**Department of Medical Physics**
**Bharathidasan University**

Mathematical Physics (MP101)

Date : **13-Nov-2020**

Attendance : **10**

Volume Integral :

$$V.I \Rightarrow \iiint_V \underset{\sim}{F}\, dv$$

either scalar
or
it can also be a vector

$$= \iiint_V \vec{F}\, dv$$

Example 78 :

$$\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$$

Find $\iiint_V \vec{F}\, dv$ where $V$ is the region bounded by the surfaces

$$x = 0, \quad x = 2$$

$$y = 0, \quad y = 4$$

$$z = x^2, \quad z = 2$$

$$\iiint_V \vec{f} \cdot dx\,dy\,dz = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\,dz\,\hat{i} - x\,dz\,\hat{j} + y\,dz\,\hat{k})$$

$$= \int_0^2 dx \int_0^4 dy \left( \cancel{2} \cdot \frac{z^2}{\cancel{2}}\Big|_{x^2}^2 \hat{i} - x \cdot z\Big|_{x^2}^2 \hat{j} + y\,z\Big|_{x^2}^2 \hat{k} \right.$$

$$= \int_0^2 dx \int_0^4 dy \Big[ 4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i}$$

$$+ x \cdot x^2\hat{j} - y\,x^2\hat{k} \Big]$$

$$= \int_0^2 dx \int_0^4 \Big[ 4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k} \Big]\,dy$$

$$= \int_0^2 dx \Big[ 4y\Big|_0^4 \hat{i} - 2xy\Big|_0^4 \hat{j} + 2\frac{y^2}{2}\Big|_0^4 \hat{k} - x^4y\Big|_0^4 \hat{i}$$

$$+ x^3y\Big|_0^4 \hat{j} - x^2\frac{y^2}{2}\Big|_0^4 \hat{k} \Big]$$

$$= \int_0^2 dx \left[ 16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k} \right]$$

$$= \left[ 16x\Big|_0^2 \hat{i} - 8\frac{x^2}{2}\Big|_0^2 \hat{j} + 16x\Big|_0^2 \hat{k} - 4\frac{x^5}{5}\Big|_0^2 \hat{i} + 4\cdot\frac{x^4}{4}\Big|_0^2 \hat{j} - 8\frac{x^3}{3}\Big|_0^2 \hat{k} \right]$$

$$= \left[ 16\cdot(2)\hat{i} - 4\cdot(4)\hat{j} + 16\cdot(2)\hat{k} - 4\cdot\frac{32}{5}\hat{i} + 16\hat{j} - 8\cdot\frac{8}{3}\hat{k} \right]$$

$$= 32\hat{i} - \cancel{16\hat{j}} + 32\hat{k} - \frac{128}{5}\hat{i} + \cancel{16\hat{j}} - \frac{64}{3}\hat{k}$$

$$= \left[ 32 - \frac{128}{5} \right]\hat{i} + \hat{k}\left[ 32 - \frac{64}{3} \right] \quad //$$

Green's theorem:

Line integral $\longrightarrow$ Surface Integral

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

P & Q are both fn. x and y

$$\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

$$\vec{F} \cdot d\vec{r} = \left( P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} \right) \cdot \left( dx\,\hat{i} + dy\,\hat{j} \right)$$

$$\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy$$

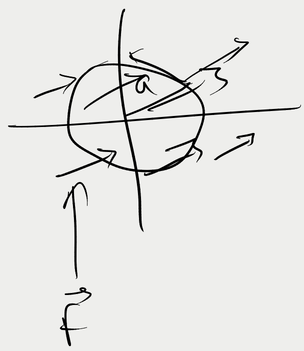
By Green's theorem we have,

$$\boxed{\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{line Integral}} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

Example 79:

$$\vec{F} = \underbrace{\sin y}_{P}\,\hat{i} + \underbrace{x(1+\cos y)}_{Q}\,\hat{j}$$

Find $\oint_C \vec{F} \cdot d\vec{r}$ where $C$ is a circular path given by $\underline{x^2 + y^2 = a^2}$




$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

where in our case $P = \sin y$

$Q = x(1+\cos y)$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$x^2 + y^2 = a^2$$

$$\boxed{y = (a^2 - x^2)^{1/2}}$$

$$0 = (a^2 - x^2)^{1/2}$$

(or) $\boxed{x = a}$

$$\frac{\partial Q}{\partial x} = 1 + \cos y$$

$$\frac{\partial P}{\partial y} = \cos y$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^a dx \int_0^{(a^2-x^2)^{1/2}} (1 + \cancel{\cos y} - \cancel{\cos y})\, dy$$

$$= \int_0^a dx \cdot \int_0^{(a^2-x^2)^{1/2}} dy$$

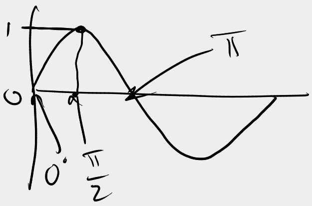
$$= \int_0^a dx \cdot y \Big|_0^{(a^2-x^2)^{1/2}}$$

$$= \int_0^a (a^2 - x^2)^{1/2}\, dx$$

$$\int (a^2 - x^2)^{1/2}\, dx = \frac{1}{2}\left[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{2}\left[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{1}{2}\left[\underbrace{a(\cancel{a^2-a^2})^{1/2}}_{\to 0} + a^2 \sin^{-1}\left(\frac{a}{a}\right) - 0 - a^2 \sin^{-1}(0)\right]$$



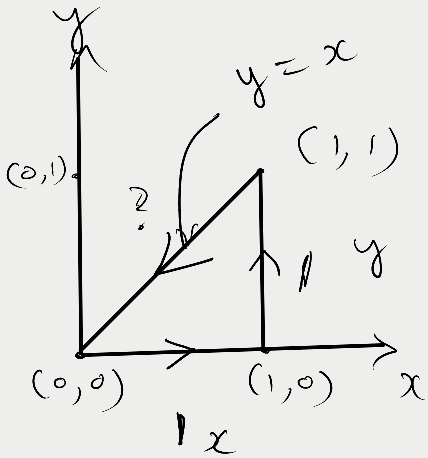

$$= \frac{1}{2}\left[a^2 \cdot \sin^{-1}(1) - a^2 \sin^{-1}(0)\right]$$

$$= \frac{1}{2}\left[a^2 \cdot \frac{\pi}{2} - a^2(0)\right]$$

$$\oint_C \vec{f} \cdot d\vec{r} = \frac{a^2 \cdot \pi}{4}$$ //

Example 80:

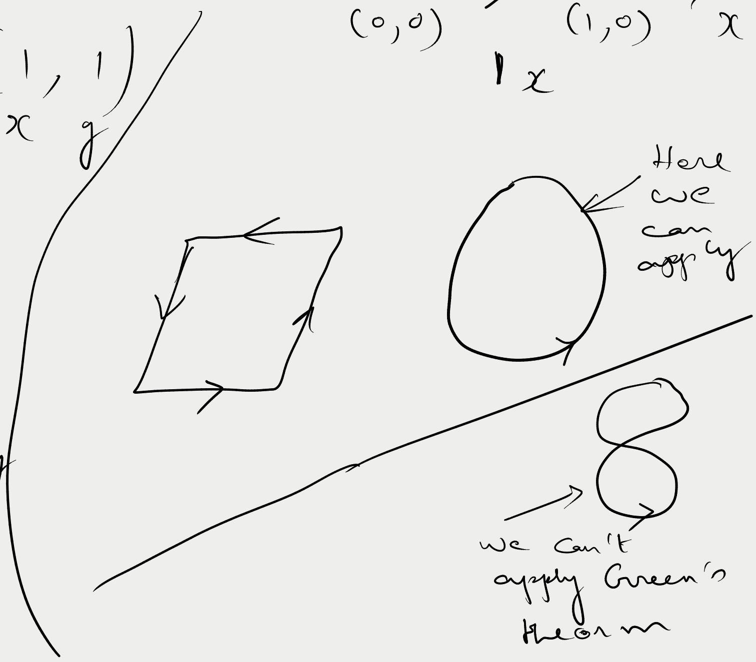
$$\oint_C x^2 y\,dx + x^2\,dy$$




$$\underset{x\ \ y}{(0,0)}\ ;\ \underset{x\ \ y}{(1,0)}\ ;\ \underset{x\ \ y}{(1,1)}$$

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$P = x^2 y \quad ; \quad Q = x^2$

$\frac{\partial P}{\partial y} = x^2 \quad ; \quad \frac{\partial Q}{\partial x} = 2x$

$y = mx + c$

$c = 0$

$1 = m \cdot 1$

$m = 1 \; ; \; \boxed{y = x}$

$$\oint_C x^2 y \, dx + x^2 dy = \int_0^1 \int_0^x (2x - x^2) \, dx \, dy$$

$$= \int_0^1 \left[ 2x \cdot y \Big|_0^x - x^2 y \Big|_0^x \right] dx$$

$$= \int_0^1 (2x^2 - x^3) \, dx$$

$$= \left[ 2 \cdot \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$\boxed{\oint_C x^2 y \, dx + x^2 dy = \frac{5}{12}}$$

$$\oint_C P dx + Q dy$$

$P(x, y)$

$Q(x, y)$

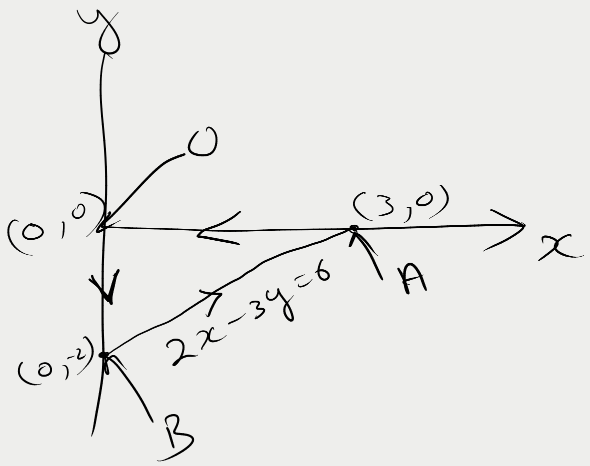
Example 81:

$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$x \geq 0$

$y \leq 0$

$2x - 3y = 6$




$-3y = 6 - 2x$

$y = \frac{6}{-3} - \frac{-2}{-3}x$

$y = \frac{2}{3}x - 2$

$m = \frac{2}{3}$

$c = -2$

If $y = 0$ in $2x - 3y = 6$

we have

$2x = 6$

$\boxed{x = 3}$

If $x = 0$ in $2x - 3y = 6$

we have

$-3y = 6$

$\boxed{y = -2}$

| $x$ | $y$ |
|---|---|
| 3 | 0 |
| 0 | -2 |

coordinate of A (3,0)

" " B (0,-2)

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

First let do the $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ using Green's theorem.

$$P = 3x^2 - 8y^2 \quad ; \quad Q = 4y - 6xy$$

$$\frac{\partial P}{\partial y} = -16y \quad ; \quad \frac{\partial Q}{\partial x} = -6y$$

$$\oint_C \sim + \sim = \iint_R \left[-6y - (-16y)\right] dx\,dy$$

$$2x - 3y = 6$$

$$-3y = 6 - 2x$$

$$\boxed{y = \frac{2}{3}x - 2}$$

If $y = 0$ in the above equation we have

$$\boxed{x = 3}$$

$$= \int_0^3 dx \int_{\frac{2}{3}x - 2}^{0} (10y)\,dy$$

$$= \int_0^3 dx \cdot \cancel{10}\,5 \cdot \frac{y^2}{2}\Bigg|_{\frac{2}{3}x-2}^{0}$$

$$= 5\int_0^3 0 - \left(\frac{2}{3}x - 2\right)^2 dx$$

$$= -5\int_0^3 \left(\frac{4}{9}x^2 + 4 - \frac{8}{3}x\right) dx$$

$$= -5 \cdot \left[\frac{4}{9}\,\frac{x^3}{3}\Bigg|_0^3 + 4\,x\Bigg|_0^3 - \frac{8}{3}\,\frac{x^2}{2}\Bigg|_0^3\right]$$

$$= -5\left[\frac{4}{\cancel{9}} \times \frac{\cancel{3\times} 3 \times \cancel{3}}{\cancel{3}} + 4 \cdot 3 - \frac{8}{\cancel{3}} \cdot \frac{\cancel{3} \times 3}{2}\right]$$

$$= -5\left[4 + 12 - 12\right]$$

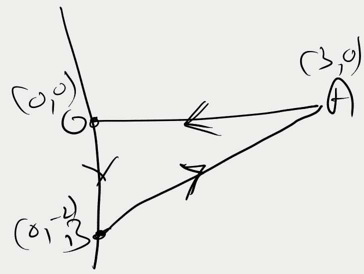
$= -20$ // $\Leftarrow$ using Green's theorem.

Again we have to evaluate

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$ by using

explicity line integral.

$$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$




$$= \int_{OB} \sim dx + \sim dy + \int_{BA} \sim dx + \sim dy$$

$$+ \int_{AO} \sim dx + dy$$

Along OB $\quad x=0 \quad \therefore \quad dx = 0$

$\qquad\qquad y=0 \; ; \quad y=-2$

$$\int_{OB} \underbrace{\cancel{(3x^2-8y^2)}}_{=0 \;\because\; dx=0}\,dx + (4y - \cancel{6xy}^{\,=0 \;\because\; x=0})\,dy$$

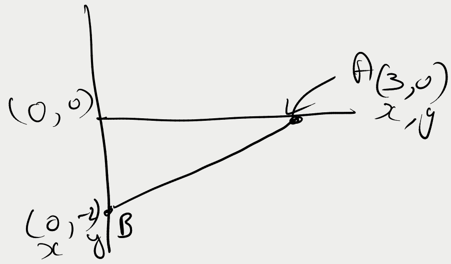
$$\int_{OB} \sim dx + \sim dy = \int_0^{-2} 4y\,dy$$

$$= 4 \cdot \frac{y^2}{2} \Big|_0^{-2}$$

$$= 2 \cdot (-2)^2$$

$$\int_{OB} \sim dx + \sim dy = 8$$

Along BA $\quad x_i = 0 \quad x_f = 3$




$y_i = -2 \quad y_f = 0$

$$2x - 3y = 6$$

$$y = \frac{2}{3}x - 2 \Rightarrow dy = \frac{2}{3}dx$$

Here $y$ varies from $-2$ to $0$

when $y_l = -2 \;;\; x = 0$

$y_u = 0 \;;\; x = 3$

$-2 = \frac{2}{3}x - 2$

$\frac{2}{3}x = 0$

$\boxed{x = 0}$

$\boxed{x = 3}$

$$\int_{BA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\int_0^3 3x^2 - 8\left(\frac{2}{3}x - 2\right)^2 dx + \left[4\left(\frac{2}{3}x - 2\right) - 6x\left(\frac{2}{3}x - 2\right)\right] \cdot \frac{2}{3}\,dx$$

$$\int_0^3 3x^2 - 8\left[\frac{4x^2}{9} + 4 - \frac{8}{3}x\right] dx$$

$$+ \left[\frac{8}{3}x - 8 - \frac{12x^2}{3} + 12x\right]\frac{2}{3}\,dx$$

$$\int_0^3 \left(3x^2 - \frac{8 \cdot 4x^2}{9} - 32 + \frac{64}{3}x\right) dx$$

$$+ \left(\frac{16}{9}x - \frac{16}{3} - \frac{24x^2}{9} + \frac{24x}{3}\right) dx$$

$$\int_0^3 3x^2dx - \frac{32x^2}{9}dx - \cancel{32dx} + \frac{64}{3}xdx + \frac{16}{9}xdx$$

$$- \frac{16}{3}dx - \frac{24}{9}x^2dx + \frac{24}{3}xdx$$

$$\int_0^3 x^2dx\left(3 - \frac{32}{9} - \frac{24}{9}\right) + xdx\left(\frac{64}{3} + \frac{16}{9} + \frac{24}{3}\right)$$

$$+ dx\left[-32 - \frac{16}{3}\right]$$

$$\int_0^3 x^2dx\left(-\frac{47}{9}\right) + xdx\left(\frac{280}{9}\right) - \frac{112}{3}dx$$

$-\frac{29}{9}$

$$= -\frac{47}{9}\left.\frac{x^3}{3}\right|_0^3 + \frac{280}{9}\left.\frac{x^2}{2}\right|_0^3 - \frac{112}{3}x\Big|_0^3$$

$$= -\frac{47}{\cancel{9}} \cdot \frac{\cancel{27}^{\,3}}{\cancel{3}} + \frac{280}{\cancel{9}} \cdot \frac{\cancel{9}}{2} - \frac{112}{\cancel{3}} \cdot \cancel{3}$$

$$= -\cancel{47}^{\,29} + 140 - 112$$

$$= \cancel{-19} \quad -1$$

$A(3,0)$
$(x, y)$

$$\int_{AO} \sim dx + \sim dy$$

$x_i = 3$ ; $x_f = 0$

$y_i = 0$ ; $y_f = 0$

$dy = 0$

$$\int_{AO} (3x^2 - 8y^2)\,dx + \cancel{(4y - 6xy)\,dy} = \int_3^0 (3x^2 - \cancel{8y^2})\,dx \qquad \because dy = 0, \quad \because y = 0$$

$$= \int_3^0 3x^2\,dx$$

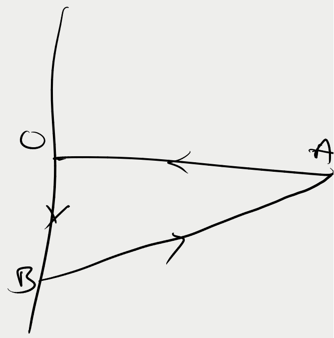
$$= \cancel{3}\cdot\frac{x^3}{\cancel{3}}\Bigg|_3^0$$

$$= -x^3\Bigg|_0^3$$

$$= -27$$

$$\int_{AO} \sim$$

$$\oint_{OBA} \sim = \int_{OB} + \int_{BA} + \int_{AO}$$




$$= 8 - \cancel{19}^{\,=1} - 27$$

$$= 8 - 1 - 27$$

$$= -20 \;//$$

Ex 5.13

1, 2, 3, 5

Ex 5.9

5.10

5.11

~~5.12~~

5.13