Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

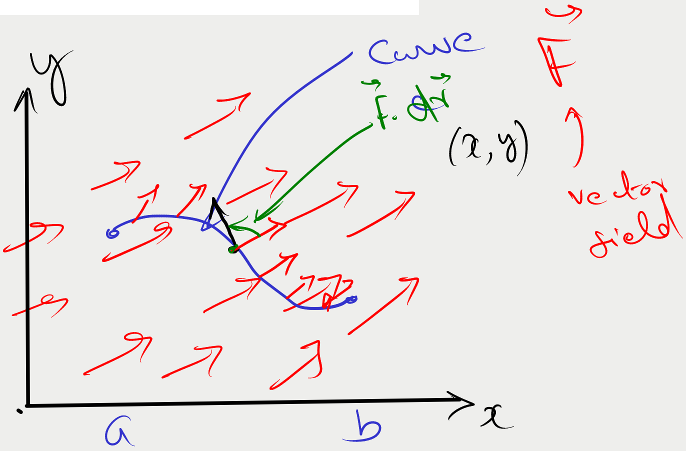
Date : **11-Nov-2020**

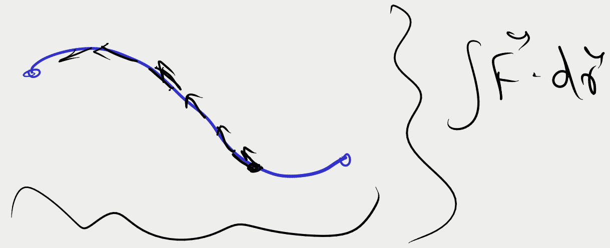
Attendance : **10**

Line Integral :

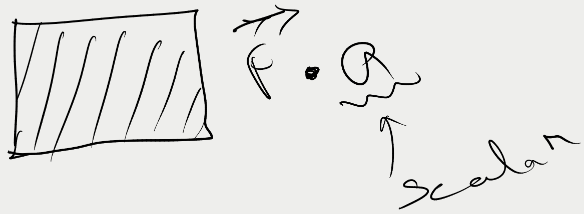
$$\int_C \vec{F} \cdot d\vec{r}$$

tangential vector

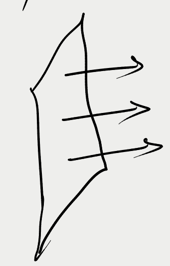




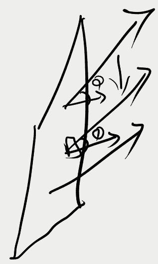
Surface Integral :



$$\phi = \vec{F} \cdot \vec{Q}$$

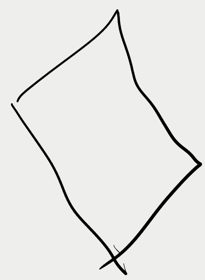


$$(Q \cdot \hat{n}) \cdot \vec{F}$$



$$d\phi = \vec{F} \cdot \hat{n}\, ds$$

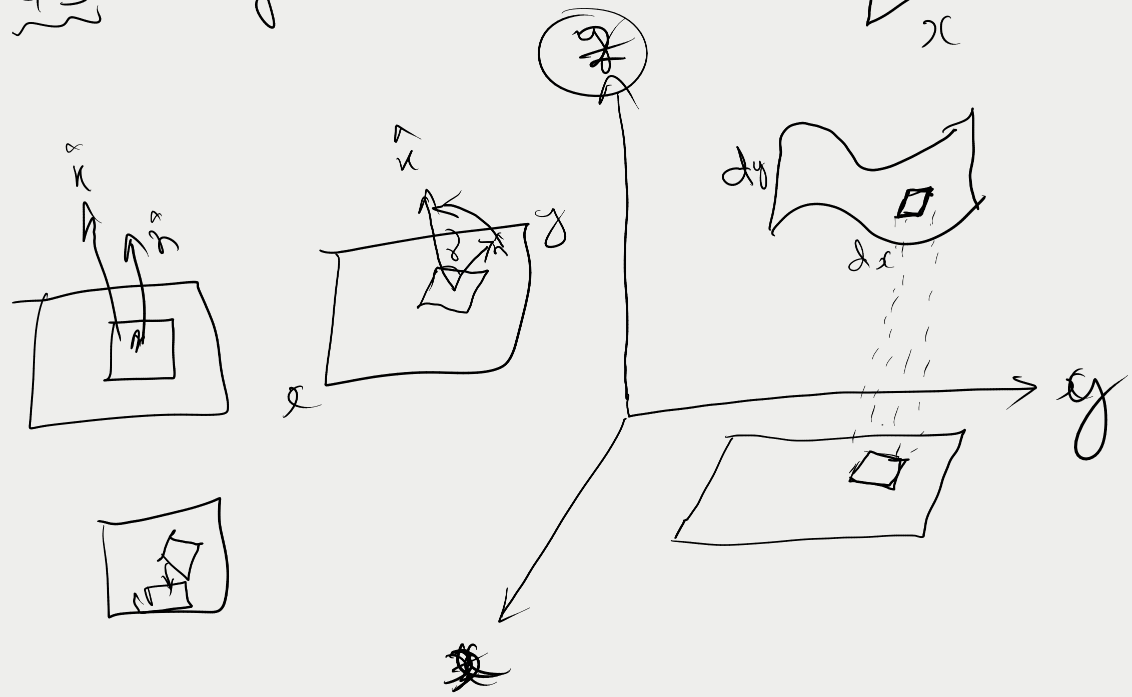
$$\boxed{\phi = \iint_S \vec{F} \cdot \hat{n}\, ds}$$

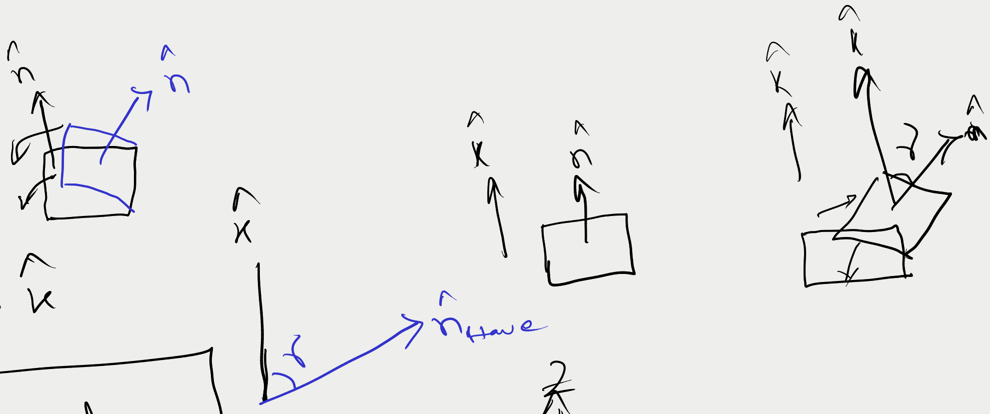


ds — scalar quantity

What is ds?

$$\underset{\sim}{ds} = dy \cdot dx$$





$$ds = dx\,dy\,\hat{n}\cdot\hat{k}$$

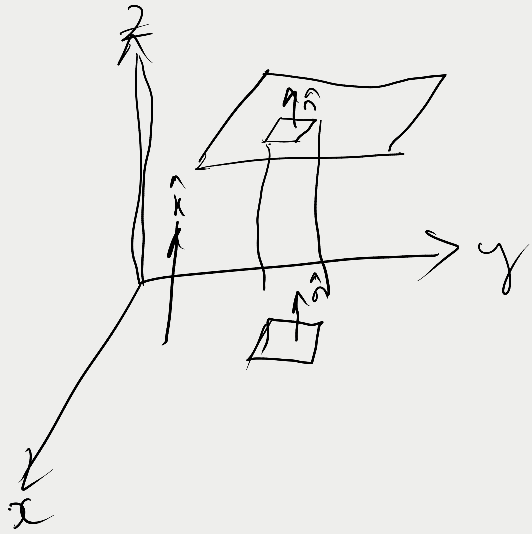
(or) $\boxed{dx\,dy = \dfrac{ds}{\hat{n}\cdot\hat{k}}}$

$$\hat{n}\cdot\hat{k} = |n||k|\cos\theta$$

$\hat{n}$ $\|^{el}$ $\hat{k}$ then $\theta = 0$

$$\underset{\sim}{\hat{n}}\cdot\hat{k} = |n||k|\cdot 1$$

$\hat{n}\cdot\hat{k} = 1$ when $\hat{n}$ $\|^{el}$ $\hat{k}$



$$dx\,dy = \underline{ds}$$

$$\iint \vec{F}\cdot\hat{n}\, ds \Rightarrow \iint \vec{F}\cdot\hat{n}\cdot \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

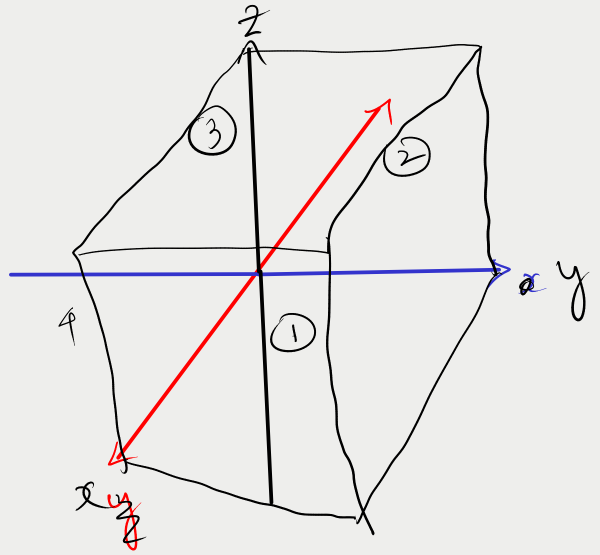
S.I

$$\Rightarrow \iint_{x\,y} \vec{F}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

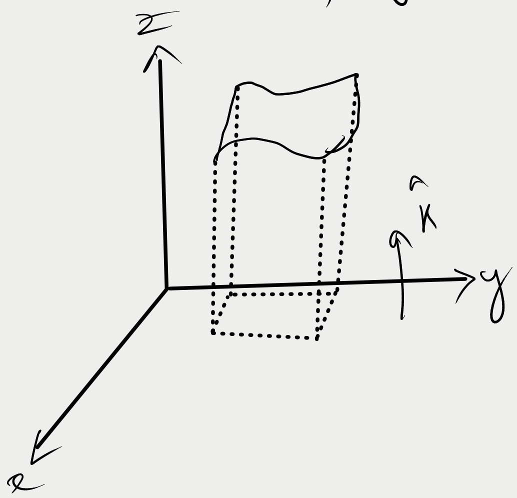
↑ first octant

x, y, z

$+ \quad + \quad + \quad z$



x, y



Example 70. *Evaluate* $\iint_S \vec{A}.\hat{n}\, ds$ *where* $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (Nagpur University, Summer 2000)

$$\vec{A} = (x+y^2)\,\hat{i} - 2x\,\hat{j} + 2yz\,\hat{k}$$

$$\phi = 2x + y + 2z - 6$$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_S \vec{A}\cdot\hat{n}\, \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z - 6)$$

$$\vec{n} = \vec{\nabla}\phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right)\cdot\hat{k} = \frac{2}{3} \text{ A.}$$

$$\vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\frac{1}{3}\left[2\hat{i} + \hat{j} + 2\hat{k}\right]$$

$$\vec{A}\cdot\hat{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$\vec{A}\cdot\hat{n} = \cancel{\frac{2}{3}x} + \frac{2}{3}y^2 - \cancel{\frac{2}{3}x} + \frac{4}{3}yz$$

$$\iint \frac{\vec{A}\cdot\hat{n}\,dx\,dy}{\hat{n}\cdot\hat{k}} = \iint_S \left(\frac{2}{3}y^2 + \frac{4}{3}yz\right)\frac{3}{2}\,dx\,dy$$

$$= \iint_S (y^2 + 2yz)\,dx\,dy \quad \text{--- (1)}$$

$$2x+y+2z=6$$

$$z=\frac{6-2x-y}{2} \quad \text{—— } (2)$$

Sub value of z in (1) we have

$$=\iint_S \left[y^2+\cancel{\tfrac{2}{2}}y\left(6-2x-y\right)\right] dx\,dy$$

$$=\iint_S \left[y^2+6y-2xy-y^2\right) dx\,dy$$

$$=\int_0^3\int_0^{6-2x}\left(\cancel{y^2}+6y-2xy-\cancel{y^2}\right)dx\,dy$$

$$=\int_0^3\int_0^{6-2x}\left(6y-2xy\right)dx\,dy$$

$$=\int_0^3 dx\int_0^{6-2x}\left(6y-2xy\right)dy$$

$$=\int_0^3 dx\left[6\int_0^{6-2x}y\,dy-2x\int_0^{6-2x}y\,dy\right]$$

$$=\int_0^3 dx\left[6\cdot\frac{y^2}{2}\Bigg|_0^{6-2x}-2x\cdot\frac{y^2}{2}\Bigg|_0^{6-2x}\right.$$

$=$

$z=0$ in eqn (2)

$$0=6-2x-y$$

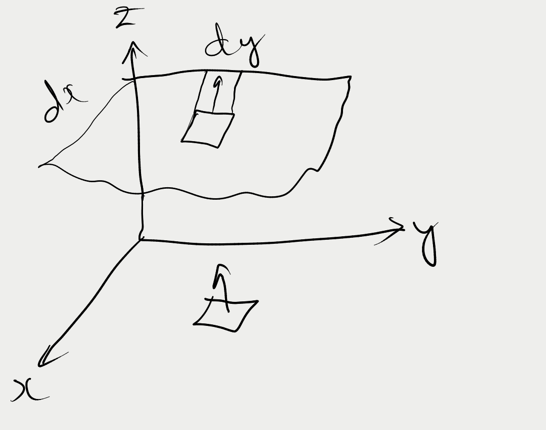
$$\boxed{y=6-2x} \quad \text{—— } (3)$$

$y=0$ in eqn (3)

$$0=6-2x$$

$$x=\frac{6}{2}$$

$$\boxed{x=3} \qquad 0 \text{ to } 3$$



Example 75. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\, d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

$$\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$\phi = 2x + 3y + 6z - 12$$

① $\vec{\nabla}\phi$; ② $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$; ③ $\vec{A}\cdot\hat{n}$; ④ $\hat{n}\cdot\hat{k}$

⑤ $$\iint_S \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Example 76. *Evaluate* $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\vec{ds}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* (*U.P., I Semester, Dec. 2004*)

$$\phi = x^2 + y^2 + z^2 - a^2$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{2\sqrt{\underbrace{x^2 + y^2 + z^2}_{a^2}}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{a^2}}$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{z}{a} \quad ; \quad \cancel{\vec{A}\cdot\hat{n} =}$$

$$\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{A}\cdot\hat{n} = \frac{xy^2}{a} + \frac{yzx}{a} + \frac{xyz}{a}$$

$$\vec{A}\cdot n = 3xyz/a$$

$$z = (a^2 - x^2 - y^2)^{1/2} \quad \text{——} \quad 1$$

$$\vec{A} \cdot \hat{n} = \frac{3xy}{a} (a^2 - x^2 - y^2)^{1/2}$$

$$\iint_S \vec{A} \cdot \hat{n} \, ds = \iint_S \frac{3xy}{\cancel{a}} \underbrace{\cancel{(a^2 - x^2 - y^2)^{1/2}}}_{z} \cdot \cancel{a} \frac{dx \, dy}{\cancel{z}}$$

$$= \int_0^a dx \int_0^{(a^2 - x^2)^{1/2}} 3xy \, dy$$

$$= \int_0^a \left[3x \frac{y^2}{2} \Big|_0^{(a^2-x^2)^{1/2}} \right] \cdot dx$$

$$= \int_0^a \frac{3x}{2} (a^2 - x^2) \, dx$$

$$= \frac{3}{2} \int_0^a (a^2 x - x^3) \, dx$$

$$= \frac{3}{2} \left[a^2 \cdot \frac{x^2}{2} \Big|_0^a - \frac{x^4}{4} \Big|_0^a \right]$$

$$= \frac{3}{2} \left[a^2 \cdot \frac{a^2}{2} - \frac{a^4}{4} \right]$$

$$= \frac{3}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \Rightarrow \frac{3}{2} \cdot a^4 \left[\frac{1}{2} - \frac{1}{4} \right]$$

$z = 0$

$\underline{y = (a^2 - x^2)^{1/2}}$

$y = 0$

$x^2 = a^2$

$\boxed{x = a}$

$$= \frac{3}{2} \cdot a^4 \cdot \frac{1}{4}$$

$$= \frac{3}{8} a^4 \quad a.$$

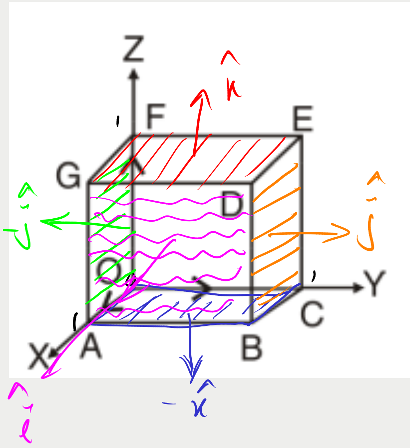
Example 77. *Show that* $\iint_S \vec{F} \cdot \hat{n}\, ds = \frac{3}{2}$, *where* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by the planes,* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$\hat{n} \cdot \hat{k} = 1$

$\hat{n} \cdot \hat{k} = 1$

Plane	Outward normal	ds
OABC	$-\hat{k}$	$dx\,dy$
DEFG	$\hat{k}$	$dx\,dy$
OAGF	$-\hat{j}$	$dx\,dz$
BCED	$\hat{j}$	$dx\,dz$
ABDG	$\hat{i}$	$dy\,dz$
OCEF	$-\hat{i}$	$dy\,dz$



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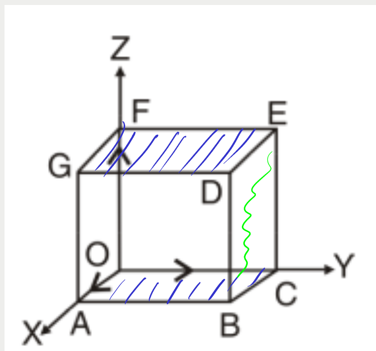

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{OABC} \vec{F} \cdot \hat{n}\, ds + \iint_{DEFG} \vec{F} \cdot \hat{n}\, ds + \iint_{OAGF} \vec{F} \cdot \hat{n}\, ds$$

S: bounded by unit cube

$$+ \iint_{BCED} \vec{F} \cdot \hat{n}\, ds + \iint_{ABDG} \vec{F} \cdot \hat{n}\, ds + \iint_{OCEF} \vec{F} \cdot \hat{n}\, ds$$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$OABC \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{k}) = -yz \quad ; \quad ds = dx\,dy$$

$$DEFG \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{k}) = yz \quad ; \quad ds = dx\,dy$$

$$OAGF \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{j}) = y^2$$

$$BCED \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{j}) = -y^2$$

$$ABDG \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{i}) = 4xz$$

$$OCEF \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{i}) = -4xz$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint -yz\,dx\,dy \quad ; \quad z = 0$$

$$\therefore \iint_{OABC} \vec{F}\cdot\hat{n}\,ds = 0$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint yz\,dx\,dy \quad ; \quad z = 1$$

$$= \int_0^1\int_0^1 y\,dx\,dy \Rightarrow \int_0^1 dx \int_0^1 y\,dy$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\Big|_0^1\right]$$

$$= \int_0^1 \frac{1}{2}\,dx \Rightarrow \frac{1}{2}x\Big|_0^1$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \frac{1}{2}$$

$$\iint_{OAGF} \vec{F}\cdot\hat{n}\,ds = \iint y^2\,dx\,dz \quad ; \; y=0$$

$$= 0 \quad \because y = 0$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = \iint -y^2\,dx\,dz$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = -1$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4xz\,dy\,dz \quad ; \; x=1$$

$$= \int_0^1 dy \int_0^1 4z\,dz$$

$$= \int_0^1 dy\left[4\cdot\frac{z^2}{2}\Big|_0^1\right]$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = 2\int_0^1 dy = 2$$

$$\iint_{OCEF} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -4xz\,dy\,dz \; ; \; x$$

$$= 0$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = 0 + \left(\frac{1}{2}\right) + (0) + (-1) + 2 + (0)$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$ //.

Ex. 5.10

1, 3, 4,