Date : **28-Oct-2020**

Attendance : **8**

Curl of a vector field :

Curl is a also a vector operator.

Curl operates on a <u>vector</u>

$\downarrow$ results

Also a <u>vector</u>

Whereas for div $\rightarrow$ div. act on vector

$\downarrow$ results

<u>Scalar</u>.

Curl $\vec{F} = 0 \Leftarrow$ the vector field is irrotational.

whereas

div $\vec{F} = 0 \Leftarrow$ Solenodial

Curl $\vec{F}$ or $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

(if $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$)

//.

Example 41. *Find the divergence and curl of* $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at* $(2, -1, 1)$ *(Nagpur University, Summer 2003)*

$\operatorname{div} \vec{v} = ?$

$$\Rightarrow \vec{\nabla} \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(xyz\,\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}\right)$$

$$\Rightarrow yz + 3x^2 + 2xz - y^2$$

$$\vec{\nabla} \cdot \vec{v}\Big|_{2,-1,1} = (-1)(1) + 3(2)^2 + 2(2)(1) - (-1)^2$$

$$= -1 + 12 + 4 - 1$$

$$= 14 \;//$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xzy)\right)$$

$$= \hat{i}(0 - 2yz - 0) - \hat{j}(z^2 - 0 - xy) + \hat{k}(6xy - xz)$$

$$\vec{\nabla} \times \vec{F} = -2yz\,\hat{i} + \hat{j}(-z^2 + xy) + (6xy - xz)\hat{k}$$

$$\vec{\nabla} \times \vec{F}\Big|_{(2,-1,1)} = -2(-1)(1)\hat{i} + [-(-1)^2 + 2(-1)]\hat{j} + [6(2)(-1) - 2(1)]\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \,//$$

Example 42. *If* $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of curl* $\vec{V}$.

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \partial/\partial y & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)\frac{u}{v} = \frac{(x^2+y^2+z^2)^{1/2}\cdot 0 - z\left(\frac{1}{2}\cdot 2y\right)(x^2+y^2+z^2)^{-\frac{1}{2}}}{x^2+y^2+z^2}$$

$$\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\frac{1}{2}-1=?$$

$$-\frac{1}{2}$$

$$= \frac{-zy(x^2+y^2+z^2)^{-\frac{1}{2}}}{x^2+y^2+z^2}$$

$$= \frac{-zy}{(x^2+y^2+z^2)\cdot(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{-zy}{(x^2+y^2+z^2)^{\frac{1}{2}+1}}$$

$$= \frac{-zy}{(x^2+y^2+z^2)^{3/2}}$$

Contine remain steps

Example 43. *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.* (U.P., I Sem, Dec. 2008)

Let
$$\vec{v} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$

$$\vec{\nabla}\cdot\vec{v} = 0 \;; \quad \vec{\nabla}\times\vec{v} = 0$$

Do this a H.w.

Example 44. *Determine the constants a and b such that the curl of vector* $\overline{A} = (2xy+3yz)\hat{i} + (x^2+axz-4z^2)\hat{j} - (3xy+byz)\hat{k}$ *is zero.*

(*U.P. I Semester, Dec 2008*)

$$\vec{A} = (2xy+3yz)\hat{i} + (x^2+\underline{a}xz-4z^2)\hat{j} - (3xy+\underline{b}yz)\hat{k}$$

$$\vec{\nabla}\times\vec{A} = 0$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2xy+3yz) & (x^2+axz-4z^2) & -(3xy+byz) \end{vmatrix}$$

$$\vec{\nabla}\times\vec{A} = \hat{i}(-3x-bz-ax+8z) + \hat{j}(6y) + \hat{k}(\cancel{2x}+az-\cancel{2x}-3z)$$

$$\vec{\nabla}\times\vec{A} = \hat{i}[-x(3+a)+z(8-b)] + 6y\hat{j} + z(a-3)\hat{k}$$

$\vec{\nabla}\times\vec{A} = 0$; w.r.t $\hat{i}, \hat{j}, \hat{k} \neq 0$ ∵ they are unit vectors

$$\boxed{-x(3+a)+z(8-b)=0}$$

$$6y = 0$$

$$\underbrace{z(a-3)=0}$$ here $z=0$ is a trival solution.

∴ Lets assume $z \neq 0$ ∴ $\boxed{a=3}$

$$-x(3+3) + z(8-b) = 0$$

$$\left.\begin{aligned} x(-3-a) &= 0 \\ z(8-b) &= 0 \end{aligned}\right\}$$ here $\to$ $\left.\begin{aligned} x &= 0 \\ z &= 0 \end{aligned}\right\}$ trivial solution

So lets assume $x \neq 0$; $z \neq 0$

$-3 - a = 0$; $8 - b = 0$

(or) $\boxed{a = 3}$; $-b = -8$

or $\boxed{b = 8}$

Example 45. *If a vector field is given by*
$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. *Is this field irrotational ? If so, find its scalar potential.*
(*U.P. I Semester, Dec 2009*)

?

$$\vec{F} = (x^2 - y^2 + x)\hat{e} - (2xy + y)\hat{j}$$

$\vec{\nabla} \times \vec{F} = 0 \Leftarrow$ irrotational.

Any vector field which is irrotational can be given by gradient of a scalar fn.

$$\vec{F} = \vec{\nabla}\phi \leftarrow \text{scalar field.}$$

usually "ϕ" is called the Scalar potential of $\vec{F}$.

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$\vec{\nabla}\phi$

$$= \left(\hat{e}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$

If $\vec{F}$ is irrotational the $\vec{F} = \vec{\nabla}\phi$

$$d\phi = \vec{F} \cdot d\vec{r}$$

$$\phi = \int \vec{F} \cdot d\vec{r}$$

$$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)\,dx - (2xy + y)\,dy + 0$$

$$\phi = \int \vec{F} \cdot d\vec{r}$$

$$= \int x^2\,dx - \int y^2\,dx + \int x\,dx$$

$$- \int 2xy\,dy - \int y\,dy$$

$$= \frac{x^3}{3} + C_1 - (y^2 x + C_2) + \left(\frac{x^2}{2} + C_3\right) - \left(2x\frac{y^2}{2} + C_4\right)$$

$$- \left(\frac{y^2}{2} + C_5\right)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - xy^2 - \frac{y^2}{2} + (C_1 + C_2 \cdots + C_5)$$

$$\boxed{\phi_1 = \frac{x^3}{3} + \frac{x^2}{2} - 2xy^2 - \frac{y^2}{2} + c \;//.}$$

$$\int \vec{F} \cdot d\vec{r} = \phi \;//$$

x Assignment Doubt

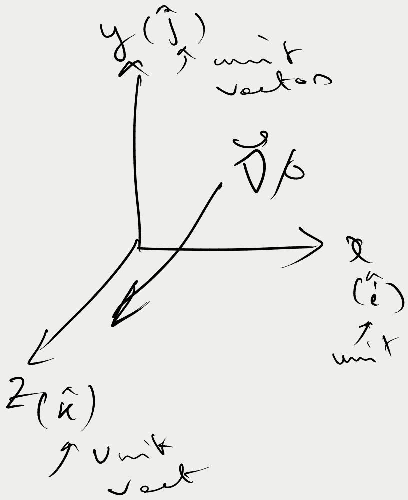
11. Find the values of constants a, b, c so that the maximum value of the directional directive of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the axis of z.

Ans. $a = b$, $b = 24$, $c = -8$

$$\phi = axy^2 + byz + cz^2x^3$$

$$\vec{\nabla}\phi = (ay^2 + 3cz^2x^2)\hat{i} + (2yax + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

$$\vec{\nabla}\phi\Big|_{1,2,-1} = (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k}$$



DD of $\vec{\nabla}\phi$ along x axis "$\hat{i}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{i} = 4a + 3c \quad \text{—— (1)}$$

DD of $\vec{\nabla}\phi$ along y axis "$\hat{j}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{j} = 4a - b \quad \text{—— (2)}$$

DD of $\vec{\nabla}\phi$ along z axis "$\hat{k}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{k} = 2b - 2c \quad \text{—— (3)}$$

Since $\vec{\nabla}\phi$ is parallel to z axis

$$\left.\begin{aligned} \vec{\nabla}\phi\big|_p \cdot \hat{i} \\ \vec{\nabla}\phi\big|_p \cdot \hat{j} \end{aligned}\right\} = 0$$

$$\vec{\nabla}\phi\big|_p \cdot \hat{k} = 64$$

$$4a + 3c = 0$$

$$4a - b = 0$$

$$2b - 2c = 64$$