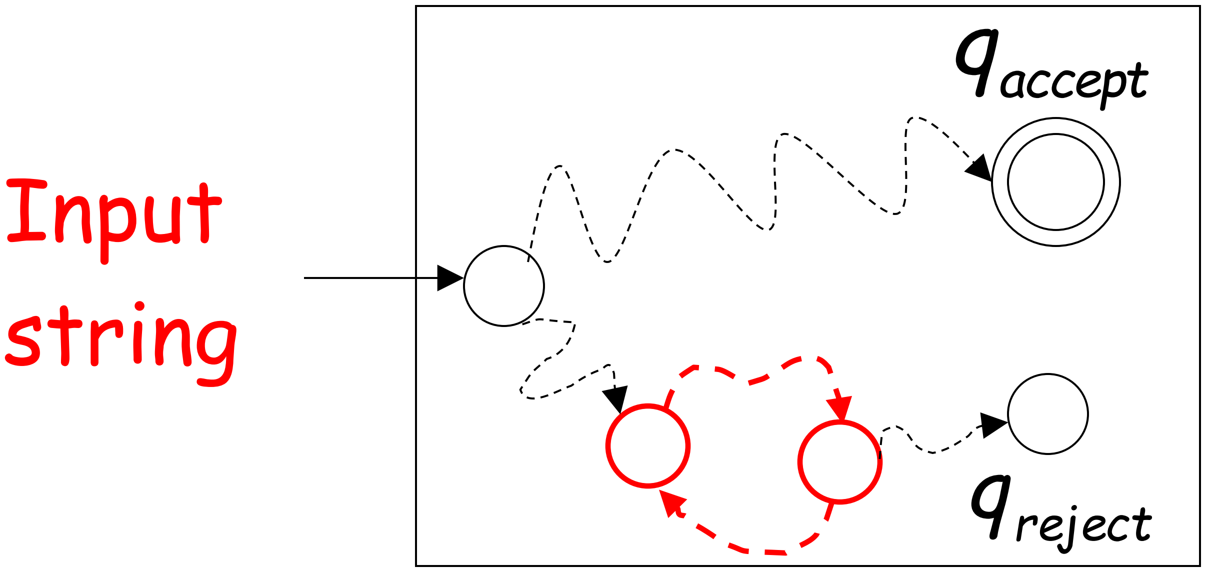
For a decidable language L :



For each input string, the computation halts in the accept or reject state

For a Turing-Acceptable language L :

Turing Machine for L



It is possible that for some input string the machine enters an infinite loop

Problem: Is number x prime?

Corresponding language:

$$PRIMES = \{1, 2, 3, 5, 7, \ldots\}$$

We will show it is decidable

Decider for $PRIMES$:

On input number x :

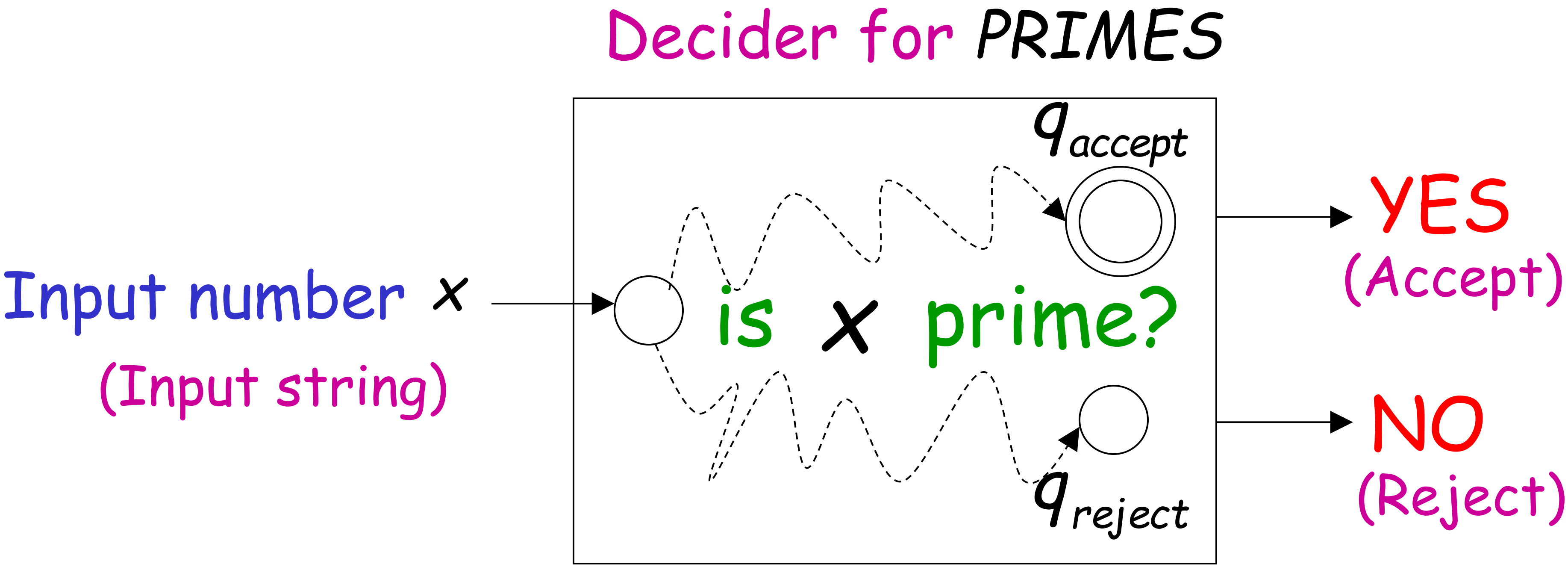
Divide x with all possible numbers between 2 and $\sqrt{x}$

If any of them divides x

<u>Then</u> reject

<u>Else</u> accept

the decider for the language solves the corresponding problem



Theorem:

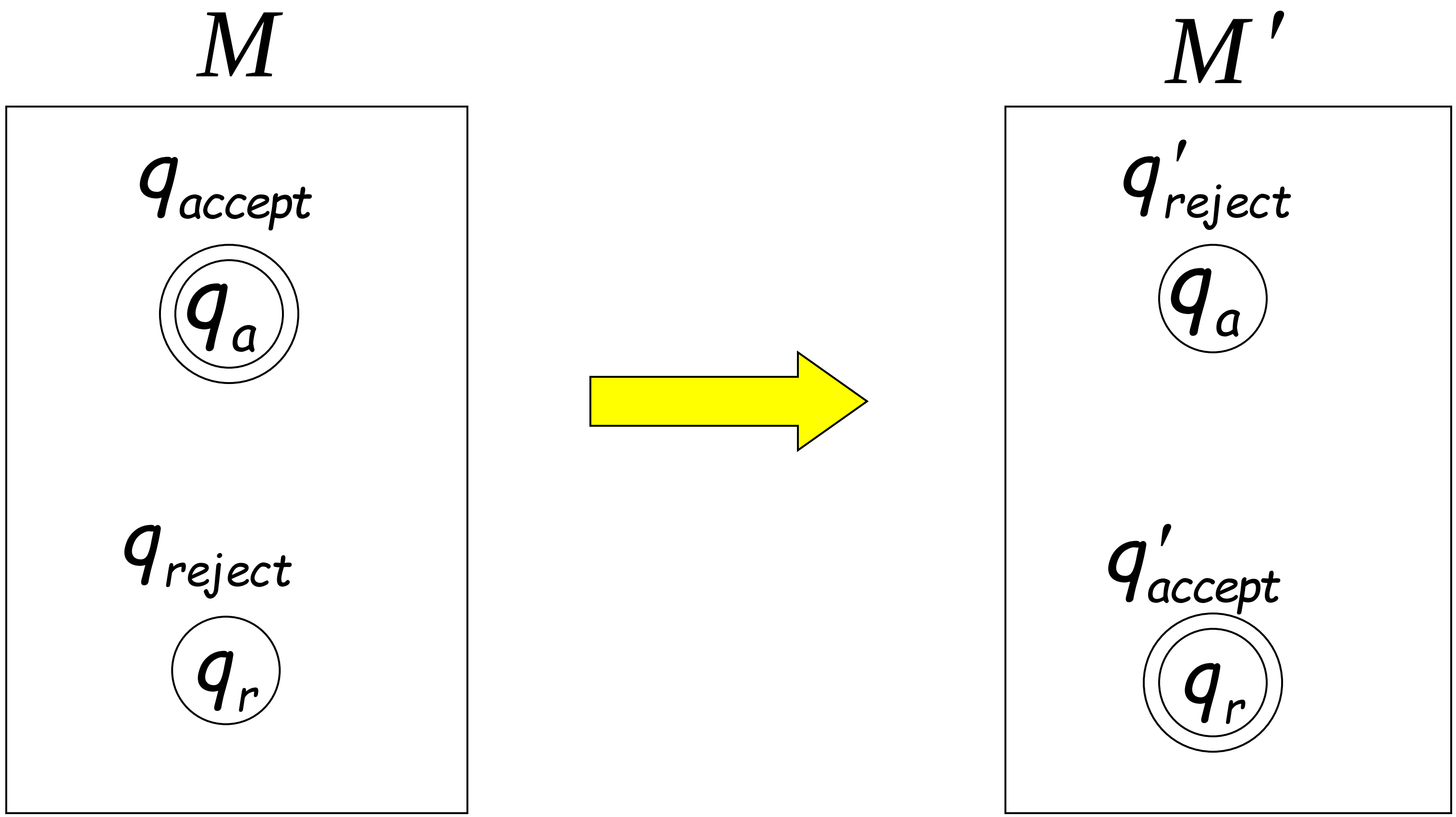
If a language L is decidable,
then its complement $\overline{L}$ is decidable too

Proof:

Build a Turing machine M' that
accepts $\overline{L}$ and halts on every input string

(M' is decider for $\overline{L}$)

Transform accept state to reject and vice-versa



Turing Machine M'

On each input string w do:

1. Let M be the decider for L

2. Run M with input string w

   If M accepts then reject

   If M rejects then accept

Accepts $\overline{L}$ and halts on every input string

END OF PROOF

Undecidable Languages

An undecidable language has no decider:

each Turing machine that accepts L

does not halt on some input string

We will show that:

There is a language which is

Turing-Acceptable and undecidable

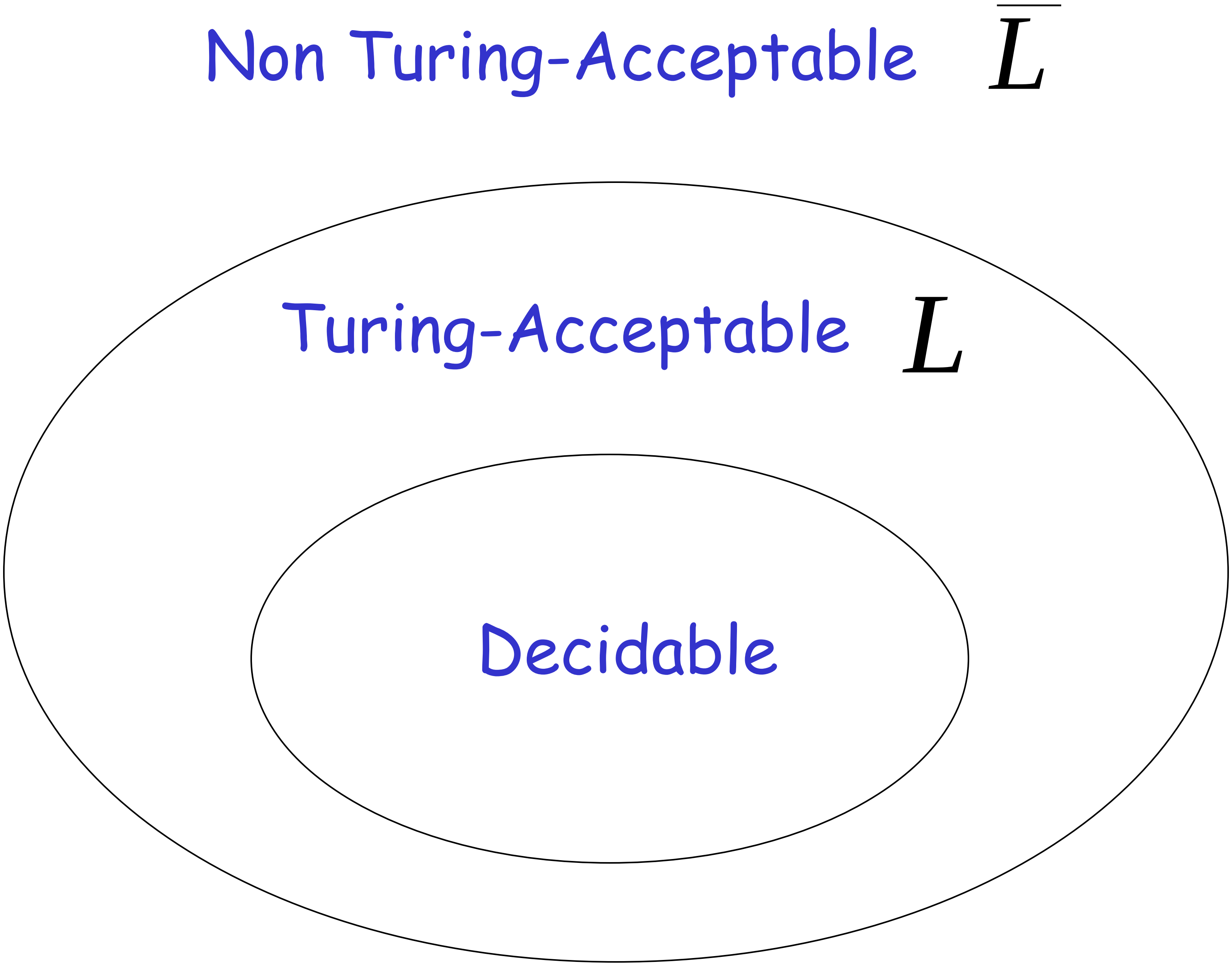
We will prove that there is a language L:

- $\overline{L}$ is not Turing-acceptable
  (not accepted by any Turing Machine)

- L is Turing-acceptable

the complement of a decidable language is decidable

Therefore, L is undecidable


Non Turing-Acceptable $\overline{L}$
Turing-Acceptable L
Decidable

A Language which is not Turing Acceptable

Consider alphabet $\{a\}$

Strings of $\{a\}^{+}$:

$$a, \ aa, \ aaa, \ aaaa, \ \ldots$$

$$a^1 \quad a^2 \quad a^3 \quad a^4 \quad \ldots$$

Consider Turing Machines
that accept languages over alphabet $\{a\}$

They are countable:

$$M_1,\ M_2,\ M_3,\ M_4,\ \ldots$$

(There is an enumerator that generates them)

Each machine accepts some language over $\{a\}$

$$\begin{tikzcd}[column sep=small]
M_1, \arrow[d] & M_2, \arrow[d] & M_3, \arrow[d] & M_4, \arrow[d] & \ldots \\
L(M_1), & L(M_2), & L(M_3), & L(M_4), & \ldots
\end{tikzcd}$$

Note that it is possible to have

$$L(M_i) = L(M_j) \quad \text{for} \quad i \neq j$$

Since, a language could be accepted by more than one Turing machine

Example language accepted by M_i

$$L(M_i) = \{aa, aaaa, aaaaaa\}$$

$$L(M_i) = \{a^2, a^4, a^6\}$$

Binary representation

	a^1	a^2	a^3	a^4	a^5	a^6	a^7	$\cdots$
$L(M_i)$	0	1	0	1	0	1	0	$\cdots$

Example of binary representations

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Consider the language

$$L = \{a^i : a^i \in L(M_i)\}$$

L consists of the **1's** in the diagonal

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$L = \{a^3, a^4, \ldots\}$$

Consider the language $\overline{L}$

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

$$L = \{a^i : a^i \in L(M_i)\}$$

$\overline{L}$ consists of the 0's in the diagonal

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$\overline{L} = \{a^1, a^2, \ldots\}$$

Theorem:

Language $\overline{L}$ is not Turing-Acceptable

Proof:

Assume for contradiction that

$\overline{L}$ is Turing-Acceptable

There must exist some machine M_k that accepts $\overline{L}$: $L(M_k) = \overline{L}$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_1$?

$L(M_k) = \overline{L}$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^1 \in L(M_k)$$

Answer: $M_k \neq M_1$

$$a^1 \notin L(M_1)$$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_2$?

$L(M_k) = \overline{L}$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^2 \in L(M_k)$$

Answer: $M_k \neq M_2$

$$a^2 \notin L(M_2)$$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_3$?

$L(M_k) = \overline{L}$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^3 \notin L(M_k)$$

Answer: $M_k \neq M_3$

$$a^3 \in L(M_3)$$

Similarly: $M_k \neq M_i$ for any i

Because either:

$a^i \in L(M_k)$ or $a^i \notin L(M_k)$

$a^i \notin L(M_i)$ $a^i \in L(M_i)$

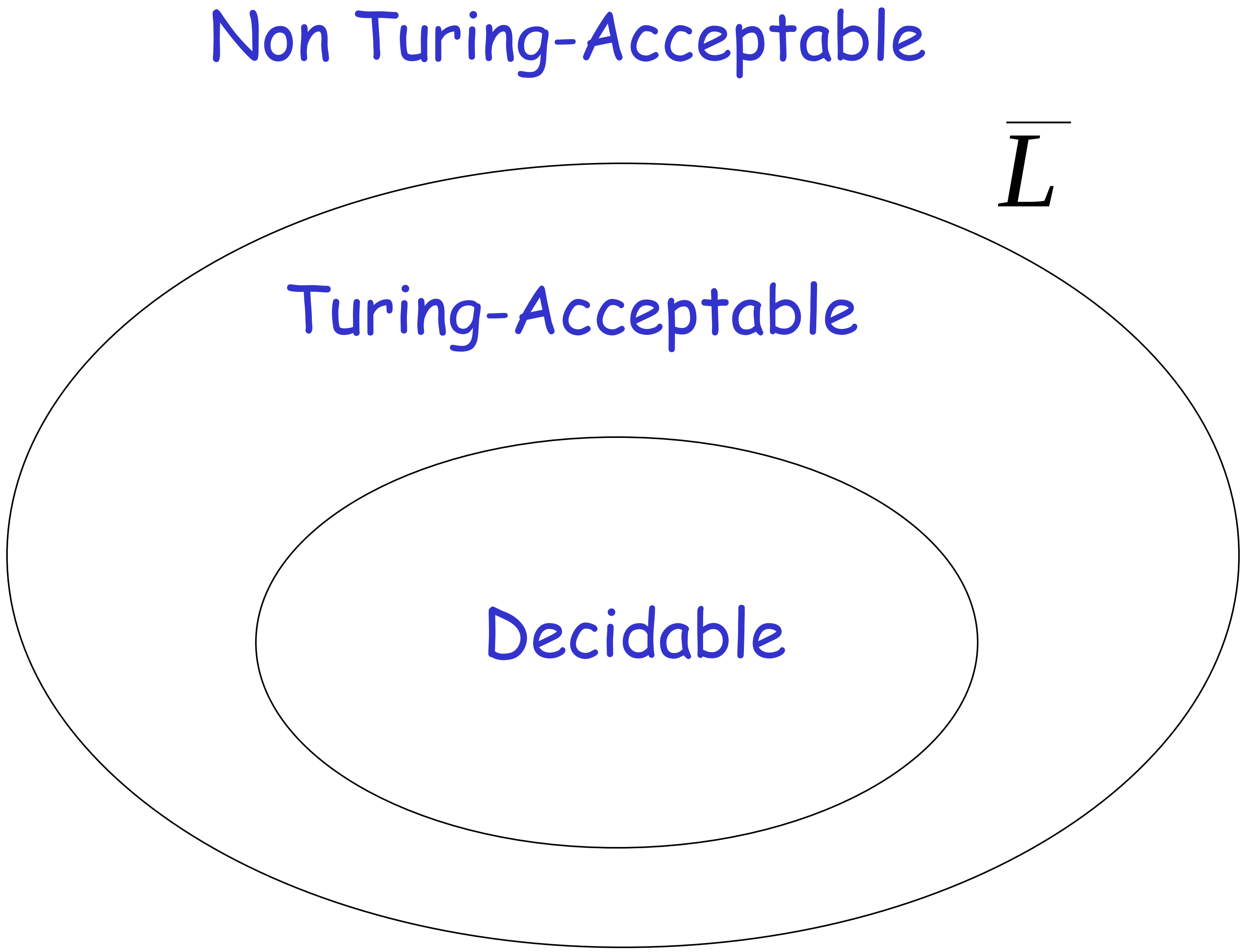
$\Downarrow$

the machine M_k cannot exist

$\Downarrow$

$\overline{L}$ is not Turing-Acceptable

End of Proof


Non Turing-Acceptable
$\overline{L}$
Turing-Acceptable
Decidable

A Language which is Turing-Acceptable and Undecidable

We will prove that the language

$$L = \{a^i : a^i \in L(M_i)\}$$

Is Turing-Acceptable

There is a Turing machine that accepts L

Undecidable

Each machine that accepts L doesn't halt on some input string

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$L = \{a^3, a^4, \ldots\}$$

Theorem: The language

$$L = \{a^i : a^i \in L(M_i)\}$$

Is Turing-Acceptable

Proof: We will give a Turing Machine that accepts L

Turing Machine that accepts L

For any input string w

- Compute i, for which $w = a^i$
- Find Turing machine M_i

  (using the enumerator for Turing Machines)
- Simulate M_i on input a^i
- If M_i accepts, then accept w

End of Proof

Observation:

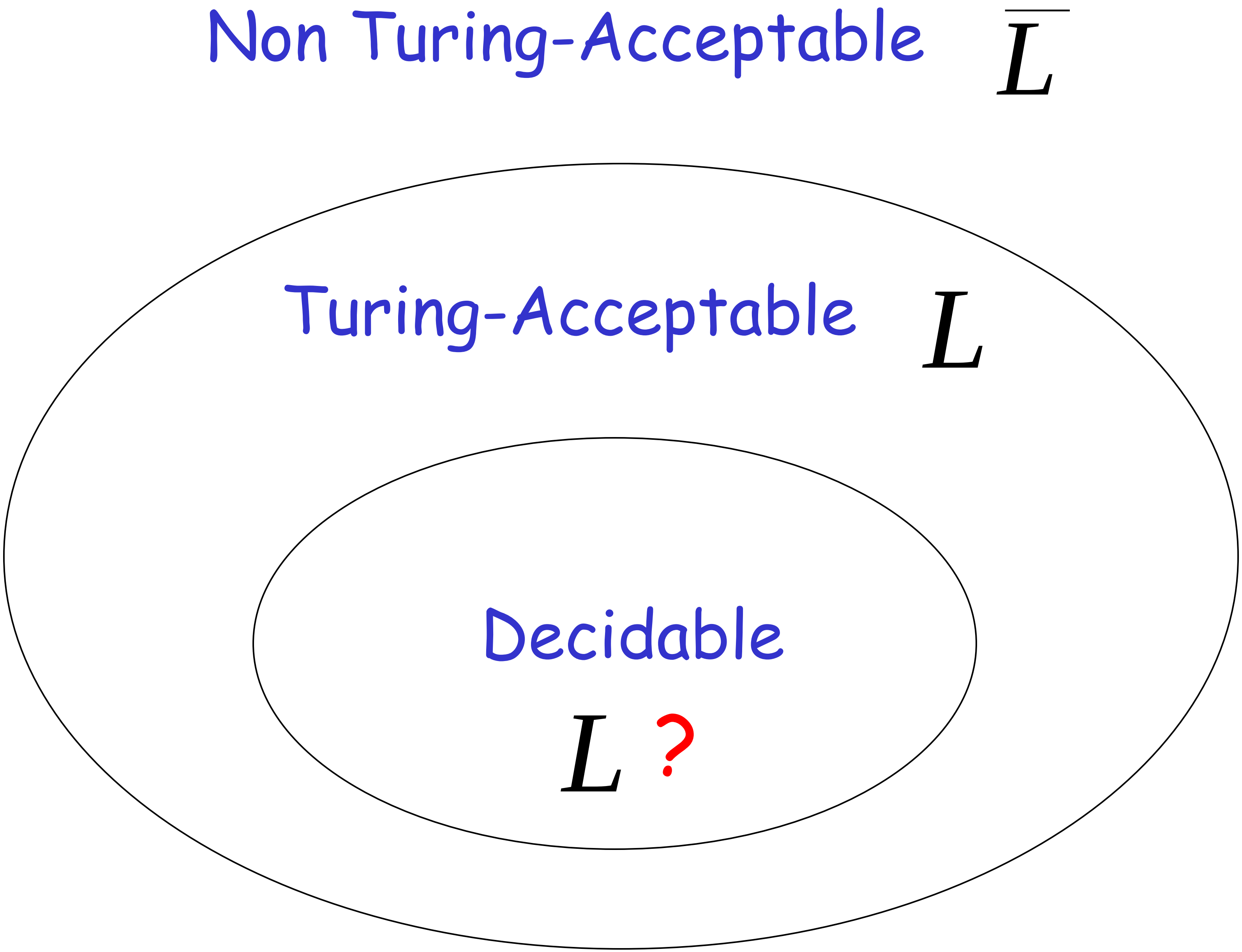
Turing-Acceptable

$$L = \{a^i : a^i \in L(M_i)\}$$

Not Turing-acceptable

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

(Thus, $\overline{L}$ is undecidable)



Non Turing-Acceptable $\overline{L}$
Turing-Acceptable L
Decidable
L ?

Theorem: $L = \{a^i : a^i \in L(M_i)\}$

is undecidable

Proof: If L is decidable

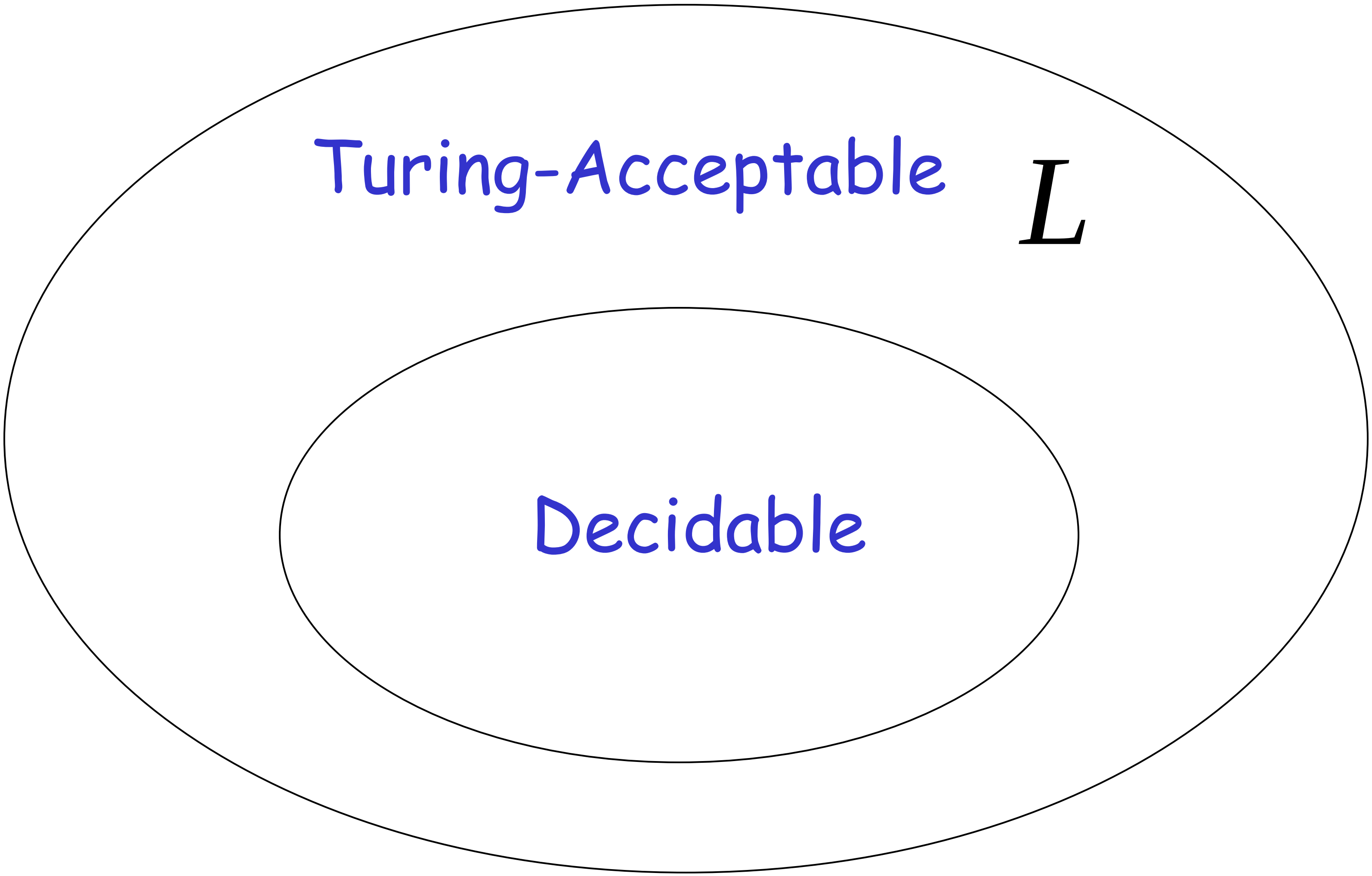
the complement of a decidable language is decidable

Then $\overline{L}$ is decidable

However, $\overline{L}$ is not Turing-Acceptable!

Contradiction!!!!

Not Turing-Acceptable $\overline{L}$



Turing acceptable languages and Enumerators

We will prove:

(weak result)

- If a language is decidable then there is an enumerator for it

(strong result)

- A language is Turing-acceptable if and only if there is an enumerator for it

Theorem:

if a language L is decidable then there is an enumerator for it

Proof:

Let M be the decider for L

Use M to build the enumerator for L

Let $\tilde{M}$ be an enumerator that prints all strings from input alphabet in proper order

Example:

alphabet is $\{a, b\}$

$$a$$
$$b$$
$$aa$$
$$ab$$
$$ba \quad \text{(proper order)}$$
$$bb$$
$$aaa$$
$$aab$$
$$\ldots\ldots$$

Enumerator for L

Repeat:

1. $\tilde{M}$ generates a string w

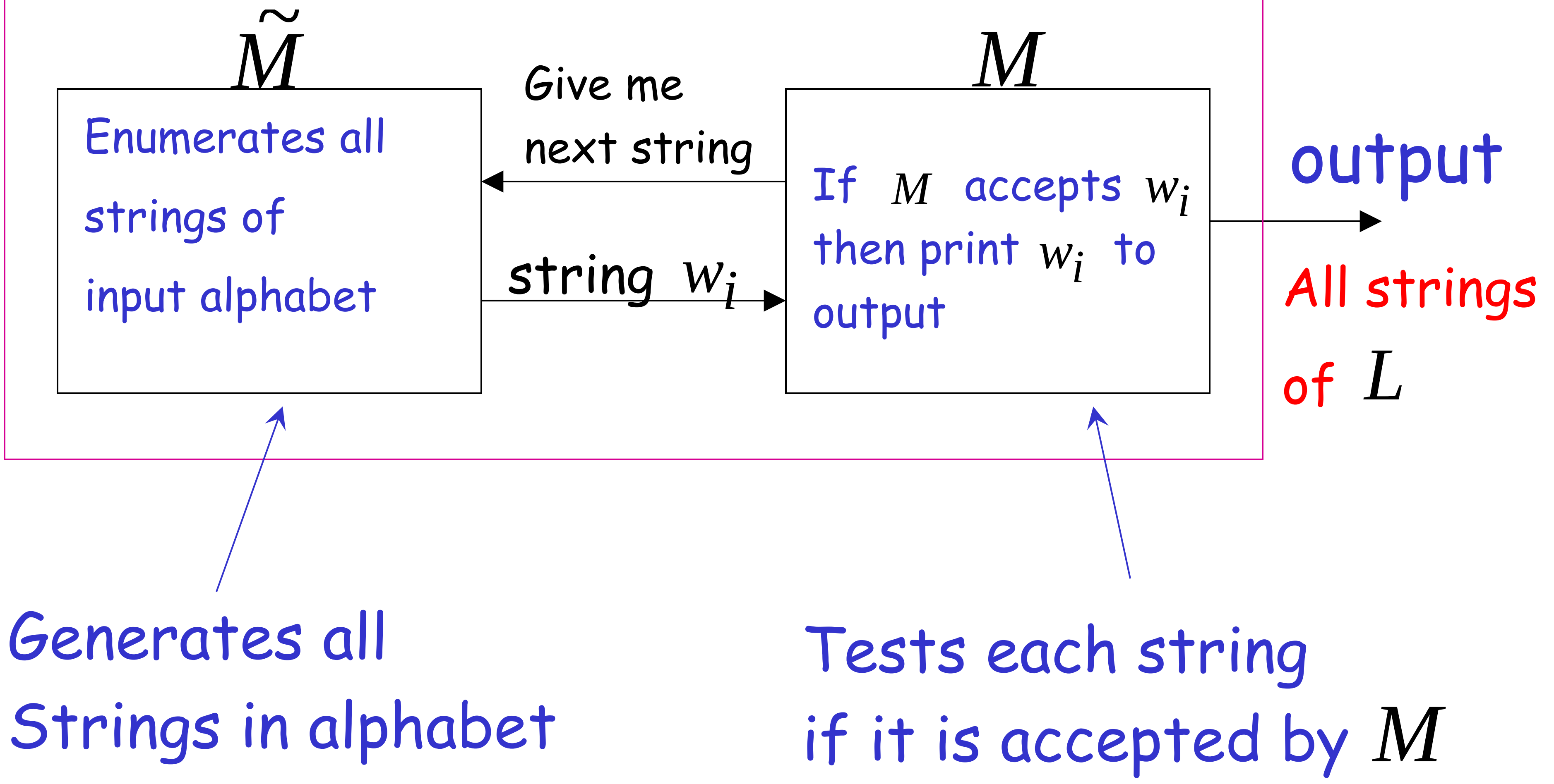
2. M checks if $w \in L$

YES: print w to output

NO: ignore w

This part terminates,
because L is decidable

Enumerator for L



Example: $L = \{b, ab, bb, aaa, ...\}$

	$\tilde{M}$	M	Enumeration Output
w_1	a	reject	
w_2	b	accept	b
w_3	aa	reject	
$\vdots$	ab	accept	ab
	ba	reject	
	bb	accept	bb
	aaa	accept	aaa
	aab	reject	$\vdots$
	$\vdots$	$\vdots$	

END OF PROOF

Theorem:

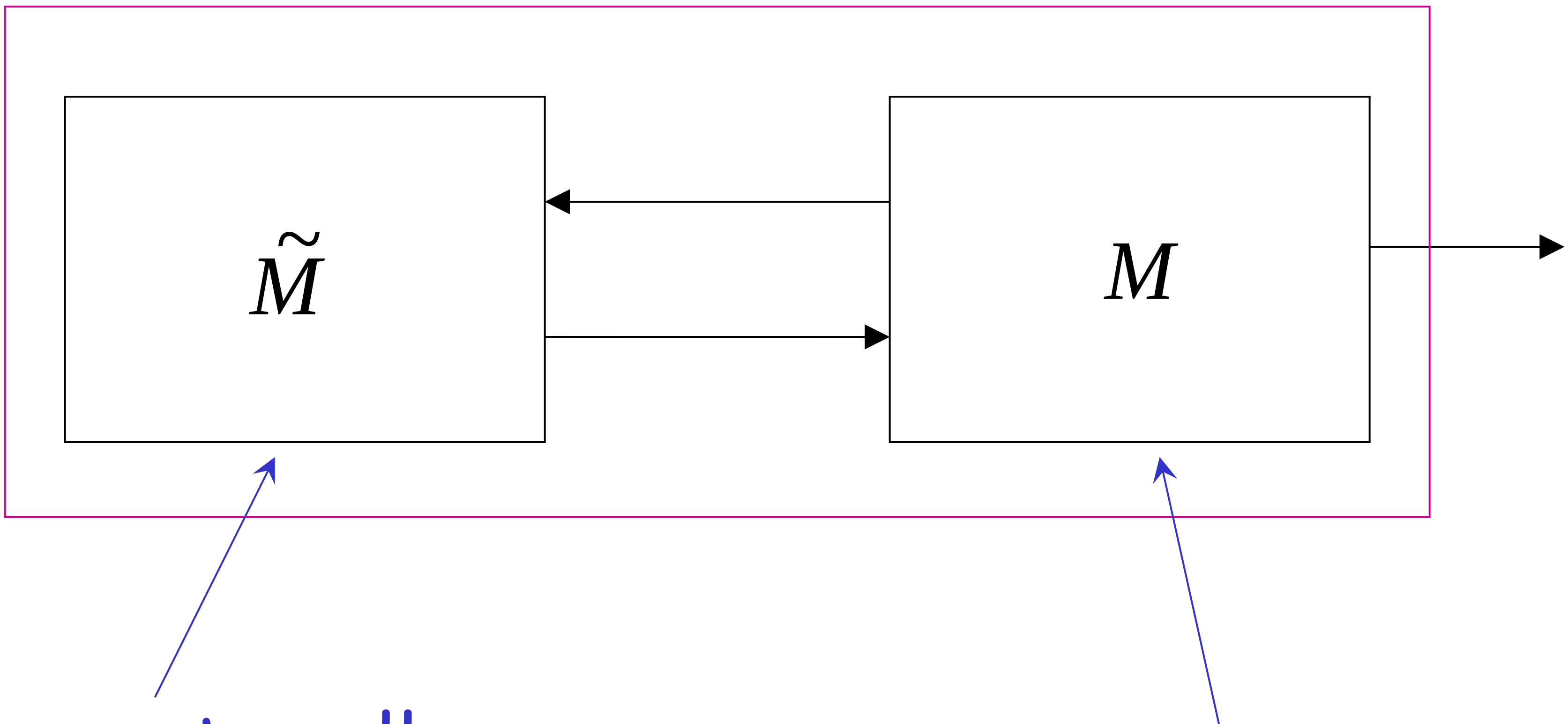
if language L is Turing-Acceptable then there is an enumerator for it

Proof:

Let M be the Turing machine that accepts L

Use M to build the enumerator for L

Enumerator for L



Enumerates all strings of input alphabet in proper order

Accepts L

NAIVE APPROACH

Enumerator for L

Repeat: $\tilde{M}$ generates a string w

M checks if $w \in L$

YES: print w to output

NO: ignore w

Problem: If $w \notin L$

machine M may loop forever

BETTER APPROACH

$\tilde{M}$ Generates first string w_1

M executes first step on w_1

$\tilde{M}$ Generates second string w_2

M executes first step on w_2

second step on w_1

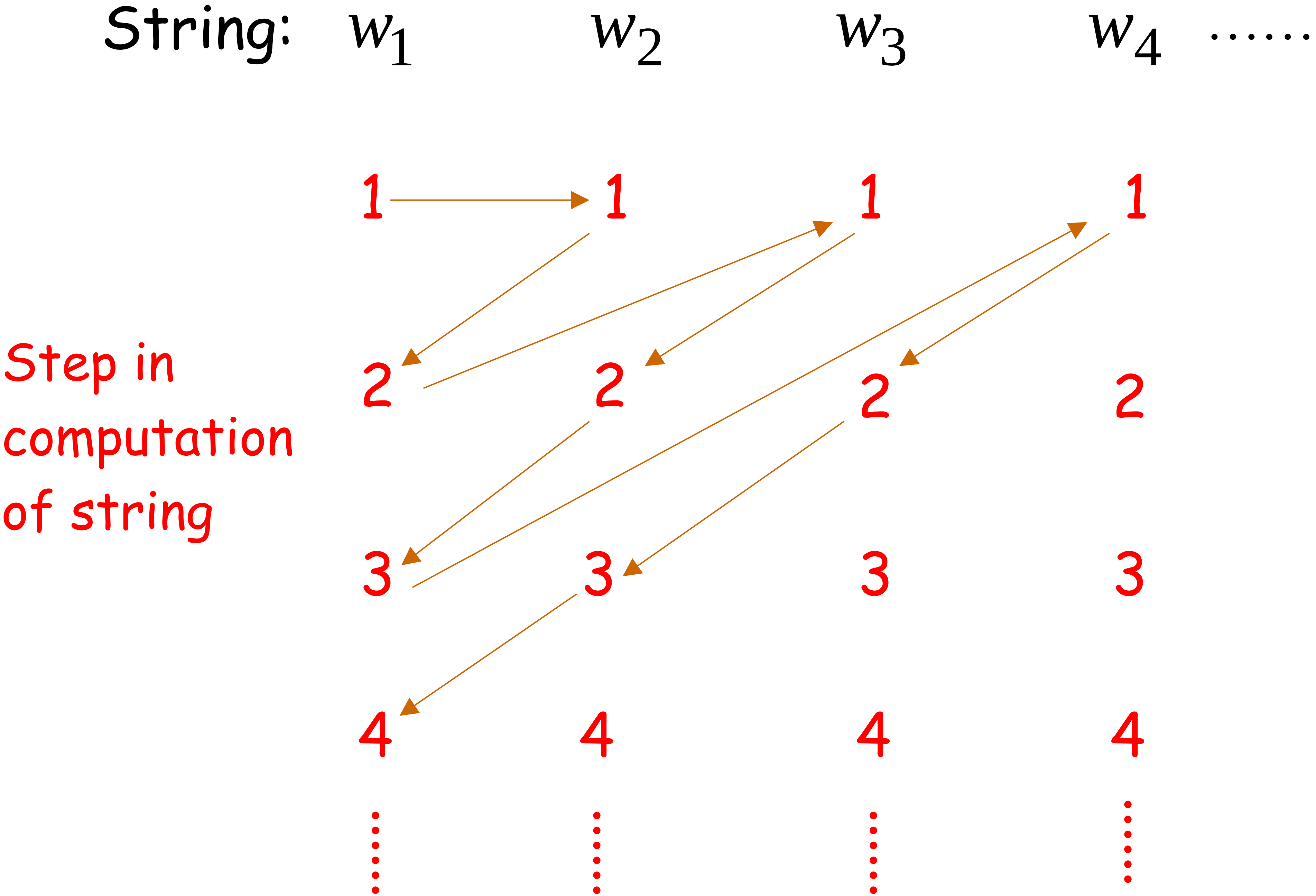
$\tilde{M}$ Generates third string w_3

M executes first step on w_3

second step on w_2

third step on w_1

And so on............


String: w_1 w_2 w_3 w_4 ......
Step in computation of string
1 1 1 1
2 2 2 2
3 3 3 3
4 4 4 4

If for any string w_i
machine M halts in an accepting state
then print w_i on the output

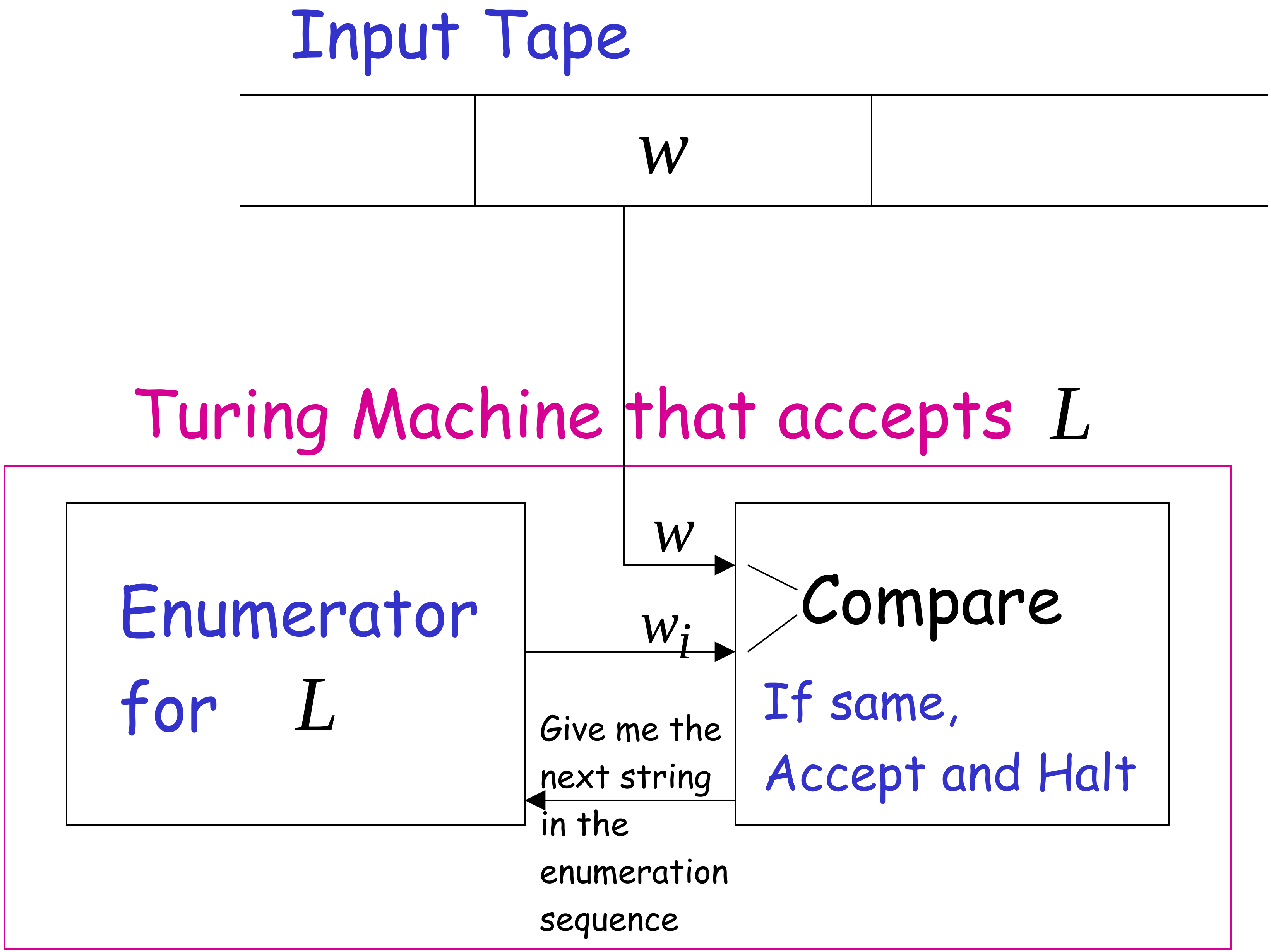
End of Proof

Theorem:

If for language L
there is an enumerator
then L is Turing-Acceptable

Proof:

Using the enumerator for L
we will build a Turing machine
that accepts L


Input Tape
w
Turing Machine that accepts L
Enumerator for L
w
w_i
Compare
If same, Accept and Halt
Give me the next string in the enumeration sequence

Turing machine that accepts L

For any input string w

Loop:

- Using the enumerator of L, generate the next string of L
- Compare generated string with w
  If same, accept and exit loop

End of Proof

By combining the last two theorems, we have proven:

A language is Turing-Acceptable
if and only if
there is an enumerator for it