Fourier transform of a fn $= f(x)$

$$F(S) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

Properties of F.T :

i) Linear Property

$$F\left[a f_1(x) + b f_2(x)\right] = a F_1(s) + b F_2(s)$$

W.K.T

$$F_1(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f_1(x)\, e^{isx}\, dx$$

$$F_2(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f_2(x)\, e^{isx}\, dx$$

$$F\{a f_1(x) + b f_2(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(a f_1(x) + b f_2(x)\right) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[a \underbrace{\int_{-\infty}^{\infty} f_1(x)\, e^{isx}\, dx}_{F_1(s)} + b \underbrace{\int_{-\infty}^{\infty} f_2(x)\, dx}_{F_2(s)} \right]$$

$$F\{a f_1(x) + b f_2(x)\} = a F_1(s) + b F_2(s) \quad //$$

2) Change of Scale property

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

W.K.T

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

$x \to \infty\,,\ t \to \infty$

$x \to -\infty\,,\ t \to -\infty$

$$ax = t$$

$$x = \frac{t}{a} \qquad dx = \frac{1}{a} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)\cdot t}\, \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) //$$

$\left[\text{because } F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \right]$

3) Shifting property

$$F\{f(x-a)\} = e^{isa} F(s)$$

W.K.T

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$t = x - a$

$x = t + a \qquad dx = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ist} dt \cdot e^{isa}$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{isa} F(s) //$$

4) $$F\{e^{iax} f(x)\} = F(s+a)$$

W.K.T $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i(s+a)x} dx$$

$$= F(s+a) //$$

5) $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

a)

W.K.T

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x) \cos ax \, dx$$

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x) \cdot \frac{e^{iax} + e^{-iax}}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right] //$$

5b)

$F\{f(x)\sin ax\} \longrightarrow \sin ax = \frac{e^{iax} - e^{-iax}}{2i}$

6) $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$

W.K.T $\quad F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$\frac{d}{ds} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot \frac{d}{ds} e^{isx} \, dx$$

$$\frac{d}{ds} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot ix \cdot e^{isx} \, dx$$

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot (ix)^n e^{isx} \, dx$$

$$= i^n \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, x^n e^{isx} \, dx$$

$$\frac{d^n}{ds^n} F(s) = i^n \; F\{x^n f(x)\}$$

$$\frac{1}{(i)^n} \frac{d^n}{ds^n} F(s) = F\{x^n f(x)\}$$

$$(-i)^n \frac{d^n}{ds^n} F(s) = F\{x^n f(x)\}$$

7) $$F\{f'(x)\} = (-is) F(s)$$

W.K.T

$$F\{f(x)\} \quad F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$$

$$F\{d f(x)\} = F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{d f(x) \, dx}_{dv} \; \underbrace{e^{isx}}_{u}$$

$$u = e^{isx} \qquad dv = d f(x) \, dx$$

$$du = is \cdot e^{isx} \, dx \qquad v = \int d f(x) \, dx = f(x)$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{e^{isx} f(x)}\Big|_{-\infty}^{\infty} - is\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-is\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \right]$$

$$F\{f'(x)\} = -is\,F(s) \quad F\{f(x)\} \;//$$

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8) $$F\left\{\int_a^x f(x)\,dx\right\} = \frac{-1}{is}\,F\{f(x)\} \;//$$

W.K.T

$$F\{f'(x)\} = F\{df(x)\} = -is\,F\{f(x)\}$$

$$f_1(x) = \int_a^x f(x)\,dx$$

$$df_1(x) = d\int_a^x f(x)\,dx$$

$$df_1(x) = f(x) \;//$$

$$F\{df_1(x)\} = -is\,F\{f_1(x)\} = -is\,F\left\{\int_a^x f(x)\,dx\right\}$$

$$F\{f(x)\} = (-is)\,F\left\{\int_a^x f(x)\,dx\right\}$$

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{-1}{is} F\{f(x)\}.$$

Similarly we can derive the properties for $F_c(s)$ and $F_s(s)$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$