Solving Partial Differential Equation Using F.T

$$F\left\{\frac{d}{dx}f(x)\right\} = -is\, F\{f(x)\}$$

$$F\left\{\frac{d^n}{dx^n}f(x)\right\} = (-is)^n F\{f(x)\}$$

If our fn = is a fn = of 2 variables then $u(x,t)$

$$F\left\{\frac{\partial^n}{\partial x^n}u(x,t)\right\} = (-is)^n F\{u(x,t)\}$$

$$\boxed{F\left\{\frac{\partial^2}{\partial x^2}u(x,t)\right\} = -s^2 F\{u(x,t)\}}$$

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = s\cdot \underbrace{(u(x,t))\Big|_{x=0}}_{u(0,t)} - s^2 F_s\{u(x,t)\}$$

$$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\underbrace{\left(\frac{\partial u}{\partial x}\right)\Big|_{x=0}}_{u'(0,t)} - s^2 F_c\{u(x,t)\}$$

$$\frac{\partial u}{\partial x} \to u'(0,t)$$

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Solution of heat conduction problems by Fourier sine Transforms

Example 31. *Solve the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x > 0, t > 0$$

subject to the conditions

(i) $u = 0$ when $x = 0,\ t > 0$

(ii) $u = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$ when $t = 0$

(iii) $u(x, t)$ is bounded.

$$u(\overset{x}{0}, t) = 0$$

$$\left.\begin{aligned} u(x,t) &= 1 \quad \text{when } x < 1 \\ u(x,t) &= 0 \quad \text{when } x \geq 1 \end{aligned}\right\} \text{ when } t = 0$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

Let take F_s on both sides in (1)

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\frac{\partial}{\partial t}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin sx\, dx\right) = s \cdot u(0,t) - s^2 \cdot F_s\{u(x,t)\}$$

$$\frac{\partial}{\partial t}\underbrace{F_s\{u(x,t)\}}_{\bar{u}} = s \cdot u(0,t) - s^2 \cdot \underbrace{F_s\{u(x,t)\}}_{\bar{u}}$$

$$\frac{\partial \bar{u}}{\partial t} = s \cdot \cancel{u(0,t)}^{\,0} - s^2 \bar{u}$$

$$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad (\text{or})$$

$$\frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \leftarrow$$ Linear first order Differential Equation

$$\frac{\partial x}{\partial t} + kx = 0$$

general solution

$$x = A \cdot e^{-kt}$$

$$\bar{u} = A e^{-s^2 t} \quad \text{———} \quad (2)$$

We also know $\Big\} \rightarrow$ $\bar{u} = F_s\left\{u(x,t)\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx$

$$\therefore \bar{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx$$

$$\bar{u}(x,t) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 u(x,t) \sin sx \, dx + \int_1^\infty \cancel{u(x,t) \sin sx \, dx} \right]$$

When $t = 0$
$$\begin{cases} u(x,0) = 1 & \text{for } x < 1 \\ u(x,0) = 0 & \text{for } x \geq 1 \end{cases}$$

by this condition

$$\bar{u}(x,0) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 u(x,0) \sin sx \, dx + \int_1^\infty \cancel{u(x,0) \sin sx \, dx} \right]$$

0

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx \Rightarrow \sqrt{\frac{2}{\pi}} \left(-\frac{\cos sx}{s} \right)_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{-\cos s}{s} + \frac{1}{s} \right)$$

$$\bar{u}(x,0) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \quad \text{——— (3)}$$

From (2) we. k.T

$$\bar{u}(x,t) = Ae^{-s^2 t}$$

$$\bar{u}(x,0) = A\cdot e^{-s^2\cdot 0}$$

$$\bar{u}(x,0) = A. \quad \text{——— (4)}$$

Comparing (3) & (4) we have

$$A = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \quad \text{——— (5)}$$

Sub. value of A in (2) we have

$$\bar{u}(x,t) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)\cdot e^{-s^2 t}$$

$$F_s\left\{\underline{u(x,t)}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)\cdot e^{-s^2 t}$$

Taking Inverse F_s Transform

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1-\cos s}{s}\right)\cdot e^{-s^2 t}\cdot \sin xs\, ds$$

Solution.

Example 33. *Solve* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for* $0 \le x < \infty, t > 0$ *given the conditions*

(i) $u(x, 0) = 0$ *for* $x \ge 0$ *(ii)* $\frac{\partial u}{\partial x}(0,t) = -a$ *(constant)*

(iii) u(x, t) is bounded.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

@ $t=0$

$$u(x,0) = 0$$

$u(\overset{x}{0},t) \leftarrow f_s$

$u'(\overset{x}{0},t) \leftarrow f_c$

& on $t>0$

$$\frac{\partial u}{\partial x}(0,t) = -a$$

Lets take f_c transform on both side of (1)

$$f_c\left\{\frac{\partial u}{\partial t}\right\} = k f_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

using the formula

$$\frac{\partial}{\partial t} f_c\{u(x,t)\} = k\left(\underbrace{-\left.\frac{\partial u}{\partial x}\right|_{x=0}}_{=-a} - s^2 f_c\{u(x,t)\}\right)$$

$$\frac{\partial}{\partial t}\underbrace{f_c\{u(x,t)\}}_{\bar{u}} = ka - s^2 k\underbrace{f_c\{u(x,t)\}}_{\bar{u}}$$

$$\frac{\partial}{\partial t}\underbrace{\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\,dx}_{\bar{u}} = ka - s^2 k\underbrace{\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\,dx}_{\bar{u}}$$

$$\frac{\partial \bar{u}}{\partial t} = ka - s^2 k\bar{u}$$

$$\underbrace{\frac{\partial \bar{u}}{\partial t}}_{} + \underbrace{s^2 k}_{P}\,\bar{u} = \underbrace{ka}_{Q} \quad \text{——— (2)}$$

General solution

$$\boxed{\begin{gathered} \frac{\partial y}{\partial x} + Py = Q \\ y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx} + c \end{gathered}}$$

For our case $P = s^2 k$; $dx = dt$

$$\therefore \int P\,dx = \int s^2 k\,dt$$

$$= s^2 k \cdot \int dt$$

$$\int P\,dx = s^2 k t$$

$$\bar{u} \cdot e^{s^2 k t} = \int ka \cdot e^{s^2 k t}\,dt + c$$

$$= ka \cdot \int e^{s^2 k t}\,dt + c$$

$$= \cancel{k} a \cdot \frac{e^{s^2 k t}}{s^2 \cancel{k}} + c$$

$$\bar{u} \cdot e^{s^2 k t} = \frac{a}{s^2} e^{s^2 k t} + c$$

$\times\ e^{-s^2 k t}$

$$\bar{u}(x,t) = \frac{a}{s^2} + c \cdot e^{-s^2 k t} \quad \text{—— (3)}$$

@ $t = 0$

$$\bar{u}(x,0) = \frac{a}{s^2} + c \quad \text{—— (4)}$$

W.K.T

$$\bar{u}(x,0) = F_c\{u(x,0)\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cancel{u(x,0)}^{\,0} \cos sx \, dx$$

$$\bar{u}(x,0) = 0 \longleftarrow \text{Defin' of } \bar{u}$$

$$\text{——— (5)}$$

Comparing (5) & (4) we have

$$0 = \frac{a}{s^2} + c$$

$$c = -\frac{a}{s^2} \quad \text{——— (6)}$$

Sub (6) in (3) we have

$$\bar{u}(x,t) = \frac{a}{s^2} - \frac{a}{s^2} \cdot e^{-s^2 k t}$$

$$\bar{u}(x,t) = \frac{a}{s^2}\left(1 - e^{-s^2 k t}\right)$$

$$F_c\{u(x,t)\} = \frac{a}{s^2}\left(1 - e^{-s^2 k t}\right)$$

Taking Inverse F_c transform we have

$$u(x,t) = a \int_0^\infty \frac{1 - e^{-s^2 k t}}{s^2} \cdot \cos sx \, ds$$