- MEAN VARIANCE ANALYSIS (Portfolio Analysis)

Random Variable

- X be a random variable with X1, X2, X3, X4 values and p1, p2, p3, p4 probabilities (respectively)

- $E(X)= X1p1+X2p2+X3p3+X4p4$
- $Var(X)= E(X^2)- [E(X)]^2$

  $=(X1^2)p1+(X2^2)p2+\ldots\ldots - [E(X)]^2$

Numerical Example

Expected value of number of spots for a rolled die

$$\tfrac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

$$\begin{aligned}\sigma^2 &= \mathrm{E}(y^2) - \bar{y}^2 \\ &= \tfrac{1}{6}[1 + 4 + 9 + 16 + 25 + 36] - (3.5)^2 = 2.92\end{aligned}$$

Properties of Expectation

Nonnegativity If x is random but never less than zero, then $\mathrm{E}(x) \geq 0$.

Linearity If y and z are random, then $\mathrm{E}(\alpha y + \beta z) = \alpha \mathrm{E}(y) + \beta \mathrm{E}(z)$ for any real values of α and β.

Certain value If y is a known value (not random), then $\mathrm{E}(y) = y$.

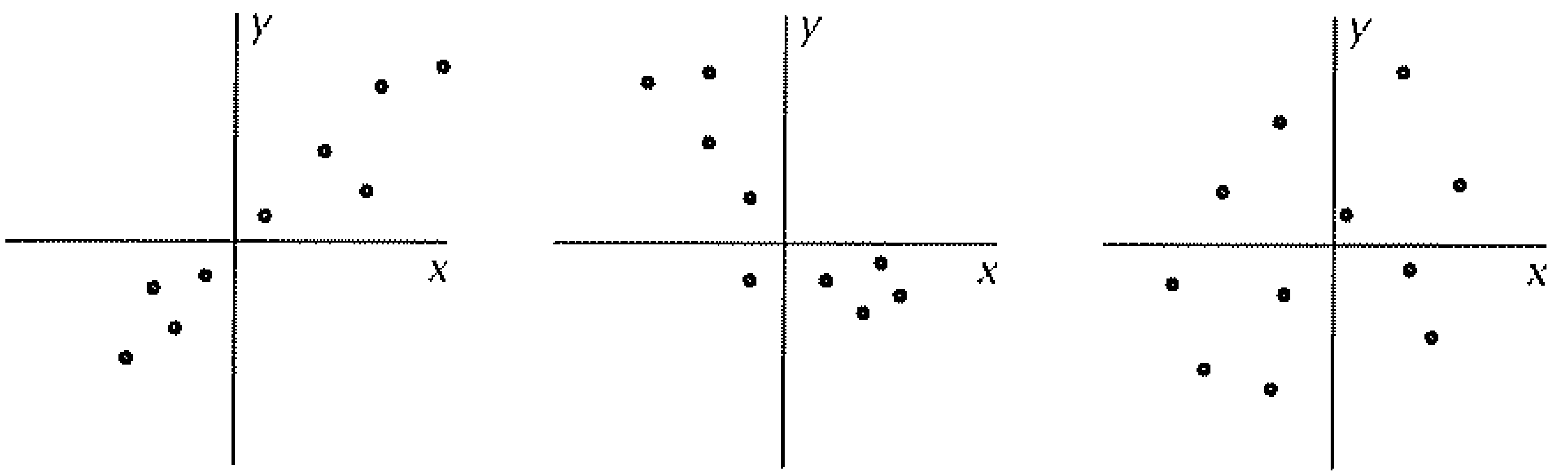
Covariance (Two random variables)

$$\mathrm{cov}(x_1, x_2) = \mathrm{E}\Big[(x_1 - \overline{x}_1)(x_2 - \overline{x}_2)\Big].$$

$$\mathrm{cov}(x_1, x_2) = \mathrm{E}(x_1 x_2) - \overline{x}_1 \overline{x}_2.$$

Correlation Coefficient

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$



(a) Positively correlated (b) Negatively correlated (c) Uncorrelated

Sum of Two Variables Variances

$$V(A) = E(A^2) - [E(A)]^2,$$

$$\begin{aligned}
V(X+Y) &= E[(X+Y)^2] - E^2(X+Y) \\
&= [E(X^2) + E(Y^2) + 2E(XY)] - [E^2(X) + E^2(Y) + 2E(X)E(Y)] \\
&= E(X^2) - E^2(X) + E(Y^2) - E^2(Y) + 2E(XY) - 2E(X)E(Y) \\
&= V(X) + V(Y) + 2\operatorname{cov}(X,Y).
\end{aligned}$$

What if two variables are statistically independent

- $V(X+Y)= V(X)+V(Y)+0$

A die is rolled twice. Average of two numbers on surface is assumed to be a random variable Z.

Calculate the mean/expectation and variance of Z.

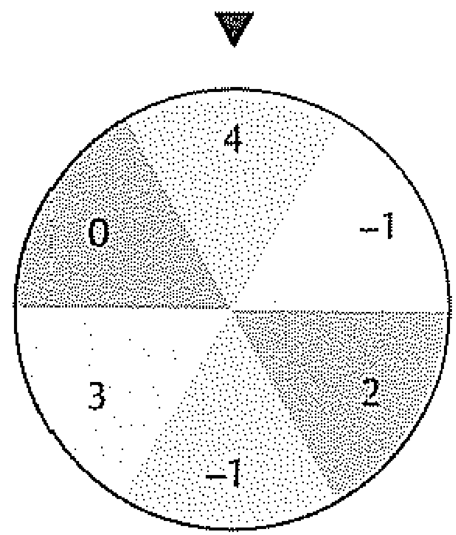
Solution

$$z = \tfrac{1}{2}(x + y).$$

$$\overline{z} = \tfrac{1}{2}(\overline{x} + \overline{y}) = 3.5,$$

$$\tfrac{1}{4}(\sigma_x^2 + \sigma_y^2) = 2.92/2 = 1.46.$$

Example (Fortune Wheel)



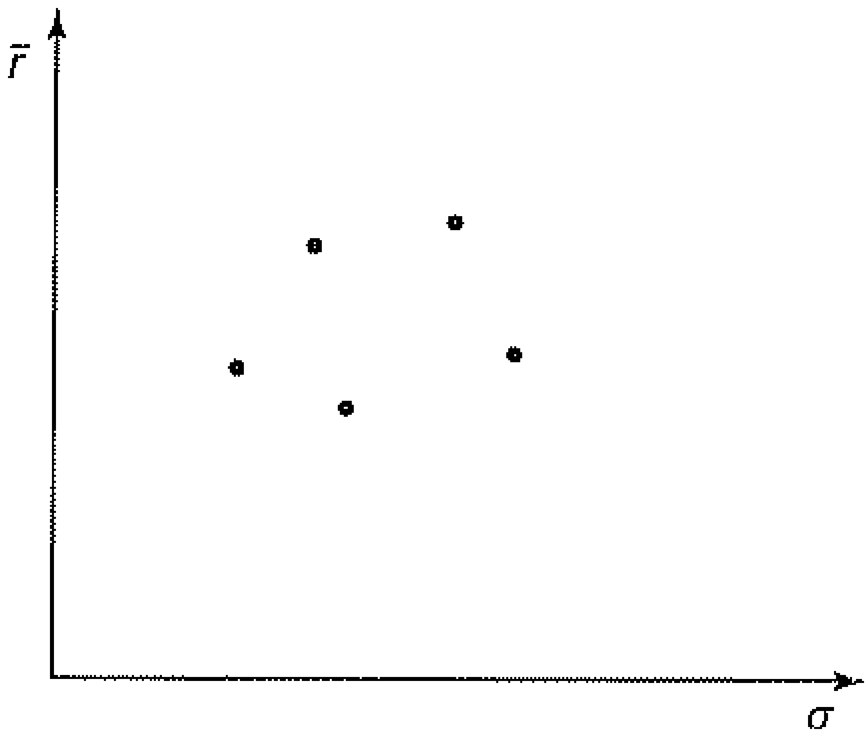
Solution

$$\overline{Q} = \sum_i p_i Q_i = \tfrac{1}{6}(4 - 1 + 2 - 1 + 3) = 7/6.$$

$$\sigma_Q^2 = \mathrm{E}(Q^2) - \overline{Q}^2 = \tfrac{1}{6}(16 + 1 + 4 + 1 + 9) - (7/6)^2 = 3.81$$

$$
\begin{aligned}
\operatorname{var}(x+y) &= \mathrm{E}[(x-\overline{x}+y-\overline{y})^2] \\
&= \mathrm{E}\left[(x-\overline{x})^2\right] + 2\mathrm{E}\left[(x-\overline{x})(y-\overline{y})\right] + \mathrm{E}\left[(y-\overline{y})^2\right] \\
&= \sigma_x^2 + 2\sigma_{xy} + \sigma_y^2.
\end{aligned}
$$

Mean–standard deviation diagram.



Expected value/mean return of a portfolio

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n .$$

$$\mathrm{E}(r) = w_1 \mathrm{E}(r_1) + w_2 \mathrm{E}(r_2) + \cdots + w_n \mathrm{E}(r_n) .$$

Variance of a portfolio

$$
\begin{aligned}
\sigma^2 &= \mathrm{E}\left[(r - \overline{r})^2\right] \\
&= \mathrm{E}\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \overline{r}_i\right)^2\right] \\
&= \mathrm{E}\left[\left(\sum_{i=1}^{n} w_i (r_i - \overline{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \overline{r}_j)\right)\right] \\
&= \mathrm{E}\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \overline{r}_i)(r_j - \overline{r}_j)\right] \\
&= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}\,.
\end{aligned}
$$

Calculate mean and variance of the portfolio (Assume w1=0.25)

- If r1= 0.12, r2=0.15 and $\sigma_1 = .20$, $\sigma_2 = .18$, and $\sigma_{12} = .01$

Solution

$$\bar{r} = .25(.12) + .75(.15) = .1425 .$$

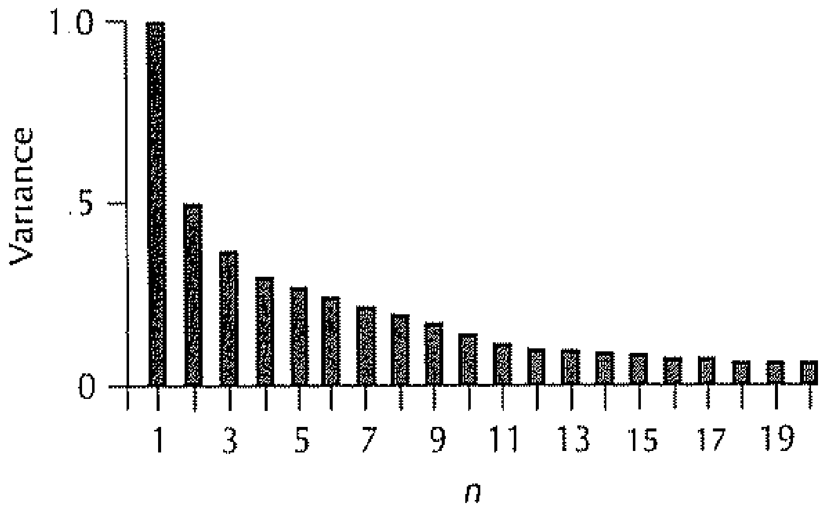
$$\sigma^2 = (.25)^2(.20)^2 + .25(.75)(.01) + .75(.25)(.01) + (.75)^2(.18)^2 = .024475 .$$

Notion of diversification

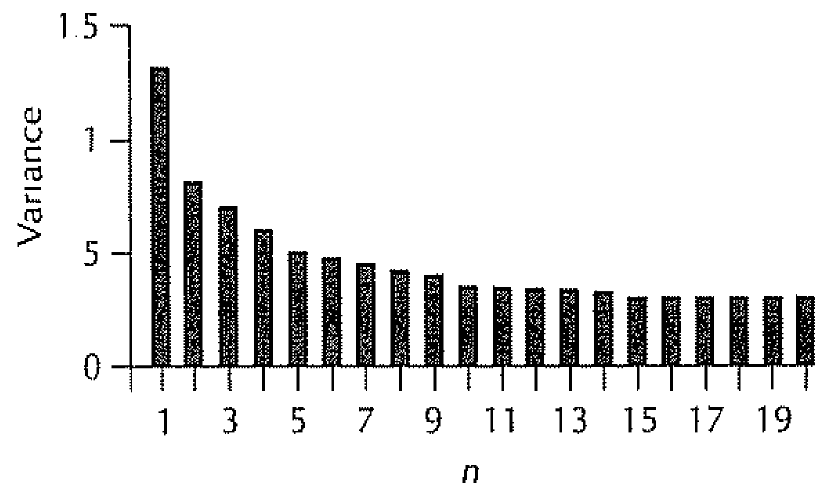
$$r = \frac{1}{n}\sum_{i=1}^{n} r_i.$$

$$\operatorname{var}(r) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$

Diversification

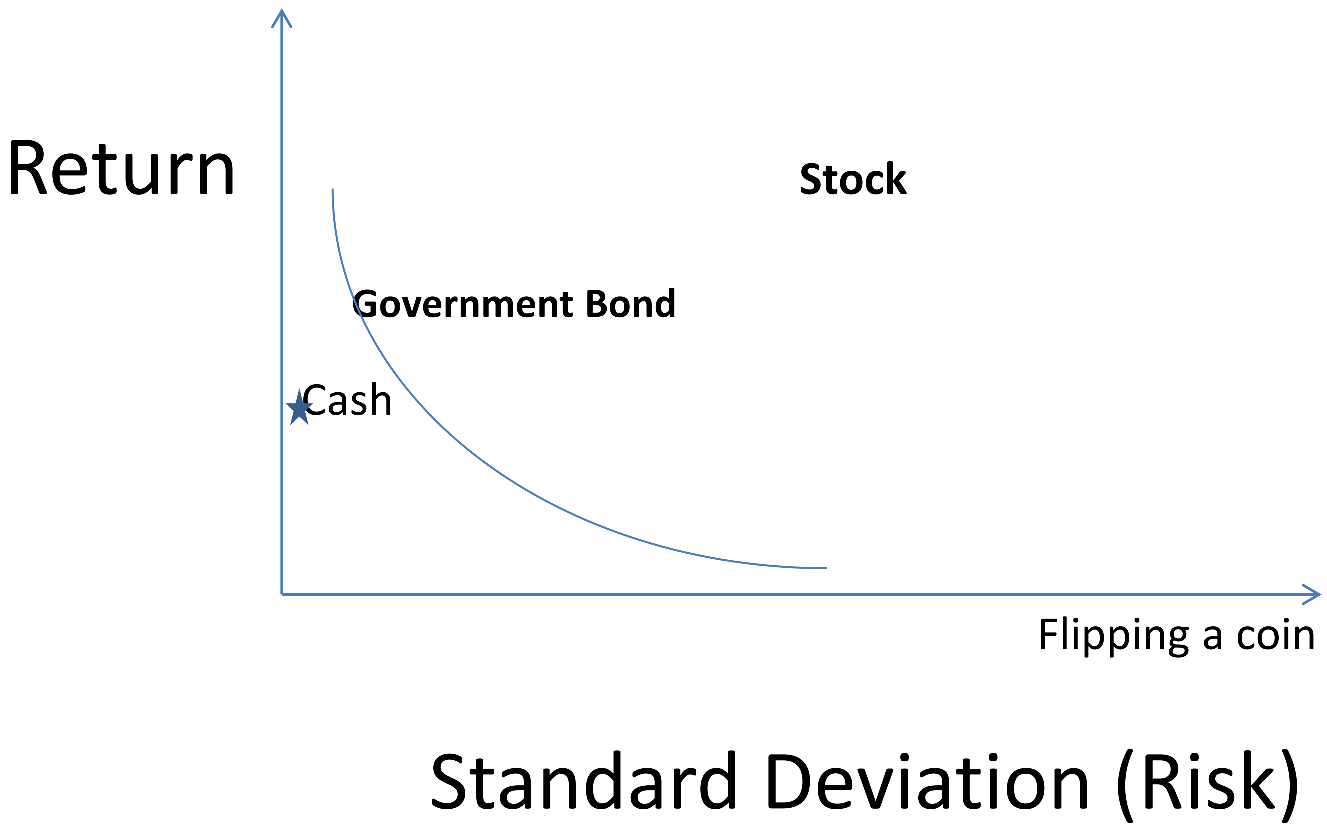


(a) Uncorrelated assets



(b) Correlated assets

Efficient Frontier



Suppose we have n stock in a portfolio with equal weightage and each with equal sd=σ. Further, covariance between each pair of stocks=0.3σ^2.
What is variance of the portfolio

$$
\begin{aligned}
\operatorname{var}(r) &= \mathrm{E}\left[\sum_{i=1}^{n} \frac{1}{n}(r_i - \bar{r})\right]^2 \\
&= \frac{1}{n^2}\mathrm{E}\left\{\left[\sum_{i=1}^{n}(r_i - \bar{r})\right]\left[\sum_{j=i}^{n}(r_j - \bar{r})\right]\right\} \\
&= \frac{1}{n^2}\sum_{i,j}\sigma_{ij} = \frac{1}{n^2}\left\{\sum_{i=j}\sigma_{ij} + \sum_{i\neq j}\sigma_{ij}\right\} \\
&= \frac{1}{n^2}\left\{n\sigma^2 + .3(n^2 - n)\sigma^2\right\} \\
&= \frac{\sigma^2}{n} + .3\sigma^2\left(1 - \frac{1}{n}\right) \\
&= \frac{.7\sigma^2}{n} + .3\sigma^2
\end{aligned}
$$

Two assets with different returns and risks (SD)

$$\bar{r}(\alpha) = (1-\alpha)\bar{r}_1 + \alpha\bar{r}_2 .$$

$$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{12} + \alpha^2\sigma_2^2} .$$

$$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\rho\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2} .$$

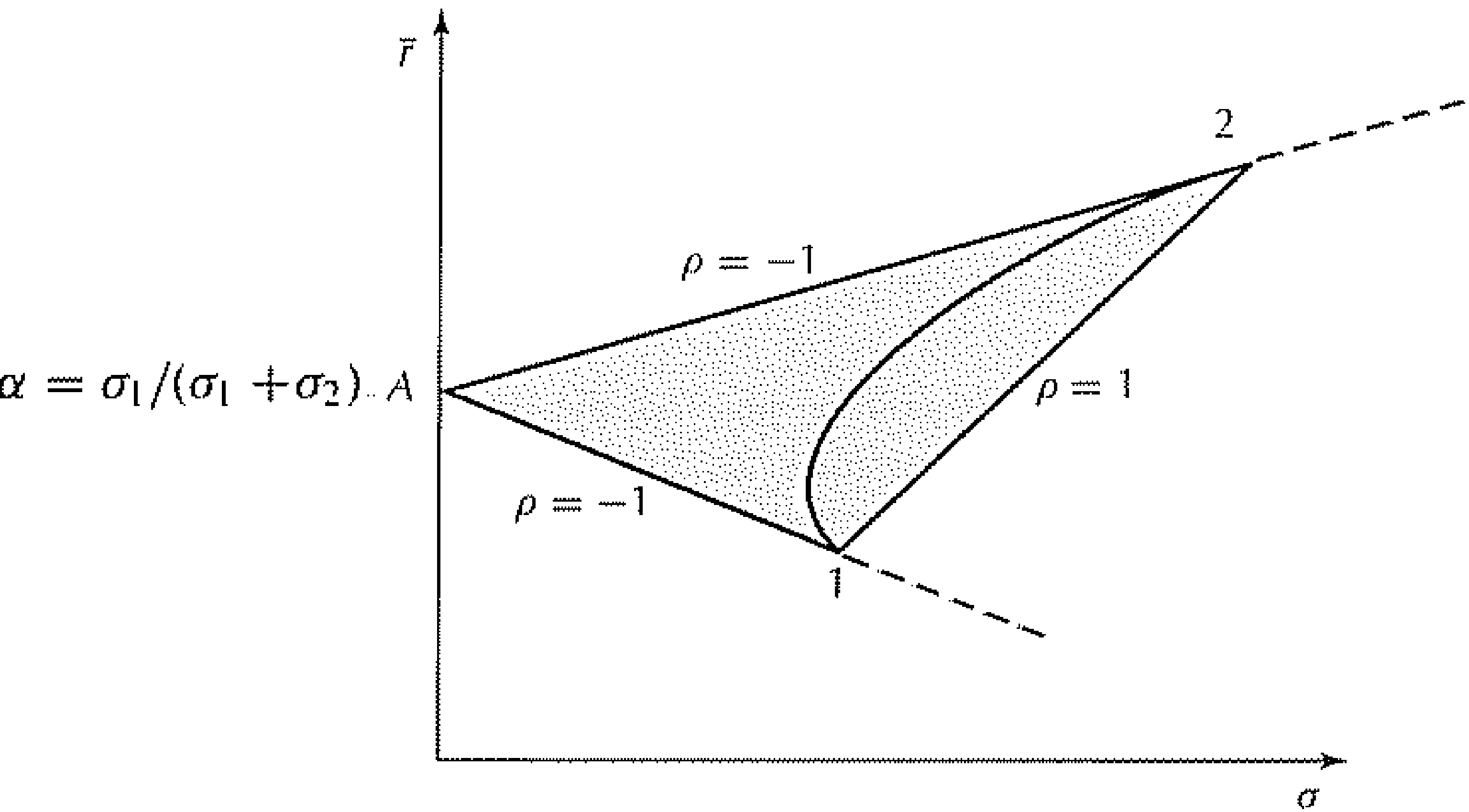
When ρ=+1 (upper bound)

$$
\begin{aligned}
\sigma(\alpha)^* &= \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2} \\
&= \sqrt{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2} \\
&= (1-\alpha)\sigma_1 + \alpha\sigma_2 .
\end{aligned}
$$

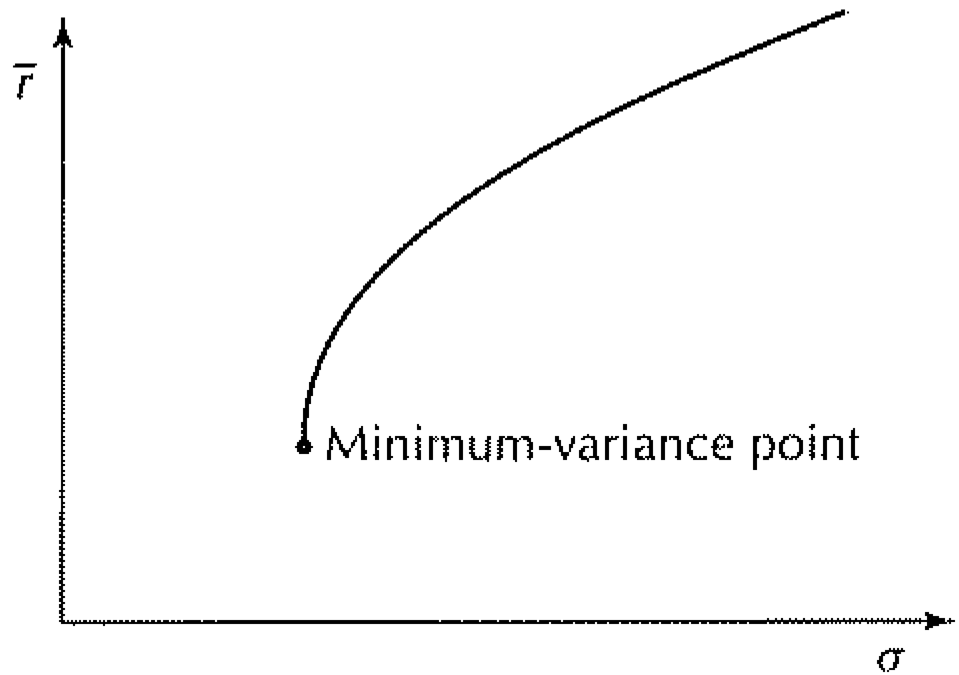
When ρ=-1 (lower bound)

$$\begin{aligned}\sigma(\alpha)_* &= \sqrt{(1-\alpha)^2\sigma_1^2 - 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2} \\ &= \sqrt{[(1-\alpha)\sigma_1 - \alpha\sigma_2]^2} \\ &= \left|(1-\alpha)\sigma_1 - \alpha\sigma_2\right|.\end{aligned}$$

Efficient Frontier



Efficient Frontier



Markowitz Model

$$\text{minimize} \quad \tfrac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$\text{subject to} \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1 \, .$$

Solution (Lagrange Formula)- Constrained Optimization

$$L = \tfrac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right).$$

$$L = \tfrac{1}{2} \left(w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2^2 \sigma_2^2 \right)$$
$$- \lambda (\bar{r}_1 w_1 + \bar{r}_2 w_2 - \bar{r}) - \mu (w_1 + w_2 - 1).$$

$$\frac{\partial L}{\partial w_1} = \tfrac{1}{2} \left(2\sigma_1^2 w_1 + \sigma_{12} w_2 + \sigma_{21} w_2 \right) - \lambda \bar{r}_1 - \mu$$

$$\frac{\partial L}{\partial w_2} = \tfrac{1}{2} \left(\sigma_{12} w_1 + \sigma_{21} w_1 + 2\sigma_2^2 w_2 \right) - \lambda \bar{r}_2 - \mu .$$

Using the fact that $\sigma_{12} = \sigma_{21}$ and setting these derivatives to zero, we obtain

$$\sigma_1^2 w_1 + \sigma_{12} w_2 - \lambda \bar{r}_1 - \mu = 0$$
$$\sigma_{21} w_1 + \sigma_2^2 w_2 - \lambda \bar{r}_2 - \mu = 0 .$$