Vectors, Tensors and Fields

Lecture 1: Review of Vectors (*D Chapters 1 & 2; RHB 6.1–6.6*)

1. 1. Definitions (Physicist's)

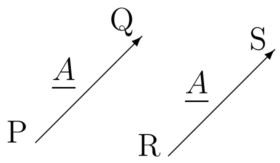
Scalar : quantity specified by a single number;

Vector : quantity specified by a number (magnitude) and a **direction**;

e.g. speed is a scalar, velocity is a vector

1. 2. Geometrical Approach

A vector is *represented* by a 'directed line segment' with a length and direction proportional to the magnitude and direction of the vector (in appropriate units). A vector can be considered as a class of equivalent directed line segments
e.g.



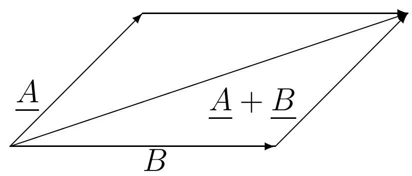
Both displacements from P to Q and from R to S are represented by the same vector

Also different quantities can be represented by the same vector *e.g.* a displacement of A cm, or a velocity of A m/s or ..., where A is the magnitude or **length** of vector $\underline{A}$

Notation: Textbooks often denote vectors by boldface ($\boldsymbol{A}$); here we use underline.
Denote a vector by $\underline{A}$ and its magnitude by $|\underline{A}|$ or A. *Always* underline a vector to distinguish from its magnitude .
A unit vector is often denoted by a hat $\underline{\hat{A}} = \underline{A} \,/\, A$ and represents a direction.

Addition of vectors—parallelogram law

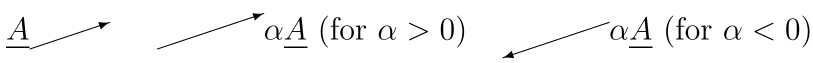
i.e.



$$\begin{aligned} \underline{A}+\underline{B} &= \underline{B}+\underline{A} && \text{(commutative) ;} \\ (\underline{A}+\underline{B})+\underline{C} &= \underline{A}+(\underline{B}+\underline{C}) && \text{(associative) .} \end{aligned}$$

Multiplication by scalars,

A vector may be multiplied by a scalar to give a new vector *e.g.*



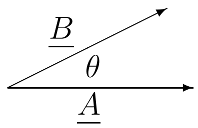
Also

$$\begin{aligned} |\alpha\underline{A}| &= |\alpha||\underline{A}| \\ \alpha(\underline{A}+\underline{B}) &= \alpha\underline{A}+\alpha\underline{B} \quad \text{distributive} \\ \alpha(\beta\underline{A}) &= (\alpha\beta)\underline{A} \quad \text{associative} \\ (\alpha+\beta)\underline{A} &= \alpha\underline{A}+\beta\underline{A}\,. \end{aligned}$$

1. 3. Scalar or dot product

The scalar product (also known as the dot product) between two vectors is defined as

$$\boxed{(\underline{A}\cdot\underline{B}) \stackrel{\text{def}}{=} AB\cos\theta, \text{ where } \theta \text{ is the angle between } \underline{A} \text{ and } \underline{B}}$$



$$\boxed{(\underline{A}\cdot\underline{B}) \text{ is a scalar — } \textit{i.e.} \text{ a single number.}}$$

Notes on scalar product

(i) $\underline{A}\cdot\underline{B} = \underline{B}\cdot\underline{A}$; $\underline{A}\cdot(\underline{B}+\underline{C}) = \underline{A}\cdot\underline{B}+\underline{A}\cdot\underline{C}$

(ii) $\underline{\hat{n}}\cdot\underline{A}$ = the scalar projection of $\underline{A}$ onto $\underline{\hat{n}}$

$(\underline{\hat{n}}\cdot\underline{A})\,\underline{\hat{n}}$ = the vector projection of $\underline{A}$ onto $\underline{\hat{n}}$

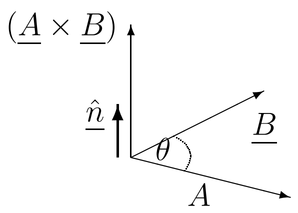
(iii) A vector may be resolved with respect to some direction $\underline{\hat{n}}$ into a parallel component $\underline{A}_{\parallel} = (\underline{\hat{n}}\cdot\underline{A})\underline{\hat{n}}$ and a perpendicular component $\underline{A}_{\perp} = \underline{A}-\underline{A}_{\parallel}$. You should check that $\underline{A}_{\perp}\cdot\underline{\hat{n}} = 0$

(iv) $\boxed{\underline{A}\cdot\underline{A} = |\underline{A}|^2}$ which defines the magnitude of a vector. For a unit vector $\underline{\hat{A}}\cdot\underline{\hat{A}} = 1$

1. 4. The vector or 'cross' product

$$\boxed{(\underline{A}\times\underline{B}) \stackrel{\text{def}}{=} AB\sin\theta\;\underline{\hat{n}}\,, \text{ where } \underline{\hat{n}} \text{ in the 'right-hand screw direction'}}$$

i.e. $\underline{\hat{n}}$ is a unit vector normal to the plane of $\underline{A}$ and $\underline{B}$, in the direction of a right-handed screw for rotation of $\underline{A}$ to $\underline{B}$ (through $< \pi$ radians).



$$\boxed{(\underline{A}\times\underline{B}) \text{ is a vector — i.e. it has a direction and a length.}}$$

[It is also called the **cross** or **wedge** product — and in the latter case denoted by $\underline{A} \wedge \underline{B}$.]

Notes on vector product

(i) $\underline{A} \times \underline{B} = -\underline{B} \times \underline{A}$

(ii) $\underline{A} \times \underline{B} = 0$ if $\underline{A}, \underline{B}$ are parallel

(iii) $\underline{A} \times (\underline{B} + \underline{C}) = \underline{A} \times \underline{B} + \underline{A} \times \underline{C}$

(iv) $\underline{A} \times (\alpha \underline{B}) = \alpha \underline{A} \times \underline{B}$

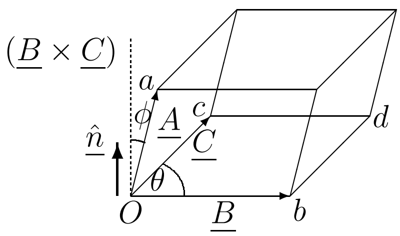
1. 5. The Scalar Triple Product

The scalar triple product is defined as follows

$$(\underline{A}, \underline{B}, \underline{C}) \stackrel{\text{def}}{=} \underline{A} \cdot (\underline{B} \times \underline{C})$$

Notes

(i) Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be three concurrent edges of a parallelepiped then the volume of the parallelepiped $= (\underline{A}, \underline{B}, \underline{C})$. To see this note that



$$\begin{aligned} \text{area of the base} &= \text{Area of parallelogram } Obdc \\ &= |B|\,|C| \sin\theta = |\underline{B} \times \underline{C}| \\ \text{height} &= A \cos\phi = \hat{\underline{n}} \cdot \underline{A} \\ \text{volume} &= \text{Area of base} \times \text{height} \\ &= |B|\,|C| \sin\theta \; \underline{A} \cdot \hat{\underline{n}} \\ &= \underline{A} \cdot (\underline{B} \times \underline{C}) \end{aligned}$$

(ii) If we choose $\underline{A}, \underline{B}$ to define the base then a similar calculation gives volume $= \underline{C} \cdot (\underline{A} \times \underline{B})$
Therefore there is a cyclic symmetry

$$(\underline{A}, \underline{B}, \underline{C}) = (\underline{B}, \underline{C}, \underline{A}) = (\underline{C}, \underline{A}, \underline{B}) = -(\underline{A}, \underline{C}, \underline{B}) = -(\underline{B}, \underline{A}, \underline{C}) = -(\underline{C}, \underline{B}, \underline{A})$$

(iii) If $\underline{A}, \underline{B}$ and $\underline{C}$ are **coplanar** ($i.e.$ all three vectors lie in the same plane) then $V = (\underline{A}, \underline{B}, \underline{C}) = 0$, and vice-versa.

1. 6. The Vector Triple Product

There are *several* ways of combining 3 vectors to form a new vector.
e.g. $\underline{A} \times (\underline{B} \times \underline{C})$; $(\underline{A} \times \underline{B}) \times \underline{C}$, etc.
Note carefully that *brackets are important*, since

$$\underline{A} \times (\underline{B} \times \underline{C}) \neq (\underline{A} \times \underline{B}) \times \underline{C} \,.$$

Expressions involving two (or more) vector products can be simplified by using the identity:–

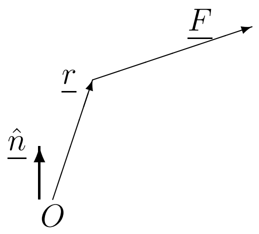
$$\boxed{\underline{A} \times (\underline{B} \times \underline{C}) \;=\; \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B})} \;.$$

This is a result you must memorise. We will prove it later in the course.

1. 7. Some examples in Physics

(i) Torque

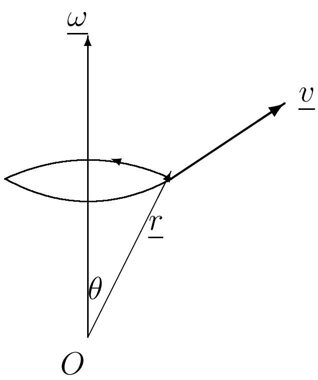
The **torque** or **moment** of a force about the origin $= \underline{r} \times \underline{F}$ where $\underline{r}$ is the position vector of the point where the force is acting and $\underline{F}$ is the force vector. Thus torque about the origin is a vector quantity



The magnitude of the torque about an axis through the origin in direction $\hat{\underline{n}}$ is given by $\hat{\underline{n}} \cdot (\underline{r} \times \underline{F})$. Note that this is a scalar quantity formed by a scalar triple product

(ii) Angular velocity and angular momentum

Consider a point $\underline{r}$ in a rigid body rotating with **angular velocity** $\underline{\omega}$ defined through: $\hat{\underline{\omega}}$ is the axis about which the body rotates; $|\underline{\omega}|$ is the angular speed of rotation measured in radians per second.



You should convince yourself that $\underline{v} = \underline{\omega} \times \underline{r}$ by checking: that this gives the right direction for $\underline{v}$; that it is perpendicular to the plane of $\underline{\omega}$ and $\underline{r}$; that the magnitude $|\underline{v}| = \omega r \sin\theta = \omega \times$ radius of circle in which the point is travelling

Now consider the **angular momentum** of the particle defined by $\underline{L} = \underline{r} \times (m\underline{v})$ where m is the mass of the particle.

Using the above expression for $\underline{v}$ we obtain

$$\underline{L} = m\underline{r} \times (\underline{\omega} \times \underline{r}) = m\left[\underline{\omega} r^2 - \underline{r}(\underline{r} \cdot \underline{\omega})\right]$$

where we have used the identity for the vector triple product. Note that only if $\underline{r}$ is perpendicular to $\underline{\omega}$ do we obtain $\underline{L} = m\underline{\omega} r^2$, which means that only then are $\underline{L}$ and $\underline{\omega}$ in the same direction. Also check that $\underline{L} = 0$ if $\underline{\omega}$ and $\underline{r}$ are parallel.