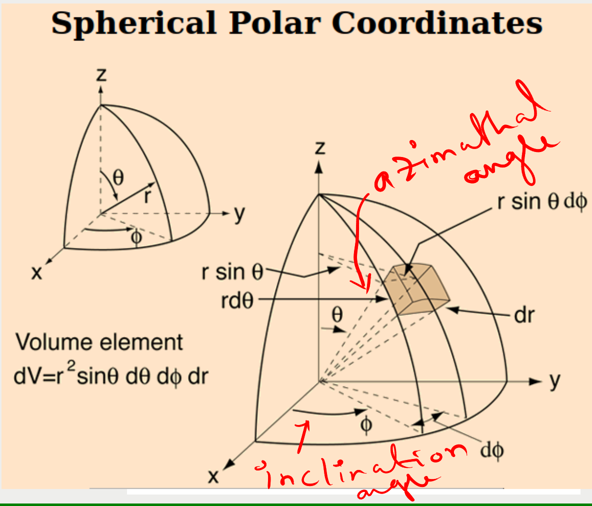
Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **09-Dec-2020**

Attendance : **9**

Spherical Polar Coordinates



$r \in [0, \infty)$ radial vector

$\theta \in [0, \pi]$ azimuth angle

$\phi \in [0, 2\pi]$ inclination angle

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$$\boxed{dv = dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

Gauss' Divergence Theorem:

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \vec{\nabla} \cdot \vec{F}\, dv$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Example 102. *State Gauss's Divergence theorem* $\iint_S \vec{F}.\hat{n}\,ds = \iiint Div\,\vec{F}\,dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$.

$$\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$$

$$x^2+y^2+z^2 = r^2$$

$$r = 4$$

$$\phi = x^2+y^2+z^2-16 \Rightarrow$$

Gauss' theorem w.k.T

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(3x\hat{i} + 4y\hat{j} + 5z\hat{k}\right)$$

$$= 3 + 4 + 5$$

$$\vec{\nabla}\cdot\vec{F} = 12$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 12\,dv$$

$$= 12\iiint_V dv$$

$$= 12\cdot\frac{4}{3}\pi\,4^3$$

$$= \cancel{12}^{4}\cdot\frac{4}{\cancel{3}}\pi\cdot 64$$

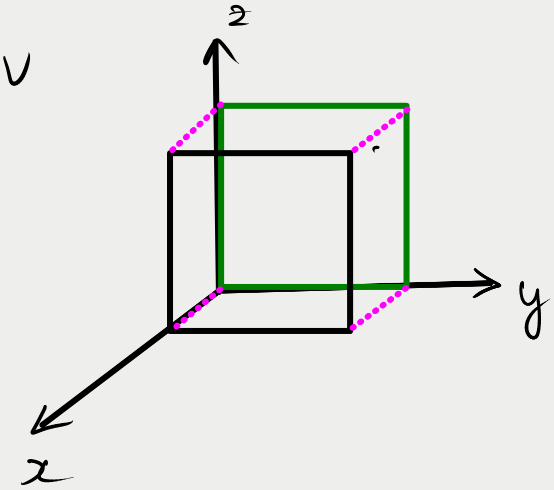
$$= 16\times 64\cdot\pi$$

$$= 1024\pi$$

Example 103. *Evaluate* $\iint_S \vec{F}\,.\,\hat{n}\,ds$ *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1.$

(U.P., Ist Semester, 2009, Nagpur University, Winter 2003)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$



$$\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = 4z - 2y + y$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 4z - y}$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)\,\overbrace{dx\,dy\,dz}^{dv}$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y)\,dz$$

$$= \int_0^1 dx \int_0^1 dy \left[\cancel{4}^{2} \cdot \frac{z^2}{\cancel{2}}\Big|_0^1 - y\cdot z\Big|_0^1 \right]$$

$$= \int_0^1 dx \int_0^1 dy\,[2 - y]$$

complete the remaining steps.

$$\vdots$$

$$= \frac{3}{2}$$ //

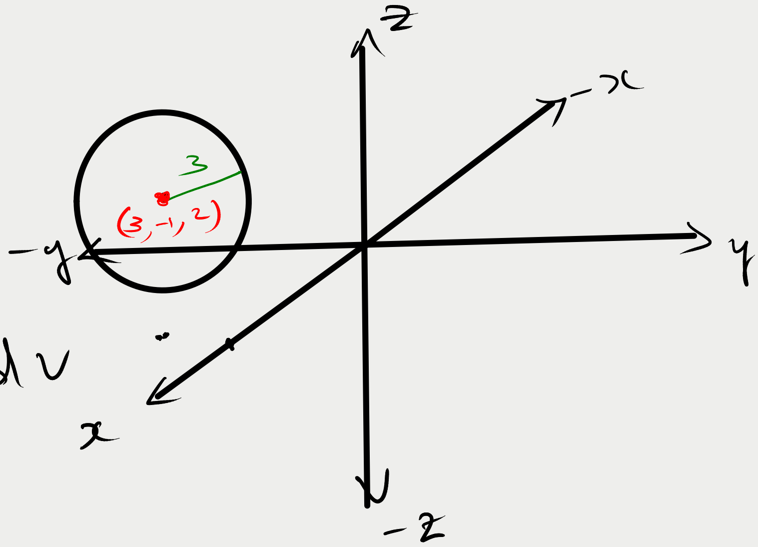
Example 104. *Find* $\iint \vec{F}\cdot\hat{n}\cdot ds$, *where* $\vec{F} = (2x+3z)\,\hat{i} - (xz+y)\,\hat{j} + (y^2+2z)\,\hat{k}$ *and S is the surface of the sphere having centre* $(3, -1, 2)$ *and radius* 3.

(AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000)

$x = 3$

$y = -1$

$z = 2$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$

$$\vec{F} = (2x+\cancel{3z})\,\hat{i} - (xz+y)\,\hat{j} + (y^2+2z)\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = 2 - 1 + 2$$

$$\underline{\vec{\nabla}\cdot\vec{F} = 3}$$

$$= \iiint_V 3\cdot dv$$

$$= 3\iiint_V dv$$

$$= 3\cdot\frac{4}{\cancel{3}}\pi\,3^{\cancel{3}}$$

$$= 12\cdot 3^2\pi$$

$$= 108\pi\ \text{Ans.}$$

Example 105. *Use Divergence Theorem to evaluate* $\iint_S \vec{A}\cdot\vec{ds}$,

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2+y^2+z^2 = a^2$.

(AMIETE, Dec. 2009)

By Gauss theorem

$$d\vec{s} = \hat{n}\,ds$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{A}\,dv$$

$$\vec{\nabla}\cdot\vec{A} = 3x^2+3y^2+3z^2$$

$$\vec{\nabla}\cdot\vec{A} = 3(x^2+y^2+z^2)$$

a^2 X you should not sub. a^2 here itself

$$= 3\iiint_V (x^2+y^2+z^2)\,dv$$

$x \to r\sin\theta\cos\phi$

$y \to r\sin\theta\sin\phi$

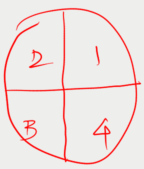
$z \to r\cos\theta$

$$dv = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\iiint_V (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\iiint_V \left[r^2\sin^2\theta\,\underbrace{(\cos^2\phi + \sin^2\phi)}_{1} + r^2\cos^2\theta\right] r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\iiint_V r^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\cdot \underbrace{r^2\sin\theta\,dr\,d\theta\,d\phi}_{dv}$$



$$= 3\iiint_V r^2\cdot r^2\sin\theta\,dr\,\underbrace{d\theta}_{0,\pi}\,d\phi \longleftrightarrow 0,2\pi$$

$$3\times 8\int_0^{\pi/2} d\phi\int_0^{\pi/2}\sin\theta\,d\theta\cdot\int_0^a r^4\,dr$$

In the book

$$= 3\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^a r^4\,dr$$

$$= 3\cdot\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\,d\theta\cdot\frac{r^5}{5}\Big|_0^a$$

$$= 3 \cdot \frac{a^5}{5} \left(-\cos\theta\right)_0^{\pi} \cdot (2\pi)$$

$$= 3 \cdot \frac{a^5}{5} \cdot (1 + 1) \cdot 2\pi$$

$$= \frac{12\pi a^5}{5} \text{ h.}$$

Example 106. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2) \cdot d\vec{s} = 6 V$$

where S is any closed surface enclosing volume V. (*U.P., I Semester, Winter 2002*)

$$\iint_S \vec{F} \cdot d\vec{r} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dv$$

$$\vec{\nabla}(x^2 + y^2 + z^2) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 + y^2 + z^2)$$

$$\vec{F} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$d\vec{s} = \hat{n} \cdot ds$$

$$\iint_S \vec{\nabla}(x^2 + y^2 + z^2) \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} \, ds$$

$$\iint_S 2(x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{n} \, ds = 2 \iiint_V \vec{\nabla} \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \, dv$$

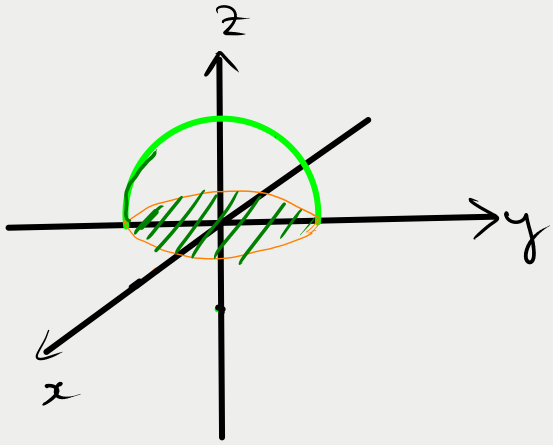
$$= 2 \cdot \iiint_V 3 \, dv$$

$$= 2 \cdot 3 \underbrace{\iiint_V dv}_{V}$$

$$= 6V \text{ h.}$$

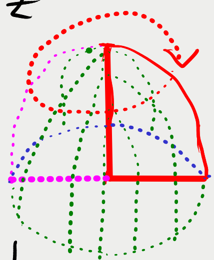
Example 107. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,\hat{n}\,dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

$$\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}$$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$\vec{\nabla}\cdot\vec{F} = 0 + 0 + y^2\cdot 2z$$

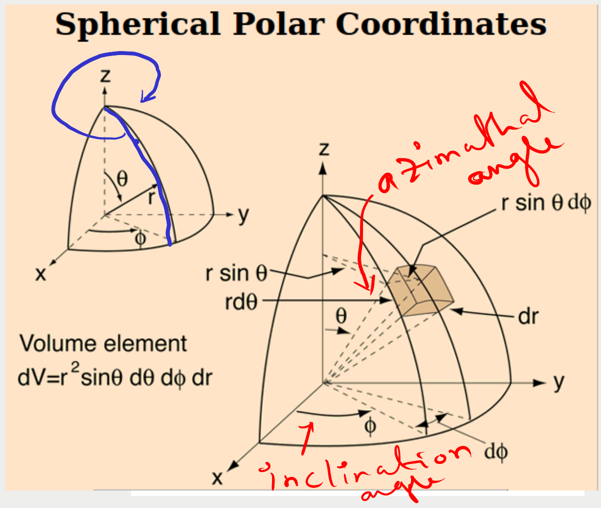


$$\boxed{\vec{\nabla}\cdot\vec{F} = 2y^2z}$$

$x \rightarrow r\sin\theta\cos\phi$

$y \rightarrow r\sin\theta\sin\phi$

$z \rightarrow r\cos\theta$



$$dv = r^2\sin\theta\,d\theta\,dr\,d\phi$$

For a hemisphere $\theta = 0$ to $\frac{\pi}{2}$; $\phi = 0$ to 2π

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$

$$= \iiint_V 2y^2z\cdot dx\,dy\,dz$$

In spherical polar coordinates we have

$$= 2\iiint_V (r^2\sin^2\theta\sin^2\phi\, r\cos\theta)\cdot r^2\sin\theta\,d\theta\,dr\,d\phi$$

$$= 2\cdot\iiint_V r^5\,dr\,\sin^3\theta\cos\theta\,d\theta\,\sin^2\phi\,d\phi$$

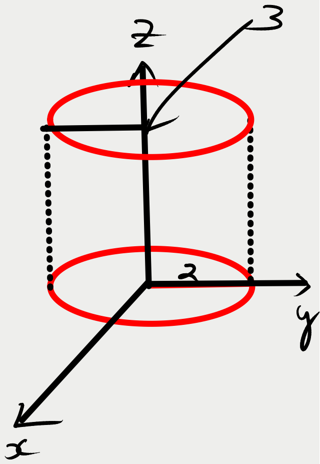
$$= 2\cdot\int_0^{2\pi}\sin^2\phi\,d\phi\cdot\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\cdot\int_0 r^5\,dr$$

Complete the remaining steps.

Example 108. *Use Divergence Theorem to evaluate* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.

(A.M.I.E.T.E., Summer 2003, 2001)

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$$



$$\vec{\nabla}\cdot\vec{F} = 4 - 4y + 2z$$

$$= \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int dx \int dy \int_0^3 (4 - 4y + 2z)\, dz$$

$$= \int dx \int dy \cdot \left[4\cdot z\Big|_0^3 - 4y\cdot z\Big|_0^3 + 2\cdot\frac{z^2}{2}\Big|_0^3 \right]$$

$$= \int dx \int dy \,[12 - 12y + 9].$$

$$= \int dx \int dy \,[21 - 12y]$$

$$= 3\iint (7 - 4y)\, dx\, dy$$

$x^2 + y^2 = 4$

$y = \sqrt{4 - x^2}$

$x^2 = 4$

$x = \pm 2$

$$= 3\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4+x^2}} (7 - 4y)\, dx\, dy$$

We can use cylindrical coor:

$x \rightarrow r\cos\theta$

$y \rightarrow r\sin\theta$

$dx\,dy \rightarrow r\,d\theta\,dr$

$$= 3\int_0^{2\pi}\int_0^{2} (7 - 4r\sin\theta)\, r\, d\theta\, dr$$

Complete the integral //.

Example 111. *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\,ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\ (\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron* $x = 0, y = 0, z = 0, x + y + z = 2$ *and n is the unit normal in the outward direction to the closed surface S.*

Eqn of plane

$$\vec{F} = (x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$$

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)+\frac{\partial}{\partial y}(x^2+y^2+z^2)+\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= 2x+2y+2z$$

$$\vec{\nabla}\cdot\vec{F} = 2(x+y+z)$$

From Gauss' theorem w.k.T

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$= 2\iiint_V (x+y+z)\,dx\,dy\,dz$$

Before proceed further let find the volume.

Eqn of plane:

$$x+y+z=2$$

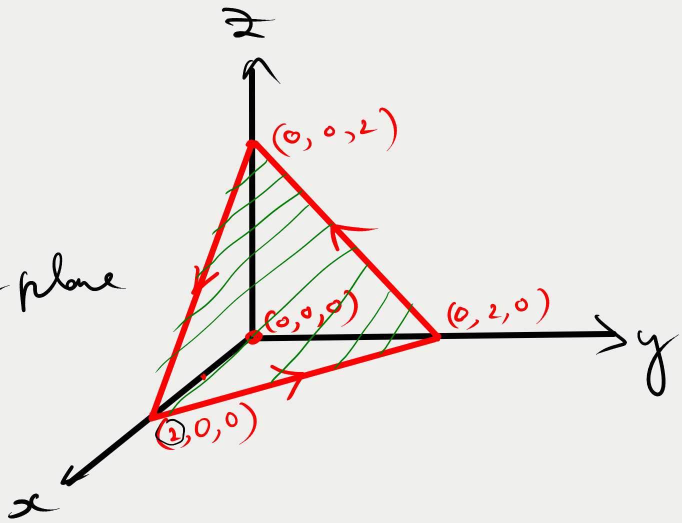
General form for Eqn of plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=1$$

$\overset{x}{0},\overset{y}{0},\overset{z}{0}$ $(0,0,0)$ $\ (\overset{x}{a},\overset{y}{b},\overset{z}{c})$

$(2,2,2)$



$$= 2 \cdot \int_0^2 dx \int_0^{2-x} dy \int_0^{2-x-y} dz\,(x+y+z)$$

$x+y+z=2$

Line on the xy plane

$z = 2-x-y$

Point on the xy plane $z=0$

$0 = 2-x-y$

$\therefore\ y = 2-x$

Point on the x axis $y=0$

$0 = 2-x$

or $\boxed{x=2}$

Complete the integral 4.

Example 112. *Use the Divergence Theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where S is the portion of the plane x + 2 y + 3 z = 6 which lies in the first Octant.

(U.P., I Semester, Winter 2003)

Given

$$\iint_S x\,dy\,dz + \iint_S y\,dz\,dx + \iint_S z\,dx\,dy \quad \text{——} \quad (1)$$

In order to use Gauss' theorem, we need $\vec{F}$.

Let assume $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

From Gauss's theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV \quad \text{——} \quad (2)$$

Lets consider L.H.S of ②

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (f_1\hat{i}+f_2\hat{j}+f_3\hat{k})\cdot\hat{n}\,ds \quad \text{--- ③}$$

In xy plane w.r.t $\hat{n} = \hat{k}$
↳ dx dy

Lets rewrite Eqn ③ for xy plane

$$\iint_S \vec{F}\cdot\hat{k}\,dx\,dy = \iint_S (f_1\hat{i}+f_2\hat{j}+f_3\hat{k})\cdot\hat{k}\,dx\,dy$$

$$\iint_S \vec{F}\cdot\hat{k}\,dx\,dy = \iint_S f_3\,dx\,dy \quad \text{--- (4a)}$$

$$\iint_S \vec{F}\cdot\hat{i}\,dy\,dz = \iint_S f_1\,dy\,dz \quad \text{--- (4b)}$$

$$\iint_S \vec{F}\cdot\hat{j}\,dz\,dx = \iint_S f_2\,\underline{dz\,dx} \quad \text{--- (4c)}$$

Comparing Eqns (4a), (4b) & (4c) with Eqn. ①

w.r.t

$$\iint_S y\,\underline{dx\,dz} = \iint_S f_2\,dx\,dz$$

$$\boxed{f_2 = y}$$

$$\boxed{f_1 = x}$$

$$\boxed{f_3 = z}$$

Sub values of f_1, f_2 & f_3 in Eqn for $\vec{F}$ we have

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

From Gauss Theorem w.k.T

$$\underbrace{\iint_S \vec{F}\cdot\hat{n}\,ds}_{} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$\iint_S \vec{F}\cdot\hat{i}\,dydz + \iint_S \vec{F}\cdot\hat{j}\,dxdz + \iint_S \vec{F}\cdot\hat{k}\,dxdy =$$

$$\boxed{\iint_S (x\,dydz + y\,dxdz + z\,dxdy)}$$

$$= \iiint_V \vec{\nabla}\cdot(x\hat{i} + y\hat{j} + z\hat{k})\cdot dV$$

$$= 3\cdot\iiint_V dV$$

Given Eqn:

$$x + 2y + 3z = 6$$

$$\frac{x}{6} + \frac{2y}{6} + \frac{3z}{6} = 1$$

comparing this Eqn

$$\frac{1}{6}(x) + \frac{1}{3}(y) + \frac{1}{2}(z) = 1 \longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

We have $a = 6$; $b = 3$; $c = 2$

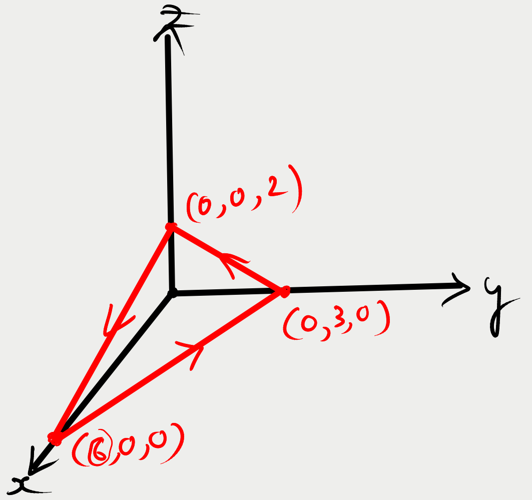
$$(0,0,0)\ (a,b,c)$$

For our case we have

$$(0,0,0)\ (6,3,2)$$

$x_i \quad y_i \quad z_i \qquad x_f \quad y_f \quad z_f$

In the xy plane Eqn of line



$$3z = 6 - 2y - x$$

$$z = 2 - \frac{2}{3}y - \frac{1}{3}x$$

Eqn of y, for that we have to put $z=0$ in the above eqn.

$$0 = 2 - \frac{2}{3}y - \frac{1}{3}x$$

$$\frac{2}{3}y = 2 - \frac{1}{3}x \Rightarrow y = \frac{2}{\frac{2}{3}} - \frac{\frac{x}{3}}{\frac{2}{3}}$$

$$= \frac{3 \times 2}{2} - \frac{3x}{3 \cdot 2}$$

$$y = 3 - \frac{x}{2}$$

For the value x we have put $y=0$

$$3 - \frac{x}{2} = 0$$

$$\text{or } x = 6$$

$$\iiint_V 3 \cdot dV = 3 \cdot \iiint_V dV$$

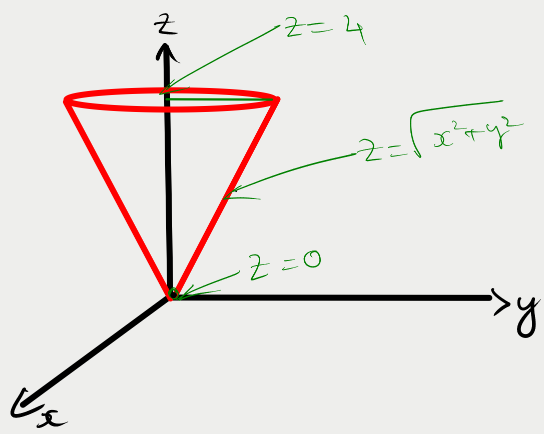
$$\iiint dx\,dy\,dz$$

$$= 3 \cdot \int_0^6 dx \int_0^{3-\frac{x}{2}} dy \int_0^{2-\frac{2}{3}y-\frac{1}{3}x} dz$$

Complete the Integral.

Example 116. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.

$$\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$$



$$\vec{\nabla} \cdot \vec{F} = 4z + xz^2 + 3$$

From Gauss' theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$$

$$= \iiint (4z + xz^2 + 3)\, dx\, dy\, dz$$

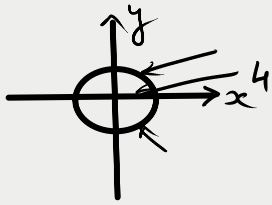
$$= \int dx \int dy \int_{\sqrt{x^2+y^2}}^{4} (4z + xz^2 + 3)\, dz$$

$$= \iint \left[44 + \frac{64}{3}x - 2(x^2+y^2) - x(x^2+y^2)^{3/2} - 3\sqrt{x^2+y^2} \right] dx\, dy$$

$x \to r\cos\theta$ $\qquad$ $r \to 0, 4$

$y \to r\sin\theta$ $\qquad$ $\theta \to 0, 2\pi$

$dx\,dy \to r\,d\theta\,dr$



Replace $x \to r\cos\theta$; $y \to r\sin\theta$; $dxdy \to r\,d\theta\,d\theta$

and do the integral for $r \to 0, 4$

$\theta \to 0, 2\pi$

Complete this integral