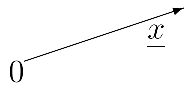
Lecture 2: Equations of Points, Lines and Planes *RHB 6.7, 6.8*

2. 1. Position vectors:

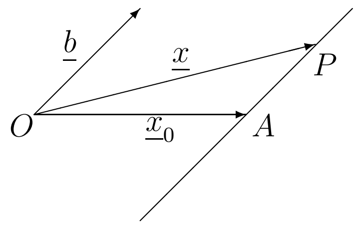
These are vectors bound to some origin and give position of a point relative to that origin. They are often denoted $\underline{x}$ or $\underline{r}$.



Equation for a point is simply $\underline{x} = \underline{a}$ where $\underline{a}$ is some vector

2. 2. The Equation of a Line

Suppose that P lies on a line which passes through a point A, so that $\underline{OA} = \underline{x}_0$, in the direction parallel to the vector $\underline{b}$.



$\underline{x}$ is the position vector of an arbitrary point P on the line
$\underline{x}_0$ is the position vector of a fixed point on the line
$\underline{b}$ is parallel to the line and passes through the origin.

We may write

$$\underline{x} = \underline{x}_0 + \lambda \underline{b}$$

which is the **parametric equation of the line** *i.e.* as we vary the parameter λ from $-\infty$ to ∞, $\underline{x}$ passes through all points on the line.

Rearranging and using $\underline{b} \times \underline{b} = 0$, we can also write this as:–

$$(\underline{x} - \underline{x}_0) \times \underline{b} \ = \ 0$$

or

$$\boxed{\underline{x} \times \underline{b} = \underline{c}}$$

where $\underline{c} = \underline{x_0} \times \underline{b}$ is normal to the plane containing the line and origin.

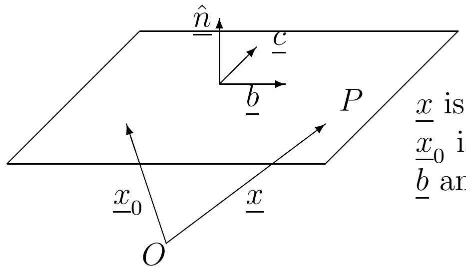
Physical example: If angular momentum $\underline{L}$ of a particle and its velocity $\underline{v}$ are known, we still don't know the position exactly because the solution of $\underline{L} = m\underline{r} \times \underline{v}$ is a line $\underline{r} = \underline{r}_0 + \lambda \underline{v}$.

Notes

(*i*) $\underline{x} \times \underline{b} = \underline{c}$ is an implicit equation for a line

(*ii*) $\underline{x} \times \underline{b} = 0$ is the equation of a line through the origin.

2. 3. The Equation of a Plane



$\underline{x}$ is the position vector of an arbitrary point P on the plane
$\underline{x}_0$ is the position vector of a fixed point in the plane
$\underline{b}$ and $\underline{c}$ are parallel to the plane but $\underline{b} \times \underline{c} \neq 0$.

Clearly

$$\underline{x} \;=\; \underline{x}_0 + \lambda \underline{b} + \mu \underline{c}$$

for some λ and μ which is the **parametric equation of the plane**.

We define the unit normal to the plane

$$\hat{\underline{n}} \;=\; \frac{\underline{b} \times \underline{c}}{|\underline{b} \times \underline{c}|} \;.$$

Since $\underline{b} \cdot \hat{\underline{n}} = \underline{c} \cdot \hat{\underline{n}} = 0$, we have the implicit equation:–

$$(\underline{x} - \underline{x}_0) \cdot \hat{\underline{n}} \;=\; 0 \;.$$

Alternatively, we can write this as:–

$$\boxed{\underline{x} \cdot \hat{\underline{n}} \;=\; p} \;,$$

where $p = \underline{x}_0 \cdot \hat{\underline{n}}$ is the perpendicular distance of the plane from the origin.
This is a very important equation which you must be able to recognise.
Note: $\underline{x} \cdot \underline{a} \;=\; 0$ is the equation for a plane through the origin (with unit normal $\underline{a}/|\underline{a}|$).

2. 4. Examples of Dealing with Vector Equations

Before going through some worked examples let us state two simple rules which will help you to avoid many common mistakes

1. **Always** check that the quantities on both sides of an equation are of the same type. e.g. any equation of the form *vector* = *scalar* is clearly wrong. (The only exception to this is if we lazily write *vector* = 0 when we mean $\underline{0}$.)

2. **Never** try to divide by a vector- there is no such operation!

Example 1: Is the following set of equations consistent?

$$\underline{x} \times \underline{b} = \underline{c} \qquad (1)$$

$$\underline{x} = \underline{a} \times \underline{c} \qquad (2)$$

Geometrical interpretation – The first equation is the (implicit) equation for a line whereas the second equation is the (explicit) equation for a point. Thus the question is whether the point is on the line.
If we insert (2) into the l.h.s. of (1) we find

$$\underline{x} \times \underline{b} = (\underline{a} \times \underline{c}) \times \underline{b} = -\underline{b} \times (\underline{a} \times \underline{c}) = -\underline{a}\,(\underline{b} \cdot \underline{c}) + \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{3}$$

Now from (1) we have that $\underline{b} \cdot \underline{c} = \underline{b} \cdot (\underline{x} \times \underline{b}) = 0$ thus (3) becomes

$$\underline{x} \times \underline{b} = \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{4}$$

so that, on comparing (1) and (4), we require

$$\underline{a} \cdot \underline{b} = 1$$

for the equations to be consistent.

Example 2
Solve the following set of equations for $\underline{x}$.

$$\underline{x} \times \underline{a} = \underline{b} \tag{5}$$
$$\underline{x} \times \underline{c} = \underline{d} \tag{6}$$

Geometrical interpretation – Both equations are equations for lines *e.g.* (5) is for a line parallel to $\underline{a}$ where $\underline{b}$ is normal to the plane containing the line and the origin. The problem is to find the intersection of two lines. (Here we assume the equations are consistent and the lines do indeed have an intersection).
Consider

$$\underline{b} \times \underline{d} = (\underline{x} \times \underline{a}) \times \underline{d} = -\underline{d} \times (\underline{x} \times \underline{a}) = -\underline{x}\,(\underline{a} \cdot \underline{d}) + \underline{a}\,(\underline{d} \cdot \underline{x})$$

which is obtained by taking the vector product of l.h.s of (5) with $\underline{d}$.
Now from (6) we see that $\underline{d} \cdot \underline{x} = \underline{x} \cdot (\underline{x} \times \underline{c}) = 0$. Thus

$$\underline{x} = -\frac{\underline{b} \times \underline{d}}{\underline{a} \cdot \underline{d}} \quad \text{for} \quad \underline{a} \cdot \underline{d} \neq 0\ .$$

Alternatively we could have taken the vector product of the l.h.s. of (6) with $\underline{b}$ to find

$$\underline{b} \times \underline{d} = \underline{b} \times (\underline{x} \times \underline{c}) = \underline{x}\,(\underline{b} \cdot \underline{c}) - \underline{c}\,(\underline{b} \cdot \underline{x})\ .$$

Since $\underline{b} \cdot \underline{x} = 0$ we find

$$\underline{x} = \frac{\underline{b} \times \underline{d}}{\underline{b} \cdot \underline{c}} \quad \text{for} \quad \underline{b} \cdot \underline{c} \neq 0\ .$$

It can be checked from (5) and (6) and the properties of the scalar triple product that for the equations to be consistent $\underline{b} \cdot \underline{c} = -\underline{d} \cdot \underline{a}$. Hence the two expressions derived for $\underline{x}$ are the same.

Case $\underline{a} \cdot \underline{d} = \underline{b} \cdot \underline{c} = 0$

In this case the above approach does not give an expression for $\underline{x}$. However from (5) we see $\underline{a} \cdot \underline{d} = 0$ implies that $\underline{a} \cdot (\underline{x} \times \underline{c}) = 0$ so that $\underline{a}$, $\underline{c}$, $\underline{x}$ are coplanar. We can therefore write $\underline{x}$ as a linear combination of $\underline{a}, \underline{c}$:

$$\underline{x} = \alpha\, \underline{a} + \gamma\, \underline{c}\,. \tag{7}$$

To determine the scalar α we can take the vector product with $\underline{c}$ to find

$$\underline{d} = \alpha\, \underline{a} \times \underline{c} \tag{8}$$

(since $\underline{x} \times \underline{c} = \underline{d}$ from (6) and $\underline{c} \times \underline{c} = 0$). In order to extract α we need to convert the vectors in (8) into scalars. We do this by taking, for example, a scalar product with $\underline{b}$

$$\underline{b} \cdot \underline{d} = \alpha\, \underline{b} \cdot (\underline{a} \times \underline{c})$$

so that

$$\alpha = -\frac{\underline{b} \cdot \underline{d}}{(\underline{a}\,,\ \underline{b}\,,\ \underline{c})}\,.$$

Similarly, one can determine γ by taking the vector product of (7) with $\underline{a}$:

$$\underline{b} = \gamma\, \underline{c} \times \underline{a}$$

then taking a scalar product with $\underline{b}$ to obtain finally

$$\gamma = \frac{\underline{b} \cdot \underline{b}}{(\underline{a}\,,\ \underline{b}\,,\ \underline{c})}\,.$$

Example 3

Solve for $\underline{x}$ the vector equation

$$\underline{x} + (\underline{n} \cdot \underline{x})\, \underline{n} + 2\underline{n} \times \underline{x} + 2\underline{b} = 0 \tag{9}$$

where $\underline{n} \cdot \underline{n} = 1$.

In order to unravel this equation one can try taking scalar and vector products of the equation with the vectors involved. However straight away one sees that taking various products with $\underline{x}$ will not help, since it will produce terms that are quadratic in $\underline{x}$. Instead, we want to eliminate $(\underline{n} \cdot \underline{x})$ and $\underline{n} \times \underline{x}$ thus we try taking scalar and vector products with $\underline{n}$. Taking the scalar product one finds

$$\underline{n} \cdot \underline{x} + (\underline{n} \cdot \underline{x})(\underline{n} \cdot \underline{n}) + 0 + 2\underline{n} \cdot \underline{b} = 0$$

so that, since $(\underline{n} \cdot \underline{n}) = 1$, one has

$$\underline{n} \cdot \underline{x} = -\underline{n} \cdot \underline{b} \tag{10}$$

Taking the vector product of (9) with $\underline{n}$ gives

$$\underline{n} \times \underline{x} + 0 + 2\,[\, \underline{n}(\underline{n} \cdot \underline{x}) - \underline{x}\,] + 2\underline{n} \times \underline{b} = 0$$

so that

$$\underline{n} \times \underline{x} = 2\,[\, \underline{n}(\underline{b} \cdot \underline{n}) + \underline{x}\,] - 2\underline{n} \times \underline{b} \tag{11}$$

where we have used (10). Substituting (10) and (11) into (9) one eventually obtains

$$\underline{x} = \frac{1}{5}\,[\, -3(\underline{b} \cdot \underline{n})\, \underline{n} + 4(\underline{n} \times \underline{b}) - 2\underline{b}\,] \tag{12}$$