HS-301

End-Semester Assignment

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Q1) Given,

Price of an underlying stock at t_0 (S_0) = 49 \$

Strike price of the stock (K) = 50 \$

Risk free rate (r) = 5% per annum

Volatility of stock (σ) = 20%

Time maturity (τ) = 20 weeks = ~ 0.3836 years

Required to find Theta (Θ) and Gamma (Γ) of the call option of above mentioned stock.

From Black-Scholes equation, optimal risk free call option price is

cumulative normal distn

$$C = S_0 N(d_1) - K e^{-r\tau} N(d_2)$$

where,

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$\tau = T - t$$

Theta (Θ)

Theta is the rate of change of the value of a portfolio with respect to passage of time.

i.e) $\Theta = \dfrac{\partial C}{\partial t} = \dfrac{-\partial C}{\partial \tau}$

$$\Rightarrow -\left[\frac{\partial}{\partial \tau}\left(S_0 N(d_1) - K e^{-r\tau} N(d_2)\right)\right]$$

$$\Rightarrow -S_0 \frac{\partial N(d_1)}{\partial \tau} + K \frac{\partial}{\partial \tau}\left(e^{-r\tau} N(d_2)\right)$$

$$\Rightarrow -S_0 \frac{\partial N(d_1)}{\partial d_1} \times \frac{\partial d_1}{\partial \tau} + -rK e^{-r\tau} N(d_2) + K \frac{\partial N(d_2)}{\partial d_2} \times \frac{\partial d_2}{\partial \tau} e^{-r\tau}$$

$$\Rightarrow -S_0 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \times \left[\frac{-\ln(S_0/K)}{2\sigma \tau^{3/2}} + \frac{\left(r + \frac{\sigma^2}{2}\right)}{2\sigma\sqrt{\tau}}\right]$$

$$- rK e^{-r\tau} N(d_2)$$

$$+ \cancel{K} \cancel{e^{-r\tau}} \times \left(\frac{1}{\sqrt{2\pi}} \times e^{-\frac{d_1^2}{2}} \times \frac{S_0}{\cancel{K}} \cancel{e^{r\tau}}\right)\left(\frac{-\ln\left(\frac{S_0}{K}\right)}{2\sigma T^{3/2}} + \frac{\left(r - \frac{\sigma^2}{2}\right)}{2\sigma\sqrt{\tau}}\right)$$

(From $N'(d_1) = \dfrac{S_0}{K} e^{r\tau} \times N'(d_2)$)

$$\Rightarrow -\frac{S_0}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \left(-\cancel{\frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\tau^{3/2}}} + \frac{\left(\cancel{\gamma} + \frac{\sigma^2}{2}\right)}{2\sigma\sqrt{\tau}} + \cancel{\frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\tau^{3/2}}} - \left(\frac{\cancel{\gamma} - \frac{\sigma^2}{2}}{2\sigma\sqrt{\tau}}\right)\right)$$

$$\Rightarrow \frac{-S_0}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} \left(\frac{\cancel{2}\sigma^2/2}{\cancel{2}\sigma\sqrt{\tau}}\right) - \gamma K e^{-\gamma\tau} N(d_2)$$

$$\Rightarrow \quad \Theta = \frac{-S_0 \sigma N'(d_1)}{2\sqrt{\tau}} - \gamma K e^{-\gamma T} N(d_2)$$

$$\underline{\underline{d_1}} = \frac{\ln\left(\frac{49}{50}\right) + \left(0.05 + \frac{(0.2)^2}{2}\right)0.3836}{0.2 \times \sqrt{0.3836}} \approx \boxed{0.05366}$$

$$\underline{\underline{d_2}} = d_1 - \sigma\sqrt{\tau} = 0.05366 - 0.2\sqrt{0.3836} \approx \boxed{-0.07024}$$

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \approx 0.398$$

$$N(d_2) = N(-0.07024) = 0.472$$

Substituting these values in Θ, we get

$$\boxed{\Theta = -4.3096}$$

$$\boxed{C = S_0 N(d_1) - K e^{-\gamma\tau} N(d_2) \approx 2.396\ \$}$$

Gamma (Γ)

Gamma is the double derivative of value of a portfolio with respect to the stock price.

i.e) $\Gamma = \frac{\partial^2 C}{\partial S^2} \Rightarrow \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right)$

$$\Rightarrow \frac{\partial}{\partial S}\left(\frac{\partial}{\partial S}\left(S_0 N(d_1) - K e^{-r\tau} N(d_2)\right)\right)$$

$$\Rightarrow \frac{\partial}{\partial S}\left(N(d_1) - 0\right) \Rightarrow \frac{\partial (N(d_1))}{\partial S}$$

$$\Rightarrow \frac{\partial N(d_1)}{\partial d_1} \times \frac{\partial d_1}{\partial S} \Rightarrow N'(d_1) \times \frac{\partial}{\partial S}\left[\frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right]$$

$$\Rightarrow \frac{N'(d_1)}{\sigma\sqrt{\tau}} \times \frac{\partial}{\partial S}\left(\ln\left(\frac{S}{K}\right)\right) \Rightarrow \frac{N'(d_1)}{\sigma\sqrt{\tau}\, S_0}$$

$$\Rightarrow \Gamma = \frac{N'(d_1)}{\sigma S_0 \sqrt{T}} \Rightarrow \frac{N'(0.05366)}{0.2 \times 49 \times \sqrt{0.3836}} \Rightarrow \frac{0.398}{0.2 \times 49 \times \sqrt{0.3836}}$$

$$\Rightarrow \boxed{\Gamma = 0.06564}$$

2)

Put - Call Parity

Put call parity is a principle that defines the relationship between the price of European put and European call options of the same class (same underlying asset, strike price and expiration date).

In other words, this principle requires that the puts and calls are the same strike, same expiration and have the same underlying future contract. The put call relationship is highly correlated, so if put call parity is violated, an arbitrage opportunity exists.

The put-call parity equation is

$$\boxed{C + Ke^{-r\tau} = S_0 + P}$$

We already know that :

$$C = S_0 N(d_1) - Ke^{-r\tau} N(d_2)$$

$$P = Ke^{-r\tau} N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where,

$C =$ call option price

$P =$ put option price

$S_0 =$ stock price at $t=0$

$r =$ risk free interest rate

$\tau =$ Time of maturity

$N =$ cumulative normal dist^n

$$\text{LHS} = C + Ke^{-r\tau} = S_0 N(d_1) - Ke^{-r\tau} N(d_2) + Ke^{-r\tau}$$

$$= S_0 (1 - N(-d_1)) - Ke^{-r\tau}(\cancel{1} - N(-d_2)) + \cancel{Ke^{-r\tau}}$$

$$= Ke^{-r\tau} N(-d_2) - S_0 N(-d_1) + S_0$$

$$= P + S_0$$

$\therefore$ LHS = RHS

$$\Rightarrow \boxed{C + Ke^{-r\tau} = S_0 + P}$$

3) ## Ito's process

Ito's process is a type of stochastic process described by Japanese mathematician Kiyoshi Ito, which can be written as the sum of the integral of a process over time and of another process over a Brownian motion.

This is a generalized weiner process where 'a' and 'b' parameters are not constant but they are dependent on time and underlying assets value.

Algebraically, $dx = a(x,t)\,dt + b(x,t)\,dz$

$$\Delta x = a(x,t)\,\Delta t + b(x,t)\,\varepsilon\sqrt{\Delta t}$$

$a(x,t) \rightarrow$ drift term

$b(x,t) \rightarrow$ variance term

Ito's Lemma

Stock price behaves stochastically and it changes depends on time. Hence any options price depends on time and the underlying stock. Behaviour of stochastic process for an option was proposed by (Ito)

~~Ito~~

Ito's lemma says that a function h dependent on x, t variables where x follows ito's process, then h satisfies the following PDE.

$$dh = \left[\frac{\partial h}{\partial x} a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} b^2\right] dt + \frac{\partial h}{\partial x} b\, dz$$

where $dx = a(x,t)\, dt + b(x,t)\, dz$

Derivation of Ito's lemma

If x (underlying asset or stock) changes by small unit Δx then

$$\Delta h \approx \frac{dh}{dx}\Delta x + \frac{\partial h}{\partial t}\Delta t$$

From taylors expansion : $\Delta h = \frac{dh}{dx}\Delta x + \frac{1}{2}\frac{d^2 h}{dx^2}\Delta x^2 + \dots + \frac{\partial h}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\Delta t^2$

$$\Rightarrow \Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 h}{\partial t^2}\Delta t^2 + \frac{\partial^2 h}{\partial x\, \partial t}\Delta x \cdot \Delta t \quad (1)$$

We know,

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

$$\Rightarrow \Delta x^2 = (a\Delta t + b\varepsilon\sqrt{\Delta t})^2$$

$$\Delta x^2 = a^2(\Delta t)^2 + b^2\varepsilon^2\Delta t + 2ab\varepsilon\,\Delta t^{3/2}$$

Ignoring higher term of Δt ($\because \Delta t \to 0$)

$$\Delta x^2 = b^2\varepsilon^2\Delta t + \text{negligible terms}$$

$\Rightarrow$ Using in 1

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\varepsilon^2 dt$$

Now, we know

$$\text{Var}(\varepsilon) = 1 \quad (\text{std Normal dist}^n)$$

$$\Rightarrow 1 = E(\varepsilon^2) - [E(\varepsilon)]^2 \Rightarrow 1 = E(\varepsilon^2)$$

(As $E(\varepsilon) = 0$, std Norm)

$\because E(\varepsilon^2) = 1$ is constant and $\text{var}(\varepsilon) = \text{const}$.
Happens when $\varepsilon^2 = 1$

$$\Rightarrow dh = \frac{\partial h}{\partial x}[a\,dt + b\,dz] + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2 dt$$

$$\Rightarrow dh = \overbrace{\left(a\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)}^{\text{new drift}}dt + \overbrace{b\frac{\partial h}{\partial x}}^{\text{new var-factor}}d$$

Hence proved.

Now, given $h(s,t) = \ln(S)$

$$\Rightarrow \quad dh = \left[a\frac{\partial h}{\partial S} + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial S^2}b^2\right]dt$$

$$+$$

$$b\frac{\partial h}{\partial S}\,dz \qquad \left(dS = \mu S\,dt + \sigma S\,dz\right)$$

$$\Rightarrow a = \mu S$$
$$b = \sigma S$$

$$\Rightarrow \quad \cancel{dh} \approx$$

$$\Rightarrow \quad dh = \left[\mu S \times \left[\frac{1}{S}\right] + 0 + (\sigma S)^2 \times \frac{1}{2} \times \left(\frac{-1}{S^2}\right)\right]dt$$

$$+$$

$$\sigma S \times \left[\frac{1}{S}\right]dz$$

$$\Rightarrow \quad dh = \left[\mu - \frac{\sigma^2}{2}\right]dt + \sigma\,dz$$

$$\boxed{\Rightarrow \quad h(s,t) = \int_0^t \left[\mu - \frac{\sigma^2}{2}\right]dt + \int_0^t \sigma\,dz}$$