Lecture 21: More on Divergence Theorem; Line Integral Definition of Curl *(RHB 9.8, 9.7)*

21. 1. Worked Examples of use of Divergence Theorem

Let $\underline{A} = (y - x)\underline{e}_1 + x^2 z\underline{e}_2 + z\underline{e}_3$. Calculate the surface integral $I = \int_S \underline{A} \cdot \underline{dS}$ over the *open* curved surface of the cylinder $x^2 + y^2 = a^2$ bounded by $z = 0$ and $z = 1$.

Writing the *closed* surface of the cylinder as the open curved surface S_C plus the top face S_T and bottom face S_B, the divergence theorem tells us

$$\int_V \underline{\nabla} \cdot \underline{A}\, dV = \int_{S_C} \underline{A} \cdot \underline{dS} + \int_{S_T} \underline{A} \cdot \underline{dS} + \int_{S_B} \underline{A} \cdot \underline{dS}$$

But for the given $\underline{A}$ we find $\underline{\nabla} \cdot \underline{A} = -1 + 0 + 1 = 0$ therefore we find

$$\int_{S_C} \underline{A} \cdot \underline{dS} = -\int_{S_T} \underline{A} \cdot \underline{dS} - \int_{S_B} \underline{A} \cdot \underline{dS}$$

At the top ($z = 1$) of the cylinder $\underline{dS} = \underline{e}_3\, dS$ and $\underline{A} \cdot \underline{dS} = z\, dS = dS$

$$\therefore \quad \int_{S_T} \underline{A} \cdot \underline{dS} = \pi a^2$$

Similarly the integral over the bottom face where $z = 0$ gives zero. Therefore our result is

$$\int_{S_C} \underline{A} \cdot \underline{dS} = -\pi a^2$$

Volume of a body

Consider the volume of a body:

$$V = \int_V dV$$

Recalling that $\underline{\nabla} \cdot \underline{r} = 3$ we can write

$$V = \frac{1}{3} \int_V \underline{\nabla} \cdot \underline{r}\, dV$$

which using the divergence theorem becomes

$$V = \frac{1}{3} \int_S \underline{r} \cdot \underline{dS}$$

Example Consider the hemisphere $x^2 + y^2 + z^2 \leq a^2$ centered on $\underline{e}_3$ with bottom face at $z = 0$. Recalling that the divergence theorem holds for a *closed* surface, the above equation for the volume of the hemisphere tells us

$$\frac{2\pi a^3}{3} = \frac{1}{3} \left[\int_{hemisphere} \underline{r} \cdot \underline{dS} + \int_{bottom} \underline{r} \cdot \underline{dS} \right] .$$

On the bottom face $\underline{dS} = -\underline{e}_3\, dS$ therefore $\underline{r} \cdot \underline{dS} = -z\, dS = 0$ since $z = 0$. Hence the only contribution comes from the (open) surface of the hemisphere and we see

$$2\pi a^3 = \int_{hemisphere} \underline{r} \cdot \underline{dS} .$$

We can verify this directly by using spherical polars to evaluate the surface integral. As was derived in lecture 19, for a hemisphere of radius a

$$\underline{dS} = a^2 \,\sin\theta \, d\theta \, d\phi \, \underline{e}_r \ .$$

On the hemisphere $\underline{r} \cdot \underline{dS} = a^3 \,\sin\theta \, d\theta \, d\phi$

$$\therefore \quad \int_S \underline{r} \cdot \underline{dS} = a^3 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 2\pi a^3$$

as predicted by the divergence theorem.

21. 2. Line Integral Definition of Curl

Let ΔS be a small planar surface containing the point P, bounded by a **closed** curve C, with unit normal $\underline{n}$ and (scalar) area ΔS.

Let $\underline{A}$ be a vector field defined on ΔS, then the component of $\underline{\nabla} \times \underline{A}$ parallel to $\underline{n}$ is defined to be

$$\boxed{\underline{n} \cdot (\underline{\nabla} \times \underline{A}) \ = \ \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \underline{A} \cdot \underline{dr}}$$

NB The integral around C is taken in the right-hand sense with respect to the normal $\underline{n}$ to the surface – as in the figure above.

This definition of curl is **independent of the choice of basis**.

21. 3. Cartesian form of **curl** $\underline{A}$

Let P be a point with cartesian coordinates (x_0, y_0, z_0) situated at the *centre* of a small rectangle $ABCD$ of size $\delta_1 \times \delta_2$, area $\Delta S = \delta_1 \, \delta_2$, in the $(\underline{e}_1 - \underline{e}_2)$ plane.

The line integral around C is given by the sum of four terms

$$\oint_C \underline{A} \cdot \underline{dr} \ = \ \int_A^B \underline{A} \cdot \underline{dr} + \int_B^C \underline{A} \cdot \underline{dr} + \int_C^D \underline{A} \cdot \underline{dr} + \int_D^A \underline{A} \cdot \underline{dr}$$

Since $\underline{r} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$, we have $\underline{dr} = \underline{e}_1 \, dx$ along DA and CB, and $\underline{dr} = \underline{e}_2 \, dy$ along AB and DC. Therefore

$$\oint_C \underline{A} \cdot \underline{dr} \ = \ \int_A^B A_2 \, dy \ - \int_C^B A_1 \, dx \ - \int_D^C A_2 \, dy \ + \int_D^A A_1 \, dx$$

For small δ_1 & δ_2, we can Taylor expand the integrands, viz

$$\begin{aligned}
\int_D^A A_1\,dx &= \int_D^A A_1(x, y_0 - \delta_2/2, z_0)\,dx \\
&= \int_{x_0-\delta_1/2}^{x_0+\delta_1/2} \left[A_1(x, y_0, z_0) - \frac{\delta_2}{2}\,\frac{\partial A_1(x, y_0, z_0)}{\partial y} + O(\delta_2^2) \right] dx \\
\int_C^B A_1\,dx &= \int_C^B A_1(x, y_0 + \delta_2/2, z_0)\,dx \\
&= \int_{x_0-\delta_1/2}^{x_0+\delta_1/2} \left[A_1(x, y_0, z_0) + \frac{\delta_2}{2}\,\frac{\partial A_1(x, y_0, z_0)}{\partial y} + O(\delta_2^2) \right] dx
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{\Delta S}\left[\int_D^A \underline{A}\cdot\underline{dr} + \int_B^C \underline{A}\cdot\underline{dr}\right] &= \frac{1}{\delta_1\,\delta_2}\left[\int_D^A A_1\,dx - \int_C^B A_1\,dx\right] \\
&= \frac{1}{\delta_1\delta_2}\int_{x_0-\delta_1/2}^{x_0+\delta_1/2}\left[-\delta_2\frac{\partial A_1(x, y_0, z_0)}{\partial y} + O(\delta_2^2)\right] dx \\
&\to -\frac{\partial A_1(x_0, y_0, z_0)}{\partial y} \quad \text{as} \quad \delta_1,\, \delta_2 \to 0
\end{aligned}$$

A similar analysis of the line integrals along AB and CD gives

$$\frac{1}{\Delta S}\left[\int_A^B \underline{A}\cdot\underline{dr} + \int_C^D \underline{A}\cdot\underline{dr}\right] \to \frac{\partial A_2(x_0, y_0, z_0)}{\partial x} \quad \text{as} \quad \delta_1,\, \delta_2 \to 0$$

Adding the results gives for our line integral definition of curl yields

$$\underline{e}_3\cdot(\underline{\nabla}\times\underline{A}) = (\underline{\nabla}\times\underline{A})_3 = \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]\Bigg|_{(x_0,\, y_0,\, z_0)}$$

in agreement with our original definition in Cartesian coordinates.

The other components of curl $\underline{A}$ can be obtained from similar rectangles in the ($\underline{e}_2$–$\underline{e}_3$) and ($\underline{e}_1$–$\underline{e}_3$) planes, respectively.

21. 4. Physical Interpretation of Curl (Dawber 6.2)

Let's illustrate the meaning of curl in physics by means of a simple example (see figure).

Let $\underline{F} = ay\underline{e}_1$ so that $\dfrac{\partial F_1}{\partial y} \neq 0$; C is a contour in the $(\underline{e}_1 - \underline{e}_2)$ plane, enclosing an area ΔS.

The **work done** on a test particle in moving it around the closed curve C is

$$\boxed{\oint_C \underline{F} \cdot \underline{dr} \quad = \textbf{circulation} \text{ of } \underline{F} \text{ about } C}$$

(Remember that when the line integral of $\underline{F}$ around a closed curve is non-zero the force is non-conservative which implies curl $\underline{F}$ is non-zero.)

The integral is given by the sum of four terms

$$\begin{aligned} \oint_C \underline{F} \cdot \underline{dr} &= \int_A^B \underline{F} \cdot \underline{dr} + \int_B^C \underline{F} \cdot \underline{dr} + \int_C^D \underline{F} \cdot \underline{dr} + \int_D^A \underline{F} \cdot \underline{dr} \\ &= \int_A^B F_1 \, dx - \int_D^C F_1 \, dx, \end{aligned}$$

the 2nd and 4th terms vanish because $\underline{F} \cdot \underline{dr} = 0$ along the vertical lines BC and DA.

In the example (see figure), $F_1 = ay \quad \dfrac{\partial F_1}{\partial y} \neq 0$, therefore $\displaystyle\int_A^B F_1 \, dx \neq \int_D^C F_1 \, dx$. Hence

$$\oint_C \underline{F} \cdot \underline{dr} \neq 0.$$

But, from the integral definition of curl, we know that

$$\oint_C \underline{F} \cdot \underline{dr} \approx (\text{curl}\underline{F})_3 \, \Delta S$$

Therefore: $(\text{curl } \underline{F})_3 \neq 0$ (in our example it is less than zero) $\iff$ A non-zero amount of work is done in moving the test particle around the small closed path. Alternatively one can think of the non-zero circulation of $\underline{F}$ causing a small test particle to **rotate** about its centre with axis of rotation in the $-\underline{e}_3$ direction (using r.h. thumb rule).

Thus, in general, $\underline{n} \cdot (\text{curl } \underline{F})$ is a measure of the **circulation** of the vector field $\underline{F}$ about an **infinitesimal** area with normal $\underline{n}$. It can be shown that curl $\underline{F}$, when defined in this way, is independent of the shape of the infinitesimal area ΔS.