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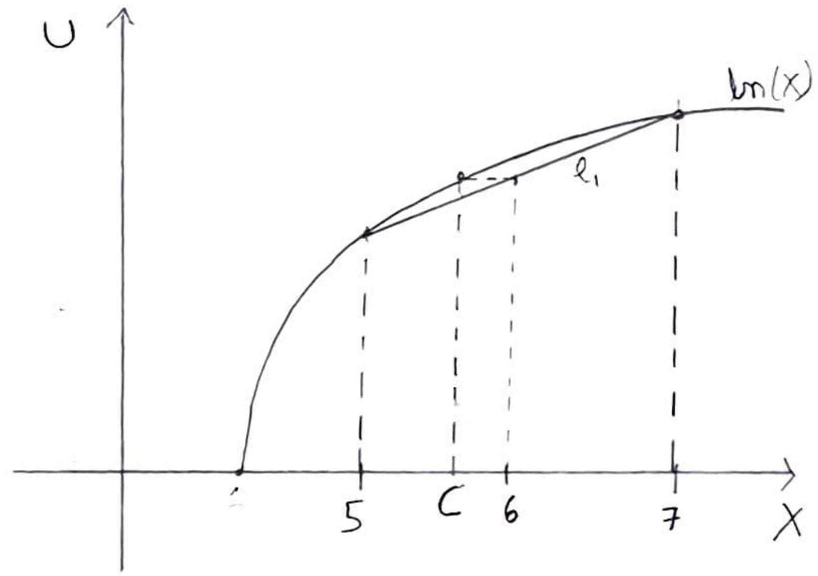
HS-301 Mid-Semester

1) Given, utility function of an investor is given by,

$U = \ln(X)$ where X is income and U is utility level.

Also given, for a gamble he/she will gain ₹5 or ₹7 with equal probability

Utility:



$$E(X) = P_1 X_1 + P_2 X_2 = \frac{1}{2}(5) + \frac{1}{2}(7) = 6$$

∴ We extend a vertical line from 6 to l_1

$$l_1: \quad y - \ln(7) = \frac{\ln(7) - \ln(5)}{7 - 5} \times (x - 7)$$

$$y \text{ at } 6 \Rightarrow y - \ln(7) = \frac{\ln(7) - \ln 5}{2} \times (6 - 7)$$

$$\Rightarrow y = \ln(7) - \frac{\ln(7)}{2} + \frac{\ln(5)}{2} = \frac{\ln(7)}{2} + \frac{\ln(5)}{2}$$

$\boxed{y = \ln(\sqrt{35})}$

Certainity equivalence is the amount of guaranteed return that someone would accept now, rather than taking a chance on a higher in future.

So, a neutral risk taker would take the expected returns at utility value of $\ln\sqrt{35}$. Now, this particular investor must achieve the same level of utility.
Therefore we find the corresponding x value for the utility functi

ie) $y = \ln(x) = \ln\sqrt{35} \Rightarrow x = \sqrt{35} = \boxed{₹5.916}$

Now, we can see that the utility curve is concave in nature indicating the investor is a risk averse person.
Meaning, they would choose a lower but certain return over a higher but uncertain return.

Arrow-Pratt measure of absolute aversion:

$$A = -\frac{U''(x)}{U'(x)} = \frac{-\left(\frac{-1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$= \frac{1}{\sqrt{35}}$$

2) Given, two assets with same return but different risks.

Assume returns, $R_1 = R_2 = R$ and risks σ_1, σ_2.

In general, expected mean value of a portfolio is given by,

$$\boxed{\bar{r} = E(r) = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n}$$

where $r_1, r_2, \cdots r_n$ are average returns of each stock.

$w_1, w_2, \cdots w_n$ are corresponding weights of each stock.

Variance of portfolio is given by,

$$\sigma^2 = E[(r-\bar{r})^2]$$

$$\Rightarrow E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{j=1}^{n} w_j \bar{r}_j\right)^2\right]$$

$$\Rightarrow E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \bar{r}_j)\right)\right]$$

$$\Rightarrow E\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right]$$

$$= \boxed{\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}}$$

Considering just 2 stocks

$$\bar{r} = w_1 r_1 + w_2 r_2$$

$$\sigma^2 = \sum_{i,j=1}^{2} w_i w_j \sigma_{ij} = w_1 w_1 \sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_2^2$$

$$\boxed{\sigma^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2}$$

Considering $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

$$\Rightarrow \sigma^2 = w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2$$

$$w_1 + w_2 = 1$$

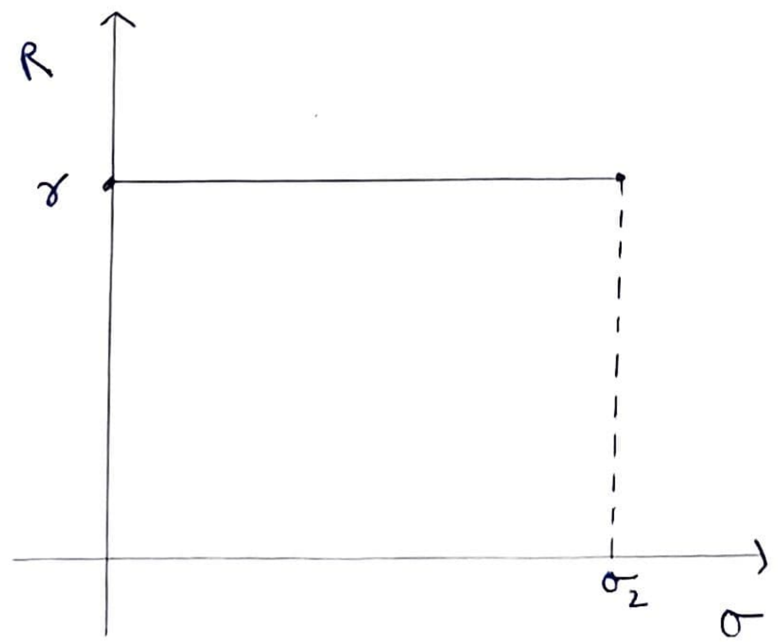
ρ varies from -1 to 1 and $\therefore$ σ varies from $|w_1\sigma_1 - w_2\sigma_2|$ to $w_1\sigma_1 + w_2\sigma_2$

min of $|w_1\sigma_1 - w_2\sigma_2| = 0$ when $w_1\sigma_1 = w_2\sigma_2$ and

max of $w_1\sigma_1 + w_2\sigma_2 = \sigma_2$ (If $\sigma_2 > \sigma_1$)

$\therefore$ σ lies in between $0, \sigma_2$ on a constant return r.

risk-return frontier.



This is a generic locus for all possible ρ values,

But for a fixed ρ.

$$\sigma = \left(w_1^2\sigma_1^2 + 2w_1w_2\sigma_1\sigma_2\rho + w_2^2\sigma_2^2 \right)^{1/2} \quad -(2)$$

should be minimized over $w_1 + w_2 = 1$

$$\frac{d\sigma}{dw_1} = \frac{d}{dw_1}\left(w_1^2\sigma_1^2 + 2w_1(1-w_1)\sigma_1\sigma_2\rho + (1-w_1)^2\sigma_2^2 \right)^{1/2}$$

$$= \frac{1}{2} \times \frac{2w_1\sigma_1^2 + 2\sigma_1\sigma_2\rho - 4w_1\sigma_1\sigma_2\rho - 2(1-w_1)\sigma_2^2}{\left(w_1^2\sigma_1^2 + 2w_1(1-w_1)\sigma_1\sigma_2\rho + w_2^2\sigma_2^2 \right)^{1/2}} = 0$$

$$\Rightarrow 2w_1\sigma_1^2 - 4w_1\sigma_1\sigma_2\rho + 2w_1\sigma_2^2 = 2\sigma_2^2 - 2\sigma_1\sigma_2\rho$$

$$\Rightarrow w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

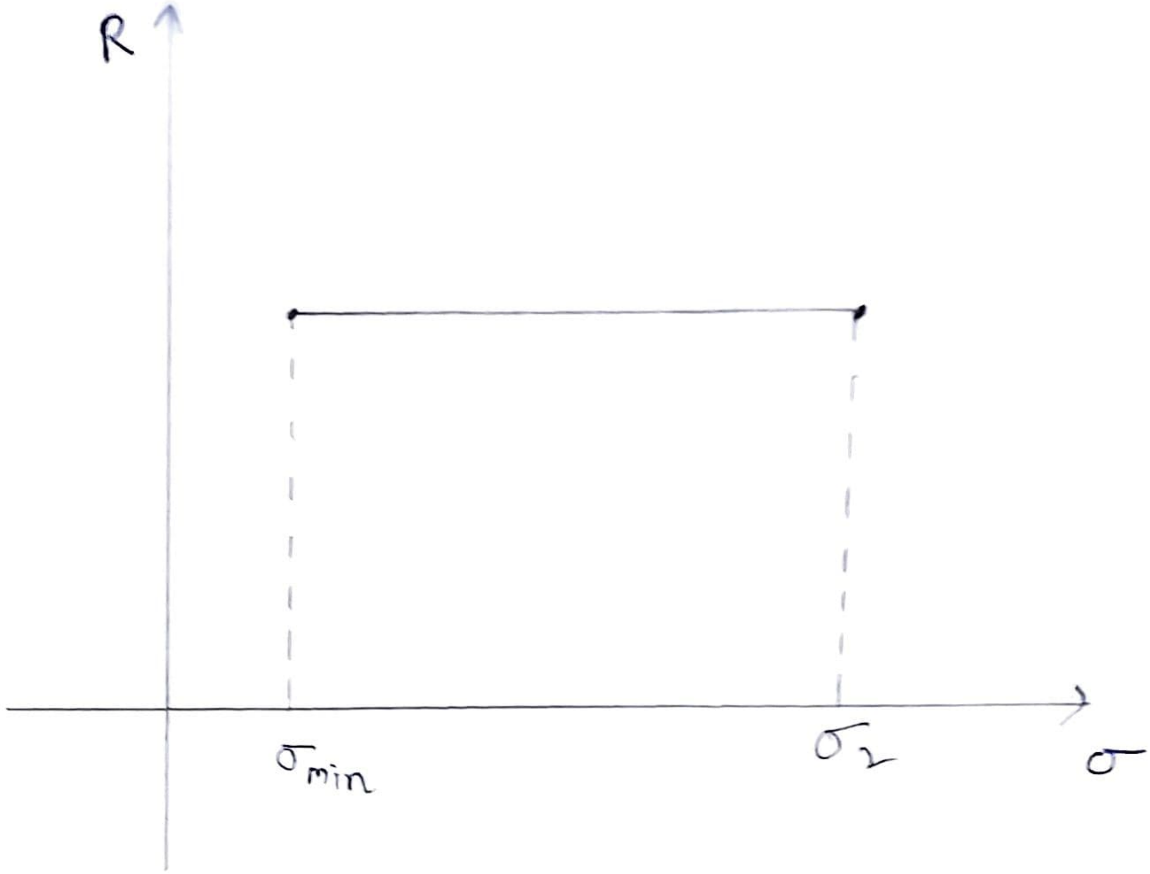
$$\left(\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} \right)$$

$$\text{111}^{\text{ly}} \quad w_2 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

This value substituted in eq^n (2) will give σ_{min}

And the horizontal line will go from σ_{min} to σ_2

(where $\sigma_2 > \sigma_1$)



$$\sigma_{min} = \left(w_1^2 \sigma_1^2 + 2 w_1 w_2 \sigma_{12} + w_2^2 \sigma_2^2 \right)^{1/2}$$

where $$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - \sigma_{12} * 2}$$

$$w_2 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

3) CAPM (Capital asset pricing model)

The capital asset pricing model describes the relationship between systematic risk and expected return for assets.

If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset 'i' satisfies

$$\bar{r}_i - r_f = \beta_i (\bar{r}_M - r_f) \quad \text{where } \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

$\sigma_{iM} \Rightarrow$ covariance between i^{th} stock and total market

Proof:

Assume 'α' is invested in asset 'i'.

and $(1-\alpha)$ is invested in market M.

$0 \le \alpha \le 1$

Portfolio's return $= \alpha \bar{r}_i + (1-\alpha)\bar{r}_M = \bar{r}_\alpha$

Portfolio's risk $= \left[\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right]^{\frac{1}{2}} = \sigma_\alpha$

$$\frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$$

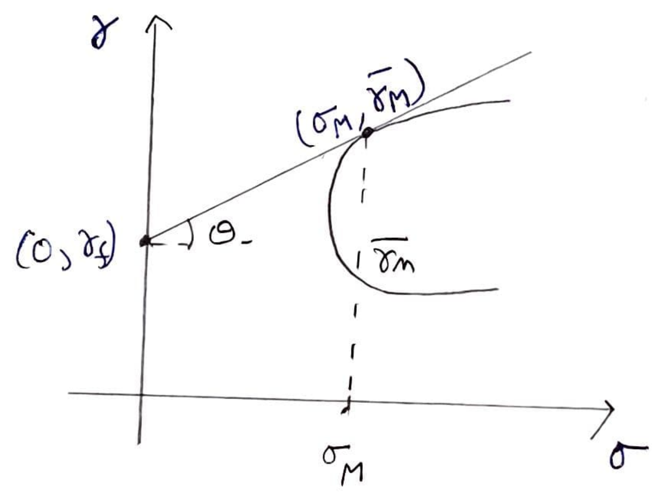
$$\cancel{\frac{d\sigma_\alpha}{d\alpha}} = \cancel{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)^2\sigma_M^2}$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{(2\alpha\sigma_i^2 + 2\sigma_{iM} - 4\alpha\sigma_{iM} + 2\alpha\sigma_M^2 - 2\sigma_M^2)}{2(\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha^2)\sigma_M^2)^{1/2}}$$

$$\Rightarrow \left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{2\sigma_{iM} - 2\sigma_M^2}{2\sigma_M} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$$

$$\frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{d\bar{r}_\alpha/d\alpha}{d\sigma_\alpha/d\alpha} = \frac{(\bar{r}_i - \bar{r}_M)}{\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

Capital Market line (M)



When the curves touch each other, their slopes will be equal

ie) Equation of capital market line ;

$$(\bar{r} - r_f) = \frac{\bar{r}_M - r_f}{\sigma_M}(\sigma - 0)$$

$$\Rightarrow \bar{r} = \left(\frac{\bar{r}_M - r_f}{\sigma_M}\right)\sigma + r_f \Rightarrow \text{Slope} = \frac{\bar{r}_M - r_f}{\sigma_M} \quad -(1)$$

slope of curve $= \dfrac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{(\sigma_{iM} - \sigma_M^2)}$ — (2)

$$\frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{(\sigma_{iM} - \sigma_M^2)} = \frac{\bar{r}_M - r_f}{\sigma_M} \qquad \text{(Equating (1) (2))}$$

$$\Rightarrow (\bar{r}_i - \bar{r}_M)\,\sigma_M^2 = (\bar{r}_M - r_f)(\sigma_{iM} - \sigma_M^2)$$

$$\Rightarrow \bar{r}_i\,\sigma_M^2 - \cancel{\bar{r}_M\,\sigma_M^2} = \bar{r}_M\,\sigma_{iM} - \cancel{\sigma_M^2\,\bar{r}_M} - r_f\,\sigma_{iM} + r_f\,\sigma_M^2$$

$$\Rightarrow (\bar{r}_i - r_f)\,\sigma_M^2 = \sigma_{iM}(\bar{r}_M - r_f)$$

$$\Rightarrow \bar{r}_i - r_f = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f)$$

$$\Rightarrow \boxed{\bar{r}_i - r_f = \beta_i\,(\bar{r}_M - r_f)}$$

Excess return for stock i

Risk profile of stock i

Excess return of Market

$$= \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Hence proved.

$$\beta \begin{cases} <1, & \text{lesser return at lower risk} \\ =1, & \text{standard market condition} \\ >1, & \text{higher return at higher risk} \end{cases}$$

By definition of Jensen's Index:

$$r_i - r_f = \beta(r_M - r_f) + J$$

$J = 0$ as per CAPM but in practical cases J may not be 0.

In $J > 0$, stock is said to perform better than the market
$J < 0$, stock is said to perform ~~bett~~ worse than market

From CAPM formula,

$$r_i = r_f + \beta_i(r_m - r_f) + \varepsilon_i \quad \text{(1)}$$

ε_i ← random idiosyncratic error

Assumptions $E(\varepsilon_i) = 0$; $\operatorname{Cov}(\varepsilon_i, \sigma_M) = 0$

Then from (1), $E(r_i) = E(r_f) + \beta_i[E(r_m) - E(r_f)] + \cancel{E(\varepsilon_i)}^{\,0}$

$\Rightarrow$ ~~E(r_f = E(r_f))~~ and $\sigma_i^2 = \beta_i^2 \sigma_M^2 + \operatorname{var}(\varepsilon_i)$

$\{\because \operatorname{var}(r_f) = 0$ ← risk free $\}$

$$\Rightarrow E(r_i - E(r_i))^2 = E[\beta_i(r_m - E(r_m))]^2 + \operatorname{var}(\varepsilon_i)$$

$$\therefore \boxed{\sigma_i^2 = \underbrace{\beta_i \sigma_M^2}_{\text{Systematic risk}} + \underbrace{\operatorname{var}(\varepsilon_i)}_{\text{Non-systematic / Idiosyncratic risk}}}$$

Systematic risk (Cannot be reduced)

Non-systematic / Idiosyncratic risk (Uncorrelated with market)

Idiosyncratic risk

It refers to the inherent factors that can negatively impact individual securities as a very specific group of assets.

Eg:- Company management's decision on financial policy, Investment strategy and operations are all idiosyncratic risk specific to a particular company stock

Market Risk

It refers to the possibility of an investor experiencing losses due to factors that affect the entirety of the financial market and its overall performance, in which the investor is involved. It is also known as 'systematic risk' and is the opposite of Idiosyncratic risk. It canot be reduced even by diversification

Eg:- The great recession of 2008