TIME VALUE OF MONEY

PART-2

Present Value of Ordinary Annuity

PVA = PMT/(1+r)+ PMT/$(1+r)^2$ + PMT/$(1+r)^3$ +…
PMT/$(1+r)^n$

Here PMT stands for payment amount payed every period
r is rate of interest
PVA is present value of Annuity

Alternative formula= PMT [(1/r)- {1/r(1+r)}]

Example and Excel Command

- PMT= 100 INR

  r= 5%

  n=3 years

  PVA= PMT*[(1/r)-1/r(1+r)]

  = 100 * [(1/0.05- 1/0.05*$(1+0.05)^3$]

  =272.32 INR

Excel Command

=PV(r, n, PMT, FV)

=PV (0.05, 3,-100,0) -----(Aforementioned example)

=272.32 INR

Present Value of Annuity Due

- PVA of Annuity Due= PVA of Ordinary Annuity (1+r)

  = 272.32 (1+0.05)

  = 285.94 INR

Excel command

=PV (r, n, PMT, FV, Type)

= PV(0.05, 3, -100, 0,1)

=285.94 INR

Finding Annuity Payments, Periods and Interest Rates

- Excel Commands to find Payment

  = PMT(r, n, PV,FV)

  = PMT(0.06 ,5, 0,1000)

  =1773.96 INR

Excel Command to find Periods

= NPER(0.06,-1200, 0,10000)

= 6.96 (say 7 years)

Excel Command to find interest rate

= RATE(3, -1200, 0, 10000)

= 25.78%

Semiannual and other compounding periods

Instead of once in a year compounding, if we compound twice a year (semiannually) then we can express the FV and PV relation in modified fashion-

$$FV = PV\ (1 + r/M)^{Mn}$$

Where M=2 (if semiannually compounded)
M=4 (if quarterly compounded)

and n= number of years
r= interest rate

Numerical Example

- If one person borrows 100 INR for 2 years and if he has to pay 12% annual interest on a quarterly basis (compounding 4 times a year) then total payment (principle and interest) will be= 100 $(1 + 0.12/4)^{4*2}$

  =126.68 INR

An interesting interpretation of 'e'

$(1 + 1/n)^n$ and if n $\rightarrow$ ∞

=e (approx. 2.718)

Effective Annual Rate

This is the rate which will yield same result if we had compounded at a given periodic rate M times per year.

EAR= $(1 + r/M)^M - 1$

Here r/M is the periodic rate and M is the number of periods per year.

Numerical Example

- Suppose we have two choices-
- 1) Borrow using a credit card that charges 1% per month

OR

- 2) Borrow from a bank with 12% interest rate which is compounded quarterly.

Credit card- EAR= $(1 + 0.01)^{12}$ -1 = 12.685%

Bank loan- EAR= $(1 + 0.03)^{4}$- 1= 12.5509%

- Credit card is more expensive

Time value and Inflation

- So far in our PV, FV analysis we did not incorporate the notion of inflation/deflation. However, in real life such factor plays important role
- To determine the valuation of money over time we need to factor
  inflation
- Inflation is characterized as continuous increase in general price level.
  If we assume the inflation (rate of change in price)
as f, then price
  worth 1 INR will become 1*(1+f) INR in future
(say after 1 period)

Time value and Inflation

- The rule is applicable for multiple years/periods and we may write the
  value after n years will be

$$(1 + f)^n$$

Hence what we can buy today for 1 INR, we need to pay $(1 + f)^n$ INR in future after n years.

So purchasing power will shrink by the factor $(1 + f)^n$

Time value and Inflation

- If a person has one rupee and if bank offers r% annual interest rate then after 1 year what will be the purchasing power of his money. Assume inflation rate is ‘f’.

- After 1 year, bank will give back (1+r) rupee to that person.

However with this (1+r) he cannot buy the same thing what he used to buy earlier. Because of inflation his purchasing power will go down by the factor (1+f)

So, the money (1+r) can afford (1+f) fraction of old goods and services.

Hence *real value* in market of his money is (1+r)/(1+f)

Real and Nominal interest rate

From previous example how can we interpret the concept of real rate of interest and its distinction from nominal interest rate?

If we assume R is the real interest rate and r is the nominal interest rate the after 1 year real value of 1 INR becomes (1+r)/(1+f).

If we equate this value and assume real interest rate is inflation-adjusted rate then $1+R = (1+r)/(1+f)$

$$\begin{aligned} \text{or } R &= (1+r)/(1+f) - 1 \\ &= (1+r-1-f)/(1+f) \\ &= (r-f)/(1+f) \end{aligned}$$

Numerical example:

- If nominal interest rate is 5% and inflation rate is 3% then what will
  be the real interest rate?

$R = (r-f)/(1+f) = (0.05-0.03)/(1+0.03) = 0.0194$

$R = 1.94\%$

Though nominal interest rate is as high as 5%, real value is only 1.94% due to inflation.

Numerical example

- Suppose one wants to grow his money up to 100000 INR in 10 years. The person wants to make 10 deposits (each at the beginning of every year) in a bank which offers 6% interest rate and inflation rate in economy is 2%. How much you need to deposit initially to get 100000INR at the end of 10 years?

- Real rate of interest R= (1+0.06)/(1+0.02)= 3.912%

- In 10 years the target 100000INR will have purchasing power of 100000/(1+0.02)^10

= 82034.83INR

Hence to get FV=82034.83INR in 10 years with 6% interest rate we can calculate PMT value.

Excel Command = PMT(R,N,PV,FV)= -6598.87INR (Initial payment to be made)

Notion of IRR

- IRR – Internal Rate of Return
- An interest rate which yield a PV=0 for a cash flow over a period of time

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} .$$

- Where r= IRR

Consider again the cash flow sequence $(-2, 1, 1, 1)$

- Calculate the IRR

Solution

$$0 = -2 + c + c^2 + c^3.$$

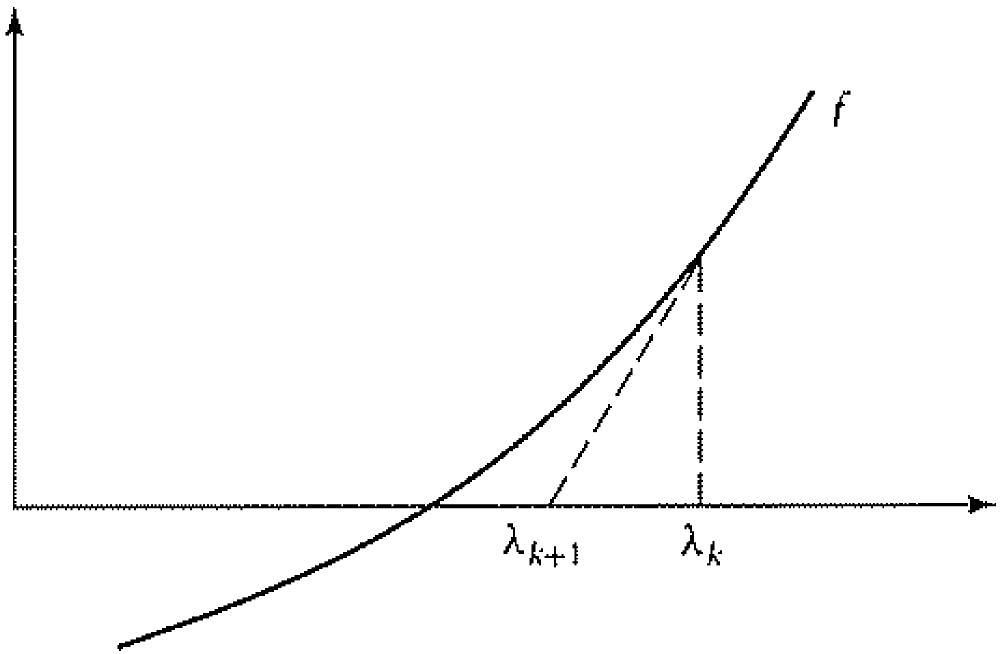
The solution can be found (by trial and error) to be $c = .81$, and thus $\mathrm{IRR} = (1/c) - 1 = .23$.

Newton Raphson Method

$$f(\lambda) = -a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n.$$

$$f'(\lambda_k) = a_1 + 2a_2\lambda_k + 3a_3\lambda_k^2 + \cdots + na_n\lambda_k^{n-1}$$

$$\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}$$



Numerical Example

Find the root of the equation $x^2 - 4x - 7 = 0$ near $x = 5$

Using Newton's method, we get the following sequence of approximations:

$$x_1 = 5 - \frac{5^2 - 4 \times 5 - 7}{2 \times 5 - 4} = 5 - \left(\frac{-2}{6}\right) = \frac{16}{3} \approx 5.33333$$

$$x_2 = \frac{16}{3} - \frac{\left(\frac{16}{3}\right)^2 - 4\left(\frac{16}{3}\right) - 7}{2\left(\frac{16}{3}\right) - 4} = \frac{16}{3} - \frac{\frac{1}{9}}{\frac{20}{3}} = \frac{16}{3} - \frac{1}{60} = \frac{319}{60} \approx 5.31667$$

$$x_3 = \frac{319}{60} - \frac{\left(\frac{319}{60}\right)^2 - 4\left(\frac{319}{60}\right) - 7}{2\left(\frac{319}{60}\right) - 4} = \frac{319}{60} - \frac{\frac{1}{3600}}{\frac{398}{60}} \approx 5.31662.$$

Investment Decision (NPV vs IRR)

- Suppose you have an opportunity to plant a tree which you can sell to lumber in future. You have two options. Grow the tree for one year and then sell for 2 INR or grow it for two years and then sell it for 3 INR. Initial cost to plant the tree is 1 INR.

- What will you do?

Assume r=10% (Let's follow NPV concept)

(a) $(-1, 2)$ cut early

(b) $(-1, 0, 3)$ cut later

(a) $\text{NPV} = -1 + 2/1.1 = .82$

(b) $\text{NPV} = -1 + 3/(1.1)^2 = 1.48.$

Let’s follow IRR Method

$(a)\ -1 + 2c = 0$

$(b)\ -1 + 3c^2 = 0.$

As usual, $c = 1/(1+r)$. These have the following solutions:

$$(a)\ c = \frac{1}{2} = \frac{1}{1+r}; \qquad r = 1.0$$

$$(b)\ c = \frac{\sqrt{3}}{3} = \frac{1}{1+r}; \qquad r = \sqrt{3} - 1 \approx .7.$$

Cyclical Problem

A 20000 INR car is expected to have low maintenance cost of 1000 INR per year and mileage up to 4 years. Second car costs 30000 INR and maintenance cost is 2000 INR per year with longevity of 6 years. Which car will you buy? (r=10%)

* Let’s assume 12 year long decision horizon for both cars

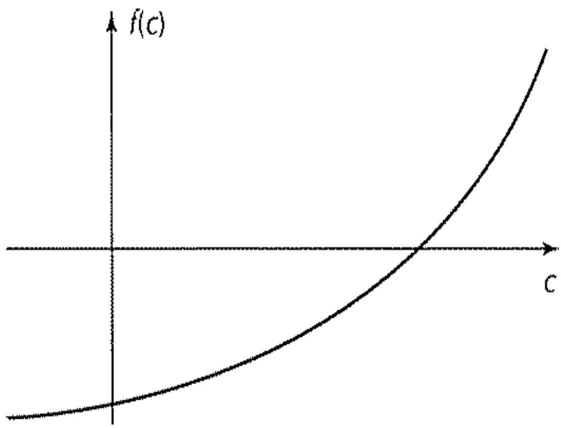
Car A:

$$\text{One cycle} \qquad PV_A = 20,000 + 1,000 \sum_{k=1}^{3} \frac{1}{(1.1)^k}$$
$$= \$22,487$$
$$\text{Three cycles} \qquad PV_{A3} = PV_A \left[1 + \frac{1}{(1.1)^4} + \frac{1}{(1.1)^8} \right]$$
$$= \$48,336$$

Car B:

$$\text{One cycle} \qquad PV_B = 30,000 + 2,000 \sum_{k=1}^{5} \frac{1}{(1.1)^k}$$
$$= \$37,582$$
$$\text{Two cycles} \qquad PV_{B2} = PV_B \left[1 + \frac{1}{(1.1)^6} \right]$$
$$= \$58,795$$

A simple proof of IRR>=0



$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n .$$

$$r = (1/c) - 1$$

$$f(0) < 0.$$

$$f(1) > 0.$$

- MEAN VARAINCE ANALYSIS (Portfolio Analysis)

Random Variable

- X be a random variable with X1, X2, X3, X4 values and p1, p2, p3, p4 probabilities (respectively)

- $E(X) = X_1p_1+X_2p_2+X_3p_3+X_4p_4$
- $Var(X) = E(X^2) - [E(X)]^2$

  $= (X_1^2)p_1+(X_2^2)p_2+\ldots\ldots - [E(X)]^2$

Numerical Example

Expected value of number of spots for a rolled die

$$\tfrac{1}{6}(1+2+3+4+5+6) = 3.5$$

$$\begin{aligned}\sigma^2 &= \mathrm{E}(y^2) - \overline{y}^2 \\ &= \tfrac{1}{6}[1+4+9+16+25+36] - (3.5)^2 = 2.92\end{aligned}$$

Properties of Expectation

Nonnegativity If x is random but never less than zero, then $\mathrm{E}(x) \geq 0$.

Linearity If y and z are random, then $\mathrm{E}(\alpha y + \beta z) = \alpha \mathrm{E}(y) + \beta \mathrm{E}(z)$ for any real values of α and β.

Certain value If y is a known value (not random), then $\mathrm{E}(y) = y$.

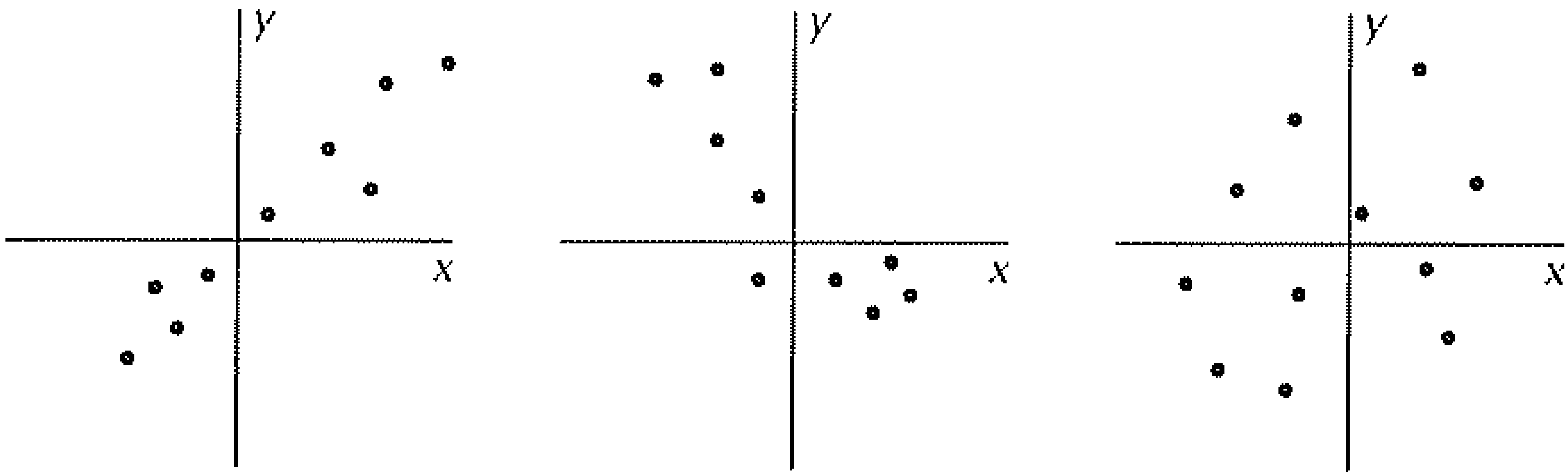
Covariance (Two random variables)

$$\text{cov}(x_1, x_2) = \text{E}\big[(x_1 - \overline{x}_1)(x_2 - \overline{x}_2)\big] \, .$$

$$\text{cov}(x_1, x_2) = \text{E}(x_1 x_2) - \overline{x}_1 \overline{x}_2 \, .$$

Correlation Coefficient

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}.$$



(a) Positively correlated (b) Negatively correlated (c) Uncorrelated