Indian Institute of Technology, Delhi

Fall 2023

COD892
MTP-1 : Wrinkle Homogenization

Discussion Guided by

Prof. Rahul Narain

Students:

Siddhesh Kalekar
(2019CS50436)

Sriram Varma
(2019CS50426)

Topic 1

Aug 1 - Aug 31 : Equations for Energy based iterative simulation of Rope

Approach

$x : R \to R$ where x (curve equation) is parameterised over arc-length s.
Let the arc-length s changes over 0 to l along the curve. And the discrete points under consideration be $x_0 = x(s_0 = 0)$, $x_1 = x(s_1)$, ... $x_n = x(s_n = l)$. Hence, $s_i = i * l/n$.

Unit tangent vector $\hat{T}$:
Curvature k calculations

$$k(s) = \left| \frac{d(\frac{d\mathbf{x}}{ds})}{ds} \right| = \left| \frac{d^2\mathbf{x}}{ds^2} \right|$$

Structural spring energy calculations : We have spring force $F_i = k_s(|x_i - x_{i-1}| - l)\hat{\mathbf{T}}$. Where l is curve length between points x_i and x_{i-1}. Thus, stress $\sigma(s)$ and Energy per unity length $e(s)$can be written as

$$\sigma(s) = \left| \lim_{l \to 0} \frac{k_s(|x_i - x_{i-1}| - l)\hat{\mathbf{T}}}{l} \right| = \left| k_s \left(\left| \frac{dx}{ds} \right| - 1 \right) \hat{\mathbf{T}} \right| = k_s \left(\left| \frac{dx}{ds} \right| - 1 \right)$$

$$\frac{dU}{ds} = e(s) = \frac{\sigma^2}{2k} = \frac{k_s}{2} \left(\left| \frac{dx}{ds} \right| - 1 \right)^2$$

Bending energy calculations : (Couldn't find proof but got the gist)

$$\frac{dU}{ds} = e(s) = \frac{k_b}{2} \left| \frac{d^2x}{ds^2} \right|^2$$

Total energy calculation U and here $\delta = l/n$

$$U = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \left(\frac{k_s}{2} \left(\left| \frac{\vec{x}_i - \vec{x}_{i-1}}{\delta} \right| - 1 \right)^2 + \frac{k_b}{2} \left| \frac{\vec{x}_{i+1} + \vec{x}_{i-1} - 2\vec{x}_i}{\delta^2} \right|^2 \right) \delta$$

Force calculations (F_i here is internal force):

$$\vec{F}_i = -\frac{\delta U}{\delta x_i}$$

Here we need to find terms in U with variable x_i in them

$$\vec{F}_{i,struct} = -k_s \left((|\vec{x}_i - \vec{x}_{i-1}| - \delta) \frac{\vec{x}_i - \vec{x}_{i-1}}{|\vec{x}_i - \vec{x}_{i-1}|} + (|\vec{x}_i - \vec{x}_{i+1}| - \delta) \frac{\vec{x}_i - \vec{x}_{i+1}}{|\vec{x}_i - \vec{x}_{i+1}|} \right)$$

$$\vec{F}_{i,bend} = \frac{-k_b}{\delta^3} \left(2 * (2\vec{x}_i - \vec{x}_{i+1} - \vec{x}_{i-1}) + (\vec{x}_i + \vec{x}_{i-2} - 2\vec{x}_{i-1}) + (\vec{x}_i + \vec{x}_{i+2} - 2\vec{x}_{i+1}) \right)$$

$\vec{F}_{i,str}$ and $\vec{F}_{i,bend}$ consists of 2 and 3 terms respectively. But, depending on the edges cases some of the terms in $\vec{F}_{i,str}$ and $\vec{F}_{i,bend}$ will be ignored.

$$\vec{F}_i = \vec{F}_{i,str} + \vec{F}_{i,bend}$$

acceleration

$$\vec{a_i} = \frac{\vec{F}_i + \vec{F}_{ext}}{m_i}$$

velocity : $\vec{v}_i^k$ is velocity of ith point in kth iteration

$$\vec{v}_i^k = \vec{a}_i^k t + \vec{v}_i^{k-1}$$

position : $\vec{x}_i^k$ is position of ith point in kth iteration

$$\vec{x}_i^k = \vec{v}_i^k t + \vec{x}_i^{k-1}$$

$\blacksquare$

Topic 2

Sept 5 - Present : Position, Amplitude and Phase variation mechanism for simulation

Approach

Equation of rope looks like

$$x(s,t) = sx_n(t) + (1-s)x_0(t) + (sa_n(t) + (1-s)a_0(t))sin(s\phi_n(t) + (1-s)\phi_0(t)) * \hat{n}$$

Potential Energy evaluation will look like,

$$\frac{dU}{ds} = \frac{k_s}{2}\left(\left|\frac{dx}{ds}\right| - 1\right)^2 + \frac{k_b}{2}\left|\frac{d^2x}{ds^2}\right|^2$$

Assuming, derivative of $x_0, x_n, a_0, a_n, \phi_0, \phi_n, \hat{n}$ w.r.t s is 0,

$$\begin{aligned}\frac{dx}{ds} &= -x_0(t) + x_1(t) + \\ &\hat{n}(-\phi_0(t) + \phi_n(t))(sa_n(t) + (1-s)a_0(t))cos(s\phi_n(t) + (1-s)\phi_0(t)) + \\ &\hat{n}(a_n(t) - a_0(t))sin(s\phi_n(t) + (1-s)\phi_0(t)) \qquad (1)\end{aligned}$$

$$(2)$$

$$\begin{aligned}\left|\frac{dx}{ds}\right|^2 &= |x_0|^2 - 2x_0.x_n + |x_n|^2 + \\ &(\phi_n - \phi_0)^2 * (sa_n + (1-s)a_0)^2 cos^2(s\phi_n + (1-s)\phi_0) + \\ &(a_n - a_0)^2 sin^2(s\phi_n + (1-s)\phi_0)\end{aligned} \qquad (3)$$

$$\left|\frac{d^2x}{ds^2}\right|^2 = (2(a_n - a_0) * (\phi_n - \phi_0) * cos(s\phi_n + (1-s)\phi_0) - (\phi_n - \phi_0)^2 * (sa_n + (1-s)a_0) * sin(s\phi_n + (1-s)\phi_0))^2$$

$$U = \int_0^1 \frac{k_s}{2}\left(\left|\frac{dx}{ds}\right| - 1\right)^2 ds + \int_0^1 \frac{k_b}{2}\left|\frac{d^2x}{ds^2}\right|^2 ds$$

Evaluating integral $\int |\frac{dx}{ds}|^2 ds$, from this point $lerp(a) = lerp(a_0, a_n, s)$ and $\Delta a = a_n - a_0$

$$\begin{aligned}\int \left|\frac{dx}{ds}\right|^2 ds &= (|x_0|^2 - 2x_0.x_n + |x_n|^2)s + \\ &\Delta^2\phi\Bigg\{lerp^2(a)\left(\frac{s}{2} + \frac{sin(lerp(2\phi))}{4\Delta\phi}\right) + s^2\Delta a\left(\frac{s\Delta a}{3} + \frac{a_0}{2}\right) + \\ &cos(lerp(2\phi))\left(\frac{a_0\Delta a}{4\Delta^2\phi} + \frac{\Delta^2 a}{4\Delta^2\phi}\right) - sin(lerp(2\phi))\frac{\Delta^2 a}{8\Delta^3\phi}\Bigg\} + \\ &\Delta^2 a\left(\frac{s}{2} - \frac{sin(lerp(2\phi))}{4\Delta\phi}\right)\end{aligned} \qquad (4)$$

$$
\begin{aligned}
\int \left|\frac{d^2x}{ds^2}\right|^2 ds = 4\Delta^2 a\Delta^2\phi\left(\frac{s}{2} + \frac{sin(lerp(2a))}{4\Delta a}\right) + \Delta^4\phi\left(sa_0^2 + s^2 a_0\Delta a + \frac{s^3\Delta a^2}{3}\right) \\
+ 2\Delta a\Delta^3\phi\left(\frac{a_0cos(lerp(2\phi))}{2\Delta\phi} + \frac{s\Delta acos(lerp(2\phi))}{2\Delta\phi} - \frac{\Delta asin(lerp(2\phi))}{4\Delta\phi^2}\right) \\
- \Delta^4\phi\Bigg\{\mathrm{lerp}^2(a)\left(\frac{s}{2} + \frac{\sin(\mathrm{lerp}(2\phi))}{4\Delta\phi}\right) + s^2\Delta a\left(\frac{s\Delta a}{3} + \frac{a_0}{2}\right) + \\
\cos(\mathrm{lerp}(2\phi))\left(\frac{a_0\Delta a}{4\Delta^2\phi} + \frac{\Delta^2 a}{4\Delta^2\phi}\right) - \sin(\mathrm{lerp}(2\phi))\frac{\Delta^2 a}{8\Delta^3\phi}\Bigg\}
\end{aligned}
\tag{5}
$$

If $|\frac{dx}{ds}| > 1$ always then $\int_0^1 |\frac{dx}{ds}|^2 ds$?? $1 + \int_0^1 |(\frac{dx}{ds}| - 1)^2 ds$ ■

Topic 3

Oct 1 - Present : Lagrangian mechanics:Potential Energy

Approach

Equation of rope looks like

$$x(s,t) = sx_1(t) + (1-s)x_0(t) + (sa_1(t) + (1-s)a_0(t))sin(s\phi_1(t) + (1-s)\phi_0(t)) * \hat{n}$$

$\Delta\phi = \phi_1 - \phi_0$ and $\phi_s = lerp(\phi_0, \phi_1, s)$.Similarly for a:

$$\frac{dx}{ds} = x_1 - x_0 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))) * \hat{n}$$

$$\left|\frac{dx}{ds}\right| = \sqrt{|x_1 - x_0|^2 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2}$$

$$\left|\frac{dx}{ds}\right| \approx |x_1 - x_0| + \frac{((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2}{2|x_1 - x_0|}$$

$$\left|\frac{dx}{ds}\right|^2 = |x_1 - x_0|^2 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2$$

$$\left|\frac{d^2x}{ds^2}\right| = 2(\Delta a)(\Delta\phi)cos(\phi_s) - (\Delta\phi)^2(a_s)sin(\phi_s)$$

$$\left|\frac{d^2x}{ds^2}\right|^2 = 4(\Delta a)^2(\Delta\phi)^2(cos^2(\phi_s)) + (\Delta\phi)^4(a_s)^2 sin^2(\phi_s) - 2*(\Delta a)(\Delta\phi)^3 sin(2\phi_s)$$

Potential Energy evaluation is,

$$\frac{dU}{ds} = \frac{k_s}{2}\left(\left|\frac{dx}{ds}\right| - 1\right)^2 + \frac{k_b}{2}\left|\frac{d^2x}{ds^2}\right|^2$$

$$U = \int_0^1 \frac{k_s}{2}\left(\left|\frac{dx}{ds}\right| - 1\right)^2 ds + \int_0^1 \frac{k_b}{2}\left|\frac{d^2x}{ds^2}\right|^2 ds$$

$$U = \int_0^1 \frac{k_s}{2}\left(\left|\frac{dx}{ds}\right|^2 - 2*\left|\frac{dx}{ds}\right| + 1\right)ds + \int_0^1 \frac{k_b}{2}\left|\frac{d^2x}{ds^2}\right|^2 ds$$

$U = \int_0^1 \frac{k_s}{2}(|x_1 - x_0|^2 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2 - 2*(|x_1 - x_0| + \frac{((\Delta\phi)(a_s)cos(\phi_s)+(\Delta a)sin(\phi_s))^2}{2|x_1 - x_0|} + 1)ds + \int_0^1 \frac{k_b}{2} 4(\Delta a)^2(\Delta\phi)^2(cos^2(\phi_s)) + (\Delta\phi)^4(a_s)^2 sin^2(\phi_s) - 2*(\Delta a)(\Delta\phi)^3 sin(2\phi_s)ds$

$U = \int_0^1 \frac{k_s}{2}\left((|x_1 - x_0| - 1)^2 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2 - \frac{((\Delta\phi)(a_s)cos(\phi_s)+(\Delta a)sin(\phi_s))^2}{|x_1 - x_0|}\right)ds + \int_0^1 \frac{k_b}{2}(4(\Delta a)^2(\Delta\phi)^2(cos^2(\phi_s)) + (\Delta\phi)^4(a_s)^2 sin^2(\phi_s) - 2(\Delta a)(\Delta\phi)^3 sin(2\phi_s))ds$

$U = \int_0^1 \frac{k_s}{2}\left((|x_1 - x_0| - 1)^2 + ((\Delta\phi)(a_s)cos(\phi_s) + (\Delta a)sin(\phi_s))^2(1 - \frac{1}{|x_1 - x_0|})\right)ds + \int_0^1 \frac{k_b}{2}(4(\Delta a)^2(\Delta\phi)^2(cos^2(\phi_s)) + (\Delta\phi)^4(a_s)^2 sin^2(\phi_s) - 2(\Delta a)(\Delta\phi)^3 sin(2\phi_s))ds$

$U = \int_0^1 \frac{k_s}{2}\Bigg((|x_1 - x_0| - 1)^2 + ((\Delta\phi)^2(a_s)^2 cos^2(\phi_s) + (\Delta a)^2 sin^2(\phi_s) + (\Delta a)(\Delta\phi)a_s sin(2\phi_s))(1 - \frac{1}{|x_1-x_0|})\Bigg)ds + \int_0^1 \frac{k_b}{2}(4(\Delta a)^2(\Delta\phi)^2(cos^2(\phi_s)) + (\Delta\phi)^4(a_s)^2 sin^2(\phi_s) - 2(\Delta a)(\Delta\phi)^3 sin(2\phi_s))ds$

$U = \int_0^1 \frac{k_s}{2}((|x_1-x_0|-1)^2 + (\frac{(\Delta\phi)^2(a_s)^2}{2} + \frac{(\Delta a)^2}{2})(1-\frac{1}{|x_1-x_0|}))ds + \int_0^1 \frac{k_b}{2}(2(\Delta a)^2(\Delta\phi)^2) + \frac{(\Delta\phi)^4(a_s)^2}{2})ds$

$U = \frac{k_s}{2}((|x_1-x_0|-1)^2 + (\frac{(\Delta\phi)^2(a_0^2+a_0a_1+a_1^2)}{6} + \frac{(\Delta a)^2}{2})(1-\frac{1}{|x_1-x_0|})) + \frac{k_b}{2}((2(\Delta a)^2(\Delta\phi)^2) + \frac{(\Delta\phi)^4(a_0^2+a_0a_1+a_1^2)}{6}))$

[Rahul: Looks OK. Can you collect the two $(\Delta\phi)^2.../6$ terms together? Actually I'm a bit surprised that the second term is there. If $\|x_1 - x_0\| < 1$ (the rope is compressed) then I would expect the terms involving $(\Delta\phi)2a^2$ to always have a negative sign (decrease the energy by buckling).]

■

Topic 3

Oct 1 - Present : Lagrangian mechanics:Kinetic Energy

Approach

Equation of rope looks like

$$x(s,t) = sx_1(t) + (1-s)x_0(t) + (sa_1(t) + (1-s)a_0(t))sin(s\phi_1(t) + (1-s)\phi_0(t)) * \hat{n}$$

$\Delta\phi = \phi_1 - \phi_0$ and $\phi_s = lerp(\phi_0, \phi_1, s)$.Similarly for a,x:

$$\frac{dx}{dt} = \dot{X} = \dot{x}_s + ((\Delta\dot{\phi}_s)(a_s)cos(\phi_s) + (\dot{a_s})sin(\phi_s)) * \hat{n}$$

$$|\dot{X}|^2 = |\dot{x_s}|^2 + ((\Delta\dot{\phi}_s)(a_s)cos(\phi_s) + (\dot{a_s})sin(\phi_s))^2$$

Kinetic Energy evaluation is,

$$T = \int_0^1 \rho|\dot{X}|^2 ds$$

$$T = \int_0^1 \rho(|\dot{x_s}|^2 + ((\Delta\dot{\phi}_s)(a_s)cos(\phi_s) + (\dot{a_s})sin(\phi_s))^2)ds$$

$$T = \int_0^1 \rho(|\dot{x_s}|^2 + (\Delta\dot{\phi}_s)^2(a_s)^2cos^2(\phi_s) + (\dot{a_s})^2sin^2(\phi_s) + (\Delta\dot{\phi}_s)(a_s)(\dot{a_s})sin(2\phi_s))ds$$

$$T = \int_0^1 \rho(|\dot{x_s}|^2 + \frac{(\Delta\dot{\phi}_s)^2(a_s)^2}{2} + \frac{(\dot{a_s})^2}{2} + (\Delta\dot{\phi}_s)(a_s)(\dot{a_s})sin(2\phi_s))ds$$

$T = \rho(\frac{|\dot{x_0}|^2+\dot{x_0}.\dot{x_1}+|\dot{x_0}|^2}{3} + \frac{a_0{}^2(6\dot{\phi_0}^2+3\dot{\phi_0}\dot{\phi_1}+\dot{\phi_1}^2)+a_0a_1(3\dot{\phi_0}^2+4\dot{\phi_0}\dot{\phi_1}+3\dot{\phi_1}^2)+a_1{}^2(\dot{\phi_0}^2+3\dot{\phi_0}\dot{\phi_1}+6\dot{\phi_1}^2)}{60} + \frac{\dot{a_0}^2+\dot{a_0}\dot{a_1}+\dot{a_1}^2}{6})$
[Rahul: Looks OK. Probably we can make some approximations here, after all even in the absence of the buckling model we usually approximate $(\|\dot{x}_0\|^2 + \dot{x}_0 \cdot \dot{x}_1 + \|\dot{x}_1\|^2)/3$ by $(\|\dot{x}_0\|^2 + \|\dot{x}_1\|^2)/2$.]

■