Lecture 5: More About Suffix Notation

5. 1. Einstein Summation Convention (BK 1.6, RHB 19.1, 19.2)

As you will have noticed, the novelty of writing out summations as in lecture 4 soon wears thin. A way to avoid this tedium is to adopt the Einstein summation convention; by adhering strictly to the following rules the summation signs are suppressed.

Rules

(i) Omit summation signs

(ii) If a suffix appears twice a summation is implied
e.g. $A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$ i is a *dummy* index

(iii) If a suffix appears only once it can take any value
e.g. $A_i = B_i$ holds for $i = 1, 2, 3$ i is a *free* index.
Note that there may be more than one free index

(iv) A suffix CANNOT appear more than twice

Examples

$\underline{A} = \underline{e}_i A_i$ here i is a dummy index.

$\underline{A} \cdot \underline{e}_i = A_j \underline{e}_j \cdot \underline{e}_i = A_j \delta_{ji} = A_i$ here i is a free index.

$\underline{A} \cdot \underline{B} = (A_i \underline{e}_i) \cdot (B_j \underline{e}_j) = A_i B_j \delta_{ij} = A_j B_j$ here j is a dummy index.

Checks: Check that free indices are the same on both sides of an equation
e.g. $A_j = B_i$ is WRONG
Check that there are no triple indices
e.g. $\underline{A} \cdot \underline{B} = A_i \underline{e}_i \cdot B_i \underline{e}_i$ is WRONG

Armed with the summation convention one can rewrite many of the equations of the previous lecture without summation signs *e.g.* the sifting property of δ_{ij} now becomes

$$\boxed{[\cdots]_j \delta_{jk} = [\cdots]_k} \, .$$

for example $\delta_{ij}\delta_{jk} = \delta_{ik}$

From now on, except where indicated, the summation convention will be assumed. You should make sure that you are completely at ease with it.

5. 2. More on matrices

The summation convention can also be used to advantage in matrix multiplication. The ik component of a product of two matrices M, N is given by

$$(M\,N)_{ik} = M_{i1}\,N_{1k} + M_{i2}\,N_{2k} + M_{i3}\,N_{3k} = M_{ij}\,N_{jk}$$

Likewise, recalling the definition of the transpose of a matrix $M^{\mathrm{T}}_{ij} = M_{ji}$

$$(M^T N)_{ik} = M^T_{ij} N_{jk} = M_{ji} N_{jk}$$

We saw in the previous lecture that δ_{ij} is represented by the identity matrix. Thus the sifting property

$$M_{ij}\delta_{jk} = M_{ik}$$

is simply interpreted as multiplication of a matrix M by the identity matrix.

The **determinant** of a matrix M, is denoted by $|M|$ or $\det M$. We state in passing some properties of determinants of matrices which you should already know *e.g.*

$$\begin{aligned} |MN| &= |M||N| = |NM| \\ |M^T| &= |M| \end{aligned}$$

5. 3. Levi-Civita Symbol ϵ_{ijk} (RHB 19.8)

We saw in the last lecture how δ_{ij} could be used to greatly simplify the writing out of the orthonormality condition on basis vectors.

We seek to make a similar simplification for the vector products of basis vectors (taken here to be right handed) *i.e.* we seek a simple, uniform way of writing the equations

$$\begin{gathered} \underline{e}_1 \times \underline{e}_2 = \underline{e}_3 \quad \underline{e}_2 \times \underline{e}_3 = \underline{e}_1 \quad \underline{e}_3 \times \underline{e}_1 = \underline{e}_2 \\ \underline{e}_1 \times \underline{e}_1 = 0 \quad \underline{e}_2 \times \underline{e}_2 = 0 \quad \underline{e}_3 \times \underline{e}_3 = 0 \end{gathered}$$

To do so we define the Levi-Cevita symbol ϵ_{ijk} (pronounced 'epsilon i j k'), where i, j and k can take on the values 1 to 3, such that:–

$$\boxed{\begin{aligned} \epsilon_{ijk} &= +1 \text{ if } (i,j,k) \text{ is an } \textit{even} \text{ permutation of } (123)\ ; \\ &= -1 \text{ if } (i,j,k) \text{ is an } \textit{odd} \text{ permutation of } (123)\ ; \\ &= \quad 0 \text{ otherwise } (\textit{i.e.} \text{ 2 or more indices are the same})\ . \end{aligned}}$$

For example,

$$\begin{aligned} \epsilon_{123} &= +1\,; \\ \epsilon_{213} &= -1 \ \{ \text{ since } (123) \to (213) \text{ under } \textit{one} \text{ permutation } [1 \leftrightarrow 2]\}\,; \\ \epsilon_{312} &= +1 \ \{(123) \to (132) \to (312);\ 2 \text{ permutations; } [2 \leftrightarrow 3][1 \leftrightarrow 3]\}\,; \\ \epsilon_{113} &= 0\,; \quad \epsilon_{111} \ = \ 0\,; \text{ etc.} \end{aligned}$$

$$\boxed{\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1\ ; \quad \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1\ ; \quad \text{all others} = 0\ .}$$

5. 4. Vector product

The equations satisfied by the vector products of the (right-handed) orthonormal basis vectors $\underline{e}_i$ can now be written uniformly as :–

$$\boxed{\underline{e}_i \times \underline{e}_j \;=\; \epsilon_{ijk}\,\underline{e}_k} \qquad (i,j = 1{,}2{,}3)\;.$$

e.g.

$$\underline{e}_1 \times \underline{e}_2 \;=\; \epsilon_{121}\,\underline{e}_1 + \epsilon_{122}\,\underline{e}_2 + \epsilon_{123}\,\underline{e}_3 \;=\; \underline{e}_3 \quad ; \quad \underline{e}_1 \times \underline{e}_1 \;=\; \epsilon_{111}\,\underline{e}_1 + \epsilon_{112}\,\underline{e}_2 + \epsilon_{113}\,\underline{e}_3 \;=\; 0$$

Also,

$$\begin{aligned} \underline{A} \times \underline{B} &= A_i\,B_j\,\underline{e}_i \times \underline{e}_j \\ &= \epsilon_{ijk}\,A_iB_j\,\underline{e}_k \end{aligned}$$

but,

$$\underline{A} \times \underline{B} = (\underline{A} \times \underline{B})_k\,\underline{e}_k\;.$$

Thus

$$\boxed{(\underline{A} \times \underline{B})_k \;=\; \epsilon_{ijk}\,A_iB_j}$$

Always recall that we are using the summation convention. For example writing out the sums

$$\begin{aligned} (\underline{A} \times \underline{B})_3 &= \epsilon_{113}\,A_1B_1 + \epsilon_{123}\,A_2B_3 + \epsilon_{133}\,A_3B_3 + \cdots \\ &= \epsilon_{123}\,A_1B_2 + \epsilon_{213}\,A_2B_1 \qquad \text{(only non-zero terms)} \\ &= A_1B_2 - A_2B_1 \end{aligned}$$

Now note a 'cyclic symmetry' of ϵ_{ijk}

$$\epsilon_{ijk} \;=\; \epsilon_{kij} \;=\; \epsilon_{jki} \;=\; -\epsilon_{jik} \;=\; -\epsilon_{ikj} \;=\; -\epsilon_{kji}$$

This holds for *any* choice of i, j and k. To understand this note that

1. If any pair of the free indices i, j, k are the same, all terms vanish;
2. If (ijk) is an even (odd) permutation of (123), then so is (jki) and (kij), but (jik), (ikj) and (kji) are odd (even) permutations of (123).

Now use the cylic symmetry to find alternative forms for the components of the vector product

$$\begin{aligned} (\underline{A} \times \underline{B})_k &= \epsilon_{ijk}\,A_iB_j = \epsilon_{kij}\,A_iB_j \\ \text{or relabelling indices} \qquad & k \to i \quad i \to j \quad j \to k \\ (\underline{A} \times \underline{B})_i &= \epsilon_{jki}\,A_jB_k = \epsilon_{ijk}\,A_jB_k\;. \end{aligned}$$

The scalar triple product can also be written using ϵ_{ijk}

$$(\underline{A}, \underline{B}, \underline{C}) = \underline{A} \cdot (\underline{B} \times \underline{C}) = A_i(\underline{B} \times \underline{C})_i$$

$$\boxed{(\underline{A}, \underline{B}, \underline{C}) = \epsilon_{ijk}\, A_i B_j C_k\,.}$$

Now as an exercise in index manipulation we can prove the cylic symmetry of the scalar product

$$\begin{aligned}
(\underline{A}, \underline{B}, \underline{C}) &= \epsilon_{ijk}\, A_i B_j C_k \\
&= -\epsilon_{ikj}\, A_i B_j C_k \qquad \text{interchanging two indices of } \epsilon_{ikj} \\
&= +\epsilon_{kij}\, A_i B_j C_k \qquad \text{interchanging two indices again} \\
&= \epsilon_{ijk}\, A_j B_k C_i \qquad \text{relabelling indices} \quad k \to i \quad i \to j \quad j \to k \\
&= \epsilon_{ijk}\, C_i A_j B_k = (\underline{C}, \underline{A}, \underline{B})
\end{aligned}$$

5. 5. Product of two Levi-Civita symbols

We state without formal proof the following identity (see tutorial questions 3.7 3.8)

$$\boxed{\epsilon_{ijk}\, \epsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}.}$$

To verify this is true one can check all possible cases *e.g.* $\epsilon_{12k}\, \epsilon_{12k} = \epsilon_{121}\, \epsilon_{121} + \epsilon_{122}\, \epsilon_{122} + \epsilon_{123}\, \epsilon_{123} = 1 = \delta_{11}\delta_{22} - \delta_{12}\delta_{21}$. More generally, note that the left hand side of the boxed equation may be written out as

- $\epsilon_{ij1}\, \epsilon_{rs1} + \epsilon_{ij2}\, \epsilon_{rs2} + \epsilon_{ij3}\, \epsilon_{rs3}$ where i, j, r, s are free indices.
- For this to be non-zero we must have $i \neq j$ and $r \neq s$
- Only one term of the three in the sum can be non-zero
- If $i = r$ and $j = s$ we have $+1$; If $i = s$ and $j = r$ we have -1

The product identity furnishes an algebraic proof for the 'BAC-CAB' rule. Consider the i^{th} component of $\underline{A} \times (\underline{B} \times \underline{C})$:

$$\begin{aligned}
\left[\underline{A} \times (\underline{B} \times \underline{C})\right]_i &= \epsilon_{ijk}\, A_j (\underline{B} \times \underline{C})_k \\
&= \epsilon_{ijk}\, A_j\, \epsilon_{klm}\, B_l C_m \;=\; \epsilon_{ijk}\, \epsilon_{lmk}\, A_j B_l C_m \\
&= (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, A_j B_l C_m \\
&= (A_j B_i C_j - A_j B_j C_i) \\
&= (\underline{A} \cdot \underline{C}) B_i - (\underline{A} \cdot \underline{B}) C_i \\
&= \left[(\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}\right]_i
\end{aligned}$$

Since i is a free index we have proven the identity for all three components $i = 1, 2, 3$.