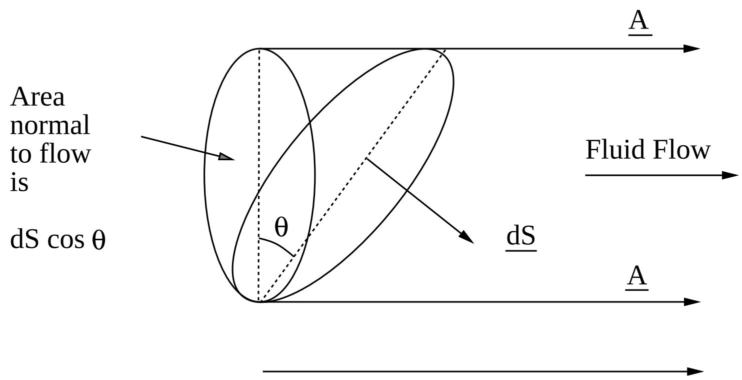
Lecture 19: More on Surface and Volume Integrals *(RHB 9.5,9.6)*

19. 1. The Concept of Flux (RHB 9.5.3)

Let $\underline{A}(\underline{r})$ be the velocity at a point $\underline{r}$ in a moving fluid.



In a region where $\underline{A}$ is approximately constant, the **volume** of fluid crossing the element of vector area $\underline{dS} = \underline{n}\, dS$ in time dt is

$$\left(\left|\underline{A}\right|\, dt\right)\ (dS\ \cos\theta)\ =\ \left(\underline{A}\cdot\underline{dS}\right)\ dt$$

since the area *normal* to the direction of flow is $\hat{\underline{A}}\cdot\underline{dS} = dS\ \cos\theta$. Therefore

$$\begin{aligned} \underline{A}\cdot\underline{dS} &= \textbf{volume per unit time} \text{ of fluid crossing } \underline{dS} \\ \text{hence}\quad \int_S \underline{A}\cdot\underline{dS} &= \textbf{volume per unit time} \text{ of fluid crossing a finite surface } S \end{aligned}$$

$$\boxed{\text{The surface integral } \int_S \underline{A}\cdot\underline{dS} \text{ is called the } \textbf{flux} \text{ of } \underline{A} \text{ through the surface } S.}$$

The concept of flux is useful in may different contexts e.g. flux of molecules in an gas; electromagnetic flux etc

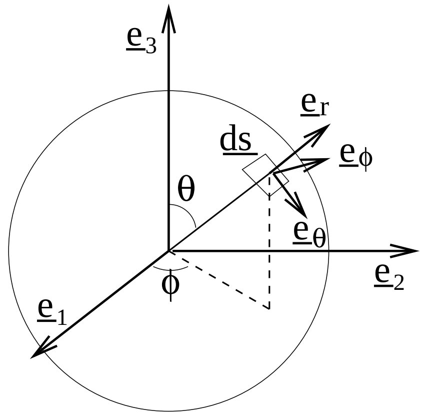
Example: Let S be the surface of sphere

$$x^2 + y^2 + z^2 = a^2$$

Find $\underline{n}$, $\underline{dS}$ and evaluate the total flux of the vector field $\underline{A} = r^{-2}\underline{\hat{r}}$ out of the sphere.

An arbitrary point $\underline{r}$ on S may be parameterised by spherical polar co-ordinates θ and ϕ

$$\begin{aligned} \underline{r} &= a\sin\theta\cos\phi\,\underline{e}_1 + a\sin\theta\sin\phi\,\underline{e}_2 + a\cos\theta\,\underline{e}_3 \qquad \{0\le\theta\le\pi,\ 0\le\phi\le 2\pi\} \\ \text{so}\qquad \frac{\partial\underline{r}}{\partial\theta} &= a\cos\theta\cos\phi\,\underline{e}_1 + a\cos\theta\sin\phi\,\underline{e}_2 - a\sin\theta\,\underline{e}_3 \\ \text{and}\qquad \frac{\partial\underline{r}}{\partial\phi} &= -a\sin\theta\sin\phi\,\underline{e}_1 + a\sin\theta\cos\phi\,\underline{e}_2 + 0\,\underline{e}_3 \end{aligned}$$



Therefore

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a\,\cos\theta\cos\phi & a\,\cos\theta\sin\phi & -a\,\sin\theta \\ -a\sin\theta\sin\phi & +a\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= a^2\,\sin^2\theta\cos\phi\,\underline{e}_1 \,+\, a^2\,\sin^2\theta\sin\phi\,\underline{e}_2 \,+\, a^2\,\sin\theta\cos\theta\,\left[\cos^2\phi + \sin^2\phi\right]\underline{e}_3 \\
&= a^2\sin\theta\,(\sin\theta\cos\phi\,\underline{e}_1 + \sin\theta\sin\phi\,\underline{e}_2 + \cos\theta\,\underline{e}_3) \\
&= a^2\sin\theta\;\hat{\underline{r}} \\
\underline{n} &= \hat{\underline{r}} \\
\underline{dS} &= \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\,d\theta\,d\phi \;=\; a^2\sin\theta d\theta\,d\phi\,\hat{\underline{r}}
\end{aligned}$$

On S

$$A(\underline{r}) \;=\; a^{-2}\hat{\underline{r}}$$

hence

$$\int_S \underline{A}\cdot\underline{dS} = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = 4\pi$$

NB The normalised vectors (shown on the figure)

$$\underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta}\bigg/\left|\frac{\partial \underline{r}}{\partial \theta}\right| \quad ; \quad \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi}\bigg/\left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad ; \quad \underline{e}_r = \hat{\underline{r}}$$

form an orthonormal set. This is the basis for spherical polar co-ordinates and is an example of a non-Cartesian basis since the $\underline{e}_\theta, \underline{e}_\phi, \underline{e}_r$ depend on position $\underline{r}$.

19. 2. Other Surface Integrals

If $f(\underline{r})$ is a scalar field, a scalar surface integral is of the form

$$\int_S f\,dS$$

For example the **surface area** of the surface S is

$$\int_S dS \;=\; \int_S \left|\underline{dS}\right| \;=\; \int_v\int_u \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right|\,du\,dv$$

We may also define vector surface integrals:-

$$\int_S f\, \underline{dS} \qquad \int_S \underline{A}\, dS \qquad \int_S \underline{A} \times \underline{dS}$$

Each of these is a double integral, and is evaluated in a similar fashion to the scalar integrals, the result being a vector in each case.

The **vector area** of a surface (see RHB 9.5.2) is defined as $\underline{S} = \int_S \underline{dS}$. For a closed surface this is always zero.

Example the vector area of an (open) hemisphere (see 16.1) of radius a is found using spherical polars to be

$$\underline{S} = \int_S \underline{dS} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} a^2 \sin\theta \underline{e}_r \, d\theta\, d\phi \, .$$

Using $\underline{e}_r = \sin\theta\cos\phi\, \underline{e}_1 + \sin\theta\sin\phi\, \underline{e}_2 + \cos\theta\, \underline{e}_3$ we obtain

$$\begin{aligned}
\underline{S} &= \underline{e}_1\, a^2 \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \cos\phi d\phi \; + \; \underline{e}_2\, a^2 \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi d\phi \\
&\quad + \underline{e}_3\, a^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \\
&= 0 + 0 + \underline{e}_3 \pi a^2
\end{aligned}$$

The vector surface of the full sphere is zero since the contributions from upper and lower hemispheres cancel; also the vector area of a *closed* hemisphere is zero since the vector area of the bottom face is $-\underline{e}_3\pi a^2$.

19. 3. Parametric form of Volume Integrals

We have already met and revised volume integrals in 16.1, Conceptually volume integrals are simpler than line and surface integrals because the elemental volume dV is a scalar quantity.

Here we discuss the *parametric* form of volume integrals. Suppose we can write $\underline{r}$ in terms of three real parameters u, v and w, so that $\underline{r} = \underline{r}(u, v, w)$. If we make a small change in each of these parameters, then $\underline{r}$ changes by

$$\underline{dr} \;=\; \frac{\partial \underline{r}}{\partial u}\, du \,+\, \frac{\partial \underline{r}}{\partial v}\, dv \,+\, \frac{\partial \underline{r}}{\partial w}\, dw$$

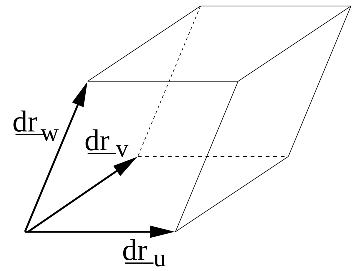
Along the curves $\{v = \text{constant},\, w = \text{constant}\}$, we have $dv = 0$ and $dw = 0$, so $\underline{dr}$ is simply:-

$$\underline{dr}_u \;=\; \frac{\partial \underline{r}}{\partial u}\, du$$

with $\underline{dr}_v$ and $\underline{dr}_w$ having analogous definitions.

The vectors $\underline{dr}_u$, $\underline{dr}_v$ and $\underline{dr}_w$ form the sides of an infinitesimal parallelepiped of volume

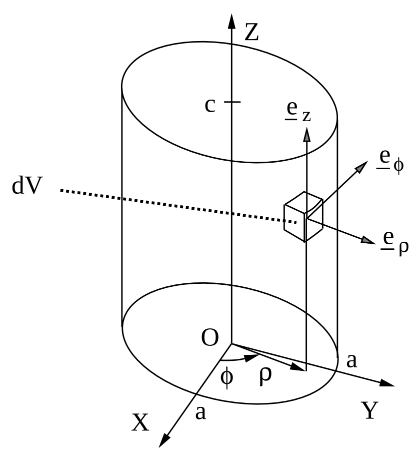
$$dV \;=\; \left| \underline{dr}_u \cdot \underline{dr}_v \times \underline{dr}_w \right|$$



$$\boxed{dV = \left|\frac{\partial \underline{r}}{\partial u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}\right| \, du\, dv\, dw}$$

Example: Consider a circular cylinder of radius a, height c. We can parameterise $\underline{r}$ using **circular cylindrical coordinates (RHB 8.9.1)**. Within the cylinder, we have

$$\begin{aligned}
\underline{r} &= \rho\cos\phi\,\underline{e}_1 + \rho\sin\phi\,\underline{e}_2 + z\underline{e}_3 \quad \{0 \le \rho \le a,\ 0 \le \phi \le 2\pi,\ 0 \le z \le c\} \\
\text{hence} \quad \frac{\partial \underline{r}}{\partial \rho} &= \cos\phi\,\underline{e}_1 + \sin\phi\,\underline{e}_2 \\
\frac{\partial \underline{r}}{\partial \phi} &= -\rho\,\sin\phi\,\underline{e}_1 + \rho\,\cos\phi\,\underline{e}_2 \\
\frac{\partial \underline{r}}{\partial z} &= \underline{e}_3 \\
\text{and so} \quad dV &= \left|\frac{\partial \underline{r}}{\partial \rho} \cdot \frac{\partial \underline{r}}{\partial \phi} \times \frac{\partial \underline{r}}{\partial z}\right| d\rho\, d\phi\, dz \;=\; \rho\, d\rho\, d\phi\, dz
\end{aligned}$$



The **volume** of the cylinder is

$$\int_V dV \;=\; \int_{z=0}^{z=c} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=a} \rho\, d\rho\, d\phi\, dz \;=\; \pi\, a^2 c.$$

NB The normalised vectors (shown on the figure) form a non-Cartesian basis where

$$\underline{e}_\rho = \frac{\partial \underline{r}}{\partial \rho} \Big/ \left|\frac{\partial \underline{r}}{\partial \rho}\right| \quad ; \quad \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi} \Big/ \left|\frac{\partial \underline{r}}{\partial \phi}\right| \quad ; \quad \underline{e}_z = \frac{\partial \underline{r}}{\partial z} \Big/ \left|\frac{\partial \underline{r}}{\partial z}\right|$$