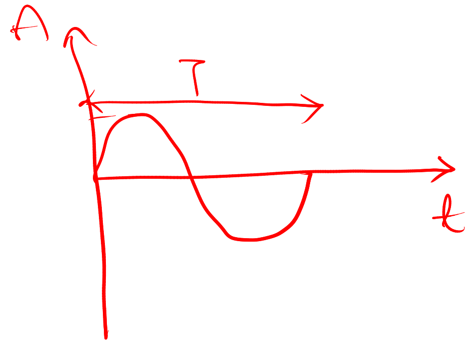
Fourier Series:



i) finite extrema

ii) absolute integrable over a period.

iii) and has only finite discontinuity

Fourier Series of a fn = $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

D.C part

a_1 and b_1 are the fundamental frequency

$a_2, a_3 \dots$ and $b_2, b_3 \dots$ are called the harmonics.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

Some Important Integral:

(i) $\int_0^{2\pi} \sin nx\,dx = 0$

ii) $\int_0^{2\pi} \cos nx\,dx = 0$

iii) $\int_0^{2\pi} \sin^2 nx\,dx = \pi$

iv) $\int_0^{2\pi} \cos^2 nx\,dx = \pi$

v) $\int_0^{2\pi} \sin nx \sin mx\,dx = 0$

vi) $\int_0^{2\pi} \cos nx \cos mx\,dx = 0$

vii) $\int_0^{2\pi} \sin nx \cos mx\,dx = 0$

viii) $\int_0^{2\pi} \sin nx \cos nx\,dx = 0$

$$\int u\,dv = uv - \int v\,du$$

Shortcut for choosing "u" for the integral

$\int u\,dv$

L I A T E

Log Inverse Algebric Trignometric Exponential

Ex: 1

Find the F.S for the fn: $f(x) = x$

$$0 < x < 2\pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots$$

a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x\,dx \Rightarrow \frac{1}{\pi}\left.\frac{x^2}{2}\right|_0^{2\pi}$$

$$a_0 = 2\pi //$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} \overset{u}{x}\,\overset{dv}{\cos nx\,dx}$$

$u = x$; $dv = \cos nx\,dx$

$du = dx$; $v = \int \cos nx\,dx = \frac{\sin nx}{n}$

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\left. x\cdot\frac{\sin nx}{n}\right|_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x \sin nx}{n}\Bigg|_0^{2\pi} - \frac{1}{n}\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n^2}\left[\cos 2\pi n - 1\right]\right]$$

$$= \frac{1}{n^2\pi}\left[\cos(2n)\pi - 1\right]$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$\cos 2n\pi = (-1)^{2n} = 1$$

$$= \frac{1}{n^2\pi}\left[1-1\right] = 0 \quad // \qquad \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} \overset{u}{x}\,\overset{dv}{\sin nx\,dx}$$

$$u = x \qquad\qquad dv = \sin nx\,dx$$

$$du = dx \qquad\qquad v = \int \sin nx\,dx = -\frac{\cos nx}{n}$$

$$\frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\,\frac{(-\cos nx)}{n} + \frac{1}{n^2}\cancelto{0}{\sin nx}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n}(-1)\right] \Rightarrow -\frac{2}{n} \quad //$$

$$\boxed{b_n = -\frac{2}{n}}$$

$$a_0 = 2\pi \quad ; \quad a_n = 0 \quad ; \quad b_n = -\frac{2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} \cancel{a_n \cos nx}^{\,0} + \sum_{n=1}^{p} b_n \sin nx$$

$$\boxed{f(x) = \frac{2\pi}{2} - 2\sum_{n=1}^{p} \frac{\sin nx}{n}}$$

Ex: 2 $\quad f(x) = x + x^2 \qquad -\pi < x < \pi$

using this

$-\pi \to 0 \to$ length π $\Big\}$ 2π

$0 \to \pi \to$ " π

P.T $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (x + x^2)\, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x + x^2) \cos nx\, dx = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (x + x^2) \sin nx\, dx = -\frac{2}{n}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{p} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{p} \frac{(-1)^n}{n} \sin nx \quad \text{——— } (1)$$

$$x+x^2=\frac{\pi^2}{3}+4\left[-\frac{1}{1}\cos 1x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\cdots\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right] \quad —② $$

$x=\pi$ in ② $\qquad \cos n\pi=(-1)^n$

$$\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad —③$$

$x=-\pi$ in ②

$$-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad —④$$

③ + ④ $\Rightarrow$

$$2\pi^2=\frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$2\pi^2-\frac{2\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\pi^2\left[2-\frac{2}{3}\right]= \quad ''$$

$$\frac{4}{3}\pi^2=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \quad //$$

Ex: 3 F-S

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\,dx + \int_{-\pi/2}^{\pi/2}(0)\,dx + \int_{\pi/2}^{\pi} 1\,dx\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\,dx + \int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} \cos nx\,dx\right.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\,dx + \int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} \sin nx\,d\right.$$

Odd & Even fn = Fourier Series

$$-\pi < x < \pi$$

If $f(x)$ is an odd fn = then

$$a_0 = 0 \; ; \; a_n = 0 \; ; \; b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

If $f(x)$ is even fn. then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Ex 4: $f(x) = x^2$ $\quad -\pi < x < \pi$ ✓

$\cancel{0 < x < 2\pi}$ ✗

a_0

a_n

Ex 5: $f(x) = x^3$ $\quad -\pi < x < \pi$

$a_0 = 0 \quad ; \quad a_n = 0$

$b_n = 2/\pi \int_0^{\pi} f(x)\sin nx\,dx$

If $f(x)$ 0 to 2π $\quad a_0, a_n$ & b_n

If $f(x)$ $-\pi$ to π

If $f(x)$ is odd find only b_n

If $f(x)$ is even find only a_0 & a_n