Derivation of Itô's Lemma

If x (underlying asset or stock) changes by small unit Δx then

$$\Delta a \approx \frac{da}{dx}\Delta x \quad \text{(from ordinary calculus)}$$

A Precise expression (From Taylor's expansion)

$$\Delta a = \frac{da}{dx}\Delta x + \frac{1}{2}\frac{d^2a}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3a}{dx^3}\Delta x^3 + \cdots \longrightarrow (1)$$

$$\Delta a = \frac{\partial a}{\partial x}\Delta x + \frac{\partial a}{\partial y}\Delta y$$

(If a is a fn. of x and y)

Similar to (1) we can write

$$\Delta a = \frac{\partial a}{\partial x}\Delta x + \frac{\partial a}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 a}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 a}{\partial y^2}\Delta y^2 + \frac{\partial^2 a}{\partial x \partial x}\Delta x \Delta y + \cdots$$

For small values of Δx and Δy,

$$\Delta a = \frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy.$$

Now if we want to assume,

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

and y is nothing but time then,

$$\Delta a = \frac{\partial a}{\partial x}\Delta x + \frac{\partial a}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 a}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 a}{\partial t^2}\Delta t^2 + \frac{\partial^2 a}{\partial x \partial t}\Delta x \Delta t$$

$\longrightarrow$ (3)

Itô Process (named after mathematician K. Itô)

This is a generalized Wiener process where 'a' and 'b' parameters are not constant but they are dependent on time and underlying asset's value.

Algebraically,

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

$$\Delta x = a(x,t)\,\Delta t + b(x,t)\,\varepsilon\sqrt{\Delta t}$$

(Small interval of time Δt)

$$\left\{\begin{array}{l} a(x,t) \rightarrow \text{drift term} \\ b(x,t) \rightarrow \text{variance term} \end{array}\right.$$

Itô's Lemma :

Stock price behaves stochastically and its changes depends on time. Hence any option's price depends on time and the underlying stock.

Behaviour of stochastic process for an option was proposed by → (K Itô)

According to Itô's Lemma, $\underset{\text{option price}}{G}$, a function of $\underset{\text{stock price}}{x}$ and $\underset{\text{time}}{t}$ will follow the dynamic →

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

where ~~dx~~ x follows a Wiener process.

Cont...

Previously, we saw →

$\Delta x = a(x,t)\,\Delta t + b(x,t)\,\Delta z$

$\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$ (a, b are fn of x, t)

$\Delta x^2 = a^2(\Delta t)^2 + 2ab\,\varepsilon\,\Delta t^{3/2} + b^2\varepsilon^2\Delta t$

Ignoring higher term of Δt ($\because \Delta t \to 0$ very small)

$\Delta x^2 = b^2\varepsilon^2\Delta t$ + negligible terms

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2 dt$$

From 3

variance$(\varepsilon) = E[\varepsilon - E(\varepsilon)]^2$

$1 = E(\varepsilon^2) - (E(\varepsilon))^2$ (standardized normal distb.)

$E(\varepsilon^2) = 1$ $[\because E(\varepsilon) = 0]$

variance = constant

$\therefore E(\varepsilon^2) = 1$ is also constant

It happens when $\varepsilon^2 = 1$

So $dh = \frac{\partial h}{\partial x}(a\,dt + b\,dz) + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2 dt$

$$= \underbrace{\left(\frac{\partial h}{\partial x}a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)}_{\text{New drift}}dt + \underbrace{\frac{\partial h}{\partial x}b}_{\text{New variance factor.}}dz$$