Lecture 10: Eigenvectors of Real, Symmetric Tensors (*RHB 7.12, 7.13*)

If T is a (2nd-rank) tensor an eigenvector $\underline{n}$ of T obeys (in any basis)

$$\boxed{T(\underline{n}) = t\underline{n}} \qquad \text{or} \qquad \boxed{T_{ij}n_j = t\, n_i}\ .$$

where t is the eigenvalue of the eigenvector.

The tensor acts on the eigenvector to produce a vector in the *same* direction.

The direction of $\underline{n}$ doesn't depend on the basis although its components do (because $\underline{n}$ is a vector) and is sometimes referred to as a **principal axis**; t is a scalar (doesn't depend on basis) and is sometimes referred to as a **principal value**.

10. 1. Construction of the Eigenvectors

Since $n_i = \delta_{ij}n_j$, we can write the equation for an eigenvector as

$$\left(T_{ij} - t\,\delta_{ij}\right) n_j \ = \ 0\,.$$

This has a non-trivial solution (i.e. a solution $\underline{n} \neq 0$) iff

$$\boxed{\det\left(T_{ij} - t\,\delta_{ij}\right) \ \equiv \ 0}\ .$$

i.e.

$$\begin{vmatrix} T_{11}-t & T_{12} & T_{13} \\ T_{21} & T_{22}-t & T_{23} \\ T_{31} & T_{32} & T_{33}-t \end{vmatrix} \ = \ 0\ .$$

This is equation, known as the 'characteristic' or 'secular' equation, is a **cubic** in t, giving 3 real solutions t_A, t_B and t_C.
For a given t we may substitute back into the original equations to find $\underline{A}$, $\underline{B}$ and $\underline{C}$.

Example

$$T_{ij} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The characteristic equation reads

$$\begin{vmatrix} 1-t & 1 & 0 \\ 1 & -t & 1 \\ 0 & 1 & 1-t \end{vmatrix} \ = \ 0.$$

Thus

$$(1-t)\{t(t-1)-1\} - \{(1-t)-0\} \ = \ 0$$

and so

$$(1-t)\{t^2-t-2\} \ = \ 0.$$

$$\boxed{\text{Solutions are } t=1,\ t=2 \text{ and } t=-1\ .}$$

Check The sum of the eigenvalues is 2, and is equal to the **trace** of the tensor; the reason for this will become apparent next lecture.

We now find the eigenvector for each of these eigenvalues, by solving $T_{ij}\, n_j = t n_i$ *i.e.*

$$\begin{array}{rcrcrcl} (1-t)\,n_1 & + & n_2 & & & = & 0 \\ n_1 & - & t\,n_2 & + & n_3 & = & 0 \\ & & n_2 & + & (1-t)\,n_3 & = & 0. \end{array}$$

for $t_A = 1$, $t_B = 2$ and $t_C = -1$.

For $t = t_A = 1$, the equations for $\underline{A}$ are:–

$$\left.\begin{array}{rcrcrcl} & & A_2 & & & = & 0 \\ A_1 & - & A_2 & + & A_3 & = & 0 \\ & & A_2 & & & = & 0 \end{array}\right\} \Longrightarrow A_2 \;=\; 0\,;\, A_3 \;=\; -A_1\,.$$

Thus $A_1 : A_2 : A_3 \;=\; 1 : 0 : -1$ and a *unit* vector in the direction of $\underline{A}$ is

$$\underline{\hat{A}} \;=\; \frac{1}{\sqrt{2}}(1, 0, -1)\ .$$

[Note that we could equally well have chosen $\underline{\hat{A}} \;=\; \frac{-1}{\sqrt{2}}(1, 0, -1)$.]

For $t = t_B = 2$, the equations for $\underline{B}$ are:–

$$\left.\begin{array}{rcrcrcl} -B_1 & + & B_2 & & & = & 0 \\ B_1 & - & 2B_2 & + & B_3 & = & 0 \\ & & B_2 & - & B_3 & = & 0 \end{array}\right\} \Longrightarrow B_2 \;=\; B_3 \;=\; B_1\,.$$

Thus $B_1 : B_2 : B_3 \;=\; 1 : 1 : 1$ and a *unit* vector in the direction of $\underline{B}$ is

$$\underline{\hat{B}} \;=\; \frac{1}{\sqrt{3}}(1, 1, 1)\ .$$

For $t = t_C = -1$, a similar calculation (exercise) gives

$$\underline{\hat{C}} \;=\; \frac{1}{\sqrt{6}}(1, -2, 1)\ .$$

Note that $\underline{\hat{A}} \cdot \underline{\hat{B}} = \underline{\hat{A}} \cdot \underline{\hat{C}} = \underline{\hat{B}} \cdot \underline{\hat{C}} = 0$ and so the eigenvectors are mutually orthogonal.

The scalar triple product of the triad $\underline{\hat{A}}$, $\underline{\hat{B}}$ and $\underline{\hat{C}}$, with the above choice of signs, is -1, and so they form a *left-handed* basis. Changing the sign of *one* (or all three) of the vectors would produce a right-handed basis.

10. 2. Important Theorem and Proof

Theorem If T_{ij} is *real* and *symmetric*, it has *three real* eigenvalues t_A (not necessarily all distinct), with corresponding eigenvectors $\underline{A}$ which are **orthogonal** (for t_A distinct).

Proof If $\underline{A}$ and $\underline{B}$ are 2 eigenvectors, with eigenvalues t_A and t_B, then:–

$$
\begin{aligned}
T_{ij}A_j &= t_A A_i && (1)\\
T_{ij}B_j &= t_B B_i && (2)
\end{aligned}
$$

We multiply equation (1) by B_i^*, and sum over i, giving:–

$$T_{ij}A_jB_i^* = t_A A_i B_i^* \qquad (3)$$

We now take the complex conjugate of equation (2), multiply by A_i and sum over i, to give:–

$$T_{ij}^*B_j^*A_i = t_B^* B_i^* A_i \qquad (4)$$

Since T_{ij} is *real* and *symmetric*, $T_{ij}^* = T_{ji}$, and so:–

$$
\begin{aligned}
\text{L.H. side of equation (4)} &= T_{ji}B_j^*A_i\\
&= T_{ij}B_i^*A_j\\
&= \text{L.H. side of equation (3).}
\end{aligned}
$$

Subtracting (4) from (3) gives:–

$$\boxed{\left\{t_A - t_B^*\right\} A_iB_i^* = 0} \; .$$

For $\underline{A} = \underline{B}$,

$$A_iA_i^* = \sum_{i=1}^{3} |A_i|^2 > 0 \text{ for all non-zero } \underline{A},$$

and so

$$\boxed{t_A = t_A^*} \; .$$

Thus, eigenvalues are real.

Since t is real and T_{ij} are real, real $\underline{A}, \underline{B}$ can be found (i.e. $B_i^* = B_i$).

For $\underline{A} \neq \underline{B}$,

$$\{t_A - t_B\} A_iB_i = 0 \; .$$

If $t_A \neq t_B$, then $A_iB_i = 0$, implying

$$\boxed{\underline{A} \cdot \underline{B} = 0} \; .$$

Thus eigenvectors are orthogonal if the eigenvalues are distinct.

10. 3. Degenerate eigenvalues

If the characteristic equation is of form

$$(t_A - t)(t_B - t)^2 = 0$$

then we have a *doubly degenerate* eigenvalue t_B.

Claim In the case of a real, symmetric tensor for this eigenvalue we can always find TWO orthogonal solutions for $\underline{B}$ (which are both orthogonal to $\underline{A}$).

Example

$$T_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

$$\left| T_{ij} - t\delta_{ij} \right| \;=\; 0 \Longrightarrow \boxed{\; t = 2 \text{ and } t = -1 \text{ (twice)} \;} .$$

For $t = t_A = 2$ $\quad$ {choose $\underline{A}$ to be the eigenvector}

$$\left.\begin{array}{rcrcrcl} -2A_1 & + & A_2 & + & A_3 & = & 0 \\ A_1 & - & 2A_2 & + & A_3 & = & 0 \\ A_1 & + & A_2 & - & 2A_3 & = & 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} A_2 & = & A_3 \;=\; A_1 \\ \underline{\hat{A}} & = & \frac{1}{\sqrt{3}}(1,1,1) \;. \end{array}\right.$$

For $t = t_B = t_C = -1$

$$B_1 + B_2 + B_3 = 0$$

is the only independent equation. This can be written as $\underline{A} \cdot \underline{B} = 0$ *i.e.* equation for a plane normal to $\underline{A}$. Thus any vector orthogonal to $\underline{A}$ is an eigenvector with eigenvalue -1.

If we choose $B_3 = 0$, then $B_2 = -B_1$ and a possible unit eigenvector is

$$\underline{\hat{B}} \;=\; \frac{1}{\sqrt{2}}(1,-1,0) \;.$$

If we require the third eigenvector $\underline{C}$ to be orthogonal to $\underline{B}$, then we must have $C_2 = C_1$. The equations then give $C_3 = -2C_1$ and so

$$\underline{\hat{C}} \;=\; \frac{1}{\sqrt{6}}(1,1,-2) \;.$$

[Alternatively, and more easily, the third eigenvector can be calculated by using $\underline{\hat{C}} = \pm\underline{\hat{A}} \times \underline{\hat{B}}$, the sign chosen determining the handedness of the triad $\underline{\hat{A}}$,$\underline{\hat{B}}$,$\underline{\hat{C}}$.]

This pair, $\underline{B}$ and $\underline{C}$, is just one of an *infinite number* of orthogonal pairs that are eigenvectors of T_{ij} — all lying in the plane normal to $\underline{A}$.

If the characteristic equation is of form

$$(t_C - t)^3 = 0$$

then we have a triply degenerate eigenvalue t_C. In fact, this only occurs if the tensor is equal to $t_C\delta_{ij}$ which means it is 'isotropic' and any direction is an eigenvector with eigenvalue t_C.