Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **28-NOV-20**

Attendance : **10**

Green's theorem :

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$\vec{F} = P\hat{i} + Q\hat{j}$$

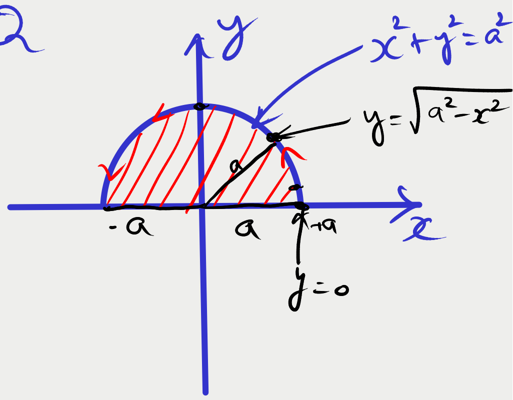
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (P\hat{i} + Q\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

Example 82. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.
(*M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004*)

$$I = \oint_C \underbrace{(2x^2 - y^2)}_{P}\,dx + \underbrace{(x^2+y^2)}_{Q}\,dy$$

$$C : x^2 + y^2 = a^2$$



$$P = 2x^2 - y^2 \quad ; \quad Q = x^2 + y^2$$

$$\frac{\partial P}{\partial y} = -2y \quad ; \quad \frac{\partial Q}{\partial x} = 2x$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (-2y) \Rightarrow 2(x+y)$$

$$= 2\iint_R (x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} (x+y)\,dy$$

$$= 2\int_{-a}^{a} dx \left[\int_0^{\sqrt{a^2-x^2}} x\,dy + \int_0^{\sqrt{a^2-x^2}} y \cdot dy\right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot y \Big|_0^{\sqrt{a^2-x^2}} + \frac{y^2}{2}\Big|_0^{\sqrt{a^2-x^2}}\right]$$

$$= 2\int_{-a}^{a} dx \cdot \left[x \cdot \sqrt{a^2-x^2} + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot \sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right]$$

$$= 2\cdot\int_{-a}^{a} \underbrace{\overbrace{x(a^2-x^2)^{1/2}}^{f_1(x)}}_{f(x)} dx \rightarrow 0 \quad \because f_1(x) \text{ is odd} \quad + 2\int_{-a}^{a} \underbrace{\overbrace{(a^2-x^2)}^{f_2(x)}}_{f(x)} dx$$

$$\boxed{\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even fn} \\ &= 0 \quad \text{if } f(x) \text{ is odd fn} \end{aligned}}$$

If $f(-x) = f(x)$ then $f(x)$ is even

If $f(-x) = -f(x)$ then $f(x)$ is odd

$f_1(x)$ is even or odd ?

$$f_1(x) = x\sqrt{a^2-x^2}$$

$$x = -x$$

$$f_1(-x) = -\overbrace{x\cdot\sqrt{a^2-x^2}}^{f_1(x)}$$

$$f_1(-x) = -f_1(x) \leftarrow \text{odd}$$

$$f_2(x) = a^2-x^2$$

$$f_2(-x) = a^2-(-x)^2$$

$$= \underbrace{a^2-x^2}_{f_2(x)}$$

$$f_2(-x) = f_2(x)$$

$\uparrow$ even

$$= 2\cdot\int_{0}^{a}(a^2-x^2)\,dx$$

$$= 2\cdot\left[a^2\cdot x\Big|_0^a - \frac{x^3}{3}\Big|_0^a\right]$$

$$= 2\cdot\left[a^2\cdot a - \frac{a^3}{3}\right] \Rightarrow 2\cdot\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\cdot\frac{2a^3}{3} \Rightarrow 4\frac{a^3}{3} \;//.$$

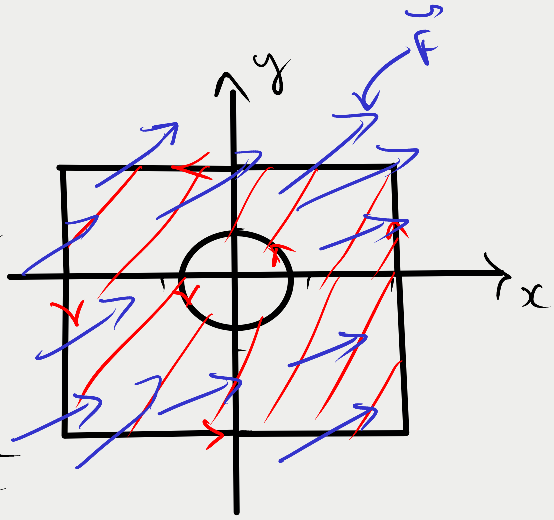
Example 83. *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2 = 1$ *and* $C_2 : x = \pm 2,\ y = \pm 2.$ *(Gujarat, I Semester, Jan 2009)*

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$C_1 : x^2+y^2=1$

$C_2 : x=\pm 2 \ ; \ y=\pm 2$

$$\vec{F} = \underbrace{\frac{-y}{x^2+y^2}}_{P}\hat{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q}\hat{j}$$



$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$P = \frac{-y}{x^2+y^2} \ ; \quad Q = \frac{x}{x^2+y^2}$$

$$P = \underbrace{-y}_{u}\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial P}{\partial y} = +y\cdot(x^2+y^2)^{-1-1}\cdot 2y + (x^2+y^2)^{-1}(-1)$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\cdot\frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{2y^2 - x^2 - y^2}{(x^2+y^2)^2}$$

$$Q = \underbrace{x}_{u}\cdot\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial Q}{\partial x} = x\cdot -1(x^2+y^2)^{-2}\cdot 2x + (x^2+y^2)^{-1}\cdot 1$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2} - \frac{(2y^2 - x^2 - y^2)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2 + x^2 + y^2 - 2y^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{0}{(x^2+y^2)^2} = 0$$

$\therefore$ from Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \cancel{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}^{\,0} dx\,dy$$

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 0$$

By Green's theorem.

From Green's Theorem w.k.T

$$\oint_C P\,dx + Q\,dy = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{= 1} dx\,dy$$

$$\left.\begin{array}{ll} P = -y \quad ; & Q = x \\ \dfrac{\partial P}{\partial y} = -1 \quad ; & \dfrac{\partial Q}{\partial x} = 1 \end{array}\right\}$$

$$\oint_C -y\,dx + x\,dy = \iint_R (1 - (-1))\,dx\,dy$$

$$\oint_C x\,dy - y\,dx = 2\iint_R dx\,dy$$

$ds = dx\,dy$

Area

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \underbrace{\iint_R dx\,dy}$$

$$\boxed{\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Example 84. *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$ (U.P. I, Semester, Dec. 2008)

$C_1 \Rightarrow y = x$ $\qquad$ $C_3 \Rightarrow y = \frac{x}{4}$

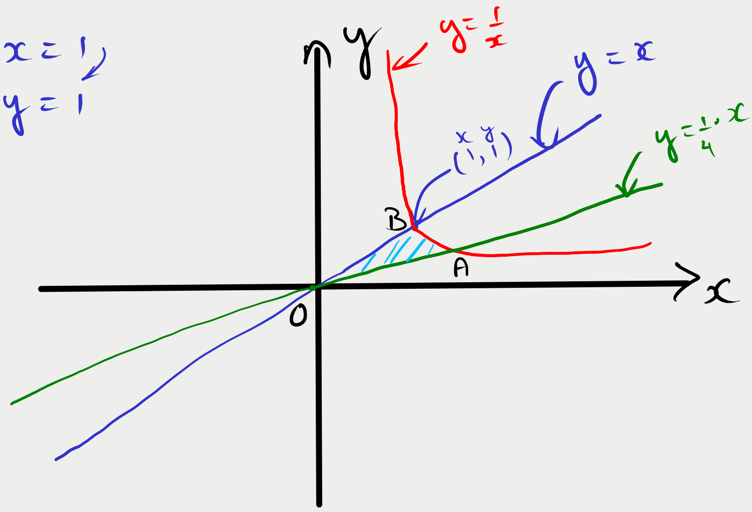
$C_2 \Rightarrow y = \frac{1}{x}$

C_1: $y = x \Rightarrow$ Slope $= 1$

$y = \underset{\substack{\uparrow \\ \text{slope}}}{m}x + c \rightarrow$ intercept

$\left.\begin{array}{l} c = 0 \\ m = 1 \end{array}\right\} \Rightarrow y = x$

when $x = 1$, $y = 1$



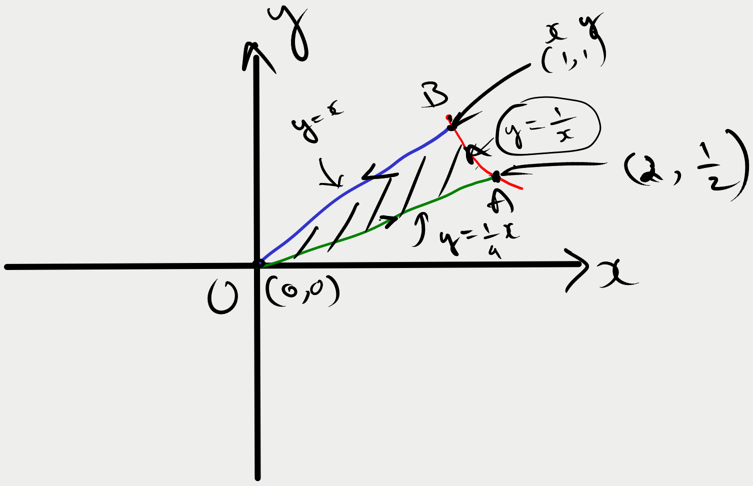
C_2: $y = \frac{1}{x}$

when $x = 0$; $y = \infty$

$x = 1$; $y = 1$

C_3: $y = \frac{1}{4}x$

Slope $m = \frac{1}{4}$



For what value of x; $y = x$ & $y = \frac{1}{x}$ are equal?

$$x = 1 \quad \left.\begin{array}{l} \overset{C_1}{y} = x \\ \& \ \underset{C_2}{y} = \frac{1}{x} \end{array}\right\} \text{are equal}$$

For what value of x; $\underbrace{y = \frac{1}{4}x}_{C_3}$ & $\underbrace{y = \frac{1}{x}}_{C_2}$ are equal?

$\boxed{x = 2}$

$y = \frac{1}{4} \cdot 2 = \frac{1}{2}$

$y = \frac{1}{2} = \frac{1}{2}$

$x = 2$

$y = \frac{1}{2}$

OA $\Rightarrow (0,0), (2, \frac{1}{2})$; AB $\Rightarrow (2, \frac{1}{2})(1, 1)$

$BO \Rightarrow (1,1), (0,0)$

$$\boxed{\text{Area} = \frac{1}{2} \oint_{OAB} x\,dy - y\,dx}$$

This is from Green's theorem formula

$$\oint_{OAB} x\,dy - y\,dx = \int_{OA} x\,dy - y\,dx + \int_{AB} x\,dy - y\,dx + \int_{BO} x\,dy - y\,dx$$

For OA $\quad C_3 : y = \frac{1}{4}x \qquad x_i = 0 \qquad y_i = 0$

$\qquad dy = \frac{1}{4}dx \qquad x_f = 2 \qquad y_f = \frac{1}{2}$

$$\int_{OA} x\,dy - y\,dx = \int_{OA} x \cdot \left(\frac{dx}{4}\right) - \left(\frac{x}{4}\right) \cdot dx$$

$$= \int_0^2 \cancel{\frac{x\,dx}{4} - \frac{x\,dx}{4}} \to 0$$

For AB $\quad C_2 : y = \frac{1}{x} \qquad x_i = 2 \; ; \; y_i = \frac{1}{2}$

$\qquad dy = \frac{-dx}{x^2} \qquad x_f = 1 \; ; \; y_f = 1$

$$\int_{AB} x\,dy - y\,dx = \int_{AB} \cancel{x} \cdot \left(\frac{-dx}{\cancel{x^2}}\right) - \left(\frac{1}{x}\right) \cdot dx$$

$$= \int_2^1 -\frac{dx}{x} - \frac{dx}{x}$$

$$= -2\int_{2}^{1} \frac{dx}{x}$$

$$= -2 \cdot \log x \Big|_{2}^{1} \Rightarrow -2 \cdot -\log x \Big|_{1}^{2}$$

$$= 2\left[\log(2) - \cancel{\log(1)}^{\,0}\right]$$

$$\int_{AB} x\,dx - y\,dy = 2 \cdot \log 2$$

for BO C_1: $y = x$ $\qquad x_i = 1 \;;\; y_i = 1$

$dy = dx \qquad x_f = 0 \;;\; y_f = 0$

$$\int_{BO} x\,dy - y\,dx = \int_{BO} x\,(dx) - (x) \cdot dx$$

$$= \int_{1}^{0} \cancel{x\,dx - x\,dx}^{\,0}$$

line integrals.

$$\oint_{OAB} x\,dy - y\,dx = \underbrace{\int_{OA}}_{0} + \underbrace{\int_{AB}}_{2 \cdot \log 2} + \underbrace{\int_{BO}}_{0}$$

$$\oint_{OAB} x\,dy - y\,dx = 0 + 2\log 2 + 0$$

$$\text{Area} = \frac{1}{2}\oint_{OAB} x\,dy - y\,dx \Rightarrow \text{Area} = \frac{1}{2} \cdot 2\log 2$$

$$\boxed{\text{Area} = \log 2}$$

Stoke's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F} \cdot d\overline{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, d\vec{s} = \iint_S curl\, \vec{F} \cdot d\vec{S}$$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (M.D.U., Dec 2009)

By Stoke's Theorem w.r.t

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \cdot \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

If our $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

and our $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (yz\,dx + zx\,dy + xy\,dz)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}[x - x] - \hat{j}[y - y] + \hat{k}[z - z]$$

$$\vec\nabla \times \vec F = 0$$

$\therefore$ By Stoke's theorem

$$\oint_c \vec F \cdot d\vec r = 0 \quad \because \quad \vec\nabla \times \vec F = 0$$

$$\oint_c \vec f \cdot d\vec r = \iint_S \cancelto{0}{\vec\nabla \times \vec f} \cdot \hat n \, \frac{dx\,dy}{|\hat n \cdot \hat k|}$$

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

$x^2 + y^2 + z^2 = 1$

where c is the circle $x^2 \cancel{+ y^2} = 1$*, corresponding to the surface of* *sphere* *of unit radius.*
(*U.P., I Semester, Winter 2001*)

$x^2 + y^2 + z^2 = 1$

$$\vec F = (2x - y)\,\hat i - yz^2\,\hat j - y^2 z\,\hat k \qquad \vec\nabla \times \vec F$$

$$\hat n = \frac{\vec\nabla \phi}{|\vec\nabla \phi|}$$

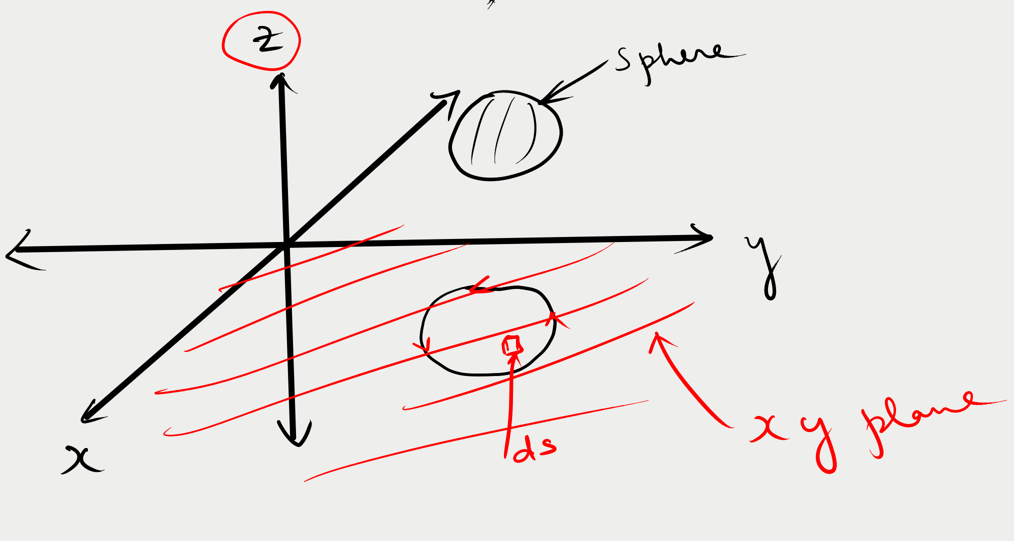
$$\phi = x^2 + y^2 + z^2 - 1$$

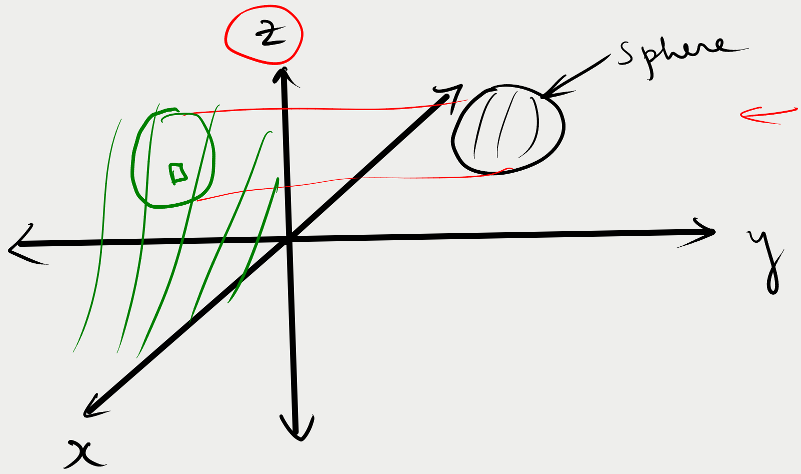
By Stoke's theorem:

$$\hat n \cdot \hat k$$

$$\oint_c \vec f \cdot d\vec r = \iint_S \vec\nabla \times \vec F \cdot \hat n \, \frac{dx\,dy}{|\hat n \cdot \hat k|} \qquad \xleftarrow{} ds$$

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix}$$





$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\cdot dz}{|\hat{n}\cdot\hat{j}|}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$0-(-1)$

$$= \hat{i}\,[\cancel{\quad}\,0] - \hat{j}\,[\cancel{\quad}\,0] + \hat{k}\,[0+1]$$

$$\boxed{\vec{\nabla}\times\vec{F} = \hat{k}}$$

$$\phi = x^2+y^2+z^2-1$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

Given:

$x^2+y^2+z^2=1$

$$= \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}} \quad \to 1$$

$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

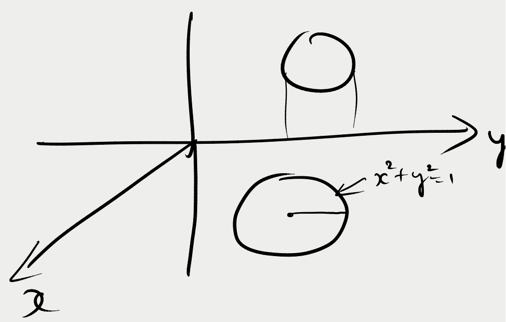
$$\boxed{\hat{n}\cdot\hat{k} = z}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

$$= \iint_S \cdot \hat{k}\cdot\cancel{\hat{n}}^{\,z} \cdot \frac{dx\,dy}{z}$$

$$= \iint_S \cancel{z} \cdot \frac{dx\,dy}{\cancel{z}}$$

$z^2 = 0$



$$\oint_C \tilde{f}\cdot d\tilde{r} = \iint_S dx\,dy$$

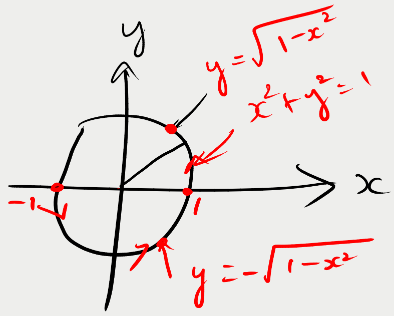
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$y=\pm\sqrt{1-x^2}$

$$= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$



$$= \int_{-1}^{1} dx \cdot y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1}\left(2\cdot\sqrt{1-x^2}\right)dx$$

$$= 2\int_{-1}^{1} \underbrace{\sqrt{1-x^2}}_{f(x)}\,dx$$

$f(x) = \sqrt{1-x^2}$ and it's a even fn $\therefore$

$$\therefore \quad = 2 \cdot 2\int_0^1 \left(\sqrt{1-x^2}\right)dx$$

Formula: ⟵ Pls chk this formula ✗

$$\int_0^1 (a^2-x^2)^{1/2}\,dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \cancel{4}\cdot\frac{1}{\cancel{2}}\left[\underbrace{x(1-x^2)^{1/2}}_{\to 0 \text{ (for both limits)}} + 1\sin^{-1}(x)\right]_0^1$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= \cancel{2}\left[\frac{\pi}{\cancel{2}} - 0\right]$$

$$\oint_c \vec{F}\cdot d\vec{r} = \pi$$

A.//