More Applications

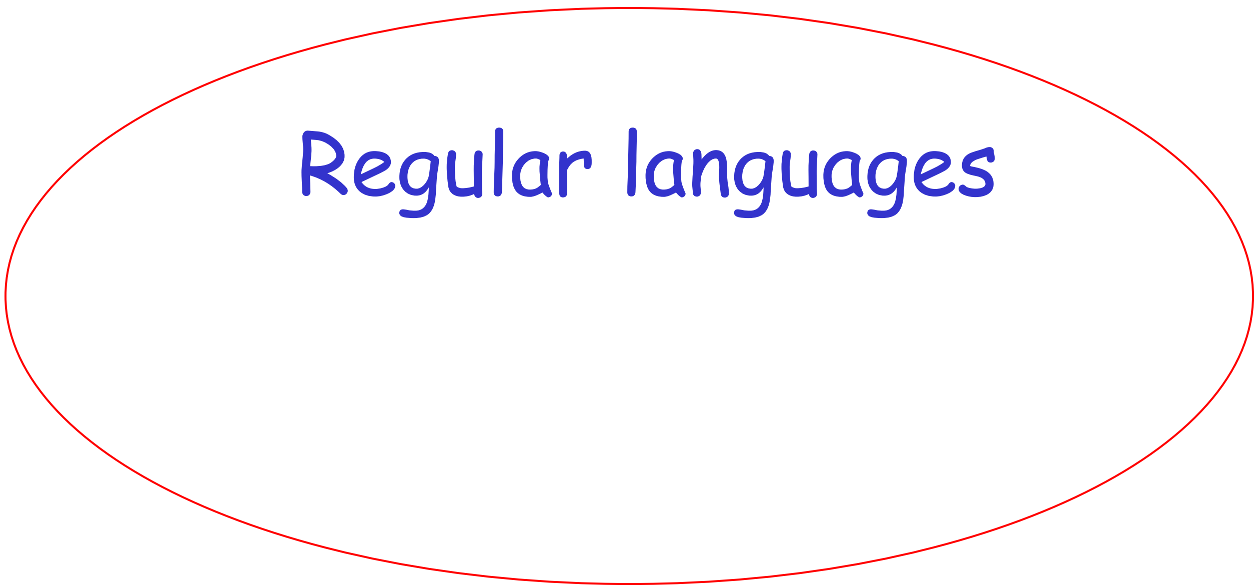
of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\ y\ z$
- with $|x\ y| \leq m$ and $|y| \geq 1$
- such that: $x\ y^i\ z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages $L = \{vv^R : v \in \Sigma^*\}$



Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

From the Pumping Lemma:

we can write: $w = a^m b^m b^m a^m = x\ y\ z$

with lengths: $|x\,y| \le m, \quad |y| \ge 1$

$$w = xyz = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\ldots a}^{m}\overbrace{b\ldots b}^{m}\overbrace{b\ldots b}^{m}\overbrace{a\ldots a}^{m}$$

(where z spans from the remaining $a\ldots a$ through the final a)

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \in L$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k,\ \ 1 \le k \le m$$

From the Pumping Lemma: $x\ y^2\ z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m} \in L$$

with $z = a...ab...bb...ba...a$

Thus: $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k} b^m b^m a^m \in L \qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k} b^m b^m a^m \notin L$$

CONTRADICTION!!!

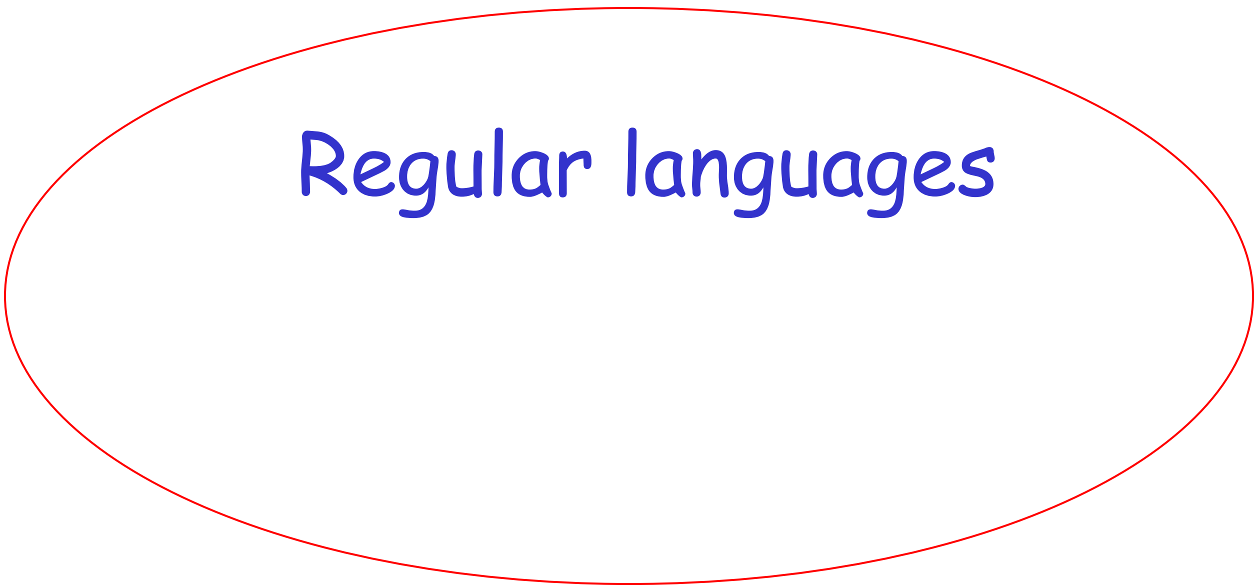
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$



Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

From the Pumping Lemma:

We can write $w = a^m b^m c^{2m} = x\ y\ z$

With lengths $|x\,y| \le m, \quad |y| \ge 1$

$$w = xyz = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}$$

$$w = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...bc...cc...c}_{z}$$

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^0\ z = xz \in L$

$x\ y\ z = a^m b^m c^{2m}$ $\quad$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \underbrace{a...a}_{x}\overbrace{a...ab...bc...cc...c}^{}_{z} \in L$$

$$xz = \overbrace{a...aa...a}^{m-k}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m} \in L$$

Thus: $\quad a^{m-k} b^m c^{2m} \in L$

$$a^{m-k}b^{m}c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \ n,l \geq 0\}$

$$\Downarrow$$

$$a^{m-k}b^{m}c^{2m} \notin L$$

CONTRADICTION!!!

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END OF PROOF

Non-regular languages $L = \{a^{n!} : \ n \geq 0\}$

Regular languages

Theorem: The language $L = \{a^{n!} : n \geq 0\}$

is not regular

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \ n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

From the Pumping Lemma:

We can write $w = a^{m!} = x\ y\ z$

With lengths $|x\ y| \leq m, \quad |y| \geq 1$

$$w = xyz = a^{m!} = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

where z is the part from the third group to the end: $z = a...aa...aa...a$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\ y\ z = a^{m!} \qquad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \in L$

$$x\ y\ z = a^{m!} \qquad\qquad y = a^{k},\ \ 1 \le k \le m$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}a\ldots a}^{m+k}\overbrace{a\ldots aa\ldots a}^{m!-m} \ \in\ L$$

(where z spans from the fourth block to the end)

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \le k \le m$$

Since: $L = \{a^{n!} : \; n \ge 0\}$

There must exist p such that:

$$m!+k = p!$$

However:

$$m! + k \le m! + m \quad \text{for} \quad m > 1$$

$$\le m! + m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m! + k < (m+1)!$$

$$\Downarrow$$

$$m! + k \ne p! \quad \text{for any} \quad p$$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $$L = \{a^{n!} : \; n \geq 0\}$$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF