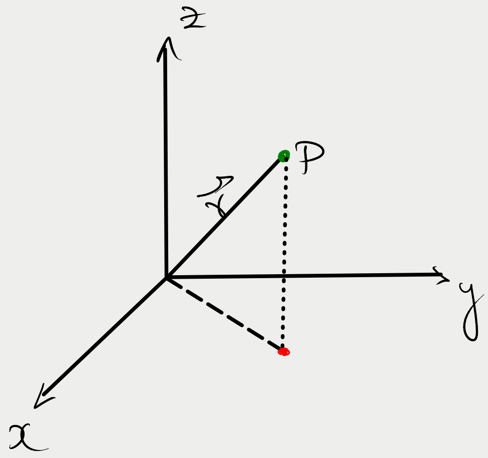
Vector Operators in Curvilinear Coordinate System

Grad, Div, Curl and Laplacian.



$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Position vector of P be $\vec{r}$

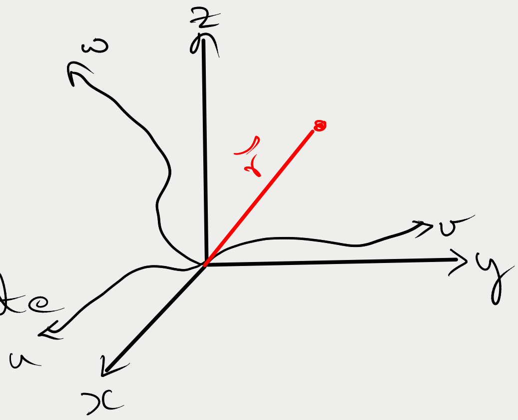
$$\vec{r} = \vec{r}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k} \quad (1)$$

$$d\vec{r} = \underbrace{\frac{\partial \vec{r}}{\partial x}}_{\text{vector}}\cdot dx + \underbrace{\frac{\partial \vec{r}}{\partial y}}_{\text{vector}}\cdot dy + \underbrace{\frac{\partial \vec{r}}{\partial z}}_{\text{vector}}\cdot dz \quad (2)$$

from (1) we can write

$$\frac{\partial \vec{r}}{\partial x} = \hat{i}\,; \quad \frac{\partial \vec{r}}{\partial y} = \hat{j}\,; \quad \frac{\partial \vec{r}}{\partial z} = \hat{k}$$

Curvilinear coordinates



In cartesian coordinat

$\vec{r}(x, y, z)$

In curvilinear coordinate

$\vec{r}(u, v, w)$

$$d\vec{r} = \underbrace{\frac{\partial \vec{r}}{\partial u}}_{\text{vector}} \cdot du + \underbrace{\frac{\partial \vec{r}}{\partial v}}_{\text{vector}} \cdot dv + \underbrace{\frac{\partial \vec{r}}{\partial w}}_{\text{vector}} \cdot dw$$

What the way to make a vector to an unit vector.

$$\hat{e}_u = \frac{\partial \vec{r}/\partial u}{|\partial \vec{r}/\partial u|} \; ; \; \hat{e}_v = \frac{\partial \vec{r}/\partial v}{|\partial \vec{r}/\partial v|}$$

$$\hat{e}_w = \frac{\partial \vec{r}/\partial w}{|\partial \vec{r}/\partial w|}$$

$$\hat{e}_u = \frac{1}{h_u} \cdot \frac{\partial \vec{r}}{\partial u} \quad \text{where} \quad h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|$$

$$\hat{e}_v = \frac{1}{h_v} \cdot \frac{\partial \vec{r}}{\partial v} \quad \quad " \quad \quad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|$$

$$\hat{e}_w = \frac{1}{h_w} \cdot \frac{\partial \vec{r}}{\partial w} \quad \quad \quad \quad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

$\left.\begin{array}{l} h_u \\ h_v \\ h_w \end{array}\right\}$ Scaling factor.

Curvilinear $\longrightarrow$ $d\vec{r} = h_u \underset{\text{unit vector}}{\hat{e}_u} du + h_v \cdot \underset{\text{unit vector}}{\hat{e}_v} dv + h_w \cdot \underset{\text{unit vector}}{\hat{e}_w} dw$

Cartesian $\longrightarrow$ $d\vec{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$

$$\hat{e}_u = \frac{1}{h_u} \cdot \frac{\partial \vec{r}}{\partial u}$$

In Cartesian system these u, v, w are nothing but x, y, z

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

w.r.t $\frac{\partial \vec{r}}{\partial x} = \hat{i}$

$$\hat{e}_x = \frac{1}{h_x} \cdot \frac{\partial \vec{r}}{\partial x}$$

$$h_x = \left|\frac{\partial \vec{r}}{\partial x}\right| = 1$$

$$\hat{e}_x = \frac{\hat{i}}{1} = \hat{i}$$

Spherical coordinate $(u = r;\ v = \theta;\ w = \phi)$

Cylindrical coordinate $(u = r;\ v = \theta;\ w = z)$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$ — Cartesian

$x \rightarrow r\sin\theta\cos\phi$

$y \rightarrow r\sin\theta\sin\phi$

$z \rightarrow r\cos\theta$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$ — Spherical coordinate

From Curvilinear coordinate w.r.t

$$d\vec{r} = h_r\hat{e}_r\,dr + h_\theta\hat{e}_\theta\,d\theta + h_\phi\hat{e}_\phi\,d\phi$$

$$\hat{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r}\ ;\quad \hat{e}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta}\ ;\quad \hat{e}_\phi = \frac{1}{h_\phi}\frac{\partial \vec{r}}{\partial \phi}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\frac{\partial \vec{r}}{\partial r} \cdot \frac{\partial \vec{r}}{\partial r}}$$

dot product

$$= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(1) + \cos^2\theta}$$

$$h_r = \sqrt{1} = 1 /.$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{\frac{\partial \vec{r}}{\partial \theta} \cdot \frac{\partial \vec{r}}{\partial \theta}}$$

$$= \left[r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\right]^{1/2}$$

$$= \left[r^2\cos^2\theta\,(\underbrace{\cancel{\cos^2\phi + \sin^2\phi}}_{1}) + r^2\sin^2\theta\right]^{1/2}$$

$$= \left[r^2[\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}]\right]^{1/2}$$

$$h_\theta = \sqrt{r^2} = r$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\,\hat{k}$$

$$= -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

$$h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{\frac{\partial \vec{r}}{\partial \phi}\cdot\frac{\partial \vec{r}}{\partial \phi}}$$

$$= \left[r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi\right]^{1/2}$$

$$= \left[r^2 \sin^2\theta \left(\cancelto{1}{\sin^2\phi + \cos^2\phi}\right)\right]^{1/2}$$

$$h_\phi = r\sin\theta$$

$$\hat{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} = \frac{1}{1}\cdot \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \frac{1}{h_\theta}\cdot\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}\left(r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}\right)$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{1}{h_\phi}\cdot\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}\cdot\left(-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}\right)$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Let find the Gradient of a Scalar fn $= \phi$ in curvilinear system.

$$\phi = \phi(u, v, w)$$

$$d\phi = \frac{\partial \phi}{\partial u}\cdot du + \frac{\partial \phi}{\partial v}\cdot dv + \frac{\partial \phi}{\partial w}\cdot dw$$

Orthogonality property

$\hat{i}\cdot\hat{i} = 1$; $\hat{j}\cdot\hat{j} = 1$; $\hat{k}\cdot\hat{k} = 1$ $\longleftarrow$ Cartesian

$\hat{e}_u\cdot\hat{e}_u = 1$; $\hat{e}_v\cdot\hat{e}_v = 1$; $\hat{e}_w\cdot\hat{e}_w = 1$ $\longleftarrow$ Curvilinear

$$d\phi = \left(\frac{\partial\phi}{\partial u}\cdot\hat{e}_u + \frac{\partial\phi}{\partial v}\cdot\hat{e}_v + \frac{\partial\phi}{\partial w}\cdot\hat{e}_w\right)\cdot\left(\underbrace{du\cdot\hat{e}_u + dv\cdot\hat{e}_v}\, \underbrace{+\, dw\cdot\hat{e}_w}\right)$$

$$d\vec{r} = h_u\hat{e}_u du + h_v\hat{e}_v dv + h_w\hat{e}_w dw$$

$$= \left(\frac{h_u}{h_u}\cdot\frac{\partial\phi}{\partial u}\hat{e}_u + \frac{h_v}{h_v}\frac{\partial\phi}{\partial v}\cdot\hat{e}_v + \frac{h_w}{h_w}\frac{\partial\phi}{\partial w}\hat{e}_w\right)$$

$$\cdot\left(du\hat{e}_u + dv\hat{e}_v + dw\hat{e}_w\right)$$

dot product

$$= \left(\frac{1}{h_u}\frac{\partial\phi}{\partial u}\cdot\hat{e}_u + \frac{1}{h_v}\cdot\frac{\partial\phi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial\phi}{\partial w}\hat{e}_w\right)\cdot$$

$$\left(h_u\hat{e}_u du + h_v\hat{e}_v dv + h_w\hat{e}_w dw\right)$$

$$d\phi = \underbrace{\left(\frac{1}{h_u}\frac{\partial}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial}{\partial v}\cdot\hat{e}_v + \frac{1}{h_w}\frac{\partial}{\partial w}\hat{e}_w\right)}_{\to\ \vec{\nabla}\text{ in curvilinear.}}\phi\cdot d\vec{r}$$

In Cartesian system

$$d\phi = \vec{\nabla}\phi\cdot d\vec{r}$$

The vector differential operator "$\vec{\nabla}$" in curvilinear system

$$\vec{\nabla} = \left(\frac{\hat{e}_u}{h_u}\frac{\partial}{\partial u} + \frac{\hat{e}_v}{h_v}\frac{\partial}{\partial v} + \frac{\hat{e}_w}{h_w}\cdot\frac{\partial}{\partial w}\right)$$

Del operator in curvilinear system.

Spherical coordinate (r, θ, ϕ) (u, v, ω)

$$h_r = 1 \; ; \; h_\theta = r \; ; \; h_\phi = r\sin\theta$$

$$\vec{\nabla} f(r, \theta, \phi) = \frac{\hat{e}_r}{1} \cdot \frac{\partial f}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{\hat{e}_\phi}{r\sin\theta} \frac{\partial f}{\partial \phi}$$

Let $\vec{F}$ be a vector field in a curvilinear system:

$$\vec{F} = \vec{F}(u, v, w) = F_1 \hat{e}_u + F_2 \hat{e}_v + F_3 \cdot \hat{e}_\omega \longleftarrow \text{curvilinear}$$

$$\vec{F} = \vec{F}(x, y, z) = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \longleftarrow \text{Cartesian}$$

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\hat{e}_u}{h_u} \frac{\partial}{\partial u} + \frac{\hat{e}_v}{h_v} \cdot \frac{\partial}{\partial v} + \frac{\hat{e}_\omega}{h_\omega} \frac{\partial}{\partial \omega} \right) \cdot \left(F_1 \hat{e}_u + F_2 \hat{e}_v + F_3 \hat{e}_\omega \right)$$

$$\hat{e}_u \cdot \hat{e}_u = 1 \; \ldots \; \hat{e}_\omega \cdot \hat{e}_\omega = 1$$

$$= \left(\frac{1}{h_u} \cdot \frac{\partial F_1}{\partial u} + \frac{1}{h_v} \frac{\partial F_2}{\partial v} + \frac{1}{h_\omega} \frac{\partial F_3}{\partial \omega} \right)$$

$$\underbrace{\vec{\nabla} \cdot \vec{F}}_{\text{Curvilinear coordinate..}} = \frac{1}{h_u h_v h_\omega} \left[\frac{\partial F_1}{\partial u} \cdot h_v \cdot h_\omega + \frac{\partial F_2}{\partial v} \cdot h_u \cdot h_\omega + \frac{\partial F_3}{\partial \omega} \cdot h_u \cdot h_v \right]$$

Example spherical coordinate $(u = r \; ; \; v = \theta \; ; \; \omega = \phi)$ $\quad h_r = 1 \; ; \; h_\theta = r \; ; \; h_\phi = r\sin\theta$

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{h_r h_\theta h_\phi} \cdot \left[\frac{\partial F_1}{\partial r} h_\theta h_\phi + \frac{\partial F_2}{\partial \theta} h_r \cdot h_\phi + \frac{\partial F_3}{\partial \phi} h_r \cdot h_\theta \right]$$

$$= \frac{1}{r^2 \sin\theta} \left[\frac{\partial f_1}{\partial r} \cdot r^2 \sin\theta + \frac{\partial f_2}{\partial \phi} \cdot r \sin\theta + \frac{\partial f_3}{\partial \phi} \cdot r \right]$$

$$= \frac{1}{r^2 \cancel{\sin\theta}} \frac{\partial f_1}{\partial r} \cdot r^2 \cancel{\sin\theta} + \frac{1}{r^2 \sin\theta} \frac{\partial f_2}{\partial \phi} \cancel{r} \sin\theta$$

$$+ \frac{1}{r^2 \sin\phi} \frac{\partial f_3}{\partial \phi} \cdot \cancel{r} \Big]$$

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial f_1}{\partial r} r^2 + \frac{1}{r} \frac{\partial f_2}{\partial \phi} \cdot \sin\theta + \frac{1}{r \sin\theta} \frac{\partial f_3}{\partial \phi}$$

Similarly the curl of $\vec{F}$ in curvilinear system

$$\vec{\nabla} \times \vec{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u f_1 & h_v f_2 & h_w f_3 \end{vmatrix}$$