Department of Medical Physics
Bharathidasan University

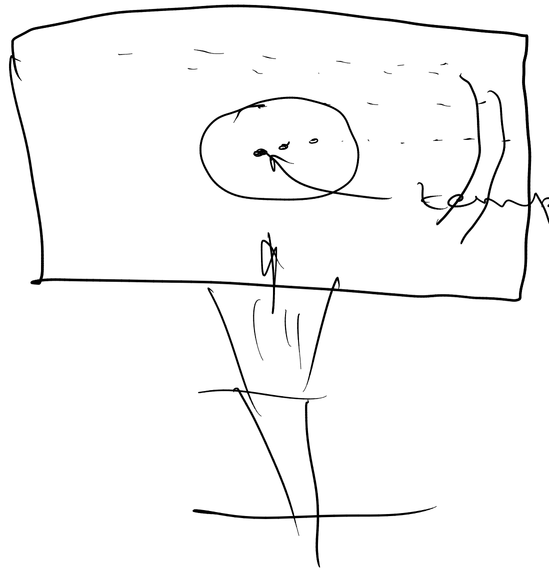
Date : **08-10-2020**

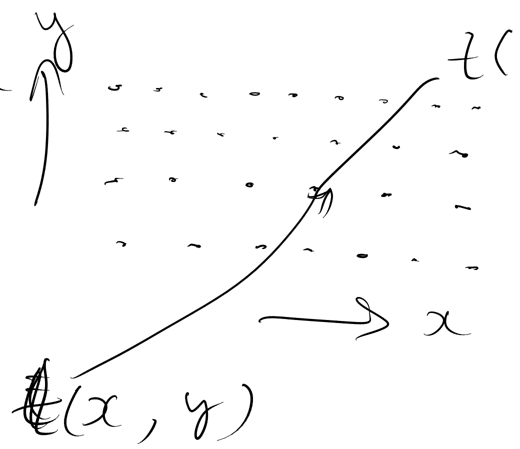
Attendance : **8**

Vector Calculus:

- Coordinate System
- Scalar and Vector fn.
- Scalar and vector product
- Gradient, Divergence, Curl and Laplacian.

Value:

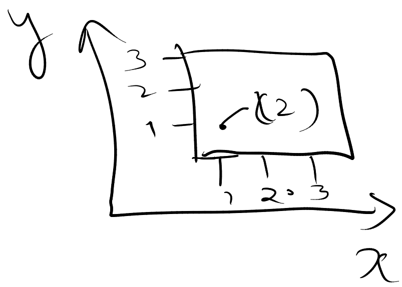




"t" temperature

"p" pressure

"D" density "m" mass



$$\boxed{t(x, y) = x^2 + y^2}$$

$x = 1$
$y = 1$

$\boxed{t = 2}$ $t(1, 1) = 2$

Magnitude & direction (vector quantities)

Eg: 1) Velocity
2) momentum
3) Force
4) Magnetization etc.. } Ex for Vector quan.

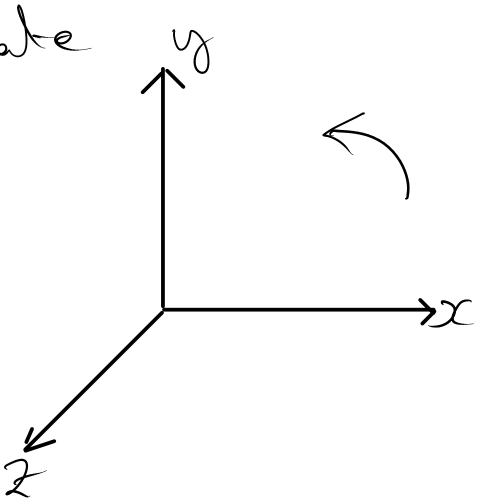
$M(x, y) = x^2 + y^2$

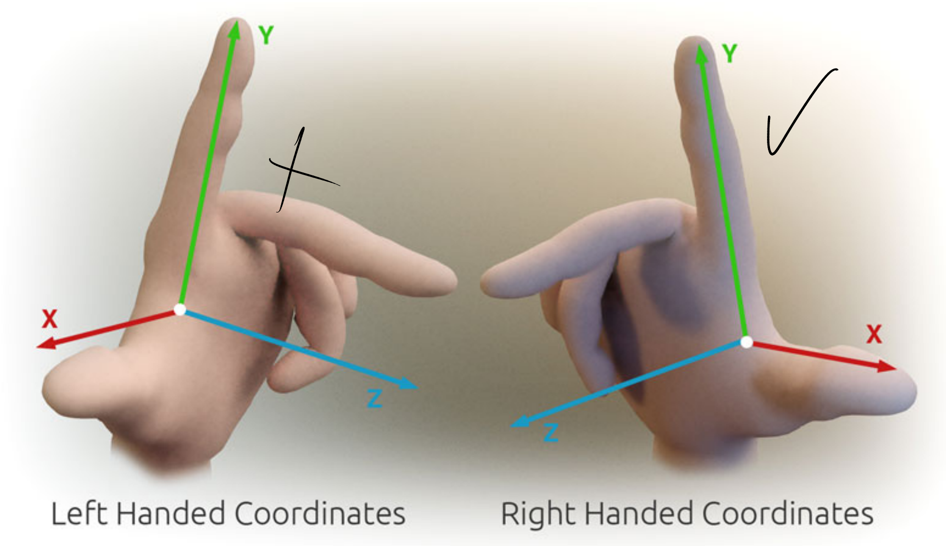
1) Scalar & Vector fn :

Coordinate System :

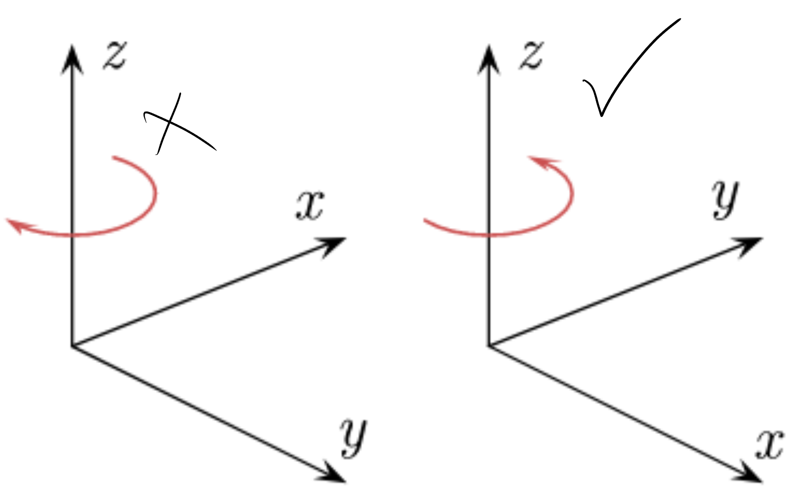
Cartesian Coordinate

Right handed Rule





for our convention



for our convention

Unit vector along

$x \rightarrow \hat{i}$

$y \rightarrow \hat{j}$

$z \rightarrow \hat{k}$

→ Scalar & Vector Int.

→ Coordinate System (Cartesian System)

→ Axis labelling convection (Right handed ~~labelling~~ Rule)

— x —

→ Scalar & Vector production:

$\vec{a}$ & $\vec{b}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\hat{i}\cdot\hat{i} = 1$	$\hat{i}\cdot\hat{j} = 0$
$\hat{j}\cdot\hat{j} = 1$	$\hat{j}\cdot\hat{k} = 0$
$\hat{k}\cdot\hat{k} = 1$	$\hat{k}\cdot\hat{i} = 0$

$\vec{a} \cdot \vec{b}$ = Scalar

$\vec{a} \cdot \vec{b} \Rightarrow$ Scalar quantity

dot product

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

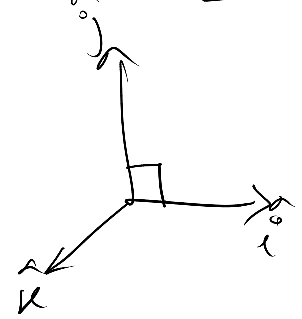
$$= a_1\hat{i}\cdot b_1\hat{i} + 0 + 0 + 0 + a_2\hat{j}\cdot b_2\hat{j} + 0 + 0$$
$$+ 0 + a_3\hat{k}\cdot b_3\hat{k}$$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\hat{i} \cdot \hat{j}$ or $\hat{j} \cdot \hat{k}$ or $\hat{k} \cdot \hat{i} = 0$?

Angle btn $\hat{i}$ & $\hat{j}$ = $\underline{90^\circ}$

$\angle \hat{i}$ & $\hat{j} = 90^\circ$

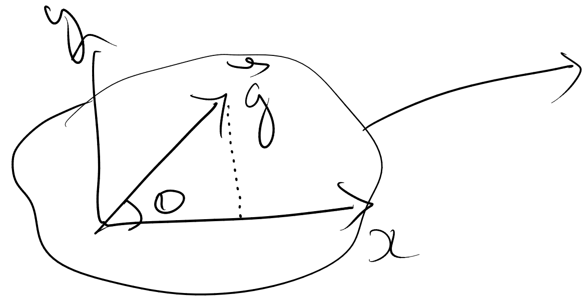


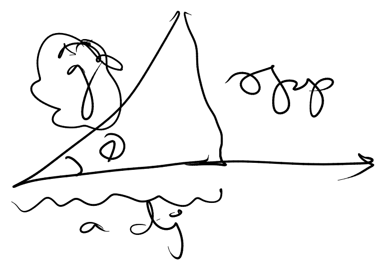
$\hat{i} \cdot \hat{j} = 0$

$\hat{i} \cdot \hat{j} = \hat{i}\hat{j} \cos\theta$

$= \hat{i}\hat{j} \cos 90^\circ$

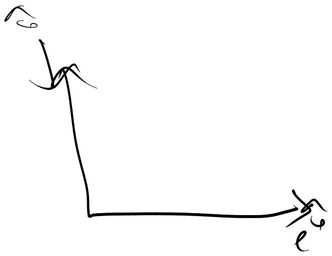
Dot product $\Rightarrow$ projection of one vector on the other.

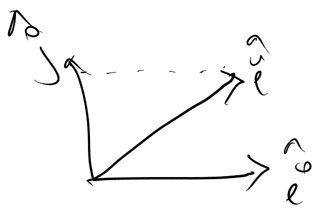




adj = hy. $\cos\theta$

$= \vec{g} \cos\theta$





$$\boxed{\hat{i} \cdot \hat{j} = 0}$$

$\hat{k} \cdot \hat{j} = 0$

$\hat{k} \cdot \hat{i} = 0$

$$\hat{i}\times\hat{j}=\hat{k} \quad \text{and} \quad \hat{j}\times\hat{i}=-\hat{k}$$
$$\hat{j}\times\hat{k}=\hat{i} \quad \text{and} \quad \hat{k}\times\hat{j}=-\hat{i}$$
$$\hat{k}\times\hat{i}=\hat{j} \quad \text{and} \quad \hat{i}\times\hat{k}=-\hat{j}$$

Next we can discuss about Vector product

$$\vec{a} \;\&\; \vec{b} \Rightarrow \vec{a}\times\vec{b} \cancel{= |\vec{a}\vec{b}|}$$

Cross product

$|\vec{a}|$ modulus $\not\equiv$ Determinant

$$|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$$

modulus

Determinant:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$
$$= a_1\hat{i}\times b_1\hat{i} + a_1\hat{i}\times b_2\hat{j} + a_1\hat{i}\times b_3\hat{k} +$$
$$a_2\hat{j}\times b_1\hat{i} + a_2\hat{j}\times b_2\hat{j} + a_2\hat{j}\times b_3\hat{k} +$$
$$a_3\hat{k}\times b_1\hat{i} + a_3\hat{k}\times b_2\hat{j} + a_3\hat{k}\times b_3\hat{k}$$

$$\boxed{\begin{aligned} \hat{i}\times\hat{i} &= 0 \\ \hat{j}\times\hat{j} &= 0 \\ \hat{k}\times\hat{k} &= 0 \end{aligned}}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2 b_3 - b_2 a_3) - \hat{j}(a_1 b_3 - b_1 a_3) + \hat{k}(a_1 b_2 - b_2 a_2)$$

→ <u>Scalar & Vector product.</u>

Vector Calculus :

Vector differential operator :

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient → Vector ~~product~~ operation

Divergent → Scalar ~~product~~ operation

Curl → Vector ~~product~~ operation

Laplacian → Scalar operation

Result ⇒ quantit

—— × ——

Gradient : Vector operation on a scalar fn. ← function

$$\vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f$$

$f(x,y,z)$

$\Downarrow$

$x^3 + 2xy + 3z$

Scalar fn.

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

Divergence : Its a scalar operation on a Vector fn.

We cannot perform Divergence on a scalar fn.

We need a vector fn. to find Divergence

$$\vec{F} = \underbrace{f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}}_{\text{scalar value}}$$

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}\right)$$

$$\boxed{\underbrace{\vec{\nabla} \cdot \vec{F}}_{\text{vector operation}} = \underbrace{\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}_{\text{scalar fn.}}}$$

Curl : Its a vector operation on a Vector fn.

$$\vec{F} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\boxed{\underbrace{\vec{\nabla}\times\vec{F}}_{\text{Vector operation}} = \underbrace{\hat{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\hat{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+\hat{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)}_{\text{Vector fn.}}}$$

Laplacian: Scalar operation on a scalar fun.

$$\nabla^2 f \equiv \underbrace{\vec{\nabla}\cdot}_{\text{divergence}}\underbrace{\vec{\nabla} f}_{\text{gradient}}$$

Vector fn.

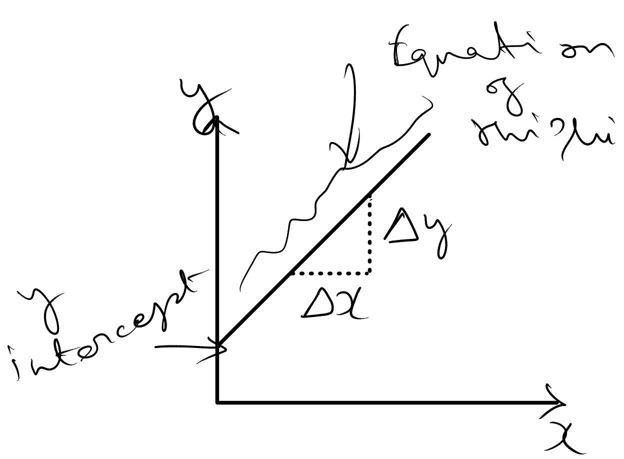
$$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\underbrace{\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)}_{\vec{\nabla} f}$$

$$\boxed{\underbrace{\nabla^2 f}_{\text{Scalar operation}} = \underbrace{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}}_{\text{Scalar fn.}}}$$

- → Gradient
- → Divergence
- → Curl
- → Laplacian.

Basics

$$\text{Slope } (m) = \frac{\Delta y}{\Delta x}$$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ \{ slope}$$

Equation of a Straight:

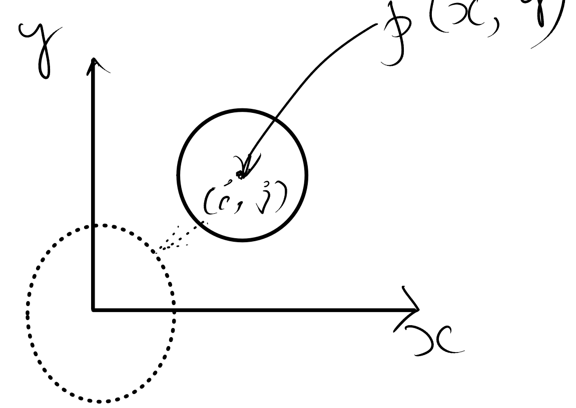
$$y = \underbrace{m}_{\text{slope}} x + \underbrace{c}_{\text{intercept}}$$

Equation for a Circle: 2D case

$$x^2 + y^2 = r^2$$

radius of the circle

$$(x-i)^2 + (y-j)^2 = r^2$$

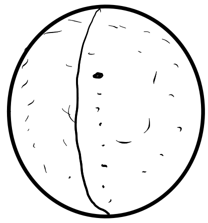


$i \; \& \; j = 0$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

3D case: In 3D we have sphere

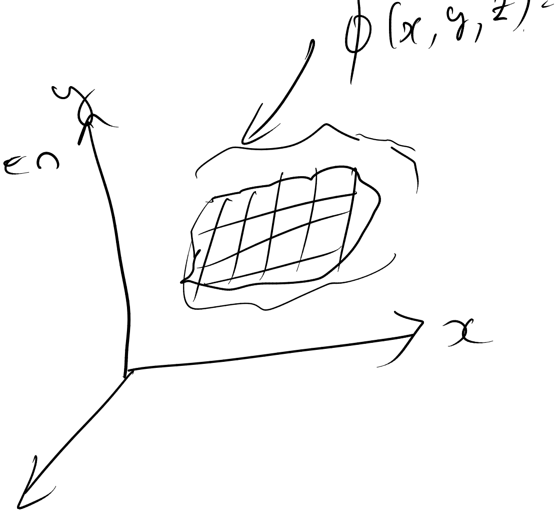


$$x^2+y^2+z^2=r^2$$

$$f(x,y,z)=c$$

$$\boxed{\phi(x,y,z)=K}$$

Represent surface by this form



Advance Engineering Mathematics
by
H.K. Dass

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **14-10-2020**

Attendance : **10**

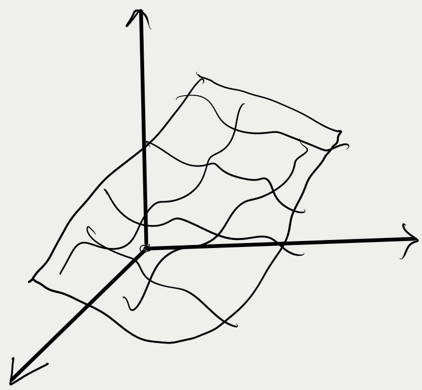
→ Gradient, Divergence, Curl & Laplacian.

So our aim is to construct a surface

$\phi_1(x, y, z) = C_1$

$\Downarrow$

Example



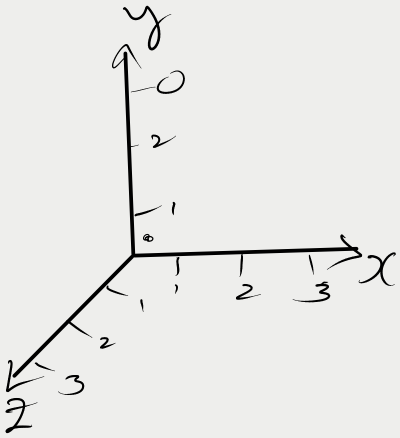
$\phi_1(x, y, z) = x + y + z$

Let the constant $C_1 = 3$ // example

$\phi_1(x, y, z) = C_1$

$\Rightarrow \boxed{x + y + z = C_1}$

$C_1 = 3$



x	y	z	C_1
1	1	1	
0	2	1	
2	0	1	
1	2	0	3
1.5	1.5	0	
1.75	0	1.25	
1.3	1.7	0	

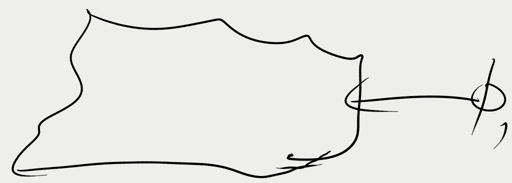
We use Gradient to study the properties of surface(s)

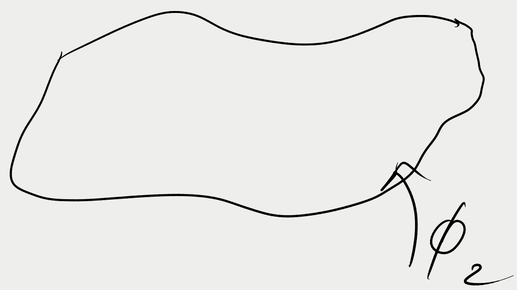
→ If we have two surface ϕ_1 & ϕ_2

→ ϕ_1 & ϕ_2 intersect?

→ Are ϕ_1 & ϕ_2 $\parallel$ell

→ Are $\phi_2 \perp^{lar} \phi_2$





$$\phi_1 \Rightarrow \phi_1(x,y,z) = c_1$$
$$\phi_2 \Rightarrow \phi_2(x,y,z) = c_2$$

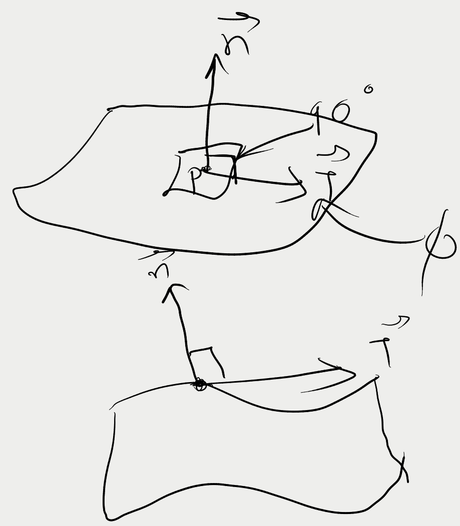
Concepts

(i) Tangent to surface @ somepoint P

(ii) Normal to surface " " P

$\phi_1 \Rightarrow \vec{n}_1$ & $\vec{T}_1$ ← Tangent

normal



$\phi_2 \Rightarrow \vec{n}_2$ & $\vec{T}_2$

$$\phi_1 \perp \phi_2 \Rightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$$

How to find normal to some surface ϕ?

This is where our "gradient" enters the picture.

Let $\phi(x,y,z)$ be some surface, then

W.K.T

$$\underbrace{d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz}$$

Can we relate

$$\underset{\text{Scalar}}{d\phi} \rightsquigarrow \underset{\text{Vector}}{\vec{\nabla}\phi} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

We want a scalar from a vector

dot product

$$\vec{\nabla}\phi \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$d\phi = \vec{\nabla}\phi \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

then we have

$$\underbrace{d\phi}_{\text{scalar}} = \underbrace{\vec{\nabla}\phi}_{\text{vector}} \cdot \underbrace{d\vec{r}}_{\text{vector}}$$

If $\phi(x, y, z) = C_1$, then $d\phi = 0$

for a surface w.r.t $\phi(x, y, z) = 0 \Rightarrow d\phi = 0$

$$\therefore \quad \vec{\nabla}\phi \cdot d\vec{r} = 0$$

$\vec{a} \cdot \vec{b} = 0$ dot then

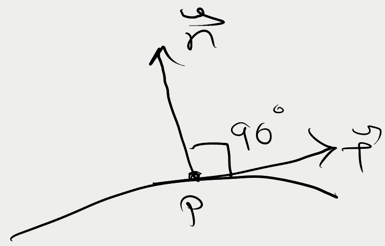
$\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$

So for our case

$$\vec{\nabla}\phi \perp d\vec{r} \in \text{Surface}$$

Generally $d\vec{r} \longrightarrow$ tangent vector to the surface.

Any vector $\perp$ar to the tangent vector is normal vector



So the conclusion is

$\vec{\nabla}\phi$ is a normal vector to the surface "ϕ" @ a point P.

$\boxed{\vec{\nabla}\phi \text{ is the normal vector}}$

If we have a surface $\phi(x, y, z) = c_1$, then

i.e. $\vec{\nabla}\phi$ is the normal to that surface.

Example 16. *If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point $(1, -2, -1)$.*

$$\phi = 3x^2y - y^3z^2$$

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$\vec{\nabla}\phi = \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\frac{\partial\phi}{\partial x} = 6xy$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2$$

$$\frac{\partial\phi}{\partial z} = -2y^3z$$

$$\vec{\nabla}\phi\Big|_{\substack{(1,\,-2,\,-1)\\ x\quad y\quad z}} = \hat{i}(-12) + \hat{j}(-9) + \hat{k}(-16)$$

$$\boxed{\vec{\nabla}\phi_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k}}$$

Example 19. *Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$.*

point

$$\phi \Rightarrow xy^3z^2 = 4$$

$$\phi = xy^3z^2 - 4$$

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\underline{\underline{\vec{\nabla}\phi}} = y^3 z^2\hat{i} + 3y^2xz^2\hat{j} + 2zxy^3\hat{k}$$

normal vector

Normal @ $\left(\frac{-1}{x}, \frac{-1}{y}, \frac{2}{z}\right)$

$$\vec{n} \rightarrow \boxed{\vec{\nabla}\phi\Big|_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}}$$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Unit vector of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\begin{array}{r} 144 \\ 32 \\ \hline \end{array}$$

Unit Normal

unit normal

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = \frac{4(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{16(1+9+1)}}$$

$$= \frac{4}{4}\frac{(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{11}}$$

$$= -\frac{1}{\sqrt{11}}\cdot(\hat{i} + 3\hat{j} - \hat{k})$$

$$\boxed{\hat{n} = \frac{-1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})}$$

Directional Derivative: (DD)

DD of "ϕ" along any vector "$\vec{d}$" is the dot product of $\vec{\nabla}\phi$ and $\vec{d}$ in otherwords

$$\text{DD of } \phi = \vec{\nabla}\phi \cdot \vec{d}$$

Example 18. *Find the directional derivative of $x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.*

Given $\phi = x^2y^2z^2$

$\vec{\nabla}\phi \cdot \vec{T} = DD$

$\vec{T} = \dfrac{d\vec{r}}{dt}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{\nabla}\phi = 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

$$\underbrace{\vec{\nabla}\phi\Big|_{(1,1,-1)} = 2\hat{i} + 2\hat{j} - 2\hat{k}}$$

$\vec{r}\Big|_{(1,1,-1)}$

$x = e^t$

$y = \sin 2t + 1$

$z = 1 - \cos t$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = \underline{e^t\hat{i}} + (\sin 2t + 1)\hat{j} + (1 - \cos t)\hat{k}$

$\vec{T} = \dfrac{d\vec{r}}{dt} = e^t\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

$\vec{T} = e^t\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

$\dfrac{de^t}{dt} = ?$

$\dfrac{d}{dt}(e^t) = ?$

$$\vec{T}\Big|_{t=0} = e^{0}\hat{i} + 2\cos(0)\hat{j} + \sin(0)\hat{k}$$

$$\boxed{\vec{T}\Big|_{t=0} = \hat{i} + 2\hat{j}}$$

Directional Derivative of ϕ along $\vec{T}$

$$DD(\phi) \text{ along } \vec{T} = \vec{\nabla}\phi \cdot \vec{T}$$

$$= (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot (\hat{i} + 2\hat{j} + 0\hat{k})$$

$$= (2\cdot 1) + (2\cdot 2) - (2\cdot 0)$$

$$= 6 \;//$$

$$DD(\phi) \text{ along } \hat{T} = \vec{\nabla}\phi \cdot \hat{T} \qquad \hat{T} = \frac{\vec{T}}{|\vec{T}|}$$

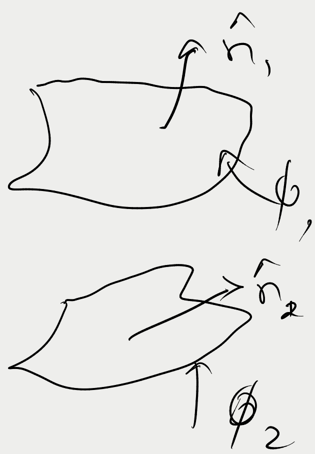
$$= 6/\sqrt{5} \;//$$

Home Work for (15-10-20)

Example 23. *Find the angle between the surfaces* $x^2 + y^2 + z^2 = 9$ *and* $z = x^2 + y^2 - 3$ *at the point* (2, –1, 2). (Nagpur University, Summer 2002)

$$\phi_1 = x^2 + y^2 + z^2 - 9$$

$$\phi_2 = x^2 + y^2 - z - 3$$



unit normal → $\hat{n}_1 \cdot \hat{n}_2 = n_1 n_2 \cos\theta$

$$\theta = \cos^{-1}\left(\frac{\hat{n}_1 \cdot \hat{n}_2}{n_1 n_2}\right)$$

$\phi_1 \xrightarrow[\text{vector}]{\text{normal}} \vec{\nabla}\phi_1 \xrightarrow[\text{normal}]{\text{unit}} \dfrac{\vec{\nabla}\phi_1}{|\vec{\nabla}\phi_1|}$

Answer : $$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

$$\vec{\nabla}\phi\Big|_{(1,-2,1)} = \hat{j} + 6\hat{k}$$

$$\vec{d} = 2\hat{i} - \hat{j} - 2\hat{k} \Rightarrow \hat{d} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

D.D of ϕ along $\hat{d}$

$$\vec{\nabla}\phi \cdot \hat{d} = (0\hat{i} + \hat{j} + 6\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$= \frac{1}{3} \cdot (-1 - 12)$$

$$\text{D.D} = \frac{-13}{3}$$

$$\vec{\nabla}\phi = \hat{j} + 6\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{1^2 + 6}$$

$$|\vec{\nabla}\phi| = \sqrt{37}$$

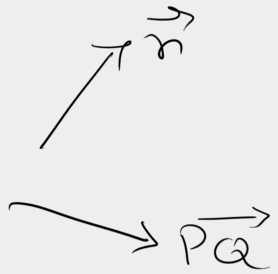
Normal $\rightarrow \vec{\nabla}\phi$

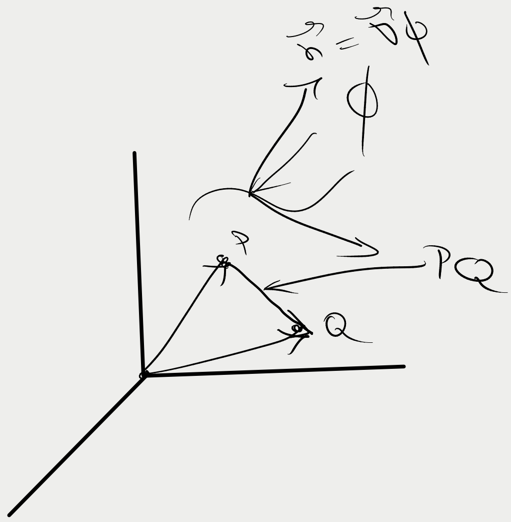
DD along $\vec{d}$ $\rightarrow \vec{\nabla}\phi \cdot \hat{d}$

Greatest rate of change of ϕ $\rightarrow |\vec{\nabla}\phi|$

Example 27: Find the DD of the fn: $\phi = x^2 - y^2 + 2z^2$ @ the pt: P $(1, 2, 3)$ in the direction of the line PQ where Q $(5, 0, 4)$

$\vec{\nabla}\phi$





$\vec{PQ} =$

$\vec{P} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{Q} = 5\hat{i} + 0\hat{j} + 4\hat{k}$

$$\underline{\overline{PQ}} = \vec{Q} - \vec{P} = 4\hat{i} - 2\hat{j} + \hat{k}$$

D.D of ϕ along $\vec{PQ}$

$\vec{\nabla}\phi$; $\hat{PQ}$

$$\vec{\nabla}\phi \cdot \hat{PQ}$$

H.W → Example 29

Example 21. *Find the constants m and n such that the surface $m x^2 - 2nyz = (m + 4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.*

$$\phi_1 = mx^2 - 2nyz = (m+4)x$$

$P = (1, -1, 2)$

$$\phi_2 = 4x^2y + z^3 - 4$$

$$\phi_1\big|_P \perp \phi_2\big|_P$$

$$\vec{\nabla}\phi_1\Big|_{P} \quad \vec{\nabla}\phi_1 = [2mx-(m+4)]\hat{i} - 2nz\hat{j} - 2ny\hat{k}$$

$$\frac{\partial \phi_1}{\partial x} = 2mx-(m+4)$$

$$\frac{\partial \phi_1}{\partial y} = -2nz$$

$$\frac{\partial \phi_1}{\partial z} = -2ny$$

$$\vec{\nabla}\phi_1\Big|_{(1,-1,2)} = (2m-m-4)\hat{i} - 4n\hat{j} + 2n\hat{k}$$

$$2m-m$$

$$\vec{\nabla}\phi_1\Big|_{P} = (m-4)\hat{i} - 4n\hat{j} + 2n\hat{k}$$

$$\vec{\nabla}\phi_2\Big|_{P} = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$

$$\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 = 0$$

$$\vec{n}_1 = \vec{\nabla}\phi_1$$

$$\vec{n}_2 = \vec{\nabla}\phi_2$$

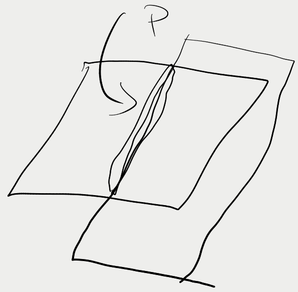
$$\underbrace{\left((m-4)\hat{i} - 4n\hat{j} + 2n\hat{k}\right) \cdot \left(-8\hat{i} + 4\hat{j} + 12\hat{k}\right) = 0}_{\text{because } \vec{\nabla}\phi_1 \text{ \& } \vec{\nabla}\phi_2 \text{ are orthogonal}}$$

$$-8(m-4) - 16n + 24n = 0$$

$$\boxed{-8m + 32 + 8n = 0} \quad \text{--- (1)}$$

2 unknowns but 1 Equation

$$\phi_1\big|_P \Rightarrow mx^2 - 2nyz - (m+4)x = 0$$



$P(1, -1, 2)$

$x \quad y \quad z$

$$\phi_1\big|_P \Rightarrow m(1)^2 - 2n(-1)(2) - (m+4)(1)$$

$$\Rightarrow \cancel{m} + 4n - \cancel{m} - 4$$

$$4n - 4 = 0$$

$$4n = 4$$

$$\boxed{n = 1}$$

Sub. Value of n in Eqn. (1) we have

$$-8m + 32 + 8(1) = 0$$

$$-8m + 40 = 0$$

$$8m = 40$$

$$m = \frac{40}{8} = 5$$

$$\boxed{m = 5;\ n = 1}$$

EXERCISE 5.7

1. Evaluate grad ϕ if $\phi = \log(x^2 + y^2 + z^2)$ **Ans.** $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

2. Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 5$ at the point (0, 1, 2). **Ans.** $\dfrac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$

(AMIETE, June 2010)

3. Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point (1, –1, 1) in the direction of (3, 1, –1) *(A.M.I.E.T.E. Winter 2009, 2000)* **Ans.** $\dfrac{5}{\sqrt{11}}$

4. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero.

(Nagpur Winter 2000) **Ans.** $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

7. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find grad ϕ at the point $(1, -2, -1)$ **Ans.** $-(16\hat{i} + 9\hat{j} + 4\hat{k})$

8. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Ans. $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

9. What is the greatest rate of increase of the function $u = xyz^2$ at the point $(1, 0, 3)$? **Ans.** 9

11. Find the values of constants a, b, c so that the maximum value of the directional directive of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the axis of z. **Ans.** $a = b$, $b = 24$, $c = -8$

12. Find the values of λ and μ so that surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$. **Ans.** $\lambda = \frac{9}{2}, \mu = 1$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **22-Oct-2020**

Attendance : **9**

→ Assignment Doubts :

1) $\underline{\underline{11^{th}}}$

Gradient : Quick recap

$\vec{\nabla}\phi$ → vector quantity

$\vec{\nabla}\phi$ → "$\vec{n}$" normal vector

$\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ → $\hat{n}$ unit normal vector

$|\vec{\nabla}\phi|$ → Greatest rate of change of ϕ

If $\vec{r}$ is some vector the $\vec{\nabla}\phi \cdot \underline{\underline{\hat{r}}}$ is the

unit vector of $\vec{r}$

Directional derivative of "ϕ" along the $\vec{r}$.

→ Scalar point fn: Example: temperature
Mass

So today we will be examining vector point fn:

Some of the example for vector point fn:

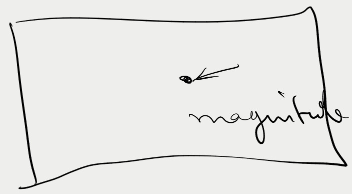
[area] = l × l

Velocity
Force
Acceleration
magnetization

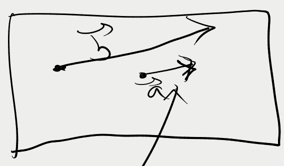
$\left.\begin{array}{l}\text{Velocity}\\ \text{Force}\\ \text{Acceleration}\\ \text{magnetization}\end{array}\right\}$ Vector → magnitude, Direction

l

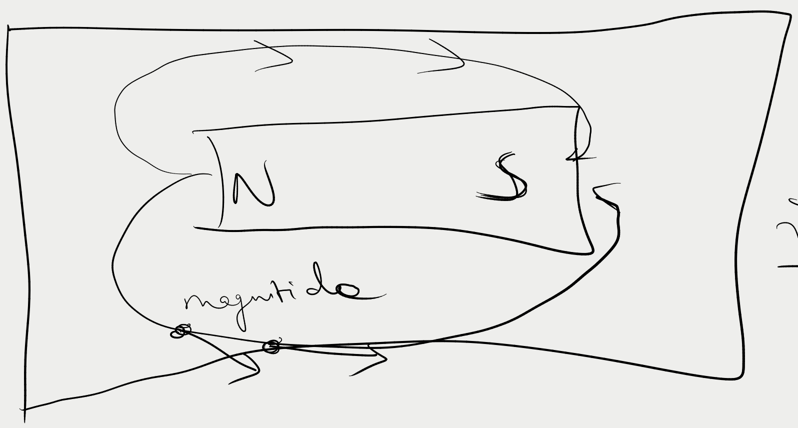
Scalar fn:

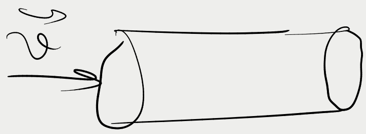


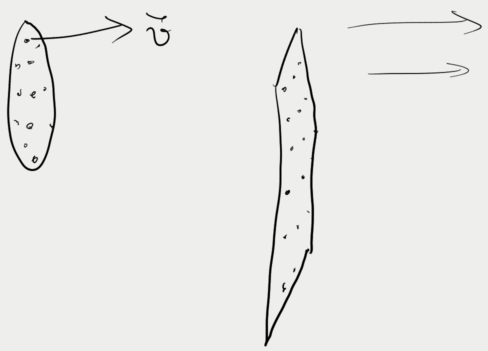
l × l



length of the arrow defines the magnitude and the head points the direction







if ρ is the density of the fluid

Speed → scalar
Velocity → vector

ρ, M, V

$$\left.\rho = \frac{M}{V}\right\} \leftarrow$$ Static case

Dynamic case :

$$\rho = \frac{M}{\frac{d\vec{V}}{dt}}$$ ← volume element

$$\rho = \frac{M}{\underbrace{dV}_{\text{decompose}}} \cdot dt$$

If we consider the flow only along the "x" direction

$$\rho = \frac{M}{\frac{dx}{dt}\,dy\,dz}$$

The mass flow along x direction :

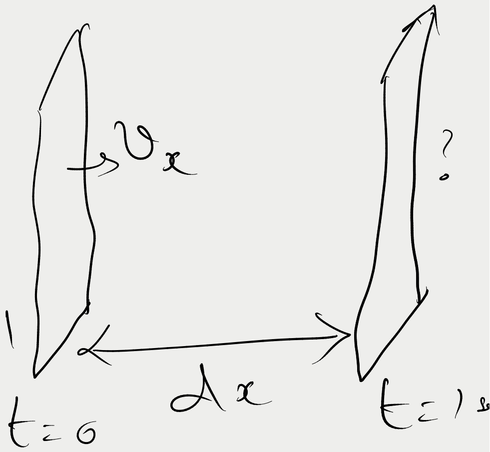
$$= \rho \cdot \overbrace{\frac{dx}{dt}}^{v_x} dy\, dz$$

$$= \rho \cdot v_x \cdot dy\, dz \qquad \text{——} \quad (1)$$

↖ velocity along x direction.

If there is a change in velocity along the length "dx"

$$v_x + \frac{\partial v_x}{\partial x} \cdot dx$$



The ~~change~~ in mass flow @ $t=1$

$$= \rho \left(v_x + \frac{\partial v_x}{\partial x} \cdot dx \right) dy\, dz \quad \text{--- (2)}$$

Change in mass flow :

(1) − (2)

$$= \rho v_x\, dy dz - \rho \left(v_x + \frac{\partial v_x}{\partial x} dx \right) \cdot dy\, dz$$

$$= \rho v_x \cancel{dy} dz - \rho v_x \cancel{dy dz} - \rho \frac{\partial v_x}{\partial x} \cdot dx\, dy\, dz$$

$$\boxed{\Delta M_x = \ominus \rho \frac{\partial v_x}{\partial x} \cdot dx\, dy\, dz}$$

$$\Delta M_y = \ominus \rho \frac{\partial v_y}{\partial y} dx\, dy\, dz$$

$$\Delta M_z = \ominus \rho \frac{\partial v_z}{\partial z} dx\, dy\, dz$$

If there in loss

The total change in mass due to flow $= \underbrace{\Delta M_x + \Delta M_y + \Delta M_z}_{\text{change}}$

$$= -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz$$

$$= -\rho\,\underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)}_{\text{scalar}} \cdot dV$$

volume element

vector → $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

vector → $\vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$

$$\vec{\nabla}\cdot\vec{v} = \text{Scalar}$$

Total change in "m" due to the flow $\Big\}$ $= -\rho\, dV \cdot \vec{\nabla}\cdot\vec{v}$

volume   velocity

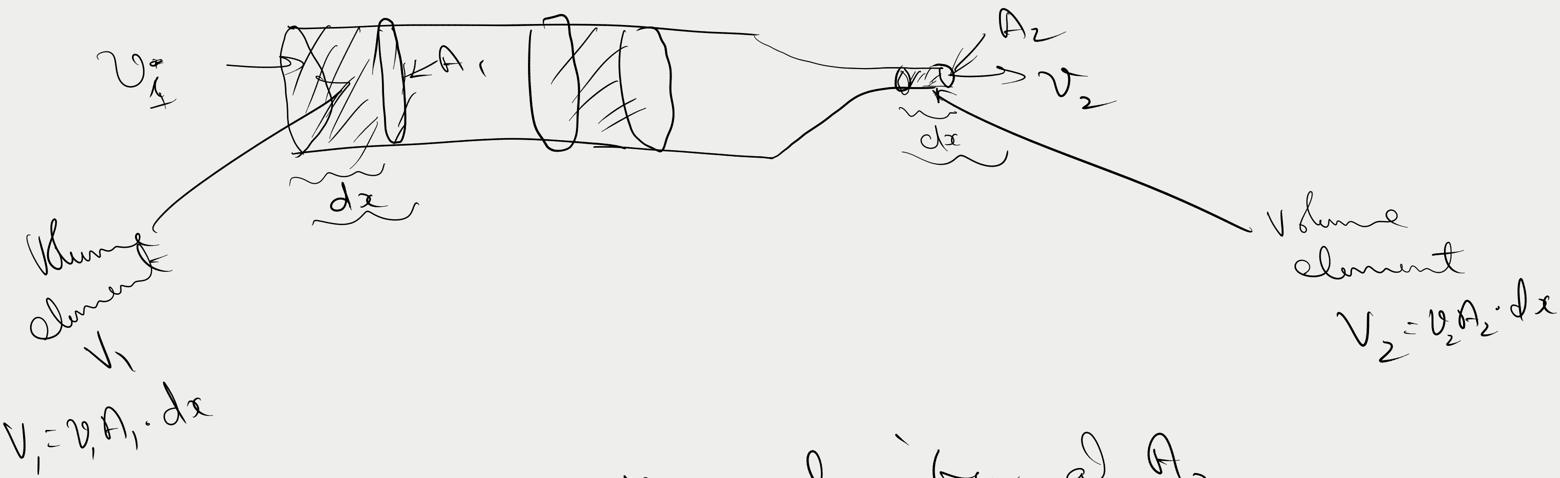
If we consider unit volume then $(dV = 1)$

$$= -\rho\,\underbrace{\vec{\nabla}\cdot\vec{v}}$$

loss

Loss of "m" due to flow $\Big\}$ $= \rho\,\vec{\nabla}\cdot\vec{v}$

constant



$v_2 \leftarrow$ is the velocity at A_2

$v_1 \leftarrow$ " " at A_1

If $A_2 < A_1$, then $v_1 < v_2$, because the fluid is incompressible V_1 must be equal to V_2

$\therefore$ If $\vec{\nabla} \cdot \vec{v} = 0$, then the fluid is incompressible

$\vec{\nabla} \cdot \vec{A} = 0$

$\vec{A}$ is solenoidal.

Example 34. *If* $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of div* $\overline{v}$.

(U.P., I Semester, Winter 2000)

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{div } \vec{v} = \vec{\nabla} \cdot \vec{v}$$

$$\vec{\nabla} \cdot \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \frac{\partial}{\partial x}\Big(\underset{u}{x}\cdot\underset{v}{(x^2+y^2+z^2)^{-1/2}}\Big) + \frac{\partial}{\partial y}\Big(y\cdot(x^2+y^2+z^2)^{-1/2}\Big)$$

$$+ \frac{\partial}{\partial z}\Big(z\cdot(x^2+y^2+z^2)^{-1/2}\Big)$$

$$\frac{\partial}{\partial x}\Big(\underset{u}{x}\,\underbrace{(x^2+y^2+z^2)^{-1/2}}_{v}\Big) = x\cdot\Big(-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}\cdot 2x\Big) +$$

$$(x^2+y^2+z^2)^{-1/2}\cdot 1$$

$$\frac{\partial}{\partial x}(\;''\;) = -x^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial y}(\;''\;) = -y^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial z}(\;''\;) = -z^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial x}(\;''\;) + \frac{\partial}{\partial y}(\;''\;) + \frac{\partial}{\partial z}(\;''\;) = 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{1-\frac{3}{2}}$$

$$= 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{-1/2}$$

$$= (x^2+y^2+z^2)^{-1/2}(3-1)$$

$$= \frac{2}{[x^2+y^2+z^2]^{1/2}}$$

Example 35. *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.* (*A.M.I.E.T.E., Summer 2004*)

Scalar fn Vector

Hllow

$$u\vec{r} = (x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= (x^2+y^2+z^2)x\hat{i} + (x^2+y^2+z^2)y\hat{j} + (x^2+y^2+z^2)z\hat{k}$$

$$\vec{\nabla}\cdot(u\vec{r})$$

Example 36. *Find the value of n for which the vector* $r^n\vec{r}$ *is solenoidal, where* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})^n$$

$$r = |\vec{r}|$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$r^n\vec{r}$

$$\vec{\nabla}\cdot(r^n\vec{r}) = 0$$ since

Since solenoidal

v u

$$r^n\vec{r} = (x^2+y^2+z^2)^{n/2}\cdot x\hat{i} + (x^2+y^2+z^2)^{n/2}\cdot y\hat{j} + (x^2+y^2+z^2)^{n/2} z\hat{k}$$

$$\vec{\nabla}\cdot(r^n\vec{r}) = nx^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$
$$+ ny^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$
$$+ nz^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (x^2+y^2+z^2)^1$$

$$+ 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1+1} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\boxed{\vec{\nabla}\cdot(r^n\vec{r}) = (n+3)(x^2+y^2+z^2)^{\frac{n}{2}}}$$

$r^n\vec{r}$ is solenoidal

$$\therefore \vec{\nabla}\cdot(r^n\vec{r}) = 0$$

$$\Rightarrow \underbrace{(n+3)}\underbrace{(x^2+y^2+z^2)^{\frac{n}{2}}} = 0$$

$$n+3 = 0 \Rightarrow \boxed{n=-3}$$

(or)

$$(x^2+y^2+z^2)^{\frac{n}{2}} = 0$$

Example 38. *Let* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$ *and* $\vec{a}$ *is a constant vector. Find the value of*

$$div\left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$$

17.ω

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

1. $$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

2. $$\frac{\vec{a} \times \vec{r}}{r^n}$$

3. $$\vec{\nabla} \cdot \left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$$

Example 39. *Find the directional derivative of div* $(\vec{u})$ *at the point (1, 2, 2) in the direction of the outer normal of the sphere* $x^2 + y^2 + z^2 = 9$ *for* $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.

$$\vec{\nabla} \cdot \vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}\right)$$

Let

$$\phi_1 = \vec{\nabla} \cdot \vec{u} = 4 \cdot x^3 + 4y^3 + 4z^3$$

Direction Vector to ϕ_1

$$\vec{\nabla}\phi_1 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(4x^3 + 4y^3 + 4z^3\right)$$

$$= (4 \cdot 3)\, x^2 \hat{i} + (4 \cdot 3)\, y^2 \hat{j} + (4 \cdot 3)\, z^2 \hat{k}$$

$$\vec{\nabla} \phi_1 = 12 x^2 \hat{i} + 12 y^2 \hat{j} + 12 z^2 \hat{k}$$

$$\vec{\nabla} \phi_1 \Big|_{1,2,2} = 12(1)^2 \hat{i} + 12(2)^2 \hat{j} + 12(2)^2 \hat{k}$$

$$\boxed{\vec{\nabla} \phi_1 \Big|_{1,2,2} = 12 \hat{i} + 48 \hat{j} + 48 \hat{k}}$$

$$\left.\begin{array}{l}\text{Outer normal to the} \\ \text{surface of spher}\end{array}\right\} = \vec{\nabla} \phi_2$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\therefore \vec{\nabla} \phi_2 = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \left(x^2 + y^2 + z^2 - 9 \right)$$

$$\vec{\nabla} \phi_2 = 2x \hat{i} + 2y \hat{j} + 2z \hat{k}$$

$$\text{Normal @ } (1,2,2) = \vec{\nabla} \phi_2 \Big|_{(1,2,2)}$$

$$= 2(1) \hat{i} + 2(2) \hat{j} + 2(2) \hat{k}$$

Let $\vec{d}$

$$\vec{d} = \vec{\nabla} \phi_2 \Big|_p = 2 \hat{i} + 4 \hat{j} + 4 \hat{k}$$

$$\therefore \hat{d} = \frac{\vec{\nabla} d_2}{|\vec{\nabla} \phi_2|} = \frac{2 \hat{i} + 4 \hat{j} + 4 \hat{k}}{\sqrt{2^2 + 4^2 + 4^2}} \Rightarrow \frac{2(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{36}}$$

$$\hat{d} = \frac{1}{3} \hat{i} + 2 \hat{j} + 2 \hat{k}$$

$\therefore$ DD of ϕ_2 along $\vec{d} \Rightarrow \vec{\nabla}\phi_1 \cdot \hat{d}$

$$= (12\hat{i} + 48\hat{j} + 48\hat{k}) \cdot \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= \frac{12 + 96 + 96}{3}$$

$$= \frac{204}{3} \Rightarrow 68$$

Example 40. *Show that div (grad r^n) = $n(n+1)r^{n-2}$, where*

$$r = \sqrt{x^2 + y^2 + z^2}$$

Hence, show that $\Delta^2\left(\frac{1}{r}\right) = 0.$ (U.P. I Semester, Dec. 2004, Winter 2002)

∇^2

$$\vec{\nabla} r^n = n \cdot r^{n-2} \vec{r}.$$

$$\operatorname{div}\cdot(\vec{\nabla} r^n) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(n r^{n-2} x\hat{i} + n r^{n-2} y\hat{j} + n r^{n-2} z\hat{k}\right)$$

$$= \underbrace{\frac{\partial}{\partial x}(\underset{u}{n r^{n-2}}\ \underset{v}{x})} + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} \cdot z)$$

$r(x)$

$$= n r^{n-2} + x \cdot n(n-2) r^{(n-3)} \frac{\partial r}{\partial x} +$$

$$n r^{n-2} + y \cdot n(n-2) r^{n-3} \frac{\partial r}{\partial y} +$$

$$n r^{n-2} + z \cdot n(n-2) r^{n-3} \frac{\partial r}{\partial z}$$

$$|r| = (x^2 + y^2 + z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x$$

$$= n r^{n-2} + x\, n(n-2) r^{n-3} \cdot \frac{x}{r} +$$

$$= \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{x}{r}$$

$$n r^{n-2} + y (n-2) n \cdot r^{n-3} \cdot \frac{y}{r} +$$

$$n r^{n-2} + z\, n (n-2)\, r^{n-3} \frac{z}{r}$$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$$= 3n r^{n-2} + x^2 \cdot n(n-2) \cdot r^{n-4} + y^2 n(n-2) r^{n-4} + z^2 n(n-2) r^{n-4}$$

$$= 3n r^{n-2} + \left(n(n-2) r^{n-4}\right)\left(x^2+y^2+z^2\right)$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} \cdot r^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= r^{n-2}\left(3n + n^2 - 2n\right)$$

$$= r^{n-2}\left(n^2 + n\right)$$

$$= r^{n-2} n (n+1)$$

$$\operatorname{div}(\operatorname{grad} r^n) = r^{n-2} n (n+1)$$

$$\operatorname{div}(\operatorname{grad} r^n) = \vec{\nabla} \cdot \vec{\nabla} r^n = \nabla^2 r^n$$

$$\boxed{\nabla^2 r^n = r^{n-2} \cdot n (n+1)} \quad \text{—— (1a)}$$

(i) prove $\nabla^2\left[\frac{1}{r}\right] = 0$

$$\nabla^2\left[\frac{1}{r}\right] \Rightarrow \nabla^2 r^{-1} \quad \text{———} \quad 2a$$

Comparing L.H.S of Eqn 1a & 2a we have

$$\nabla^2 r^n = \nabla^2 r^{-1}$$

$$\therefore \boxed{n = -1}$$

Sub: $\boxed{n = -1}$ in eqn 1a we have

$$\nabla^2 r^{-1} = \nabla^2\left[\frac{1}{r}\right] = r^{-1-2}(-1)(-1+1)$$

$$= 0$$

Hence proved.

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **28-Oct-2020**

Attendance : **8**

Curl of a vector field:

Curl is a also a vector operator.

Curl operates on a vector

$\downarrow$ results

Also a vector

Whereas for div $\rightarrow$ div. act on vector

$\downarrow$ results

Scalar.

$\text{Curl } \vec{F} = 0 \Leftarrow$ the vector field is irrotational.

whereas

$\text{div } \vec{F} = 0 \Leftarrow$ Solenodial

$$\text{Curl } \vec{F} \text{ or } \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

(if $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$)

//

Example 41. *Find the divergence and curl of* $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at (2, –1, 1)* *(Nagpur University, Summer 2003)*

$\operatorname{div} \vec{v} = ?$

$$\Rightarrow \vec{\nabla}\cdot\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xyz\,\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}\right)$$

$$\Rightarrow yz + 3x^2 + 2xz - y^2$$

$$\vec{\nabla}\cdot\vec{v}\Big|_{2,-1,1} = (-1)(1) + 3(2)^2 + 2(2)(1) - (-1)^2$$

$$= -1 + 12 + 4 - 1$$

$$= 14 \;//$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xzy)\right)$$

$$= \hat{i}\left(0-2yz-0\right)-\hat{j}\left(z^2-0-xy\right)$$
$$+\hat{k}\left(6xy-xz\right)$$

$$\vec{\nabla}\times\vec{F} = -2yz\,\hat{i}+\hat{j}\left(-z^2+xy\right)+\left(6xy-xz\right)\hat{k}$$

$$\vec{\nabla}\times\vec{F}\Big|_{(2,-1,1)} = -2(-1)(1)\hat{i}+\left[-(-1)^2+2(-1)\right]\hat{j}+\left[6(2)(-1)-2(1)\right]\hat{k}$$

$$= 2\hat{i}-3\hat{j}-14\hat{k}$$

Example 42. *If* $\vec{V} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, *find the value of curl* $\vec{V}$.

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \partial/\partial y & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)\frac{u}{v} = \frac{(x^2+y^2+z^2)^{1/2}\cdot 0 - z\left(\frac{1}{2}\cdot 2y\,(x^2+y^2+z^2)^{\frac{1}{2}-1}\right)}{x^2+y^2+z^2}$$

$$\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\frac{1}{2} - 1 = ?$$

$$-\frac{1}{2}$$

$$= \frac{-zy\,(x^2+y^2+z^2)^{-\frac{1}{2}}}{x^2+y^2+z^2}$$

$$= \frac{-zy}{(x^2+y^2+z^2)\cdot(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{-zy}{(x^2+y^2+z^2)^{\frac{1}{2}+1}}$$

$$= \frac{-zy}{(x^2+y^2+z^2)^{3/2}}$$

Continue remain steps

Example 43. *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.* (U.P., I Sem, Dec. 2008)

Let $\vec{v} = (y^2 - z^2 + 3yz - 2x)\hat{i}$
$+ (3xz + 2xy)\hat{j}$
$+ (3xy - 2xz + 2z)\hat{k}$

$\vec{\nabla}\cdot\vec{v} = 0$; $\vec{\nabla}\times\vec{v} = 0$

Do this a H.W.

Example 44. *Determine the constants a and b such that the curl of vector* $\overline{A} = (2xy+3yz)\hat{i} + (x^2+axz-4z^2)\hat{j} - (3xy+byz)\hat{k}$ *is zero.*

(*U.P. I Semester, Dec 2008*)

$$\vec{A} = (2xy+3yz)\hat{i} + (x^2+axz-4z^2)\hat{j} - (3xy+byz)\hat{k}$$

$$\vec{\nabla}\times\vec{A} = 0$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2xy+3yz) & (x^2+axz-4z^2) & -(3xy+byz) \end{vmatrix}$$

$$\vec{\nabla}\times\vec{A} = \hat{i}(-3x - bz - ax + 8z) + \hat{j}(6y) + \hat{k}(\cancel{2x} + az - \cancel{2x} - 3z)$$

$$\vec{\nabla}\times\vec{A} = \hat{i}[-x(3+a) + z(8-b)] + 6y\hat{j} + z(a-3)\hat{k}$$

$\vec{\nabla}\times\vec{A} = 0$; w.r.t $\hat{i}, \hat{j}, \hat{k} \neq 0$ $\because$ they are unit vectors

$$\boxed{-x(3+a) + z(8-b) = 0}$$

$$6y = 0$$

$z(a-3) = 0$ here $z=0$ is a trival solution.

$\therefore$ Lets assume $z \neq 0$ $\therefore$ $\boxed{a = 3}$

$$-x(3+3) + z(8-b) = 0$$

$$\left.\begin{aligned} x(-3-a) &= 0 \\ z(8-b) &= 0 \end{aligned}\right\}$$ here $\rightarrow$ $\left.\begin{aligned} x &= 0 \\ z &= 0 \end{aligned}\right\}$ trivial solution

So lets assume $x \neq 0;\ z \neq 0$

$$-3-a=0 \quad ; \quad 8-b=0$$

(or) $\boxed{a=3}$; $-b=-8$

or $\boxed{b=8}$

Example 45. *If a vector field is given by*
$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. *Is this field irrotational ? If so, find its scalar potential.*
(U.P. I Semester, Dec 2009)

?

$$\vec{F} = (x^2 - y^2 + x)\hat{e} - (2xy + y)\hat{j}$$

$\vec{\nabla} \times \vec{F} = 0 \Leftarrow$ irrotational.

Any vector field which is irrotational can be given by gradient of a scalar fn.

$$\vec{F} = \vec{\nabla}\phi \leftarrow \text{scalar field.}$$

usually "ϕ" is called the Scalar potential of $\vec{F}$.

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\vec{\nabla}\phi \quad = \left(\hat{e}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$

If $\vec{F}$ is irrotational the $\vec{F} = \vec{\nabla}\phi$

$$d\phi = \vec{F}\cdot d\vec{r}$$

$$\phi = \int \vec{F}\cdot d\vec{r}$$

$$\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F}\cdot d\vec{r} = (x^2 - y^2 + x)\,dx - (2xy + y)\,dy + 0$$

$$\phi = \int \vec{F}\cdot d\vec{r}$$

$$= \int x^2\,dx - \int y^2\,dx + \int x\,dx$$

$$- \int 2xy\,dy - \int y\,dy$$

$$= \frac{x^3}{3} + C_1 - (y^2 x + C_2) + \left(\frac{x^2}{2} + C_3\right) - \left(2x\frac{y^2}{2} + C_4\right)$$

$$- \left(\frac{y^2}{2} + C_5\right)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - xy^2 - \frac{y^2}{2} + (C_1 + C_2 \cdots + C_5)$$

$$\boxed{\phi = \frac{x^3}{3}+\frac{x^2}{2} - 2xy^2 - \frac{y^2}{2} + c}$$

$$\int \vec{F}\cdot d\vec{r} = \phi$$

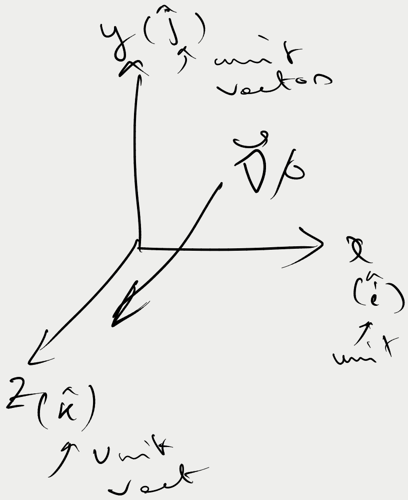
Assignment Doubt

11. Find the values of constants *a, b, c* so that the maximum value of the directional directive of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, –1) has a maximum magnitude 64 in the direction parallel to the axis of *z*. **Ans.** $a = b,\ b = 24,\ c = -8$

$$\phi = axy^2 + byz + cz^2x^3$$

$$\vec{\nabla}\phi = (ay^2 + 3cz^2x^2)\,\hat{i} + (2yax + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

$$\vec{\nabla}\phi\Big|_{1,2,-1} = (4a+3c)\,\hat{i} + (4a-b)\,\hat{j} + (2b-2c)\,\hat{k}$$



DD of $\vec{\nabla}\phi$ along x axis "$\hat{i}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{i} = 4a + 3c \quad \text{—— (1)}$$

DD of $\vec{\nabla}\phi$ along y axis "$\hat{j}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{j} = 4a - b \quad \text{—— (2)}$$

DD of $\vec{\nabla}\phi$ along z axis "$\hat{k}$"

$$\vec{\nabla}\phi\Big|_p \cdot \hat{k} = 2b - 2c \quad \text{—— (3)}$$

Since $\vec{\nabla}\phi$ is parallel to z axis

$$\left.\begin{aligned} \vec{\nabla}\phi\big|_p \cdot \hat{i} \\ \vec{\nabla}\phi\big|_p \cdot \hat{j} \end{aligned}\right\} = 0$$

$$\vec{\nabla}\phi\big|_p \cdot \hat{k} = 64$$

$$4a + 3c = 0$$

$$4a - b = 0$$

$$2b - 2c = 64$$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **4-Nov-2020**

Attendance : **10**

EXERCISE 5.8

1. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\dfrac{\vec{r}}{|\vec{r}|^3}\right) = 0$,

   (*ii*) div (grad r^n) $= n\,(n+1)\,r^{n-2}$ (*AMIETE, June 2010*) (*iii*) div $(\vec{r}\,\phi) = 3\phi + r$ grad ϕ.

2. Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal. (*R.G.P.V., Bhopal, Dec. 2003*)

3. Show that $\nabla.(\phi\, A) = \nabla\phi.A + \phi(\nabla.A)$

4. If ρ, ϕ, z are cylindrical coordinates, show that grad (log ρ) and grad ϕ are solenoidal vectors.

5. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their corresponding expression in orthogonal curvilinear coordinates.

Prove the following:

6. $\vec{\nabla}.(\phi\vec{F}) = (\vec{\nabla}\phi).\vec{F} + \phi(\vec{\nabla}.\vec{F})$

7. (*a*) $\nabla.(\nabla\phi) = \nabla^2\phi$ (*b*) $\vec{\nabla}\times\dfrac{(\vec{A}\times\vec{R})}{r^n} = \dfrac{(2-n)\vec{A}}{r^n} + \dfrac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}},\ r = |\vec{R}|$

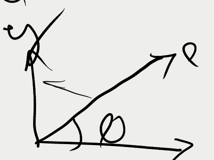
8. div $(f\,\nabla\,g)$ – div $(g\,\nabla\,f) = f\,\nabla^2 g - g\,\nabla^2 f$

4, 7b, 8

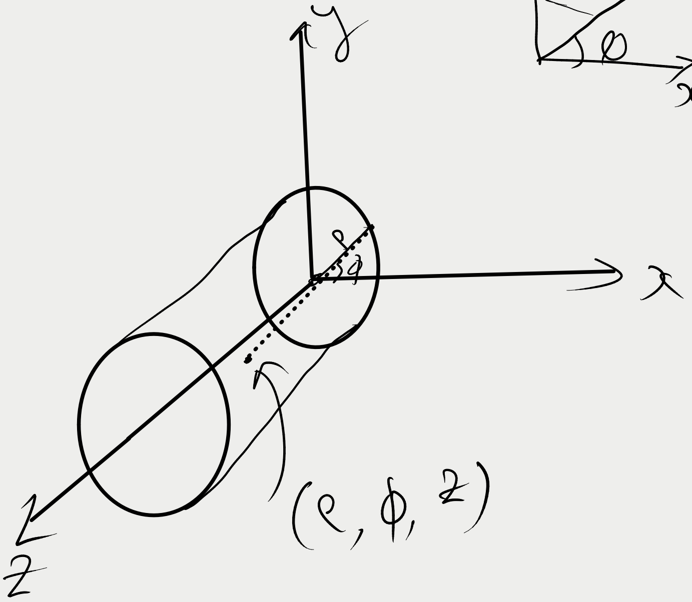
Ex 5.8, 4th problem :

$\vec{\nabla}\cdot\vec{F} = 0$ then, $\vec{F}$ is solenoidal

$\vec{\nabla}(\log \rho)$; $\vec{\nabla}(\phi)$ are Solenoidal



$\rho = \sqrt{x^2 + y^2}$



$x = \rho\cos\phi$

$y = \rho\sin\phi$

$\dfrac{y}{x} = \dfrac{\rho\sin\phi}{\rho\cos\phi} = \tan\phi$

$\therefore \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

$\rho = \sqrt{x^2 + y^2}$

$\underbrace{\vec{\nabla}(\log \rho)}_{\vec{\nabla} \cdot} = ? \quad ; \quad \underbrace{\vec{\nabla}\phi}_{\vec{\nabla} \cdot} = ?$

Ex 5.8

(b) $\vec{\nabla} \times \frac{(\vec{A} \times \vec{R})}{r^n} = \frac{(2-n)\vec{A}}{r^n} + \frac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$

$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^n = (x^2 + y^2 + z^2)^{n/2}$

$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\vec{A} \times \vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$\vec{A}\times\vec{R} = \hat{i}(A_2 z - A_3 y) - \hat{j}\,(A_1 z - A_3 x) + \hat{k}\,(A_1 y - A_2 x)$$

$$\frac{\vec{A}\times\vec{R}}{r^n} = \frac{\hat{i}\,(A_2 z - A_3 y) + \hat{j}\,(A_3 x - A_1 z) + \hat{k}\,(A_1 y - A_2 x)}{(x^2+y^2+z^2)^{n/2}}$$

$$\frac{\vec{A}\times\vec{R}}{r^n} = \left\{ \frac{A_2 z - A_3 y}{(x^2+y^2+z^2)^{n/2}}\,\hat{i} + \frac{A_3 x - A_1 z}{(x^2+y^2+z^2)^{n/2}}\,\hat{j} + \frac{A_1 y - A_2 x}{(x^2+y^2+z^2)^{n/2}}\,\hat{k} \right.$$

$$\vec{\nabla}\times\frac{\vec{A}\times\vec{R}}{r^n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{A_2 z - A_3 y}{(x^2+y^2+z^2)^{n/2}} & \frac{A_3 x - A_1 z}{(x^2+y^2+z^2)^{n/2}} & \frac{A_1 y - A_2 x}{(x^2+y^2+z^2)^{n/2}} \end{vmatrix}$$

Ex 5.8, 8 problem

$$\vec{\nabla}\cdot(f\vec{\nabla}g) - \vec{\nabla}\cdot(g\vec{\nabla}f) = f\nabla^2 g - g\nabla^2 f$$

$$\vec{\nabla}g = \vec{g} = g_1\hat{i} + g_2\hat{j} + g_3\hat{k}$$

$$f\vec{\nabla}g = f\vec{g} = fg_1\hat{i} + fg_2\hat{j} + fg_3\hat{k}$$

$$\vec{\nabla}\cdot(f\vec{\nabla}g) = \vec{\nabla}\cdot(f\vec{g}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot$$

$$\left(fg_1\hat{i} + fg_2\hat{j} + fg_3\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}fg_1 + \frac{\partial}{\partial y}fg_2 + \frac{\partial}{\partial z}fg_3\right)$$

$f(x,y,z)$

$$= f\frac{\partial g_1}{\partial x} + g_1\frac{\partial f}{\partial x} + f\frac{\partial g_2}{\partial y} + g_2\frac{\partial f}{\partial y}$$

$$+ f\frac{\partial g_3}{\partial z} + g_3\cdot\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right) + g_1\frac{\partial f}{\partial x} + g_2\frac{\partial f}{\partial y}$$

$$+ g_3\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{g}$$

$$+ \vec{g}\cdot\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\vec{\nabla}f}$$

$$= f\vec{\nabla}\cdot\vec{g} + \vec{g}\cdot\vec{\nabla}f$$

$$\vec{g} = \vec{\nabla}g$$

$$= f\vec{\nabla}\cdot\vec{\nabla}g + \vec{g}\cdot\vec{\nabla}f$$

$$\boxed{\vec{\nabla}\cdot(f\vec{\nabla}g) = f\nabla^2 g + \cancel{\vec{\nabla}g\cdot\vec{\nabla}f}} \quad \text{— (1)}$$

$$\boxed{\vec{\nabla}\cdot(g\vec{\nabla}f) = g\nabla^2 f + \cancel{\vec{\nabla}f\cdot\vec{\nabla}g}} \quad \text{— (2)}$$

$① - ②$

$$\boxed{\vec{\nabla}\cdot(f\vec{\nabla}g) - \vec{\nabla}\cdot(g\vec{\nabla}f) = f\nabla^2 g - g\nabla^2 f}$$

3. Prove that:

(*i*) $\nabla.(\phi\vec{A}) = \nabla\phi.\vec{A} + \phi(\nabla.\vec{A})$ ✓

(*ii*) $\nabla(A.B) = (A.\nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$ (*R.G.P.V. Bhopal, June 2004*)

(*iii*) $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

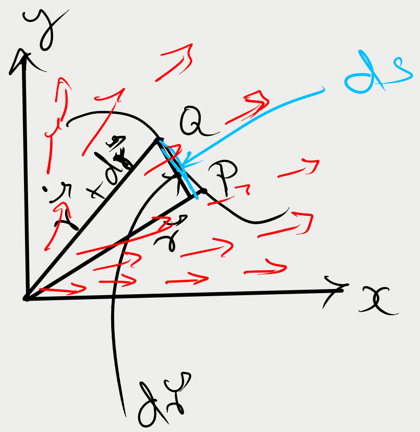
$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{A}\cdot\vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y} + \frac{\partial b_3}{\partial z}$$

Line Integral :

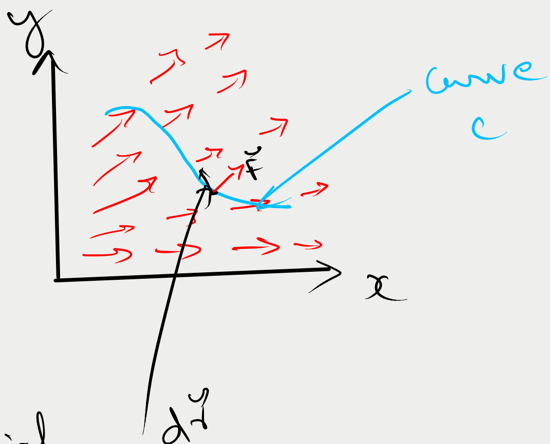
$$ds = |d\vec{r}|$$



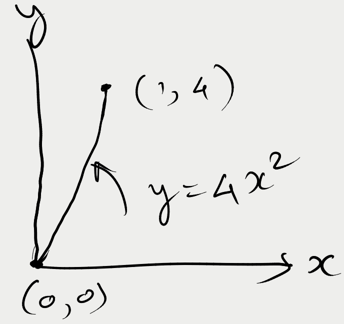
$$\boxed{\int_C \vec{F}\cdot d\vec{r}}$$

Line Integral of a Vector field $\vec{F}$ along a curve "C" $= \int \vec{F}\cdot d\vec{r}$

tangential vector to the curve C



Eg. 65



$$\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$$

$$C \Rightarrow y = 4x^2$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$W.d = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{(0,0)\; x_i\, y_i}^{(1,4)\; x_f\, y_f} 2x^2y\,dx + 3xy\,dy$$

$$\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}\cdot d\vec{r} = 2x^2y\,dx + 3xy\,dy$$

$$= \int_0^1 2x^2\underset{\uparrow\, y = 4x^2}{y}\,dx + \int_0^4 3xy\,dy$$

$$y = 4x^2$$

$$dy = 4\cdot 2x = 8x\,dx$$

$$= \int_0^1 2x^2\cdot 4x^2\,dx + 3x\cdot 4x^2\cdot 8x\,dx$$

$$= \frac{104}{5} \; //$$ check it.

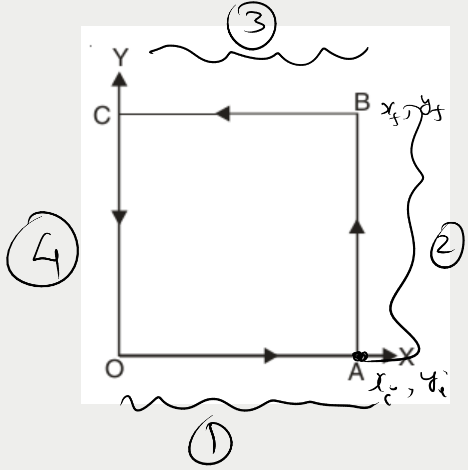
Example 66. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *where* $\vec{F} = x^2\hat{i} + xy\hat{j}$ *and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

$C = OABC$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$$

$$+ \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$



$$\vec{F} = x^2\hat{i} + xy\hat{j} \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

@ ① i.e OA $\quad x_i = 0 \quad y_i = 0$

$\underline{x_f = a} \quad y_f = 0$

$dy = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^a$$

$$= \frac{a^3}{3}$$

@ ② i.e along AB

$x_i = a \quad y_i = 0$

$x_f = a \quad y_f = a$

$dx = 0$

$\vec{F}\cdot d\vec{r} = \underline{x^2 dx} + xy\,dy$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a xy\,dy = \int_0^a ay\,dy$$

$$= a\int_0^a y\,dy \Rightarrow a \cdot \left[\frac{y^2}{2}\right]_0^a$$

$$= a \cdot \frac{a^2}{2}$$

$$= \frac{a^3}{2} \quad a.$$

@ 3 i.e along BC $\quad x_i = a \quad y_i = a$

$\qquad x_f = 0 \quad y_f = a$

$dy = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,\underset{\sim}{dy}$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 x^2\,dx = \left.\frac{x^3}{3}\right|_a^0$$

$$(\text{or}) \quad = \left.-\frac{x^3}{3}\right|_0^a$$

$$= -\frac{a^3}{3} \quad //.$$

@ 4 i.e along CO $\quad x_i = 0 \quad y_i = a$

$\qquad x_f = 0 \quad y_f = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx$

$\quad xy\,dy$

$dx = 0$

$x = 0$

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_0^a 0^2 dx + 0\cdot y\,dy$$

$$\int_{CO} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{OABC} \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \cancel{\frac{a^3}{3}} + \frac{a^3}{2} + \cancel{\left(\frac{-a^3}{3}\right)} + 0$$

$$\boxed{\int_{OABC} \vec{F}\cdot d\vec{r} = \frac{a^3}{2}}$$

Example 68. *The acceleration of a particle at time t is given by*

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at t = 0, find* $\vec{v}$ *and* $\vec{r}$ *at any point t.*

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \quad ; \quad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$$

$$\underset{\sim}{\vec{v}} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt - \hat{j}\int 8\sin 2t\,dt + 6\hat{k}\int t\,dt$$

$$= \hat{i}\,18\cdot\left(\frac{\sin 3t}{3}\right) - 8\hat{j}\left(\frac{-\cos 2t}{2}\right) + 6\hat{k}\left(\frac{t^2}{2}\right) + C$$

$$\vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \underline{\underline{c}}$$

At $t=0$; $\vec{v}=0$

$$0 = 6\sin 3(0)\,\hat{i} + 4\cos 2(0)\,\hat{j} + 0 + \underset{\sim}{c}$$

$$0 = 0\hat{i} + 4\hat{j} + 0 + c$$

$$\therefore\ c = -4\hat{j}$$

$$\vec{v} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

$$\vec{r} = \int \vec{v}\cdot dt = \hat{i}\int 6\sin 3t\, dt + 4\hat{j}\int (\underset{\sim}{\cos 2t} - 1)\, dt$$
$$+ 3\hat{k}\int t^2\, dt$$

$$= 6\hat{i}\left(-\frac{\cos 3t}{3}\right) + 4\hat{j}\left(\frac{\sin 2t}{2}\right) - 4\hat{j}(t)$$
$$+ 3\hat{k}\left(\frac{t^3}{3}\right) + c$$

$$\vec{r} = -2\hat{i}\cos 3t + 2\hat{j}\sin 2t - 4t\hat{j}$$
$$+ t^3\hat{k} + c$$

@ $t=0$; $\vec{r}=0$

$$0 = -2\hat{i}\cos 0 + 2\hat{j}\sin 0 - 0\hat{j}$$
$$+ 0\hat{k} + c$$

$$0 = -2\hat{i} + c$$

$$c = 2\hat{i}$$

$$\vec{r} = -2\hat{i}\cos 3t + 2\hat{j}\sin 2t - 4t\hat{j} + t^3\hat{k} + 2\hat{i}$$

$$\vec{r} = 2\hat{i}\left(1 - \cos 3t\right) + 2\left(\sin 2t - 2t\right)\hat{j} + t^3\hat{k}$$

Ex 5.10 ①, ②

Date : **05-Nov-2020**

Attendance : **9**

Example 69. *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\vec{dr}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*
$x = t,\ y = t^2,\ z = t^3.$ *(Uttarakhand, I Semester, Dec. 2006)*

parametric equations.

$$\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\int \vec{A}\cdot d\vec{r} = \int 3x^2\,dx + \int 6y\,dx - \int 14yz\,dy + \int 20xz^2\,dz$$

$x = t \qquad y = t^2 \qquad z = t^3$

$t = 0\ ;\ x = 0 \qquad t = 0,\ y = 0 \qquad t = 0,\ z = 0$

$t = 1\ ;\ x = 1 \qquad t = 1,\ y = 1 \qquad t = 1,\ z = 1$

$y = t^2$

$dy = 2t\cdot dt$

$z = t^3$

$dz = 3t^2\,dt$

$$\int \vec{A}\cdot d\vec{r} = \int_0^1 3t^2\,dt + \int_0^1 6t^2\,dt - \int_0^1 14\cdot(t^2)(t^3)(2t\,dt) + \int_0^1 20(t)(t^3)^2\,3t^2\,dt$$

$$= 3 \cdot \frac{t^3}{3}\Big|_0^1 + 6 \cdot \frac{t^3}{3}\Big|_0^1 - 28 \cdot \frac{t^7}{7}\Big|_0^1 + 60\,\frac{t^{10}}{10}\Big|_0^1$$

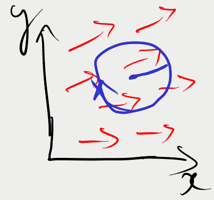
$$= 1^3 + 2 \cdot 1^3 - 4 \cdot 1^7 + 6 \cdot 1^{10}$$

$$= 3 - 4 + 6$$

$$= 3 + 2$$

$$= 5 \,//.$$

Example 71. *Compute* $\int_c \vec{F} \cdot \vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*



$$\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2 + y^2} \qquad x^2 + y^2 = 1$$

Parametric form for circle:

$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

if $\Rightarrow$ c is a unit circle the $r = 1$

$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\} \text{unit circle //.}$$

$$\vec{F} \cdot d\vec{r} = \frac{y}{\underbrace{x^2 + y^2}_{=1}}\,dx - \frac{x\,dy}{\underbrace{x^2 + y^2}_{=1}}$$

$$= y\,dx - x\,dy$$

Parametric form

$$x = \cos\theta \; ; \quad dx = -\sin\theta \, d\theta$$

$$y = \sin\theta \; ; \quad dy = \cos\theta \, d\theta$$

$$\theta \to 0 \sim 2\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta)\,d\theta - \cos\theta(\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} \left(-\sin^2\theta - \cos^2\theta\right) d\theta$$

$$= -\int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta\right) d\theta$$

$$= -\int_0^{2\pi} d\theta$$

$$= -\theta \Big|_0^{2\pi}$$

$$= -2\pi + 0$$

$$\oint \vec{F} \cdot d\vec{r} = -2\pi$$ A.

Example 72. *Show that the vector field* $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ *is conservative. Find its scalar potential and the work done in moving a particle from* (–1, 2, 1) *to* (2, 3, 4).
(*A.M.I.E.T.E. June 2010, 2009*)

$$\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$$

$$\vec{F} = \vec{\nabla}\phi$$

potential

conservative force

$\vec{F}$ depending only on the r or $\frac{1}{r}$

$\vec{F}$ depends on $\vec{v}$ $\frac{dr}{dt}$

$$\vec{\nabla}\times\vec{F} = 0$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(2x^2y)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3x^2z^2) - \frac{\partial}{\partial z}(2x(y^2+z^3))\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2x(y^2+z^3))\right]$$

$$= \hat{i}\left[\cancel{0-0}\right] - \hat{j}\left[\cancel{6xz^2 - 6xz^2}\right] + \hat{k}\left[\cancel{4xy - 4xy}\right]$$

$$\vec{\nabla}\times\vec{F} = 0$$

so $\vec{F}$ is conservative.

scalar field

$$\vec{F} = \vec{\nabla}\phi$$

$$d\phi = \left[\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz\right]$$

$$= \underbrace{\left[\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right]}\cdot\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} \qquad \vec{F} = \vec{\nabla}\phi$$

$$d\phi = \vec{F}\cdot d\vec{r}$$

$$\phi = \int d\phi = \int \vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = 2x(y^2+z^3)\,dx + 2x^2y\,dy + 3x^2z^2\,dz$$

$$\phi = \int d\phi = \int \vec{F}\cdot d\vec{r} = \int 2x(y^2+z^3)\,dx + \int 2x^2y\,dy + \int 3x^2z^2\,dz$$

$$= \int 2xy^2\,dx + \int 2xz^3\,dx + \int 2x^2y\,dy + \int 3x^2z^2\,dz$$

$$= 2y^2\int x\,dx + 2z^3\int x\,dx + 2x^2\int y\,dy + 3x^2\int z^2\,dz$$

$$= \cancel{2}y^2 \cdot \frac{x^2}{\cancel{2}} + \cancel{2}z^3 \cdot \frac{x^2}{\cancel{2}} + \cancel{2}x^2 \cdot \frac{y^2}{\cancel{2}} + \cancel{3}x^2 \cdot \frac{z^3}{\cancel{3}}$$

$$= y^2x^2 + x^2z^3 + x^2y^2 + x^2z^3 + C$$

$$\phi = 2\left(y^2x^2 + x^2z^3\right) + C$$

$$\phi\Big|_{-1,2,1}^{2,3,4} = 2\left(y^2x^2 + x^2z^3\right)\Big|_{-1,\,2,\,1}^{2,\,3,\,4}$$

$x_u\ y_u\ z_u$; $x_\ell\ y_\ell\ z_\ell$

$$= 2\cdot\left[3^2\cdot(2)^2 + 2^2\cdot 4^3 - \left(2^2\cdot(-1)^2 + (-1)^2\cdot(1)^3\right)\right]$$

$$= 2\cdot\left[9\cdot 4 + 4\cdot 64 - (\cancel{4}\ 2 + 1)\right]$$

$$= 2\cdot\left[36 + 256 - 5\right]$$

$$\phi\Big|_{-1,2,1}^{2,3,4} = 574$$

291 ← text book answer (wrong)

Ex 5.10

5. Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$, under the field of force given by $\vec{F} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$. Is the field of force conservative?
(*A.M.I.E.T.E., Winter 2000*) **Ans.** 40 π

$$\oint \vec{F}\cdot d\vec{r}$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F}\cdot d\vec{r} = (2x - y + z)\,dx + (x + y - z^2)\,dy + \cancel{(3x - 2y + 4z)\,dz}^{\,0}$$

$$z = 0 \qquad \frac{x^2}{25} + \frac{y^2}{16} = 1$$

given in the problem.

$$\vec{F}\cdot d\vec{r} = (2x - y)\,dx + (x + y)\,dy$$

Equation of circle:

$$x^2 + y^2 = r^2$$

Equation of ellipse:

General form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric form

$$x = a\cos\theta$$

$$y = b\sin\theta$$

our curve 'C'

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

For our case $a = 5$; $b = 4$

$$\therefore\ x = 5\cos\theta \ ; \quad dx = -5\sin\theta\,d\theta$$

$$y = 4\sin\theta \ ; \quad dy = 4\cos\theta\,d\theta$$

We have

$$\vec{F}\cdot d\vec{r} = (2x - y)\,dx + (x + y)\,dy$$

So,

$$\vec{F}\cdot d\vec{r} = 2\cdot(5\cos\theta\cdot(-5\sin\theta\, d\theta)) - 4\sin\theta(-5\sin\theta\, d\theta) + 5\cos\theta\cdot 4\cos\theta\, d\theta + 4\sin\theta\cdot 4\cos\theta\, d\theta$$

$$= -50\cos\theta\sin\theta\, d\theta + 20\sin^2\theta\, d\theta + 20\cos^2\theta\, d\theta + 16\cos\theta\sin\theta\, d\theta$$

$$= (16-50)\cos\theta\sin\theta\, d\theta + 20\,(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\, d\theta$$

$$\vec{F}\cdot d\vec{r} = -34\cos\theta\sin\theta\, d\theta + 20\, d\theta$$

$$W.d = \int_0^{2\pi} \vec{F}\cdot d\vec{r} = -34\int_0^{2\pi} \cos\theta\sin\theta\, d\theta + 20\int_0^{2\pi} d\theta$$

$$= -34\left.\frac{\sin^2\theta}{2}\right|_0^{2\pi} + 20\cdot\theta\Big|_0^{2\pi}$$

$$= 0 + 20\cdot 2\pi = 40\pi$$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

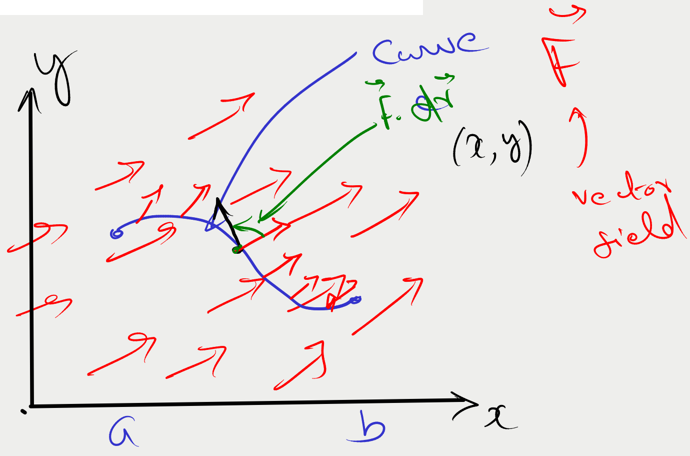
Date : **11-Nov-2020**

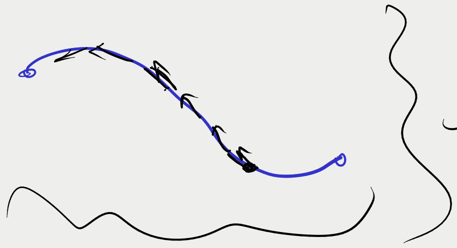
Attendance : **10**

Line Integral :

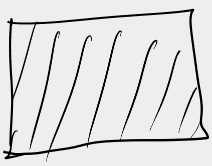
$$\int_C \vec{F} \cdot d\vec{r}$$

tangential vector





Surface Integral :

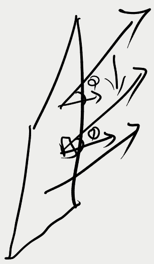


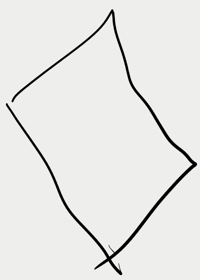
$\vec{F} \cdot Q$

scalar

$\phi = \vec{F} \cdot \vec{Q}$

$(Q \cdot \hat{n}) \cdot \vec{F}$





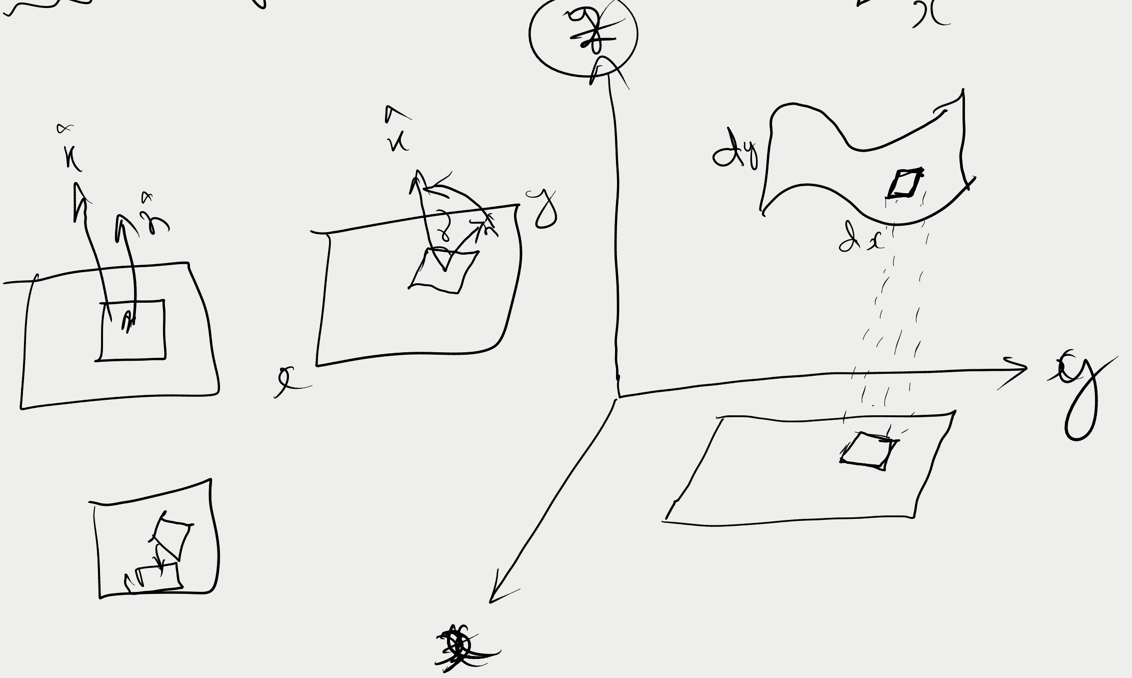
$$d\phi = \vec{F} \cdot \hat{n}\, ds$$

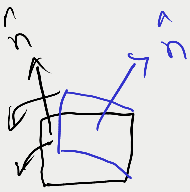
$$\boxed{\phi = \iint_S \vec{F} \cdot \hat{n}\, ds}$$

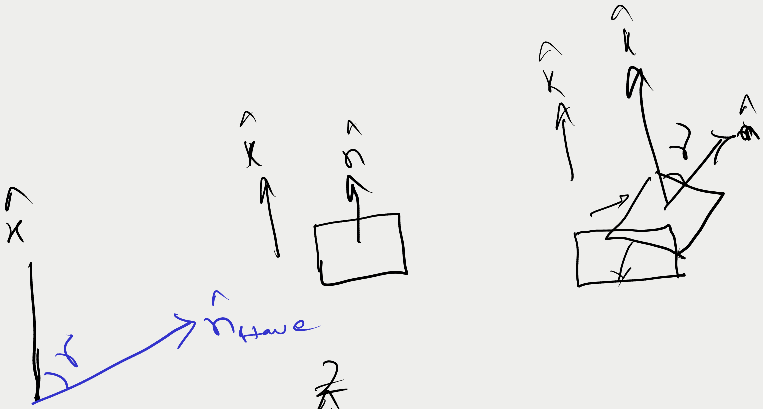
ds — scalar quantity

What is ds?

$$\underline{ds} = dy \cdot dx$$

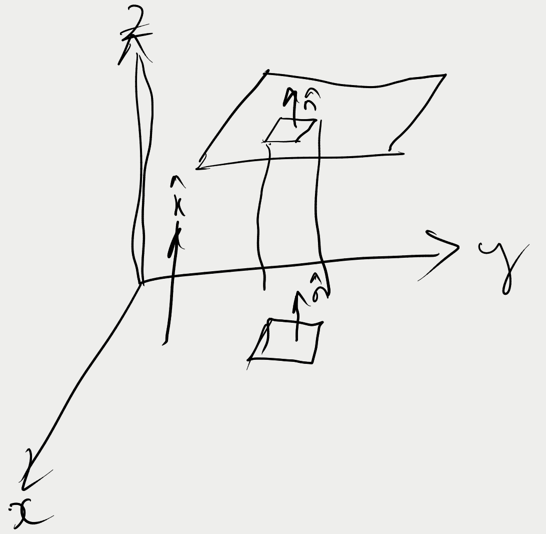






$$ds = dx\,dy\;\hat{n}\cdot\hat{k}$$

(or) $$\boxed{dx\,dy = \frac{ds}{\hat{n}\cdot\hat{k}}}$$



$$\hat{n}\cdot\hat{k} = |n||k|\cos\theta$$

$\hat{n}$ ||el $\hat{k}$ then $\theta = 0$

$$\hat{n}\cdot\hat{k} = |n||k|\cdot 1$$

$\hat{n}\cdot\hat{k} = 1$ when $\hat{n}$ ||el $\hat{k}$

$$dx\,dy = \underline{ds}$$

$$\iint \vec{F}\cdot\hat{n}\, ds \Rightarrow \iint \vec{F}\cdot\hat{n}\cdot \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

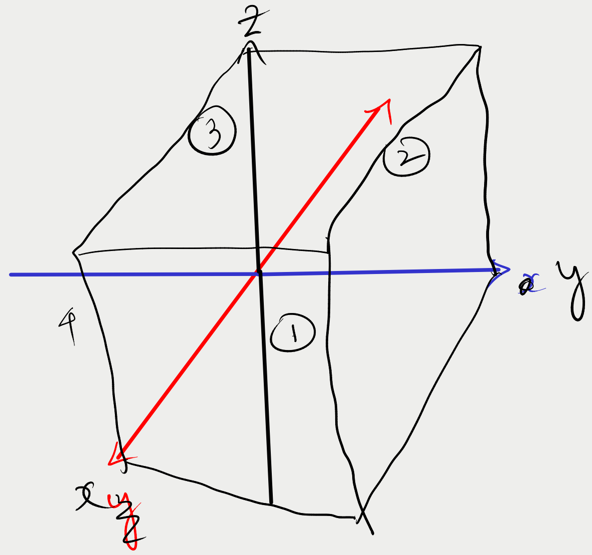
S.I

$$\Rightarrow \iint_{x\;y} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

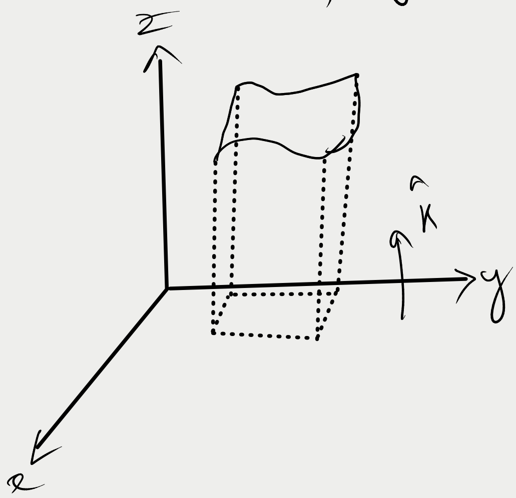
↑ first octant

x, y, z

$+ \quad + \quad + \quad z$



x, y



Example 70. *Evaluate* $\iint_S \vec{A}.\hat{n}\, ds$ *where* $\vec{A} = (x + y^2)\,\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (*Nagpur University, Summer 2000*)

$$\vec{A} = (x+y^2)\,\hat{i} - 2x\,\hat{j} + 2yz\,\hat{k}$$

$$\phi = 2x + y + 2z - 6$$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_S \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z - 6)$$

$$\vec{n} = \vec{\nabla}\phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right)\cdot\hat{k} = \frac{2}{3} \text{ A.}$$

$$\vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\frac{1}{3}\left[2\hat{i} + \hat{j} + 2\hat{k}\right]$$

$$\vec{A}\cdot\hat{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$\vec{A}\cdot\hat{n} = \cancel{\frac{2}{3}x} + \frac{2}{3}y^2 - \cancel{\frac{2}{3}x} + \frac{4}{3}yz$$

$$\iint \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint_S \left(\frac{2}{3}y^2 + \frac{4}{3}yz\right)\frac{3}{2}\,\overline{dx\,.\,dy}$$

$$= \iint_S (y^2 + 2yz)\,dx\,dy \quad \text{——— } (1)$$

$$2x + y + 2z = 6$$

$$z = \frac{6-2x-y}{2} \quad \text{—— (2)}$$

Sub value of z in (1) we have

$$= \iint_S \left[y^2 + \frac{2}{2}y\left(6-2x-y\right)\right] dx\,dy$$

$$= \iint_S \left[y^2 + 6y - 2xy - y^2\right) dx\,dy$$

$$= \int_0^3 \int_0^{6-2x} \left(\cancel{y^2} + 6y - 2xy - \cancel{y^2}\right) dx\,dy$$

$$= \int_0^3 \int_0^{6-2x} \left(6y - 2xy\right) dx\,dy$$

$$= \int_0^3 dx \int_0^{6-2x} \left(6y - 2xy\right) dy$$

$$= \int_0^3 dx \left[6\int_0^{6-2x} y\,dy - 2x\int_0^{6-2x} y\,dy\right]$$

$$= \int_0^3 dx \left[6\cdot \frac{y^2}{2}\Big|_0^{6-2x} - 2x\cdot \frac{y^2}{2}\Big|_0^{6-2x}\right.$$

$=$

$z = 0$ in eqn (2)

$$0 = 6 - 2x - y$$

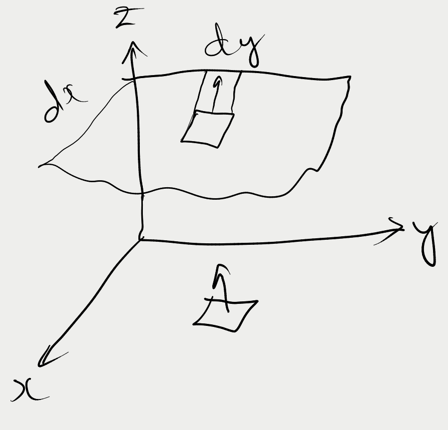
$$\boxed{y = 6 - 2x} \quad \text{—— (3)}$$

$y = 0$ in eqn (3)

$$0 = 6 - 2x$$

$$x = \frac{6}{2}$$

$$\boxed{x = 3} \qquad 0 \text{ to } 3$$



Example 75. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\, d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

$$\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$\phi = 2x + 3y + 6z - 12$$

1. $\vec{\nabla}\phi$; 2. $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$; 3. $\vec{A}\cdot\hat{n}$; 4. $\hat{n}\cdot\hat{k}$

5. $$\iint_S \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Example 76. *Evaluate* $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\vec{ds}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* (*U.P., I Semester, Dec. 2004*)

$$\phi = x^2 + y^2 + z^2 - a^2$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{a^2}}$$

$$x^2+y^2+z^2 = a^2$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{z}{a} \quad ; \quad \cancel{\vec{A}\cdot\hat{n} =}$$

$$\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{A}\cdot\hat{n} = \frac{xy\cancel{z}}{a} + \frac{yzx}{a} + \frac{xy\cancel{z}}{a}$$

$$\vec{A}\cdot\hat{n} = 3xyz/a$$

$$z = (a^2 - x^2 - y^2)^{1/2} \quad \text{———} \quad ①$$

$$\vec{A} \cdot \hat{n} = \frac{3xy}{a} (a^2 - x^2 - y^2)^{1/2}$$

$$\iint_S \vec{A} \cdot \hat{n} \, ds = \iint_S \frac{3xy}{\cancel{a}} \overbrace{\cancel{(a^2 - x^2 - y^2)^{1/2}}}^{z} \cdot \cancel{a} \frac{dx \, dy}{\cancel{z}}$$

$$= \int_0^a dx \int_0^{(a^2 - x^2)^{1/2}} 3xy \, dy$$

$z = 0$

$y = (a^2 - x^2)^{1/2}$

$y = 0$

$x^2 = a^2$

$\boxed{x = a}$

$$= \int_0^a \left[3x \frac{y^2}{2} \Big|_0^{(a^2 - x^2)^{1/2}} \right] \cdot dx$$

$$= \int_0^a \frac{3x}{2} (a^2 - x^2) \, dx$$

$$= \frac{3}{2} \int_0^a (a^2 x - x^3) \, dx$$

$$= \frac{3}{2} \left[a^2 \cdot \frac{x^2}{2} \Big|_0^a - \frac{x^4}{4} \Big|_0^a \right]$$

$$= \frac{3}{2} \left[a^2 \cdot \frac{a^2}{2} - \frac{a^4}{4} \right]$$

$$= \frac{3}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \Rightarrow \frac{3}{2} \cdot a^4 \left[\frac{1}{2} - \frac{1}{4} \right]$$

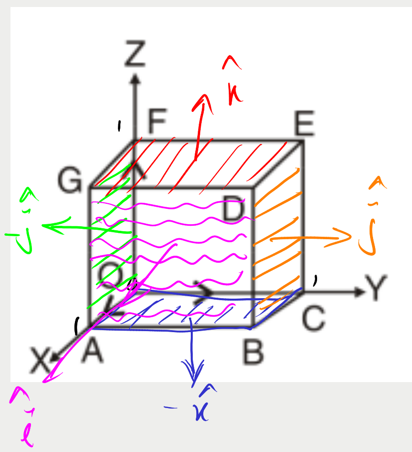
$$= \frac{3}{2}\cdot a^4 \cdot \frac{1}{4}$$

$$= \frac{3}{8}a^4$$

Example 77. *Show that* $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$, *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes,* $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1.$

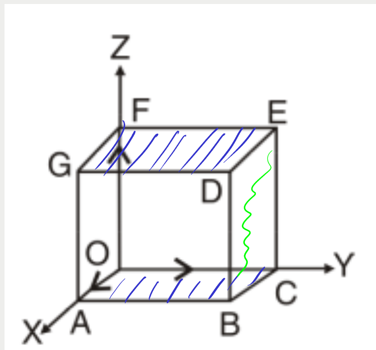
$$\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$\hat{n}\cdot\hat{k} = 1$



Plane	Outward normal	ds
OABC	$-\hat{k}$	$dx\,dy$
DEFG	$\hat{k}$	$dx\,dy$
OAGF	$-\hat{j}$	$dx\,dz$
BCED	$\hat{j}$	$dx\,dz$
ABDG	$\hat{i}$	$dy\,dz$
OCEF	$-\hat{i}$	$dy\,dz$

D



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAGF} \vec{F}\cdot\hat{n}\,ds$$

S: bounded by unit cube

$$+ \iint_{BCED} \vec{F}\cdot\hat{n}\,ds + \iint_{ABDG} \vec{F}\cdot\hat{n}\,ds + \iint_{OCEF} \vec{F}\cdot\hat{n}\,ds$$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$OABC \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{k}) = -yz \quad ; \quad ds = dx\,dy$$

$$DEFG \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{k}) = yz \quad ; \quad ds = dx\,dy$$

$$OAGF \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{j}) = y^2$$

$$BCED \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{j}) = -y^2$$

$$ABDG \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(\hat{i}) = 4xz$$

$$OCEF \Rightarrow \vec{F}\cdot\hat{n} = \vec{F}\cdot(-\hat{i}) = -4xz$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint -yz\,dx\,dy \quad ; \quad z = 0$$

$$\therefore \iint_{OABC} \vec{F}\cdot\hat{n}\,ds = 0$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint yz\,dx\,dy \quad ; \quad z = 1$$

$$= \int_0^1\int_0^1 y\,dx\,dy \Rightarrow \int_0^1 dx \int_0^1 y\,dy$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\Big|_0^1\right]$$

$$= \int_0^1 \frac{1}{2}\,dx \Rightarrow \frac{1}{2}x\Big|_0^1$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \frac{1}{2}$$

$$\iint_{OAGF} \vec{F}\cdot\hat{n}\,ds = \iint y^2 \cdot dx\,dz \quad ; \cancel{y=0}$$

$$= 0 \qquad \because y = 0$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = \iint -y^2\,dx\,dz$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = -1$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4xz\,dy\,dz \quad ; \quad x = 1$$

$$= \int_0^1 dy \int_0^1 4z\,dz$$

$$= \int_0^1 dy \left[4 \cdot \frac{z^2}{2}\Big|_0^1\right]$$

$$\iint_{ABDG} \vec{F}\cdot\hat{n}\,ds = 2\int_0^1 dy = 2$$

$$\iint_{OCEF} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -4xz\,dy\,dz \; ; \cancel{x}$$

$$= 0$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 + \left(\frac{1}{2}\right) + (0) + (-1) + 2 + (0)$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2} \;//.$$

Ex. 5.10

1, 3, 4,

Date : **13-Nov-2020**

Attendance : **10**

Volume Integral :

$$V.I \Rightarrow \iiint_V \underset{\sim}{F}\, dv$$

either Scalar
or
it can also be a Vector

$$= \iiint_V \vec{F}\, dv$$

Example 78 :

$$\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$$

Find $\iiint_V \vec{F}\, dv$ where V is the region bounded by the surfaces

$$x = 0 , \quad x = 2$$

$$y = 0 , \quad y = 4$$

$$z = x^2 , \quad z = 2$$

$$\iiint_V \vec{f} \cdot dx\,dy\,dz = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\,dz\,\hat{i} - x\,dz\,\hat{j} + y\,dz\,\hat{k})$$

$$= \int_0^2 dx \int_0^4 dy \left(\cancel{2} \cdot \frac{z^2}{\cancel{2}} \Big|_{x^2}^2 \hat{i} - x \cdot z \Big|_{x^2}^2 \hat{j} + y\,z \Big|_{x^2}^2 \hat{k} \right)$$

$$= \int_0^2 dx \int_0^4 dy \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x \cdot x^2\hat{j} - y\,x^2\hat{k} \right]$$

$$= \int_0^2 dx \int_0^4 \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k} \right] dy$$

$$= \int_0^2 dx \left[4y\Big|_0^4 \hat{i} - 2xy\Big|_0^4 \hat{j} + 2\frac{y^2}{2}\Big|_0^4 \hat{k} - x^4 y\Big|_0^4 \hat{i} + x^3 y\Big|_0^4 \hat{j} - x^2\frac{y^2}{2}\Big|_0^4 \hat{k} \right]$$

$$= \int_0^2 dx \left[16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k} \right]$$

$$= \left[16x\Big|_0^2 \hat{i} - 8\frac{x^2}{2}\Big|_0^2 \hat{j} + 16x\Big|_0^2 \hat{k} - 4\frac{x^5}{5}\Big|_0^2 \hat{i} + 4\cdot\frac{x^4}{4}\Big|_0^2 \hat{j} - 8\frac{x^3}{3}\Big|_0^2 \hat{k} \right]$$

$$= \left[16\cdot(2)\hat{i} - 4\cdot(4)\hat{j} + 16\cdot(2)\hat{k} - 4\cdot\frac{32}{5}\hat{i} + 16\hat{j} - 8\cdot\frac{8}{3}\hat{k} \right]$$

$$= 32\hat{i} - \cancel{16}\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + \cancel{16}\hat{j} - \frac{64}{3}\hat{k}$$

$$= \left[32 - \frac{128}{5}\right]\hat{i} + \hat{k}\left[32 - \frac{64}{3}\right] \quad //$$

Green's theorem:

Line integral $\longrightarrow$ Surface Integral

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

P & Q are both fn.. x and y

$$\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

$$\vec{F}\cdot d\vec{r} = \left(P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$\vec{F}\cdot d\vec{r} = P\,dx + Q\,dy$$

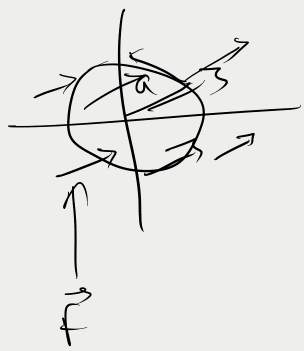
By Green's theorem we have,

$$\boxed{\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{line Integral}} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Example 79:

$$\vec{F} = \underbrace{\sin y}_{P}\,\hat{i} + \underbrace{x(1+\cos y)}_{Q}\,\hat{j}$$

Find $\oint_C \vec{F}\cdot d\vec{r}$ where C is a circular path given by $\underline{x^2+y^2=a^2}$



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

where in our case $P = \sin y$

$Q = x(1+\cos y)$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$x^2 + y^2 = a^2$$

$$\boxed{y = (a^2 - x^2)^{1/2}}$$

$$0 = (a^2 - x^2)^{1/2}$$

(or) $\boxed{x = a}$

$$\frac{\partial Q}{\partial x} = 1 + \cos y$$

$$\frac{\partial P}{\partial y} = \cos y$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^a dx \int_0^{(a^2-x^2)^{1/2}} (1 + \cancel{\cos y} - \cancel{\cos y})\,dy$$

$$= \int_0^a dx \cdot \int_0^{(a^2-x^2)^{1/2}} dy$$

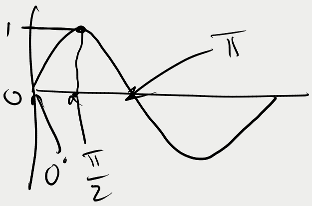
$$= \int_0^a dx \cdot y \Big|_0^{(a^2-x^2)^{1/2}}$$

$$= \int_0^a (a^2 - x^2)^{1/2} \cdot dx$$

$$\int (a^2 - x^2)^{1/2} dx = \frac{1}{2}\left[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) \right]$$

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{2}\left[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \frac{1}{2}\left[\underbrace{\cancel{a(a^2-a^2)^{1/2}}}_{}^{\to 0} + a^2 \sin^{-1}\left(\frac{a}{a}\right) - 0 - a^2\sin^{-1}(0)\right]$$



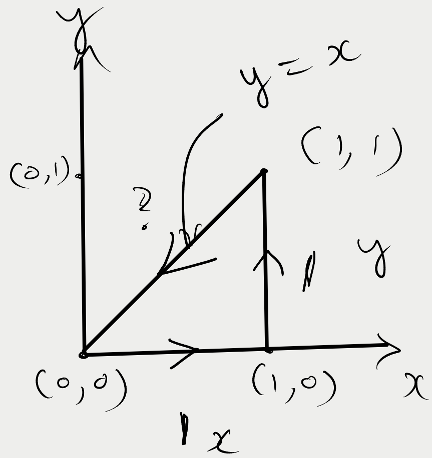
$$= \frac{1}{2}\left[a^2 \cdot \sin^{-1}(1) - a^2 \sin^{-1}(0)\right]$$

$$= \frac{1}{2}\left[a^2 \cdot \frac{\pi}{2} - a^2(0)\right]$$

$$\oint_C \vec{F}\cdot d\vec{r} = \frac{a^2 \cdot \pi}{4} \quad //$$

Example 80:

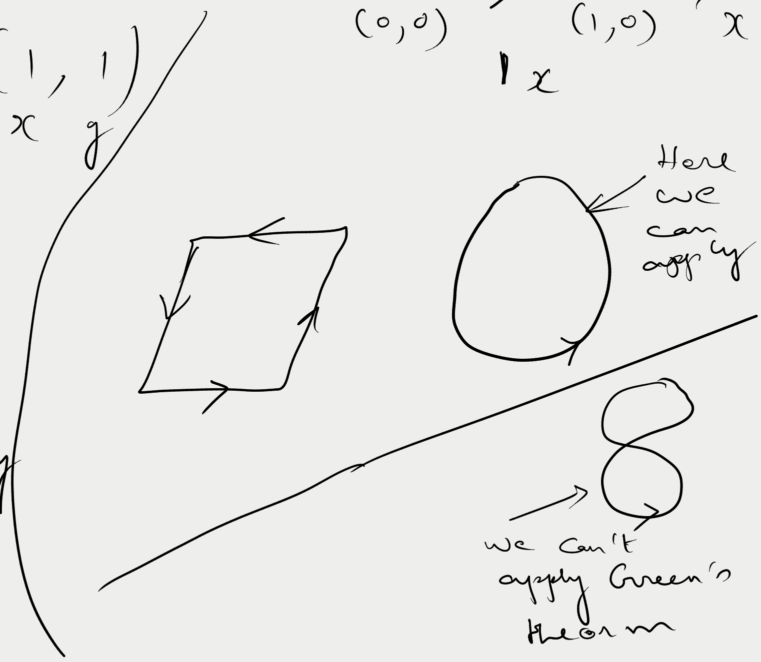
$$\oint_C x^2 y\,dx + x^2\,dy$$



$$\underset{x\ \ y}{(0,0)}\ ;\ \underset{x\ \ y}{(1,0)}\ ;\ \underset{x\ \ y}{(1,1)}$$

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$P = x^2 y$; $Q = x^2$

$\dfrac{\partial P}{\partial y} = x^2$; $\dfrac{\partial Q}{\partial x} = 2x$

$y = mx + c$

$c = 0$

$1 = m \cdot 1$

$m = 1$; $\boxed{y = x}$

$$\oint_C x^2 y\, dx + x^2 dy = \int_0^1 \int_0^x (2x - x^2)\, dx\, dy$$

$$= \int_0^1 \left\{ 2x \cdot y \Big|_0^x - x^2 y \Big|_0^x \right\} dx$$

$$= \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left[2 \cdot \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$\boxed{\oint_C x^2 y\, dx + x^2 dy = \frac{5}{12}}$$

$$\oint_C P\,dx + Q\,dy$$

$P(x, y)$

$Q(x, y)$

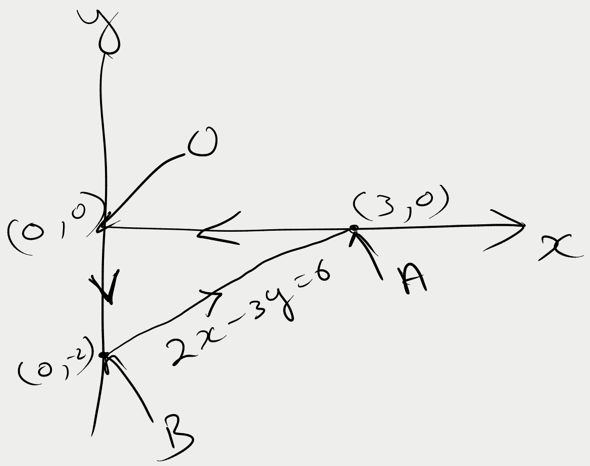
Example 81:

$$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$x \geq 0$

$y \leq 0$

$2x - 3y = 6$



$-3y = 6 - 2x$

$y = \frac{6}{-3} - \frac{-2}{-3}x$

$y = \frac{2}{3}x - 2$

$m = \frac{2}{3}$

$c = -2$

If $y=0$ in $2x-3y=6$

we have

$2x = 6$

$\boxed{x = 3}$

If $x=0$ in $2x-3y=6$

we have

$-3y = 6$

$\boxed{y = -2}$

x	y
3	0
0	-2

Coordinate of A $(3,0)$

" " B $(0,-2)$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

First let do the $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,\underline{dy}$ using Green's theorem.

$$P = 3x^2 - 8y^2 \quad ; \quad Q = 4y - 6xy$$

$$\frac{\partial P}{\partial y} = -16y \quad ; \quad \frac{\partial Q}{\partial x} = -6y$$

$$\oint_C \sim + \sim = \iint_R \left[-6y - (-16y) \right] dx\,dy$$

$$2x - 3y = 6$$

$$-3y = 6 - 2x$$

$$\boxed{y = \frac{2}{3}x - 2}$$

If $y = 0$ in the above equa= we have

$$\boxed{x = 3}$$

$$= \int_0^3 dx \int_{\frac{2}{3}x - 2}^{0} (10y)\,dy$$

$$= \int_0^3 dx \cdot \cancel{10}\,5 \cdot \frac{y^2}{2}\Big|_{\frac{2}{3}x-2}^{0}$$

$$= 5\int_0^3 0 - \left(\frac{2}{3}x - 2\right)^2 dx$$

$$= -5\int_0^3 \left(\frac{4}{9}x^2 + 4 - \frac{8}{3}x\right) dx$$

$$= -5 \cdot \left[\frac{4}{9}\frac{x^3}{3}\Big|_0^3 + 4x\Big|_0^3 - \frac{8}{3}\frac{x^2}{2}\Big|_0^3\right]$$

$$= -5\left[\frac{4}{\cancel{9}} \times \frac{\cancel{3\times} 3\times\cancel{3}}{\cancel{3}} + 4\cdot 3 - \frac{8}{\cancel{3}} \cdot \frac{\cancel{3}\times 3}{2}\right]$$

$$= -5\left[4 + 12 - 12\right]$$

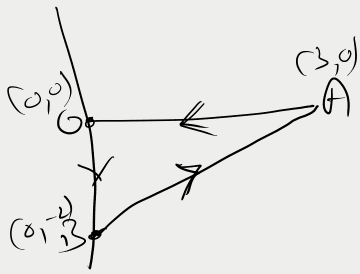
$$= -20 \Leftarrow \text{using Green's theorem.}$$

Again we have to evaluate

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$ by using

explicity line integral.

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$



$$= \int_{OB} \sim dx + \sim dy + \int_{BA} \sim dx + \sim dy$$

$$+ \int_{AO} \sim dx + dy$$

Along OB $\quad x = 0 \quad \therefore \quad dx = 0$

$y = 0 \;;\; y = -2$

$$\int_{OB} \underbrace{\cancel{(3x^2 - 8y^2)\,dx}}_{=0 \;\because\; dx = 0} + (4y - \underbrace{\cancel{6xy}}_{=0 \;\because\; x = 0})\,dy$$

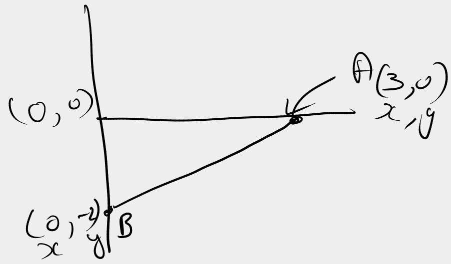
$$\int_{OB} \sim dx + \sim dy = \int_0^{-2} 4y\,dy$$

$$= 4 \cdot \frac{y^2}{2} \Big|_0^{-2}$$

$$= 2 \cdot (-2)^2$$

$$\int_{OB} \sim dx + \sim dy = 8$$

Along BA $\quad x_i = 0 \quad x_f = 3$



$$y_i = -2 \qquad y_f = 0$$

$$2x - 3y = 6$$

$$y = \underbrace{\frac{2}{3}x - 2} \Rightarrow dy = \frac{2}{3}\,dx$$

Here y varies from -2 to 0

when $y_l = -2$; $x = 0$

$y_u = 0$; $x = 3$

$$-2 = \frac{2}{3}x - 2$$

$$\frac{2}{3}x = 0$$

$$\boxed{x = 0}$$

$$\boxed{x = 3}$$

$$\int_{BA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\int_0^3 3x^2 - 8\left(\frac{2}{3}x - 2\right)^2 dx + \left[4\left(\frac{2}{3}x - 2\right) - 6x\left(\frac{2}{3}x - 2\right)\right] \cdot \frac{2}{3}\,dx$$

$$\int_0^3 3x^2 - 8\left[\frac{4x^2}{9} + 4 - \frac{8}{3}x\right] dx$$

$$+ \left[\frac{8}{3}x - 8 - \frac{12x^2}{3} + 12x\right]\frac{2}{3}\,dx$$

$$\int_0^3 \left(3x^2 - \frac{8 \cdot 4x^2}{9} - 32 + \frac{64}{3}x\right) dx$$

$$+ \left(\frac{16}{9}x - \frac{16}{3} - \frac{24x^2}{9} + \frac{24x}{3}\right) dx$$

$$\int_0^3 3x^2\,dx - \frac{32x^2}{9}dx - \cancel{32dx} + \frac{64}{3}x\,dx + \frac{16}{9}x\,dx$$

$$- \frac{16}{3}\cancel{dx} - \frac{24}{9}x^2dx + \frac{24}{3}x\,dx$$

$$\int_0^3 x^2dx\left(3 - \frac{32}{9} - \frac{24}{9}\right) + x\,dx\left(\frac{64}{3} + \frac{16}{9} + \frac{24}{3}\right)$$

$$+ dx\left[-32 - \frac{16}{3}\right]$$

$$\int_0^3 x^2dx\left(-\frac{47}{9}\right) + x\,dx\left(\frac{280}{9}\right) - \frac{112}{3}dx$$

$-\frac{29}{9}$

$$= -\frac{47}{9}\left.\frac{x^3}{3}\right|_0^3 + \frac{280}{9}\left.\frac{x^2}{2}\right|_0^3 - \frac{112}{3}\,x\Big|_0^3$$

$$= -\frac{47}{\cancel{9}} \cdot \frac{\cancel{27}^{3}}{\cancel{3}} + \frac{280}{\cancel{9}} \cdot \frac{\cancel{9}}{2} - \frac{112}{\cancel{3}} \cdot \cancel{3}$$

$$= -\cancel{47}^{29} + 140 - 112$$

$$= \cancel{-19} \quad -1$$

$A(3,0)$
(x, y)

$$\int_{AO} \sim dx + \sim dy$$

$x_i = 3$; $x_f = 0$

$y_i = 0$; $y_f = 0$

$dy = 0$

$$\int_{AO} (3x^2 - 8y^2)\,dx + \cancel{(4y - 6xy)\,dy}^{\,0} \quad \because dy = 0 \quad = \int_3^0 (3x^2 - \cancel{8y^2}^{\,0})\,dx \quad \because y = 0$$

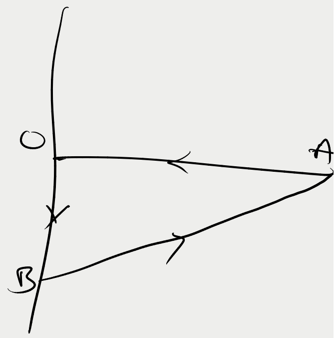
$$= \int_3^0 3x^2\,dx$$

$$= \cancel{3} \cdot \frac{x^3}{\cancel{3}} \Big|_3^0$$

$$= -x^3 \Big|_0^3$$

$$\int_{AO} \sim \qquad = -27$$

$$\oint_{OBA} \sim = \int_{OB} + \int_{BA} + \int_{AO}$$



$$= 8 - \cancel{19}^{\,-1} - 27$$

$$= 8 - 1 - 27$$

$$= -20 \; //$$

Ex 5.13

1, 2, 3, 5

Ex 5.9

5.10

5.11

~~5.12~~

5.13

Date : **28-NOV-20**

Attendance : **10**

Green's theorem:

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$\vec{F} = P\hat{i} + Q\hat{j}$$

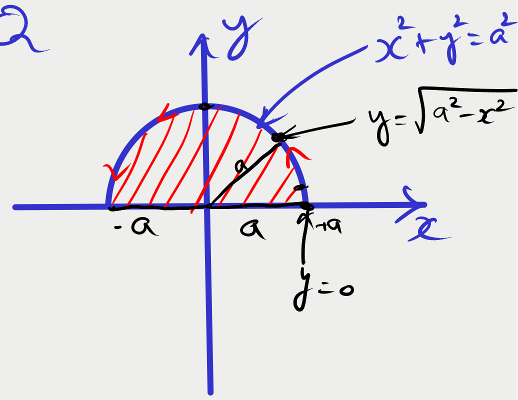
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (P\hat{i} + Q\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

Example 82. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.
(M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004)

$$I = \oint_C \underbrace{(2x^2 - y^2)}_{P}\,dx + \underbrace{(x^2+y^2)}_{Q}\,dy$$

$$C : x^2 + y^2 = a^2$$



$$P = 2x^2 - y^2 \quad ; \quad Q = x^2 + y^2$$

$$\frac{\partial P}{\partial y} = -2y \quad ; \quad \frac{\partial Q}{\partial x} = 2x$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (-2y) \Rightarrow 2(x+y)$$

$$= 2\iint_R (x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} (x+y)\,dy$$

$$= 2\int_{-a}^{a} dx \left[\int_0^{\sqrt{a^2-x^2}} x\,dy + \int_0^{\sqrt{a^2-x^2}} y \cdot dy\right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot y\Big|_0^{\sqrt{a^2-x^2}} + \frac{y^2}{2}\Big|_0^{\sqrt{a^2-x^2}}\right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot \sqrt{a^2-x^2} + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot \sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right]$$

$$= 2\cdot\int_{-a}^{a} \cancel{\underbrace{\overbrace{x\,(a^2-x^2)^{1/2}}^{f_1(x)}}_{f(x)}\,dx} \rightarrow 0 \quad \because f_1(x) \text{ is odd} \quad + \cancel{2}\int_{-a}^{a} \underbrace{\overbrace{(a^2-x^2)}^{f_2(x)}}_{f(x)}\,dx$$

$$\boxed{\begin{aligned}\int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even fn:} \\ &= 0 \qquad\qquad\qquad \text{if } f(x) \text{ is odd fn:}\end{aligned}}$$

If $f(-x) = f(x)$ then $f(x)$ is even

If $f(-x) = -f(x)$ then $f(x)$ is odd

$f_1(x)$ is even or odd ?

$$f_1(x) = x\sqrt{a^2-x^2}$$

$$x = -x$$

$$f_1(-x) = -\overbrace{x\cdot\sqrt{a^2-x^2}}^{f_1(x)}$$

$$f_1(-x) = -f_1(x) \leftarrow \text{odd}$$

$$f_2(x) = a^2 - x^2$$

$$f_2(-x) = a^2 - (-x)^2$$

$$= \underbrace{a^2 - x^2}_{f_2(x)}$$

$$f_2(-x) = f_2(x)$$

$\nwarrow$ even

$$= 2\cdot\int_{0}^{a} (a^2 - x^2)\,dx$$

$$= 2\cdot\left[a^2\cdot x\Big|_0^a - \frac{x^3}{3}\Big|_0^a\right]$$

$$= 2\cdot\left[a^2\cdot a - \frac{a^3}{3}\right] \Rightarrow 2\cdot\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\cdot\frac{2a^3}{3} \Rightarrow 4\frac{a^3}{3} \;//.$$

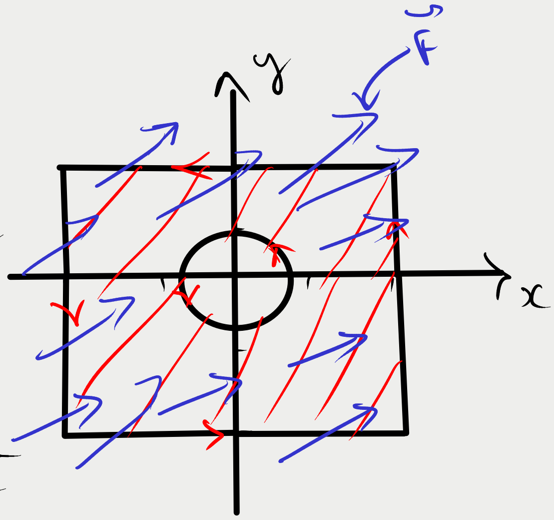
Example 83. *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2=1$ *and* $C_2 : x = \pm 2, y = \pm 2$. *(Gujarat, I Semester, Jan 2009)*

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$C_1 : x^2+y^2=1$

$C_2 : x=\pm 2 \; ; y=\pm 2$

$$\vec{F} = \underbrace{\frac{-y}{x^2+y^2}}_{P}\hat{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q}\hat{j}$$



$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$P = \frac{-y}{x^2+y^2} \quad ; \quad Q = \frac{x}{x^2+y^2}$$

$$P = \underbrace{-y}_{u}\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial P}{\partial y} = +y\cdot(x^2+y^2)^{-1-1}\cdot 2y + (x^2+y^2)^{-1}(-1)$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\cdot\frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{2y^2-x^2-y^2}{(x^2+y^2)^2}$$

$$Q = \underbrace{x}_{u}\cdot\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial Q}{\partial x} = x\cdot -1(x^2+y^2)^{-2}\cdot 2x + (x^2+y^2)^{-1}\cdot 1$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2} - \frac{(2y^2 - x^2 - y^2)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2 + x^2 + y^2 - 2y^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{0}{(x^2+y^2)^2} = 0$$

$\therefore$ from Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \underbrace{\oint_C P\,dx + Q\,dy}_{} = \iint_R \cancel{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}^{0} dx\,dy$$

$$\underbrace{\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 0}_{}$$

By Green's theorem.

From Green's Theorem w.k.T

$$\oint_C P\,dx + Q\,dy = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{= 1} dx\,dy$$

$$\left.\begin{array}{ll} P = -y & ; \quad Q = x \\ \dfrac{\partial P}{\partial y} = -1 & ; \quad \dfrac{\partial Q}{\partial x} = 1 \end{array}\right\}$$

$$\oint_C -y\,dx + x\,dy = \iint_R (1 - (-1))\,dx\,dy$$

$$\oint_C x\,dy - y\,dx = 2\iint_R dx\,dy$$

$ds = dx\,dy$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \overbrace{\iint_R dx\,dy}^{\text{Area}}$$

$$\boxed{\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Example 84. *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$

(U.P. I, Semester, Dec. 2008)

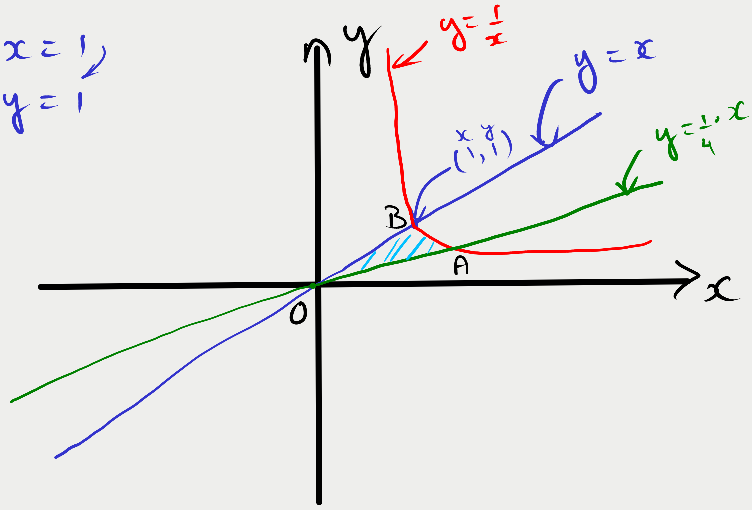
$C_1 \Rightarrow y = x$ $\qquad$ $C_3 \Rightarrow y = \frac{x}{4}$

$C_2 \Rightarrow y = \frac{1}{x}$

$C_1: \; y = x \Rightarrow \text{Slope} = 1$ when $x = 1$, $y = 1$

$y = \underset{\substack{\uparrow \\ \text{slope}}}{m}x + c \Rightarrow \text{intercept}$

$\left.\begin{array}{l} c = 0 \\ m = 1 \end{array}\right\} \Rightarrow y = x$



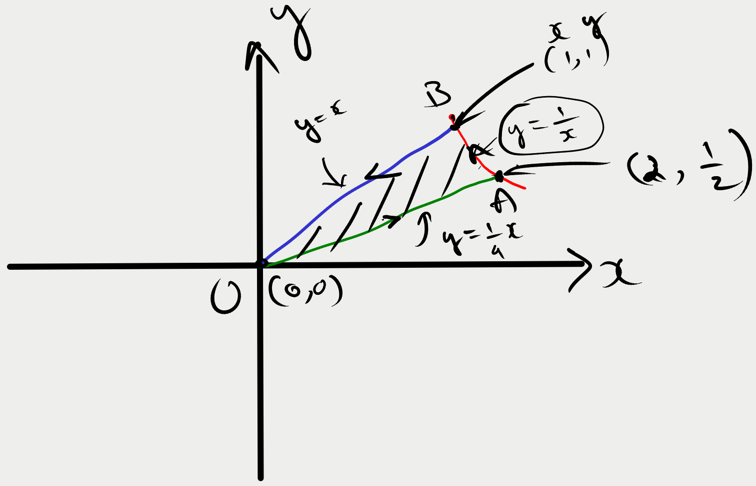
$C_2: \; y = \frac{1}{x}$

when $x = 0$; $y = \infty$

$x = 1$; $y = 1$

$C_3: \; y = \frac{1}{4}x$

Slope $m = \frac{1}{4}$



For what value of x ; $y = x$ & $y = \frac{1}{x}$ are equal?

$x = 1 \quad \left.\begin{array}{l} \overset{C_1}{y} = x \\ \& \; \underset{C_2}{y} = \frac{1}{x} \end{array}\right\}$ are equal

For what value of x ; $\underbrace{y = \frac{1}{4}x}_{C_3}$ & $\underbrace{y = \frac{1}{x}}_{C_2}$ are equal?

$\boxed{x = 2}$

$y = \frac{1}{4} \cdot 2 = \frac{1}{2}$

$y = \frac{1}{2} = \frac{1}{2}$

$x = 2$

$y = \frac{1}{2}$

OA $\Rightarrow (0,0), (2, \frac{1}{2})$; AB $\Rightarrow (2, \frac{1}{2}) (1, 1)$

$BO \Rightarrow (1,1), (0,0)$

$$\boxed{\text{Area} = \frac{1}{2} \oint_{OAB} x\,dy - y\,dx}$$

This is from Green's theorem formula

$$\oint_{OAB} x\,dy - y\,dx = \int_{OA} x\,dy - y\,dx + \int_{AB} x\,dy - y\,dx + \int_{BO} x\,dy - y\,dx$$

For OA $\quad C_3: \; y = \frac{1}{4}x \qquad x_i = 0 \qquad y_i = 0$

$$dy = \frac{1}{4}dx \qquad x_f = 2 \qquad y_f = \frac{1}{2}$$

$$\int_{OA} x\,dy - y\,dx = \int_{OA} x\cdot\left(\frac{dx}{4}\right) - \left(\frac{x}{4}\right)\cdot dx$$

$$= \int_0^2 \cancel{\frac{x\,dx}{4} - \frac{x\,dx}{4}} \rightarrow 0$$

For AB $\quad C_2: \; y = \frac{1}{x} \qquad x_i = 2 \; ; \; y_i = \frac{1}{2}$

$$dy = \frac{-dx}{x^2} \qquad x_f = 1 \; ; \; y_f = 1$$

$$\int_{AB} x\,dy - y\,dx = \int_{AB} \cancel{x}\cdot\left(\frac{-dx}{x^{\cancel{2}}}\right) - \left(\frac{1}{x}\right)\cdot dx$$

$$= \int_2^1 -\frac{dx}{x} - \frac{dx}{x}$$

$$= -2\int_{2}^{1} \frac{dx}{x}$$

$$= -2 \cdot \log x \Big|_{2}^{1} \Rightarrow -2 \cdot -\log x \Big|_{1}^{2}$$

$$= 2\left[\log(2) - \cancel{\log(1)}^{\,0}\right]$$

$$\int_{AB} x\,dx - y\,dy = 2 \cdot \log 2$$

for BO C_1: $y = x$ $\quad x_i = 1$; $y_i = 1$

$dy = dx$ $\quad x_f = 0$; $y_f = 0$

$$\int_{BO} x\,dy - y\,dx = \int_{BO} x(dx) - (x) \cdot dx$$

$$= \int_{1}^{0} \cancel{x\,dx - x\,dx}^{\,0}$$

line integrals.

$$\oint_{OAB} x\,dy - y\,dx = \underbrace{\int_{OA} \sim}_{0} + \underbrace{\int_{AB} \sim}_{2\cdot\log 2} + \underbrace{\int_{BO}}_{0}$$

$$\oint_{OAB} x\,dy - y\,dx = 0 + 2\log 2 + 0$$

$$\text{Area} = \frac{1}{2}\oint_{OAB} x\,dy - y\,dx \Rightarrow \text{Area} = \frac{1}{2} \cdot 2\log 2$$

$$\boxed{\text{Area} = \log 2}$$ //.

Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F}\cdot d\,\overline{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\,d\,\vec{s} = \iint_S curl\,\vec{F}\cdot d\,\vec{S}$$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2.$ (M.D.U., Dec 2009)

By Stoke's theorem w.r.t

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \underline{\vec{\nabla}\times\vec{F}}\cdot\hat{n}\cdot\frac{dx\,dy}{\underline{|\hat{n}\cdot\hat{k}|}}$$

If our $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

and our $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (yz\,dx + zx\,dy + xy\,dz)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}[x-x] - \hat{j}[y-y] + \hat{k}[z-z]$$

$\vec{\nabla} \times \vec{F} = 0$

$\therefore$ By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \quad \because \quad \vec{\nabla} \times \vec{F} = 0$$

$$\oint_C \vec{f} \cdot d\vec{r} = \iint_S \cancel{\vec{\nabla} \times \vec{f}}^{\,0} \cdot \hat{n} \, \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

where c is the circle $x^2 + y^2 = 1$*, corresponding to the surface of sphere of unit radius.* (*U.P., I Semester, Winter 2001*)

$x^2+y^2+z^2=1$

$x^2+y^2+z^2=1$

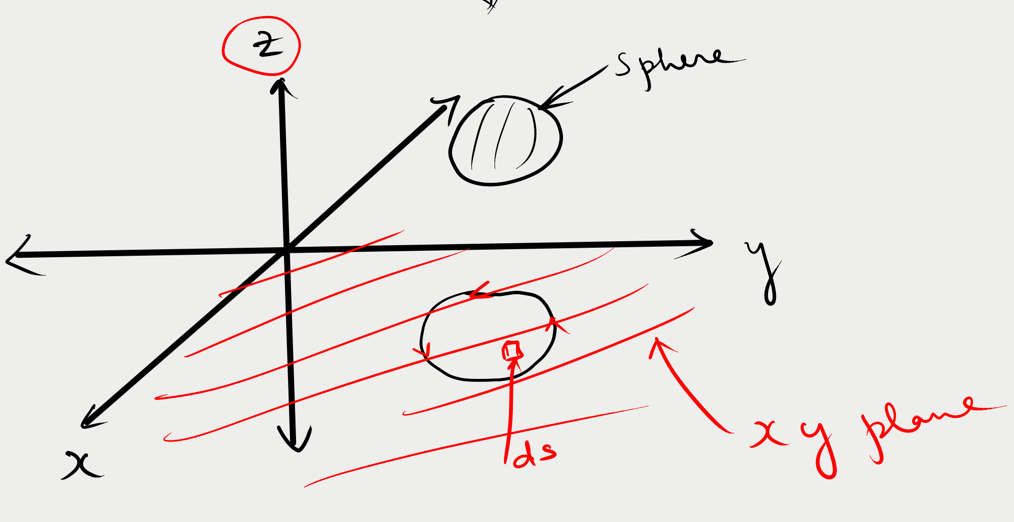
$$\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k} \qquad \vec{\nabla} \times \vec{F}$$

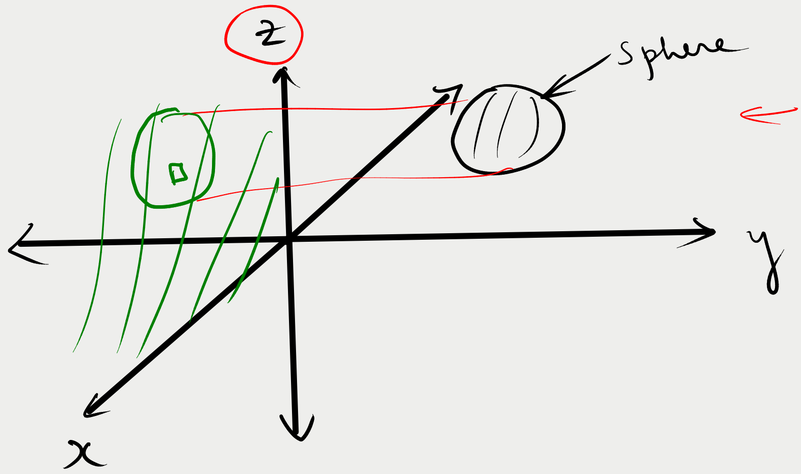
$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\phi = x^2+y^2+z^2-1$$

By Stoke's theorem: $\quad \hat{n} \cdot \hat{k}$

$$\oint_C \vec{f} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \, \frac{dx\,dy \leftarrow ds}{|\hat{n} \cdot \boxed{\hat{k}}|} \qquad \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$





$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\cdot dz}{|\hat{n}\cdot\hat{j}|}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$0-(-1)$

$$= \hat{i}\,[\cancel{}\,0] - \hat{j}\,[\cancel{}\,0] + \hat{k}\,[0+1]$$

$$\boxed{\vec{\nabla}\times\vec{F} = \hat{k}}$$

$$\phi = x^2+y^2+z^2-1$$

$$\vec{\nabla}\phi = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

Given:

$x^2+y^2+z^2=1$

$$= \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}} \to 1$$

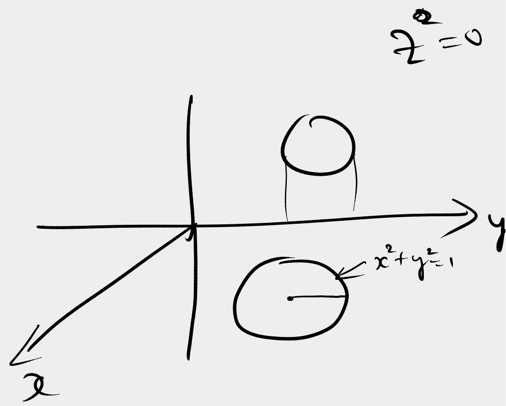
$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\boxed{\hat{n}\cdot\hat{k} = z}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

$$= \iint_S \cancelto{z}{\hat{k}\cdot\hat{n}} \cdot \frac{dx\,dy}{z}$$

$$= \iint_S \cancel{z} \cdot \frac{dx\,dy}{\cancel{z}}$$



$$\oint_C \vec{f}\cdot d\vec{r} = \iint_S dx\,dy$$

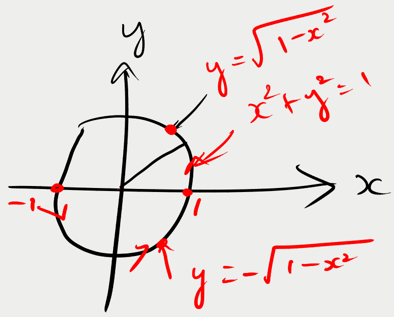
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$y=\pm\sqrt{1-x^2}$

$$= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$



$$= \int_{-1}^{1} dx \cdot y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} \left(2\cdot\sqrt{1-x^2}\right) dx$$

$$= 2\int_{-1}^{1} \underbrace{\sqrt{1-x^2}}_{f(x)}\, dx$$

$f(x) = \sqrt{1-x^2}$ and it's a even fn :

$$\therefore \quad = 2 \cdot 2 \int_0^1 \left(\sqrt{1-x^2}\right) dx$$

Formula: ← Pls chk this formula

$$\int_0^1 (a^2-x^2)^{1/2} dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$\rightarrow 0$ (for both limits)

$$= \cancel{4} \cdot \frac{1}{\cancel{2}}\left[x(1-x^2)^{1/2} + 1 \sin^{-1}(x)\right]_0^1$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= \cancel{2}\left[\frac{\pi}{\cancel{2}} - 0\right]$$

$$\oint_c \vec{F} \cdot d\vec{r} = \pi$$

Ans.

Stoke's Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,ds$$

(or)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}$$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (M.D.U., Dec 2009)

$$\phi = x^2+y^2-1$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{\cancel{2}(x\hat{i}+y\hat{j})}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{(x\hat{i}+y\hat{j})}{\sqrt{x^2+y^2}}$$

$$\vec{F}\cdot d\vec{r} = yz\,dx + zx\,dy + xy\,dz$$

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}[x-x] - \hat{j}[y-y] + \hat{k}[z-z]$$

$$\vec{\nabla}\times\vec{F} = 0$$

$\therefore$ By Stoke's theorem $\oint_C \sim dx + \sim dy + \sim dz = 0 \quad \because \vec{\nabla}\times\vec{F} = 0$

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

(U.P., I Semester, Winter 2001)

W.K.T By Stoke's theorem

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds \quad \text{or} \quad \iint_S \underbrace{\vec{\nabla} \times \vec{F}}\cdot d\vec{s}$$

$$\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \hat{i}\left[\cancel{-2yz - (-2yz)}^{\,0}\right] - \hat{j}\left[\cancel{0 - 0}^{\,0}\right] + \hat{k}\left[0 - (-1)\right]$$

$$\vec{\nabla} \times \vec{F} = \hat{k} \quad ;$$

$$\phi = x^2 + y^2 + z^2$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2} = 1}$$

$$\hat{n} \cdot \hat{k} = \frac{z}{1} \qquad \therefore\; ds = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$ds = dx\,dy$$

$$\iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} \Rightarrow \iint \cancel{z}\,\frac{dx\,dy}{\cancel{2}}$$

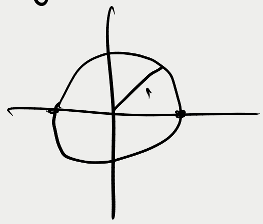
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} dx\,dy \Rightarrow$$

$$x^2+y^2=1$$

$$y=\sqrt{1-x^2}$$

$$-1\le x\le 1$$

$$-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$$



$$\int_{-1}^{1} dx\cdot y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)$$

$$2\sqrt{1-x^2}\;dx$$

f(x) is a even fn

$$2\int_{-1}^{1}(1-x^2)\,dx = 2\cdot2\int_{0}^{1}(1-x^2)\,dx$$

$$=4\cdot\int_{0}^{1}(1-x^2)dx$$

$$=4\cdot\frac{1}{2}\left((1^2-x^2)^{1/2}-a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^1$$

$$=\frac{1}{2}\left[x\cdot\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=2\cdot\left[\underbrace{\cancel{x\cdot\sqrt{1-x^2}}}_{\to 0}+1\sin^{-1}(x)\right|_0^1$$

$$=2\cdot\left[\sin^{-1}(1)-\underbrace{\cancel{\sin^{-1}(0)}}_{\to 0}\right]\Rightarrow\cancel{2}\cdot\frac{\pi}{\cancel{2}}\quad\pi.$$

Example 87. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. *(Gujarat, I sem. Jan. 2009)*

By Stoke's theorem w.k.T

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

So we have

$$\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

So given surface is

$$\phi = y + z - 2$$

$$\vec{\nabla}\phi = \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\hat{j}+\hat{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[1+2y]$$

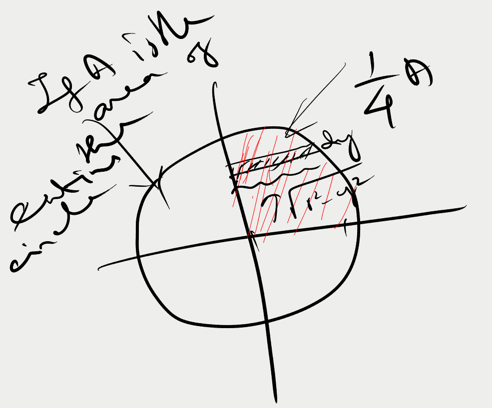
$$\vec{\nabla}\times\vec{F} = \hat{k}[1+2y]$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \hat{k}(1+2y)\cdot\frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1+2y}{\sqrt{2}}$$

By Stoke's theorem

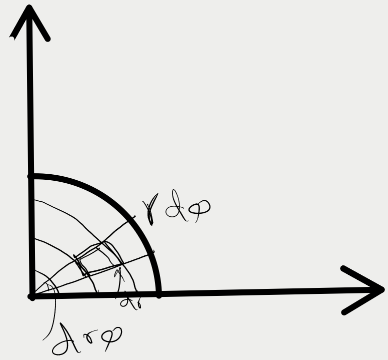
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1+2y}{\sqrt{2}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint_S (1+2y)\,dx\,dy$$



This integral is easy to solve in Spherical polar coordinates instead of Cartesian

$x = r\cos\theta \; ; \; y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

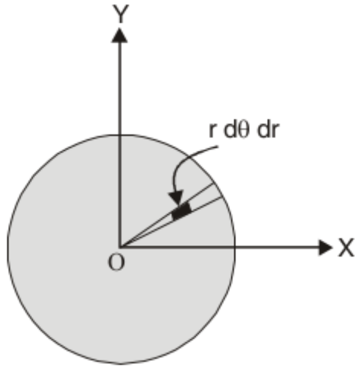


$$= \iint \frac{1+2y}{\sqrt{2}} \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

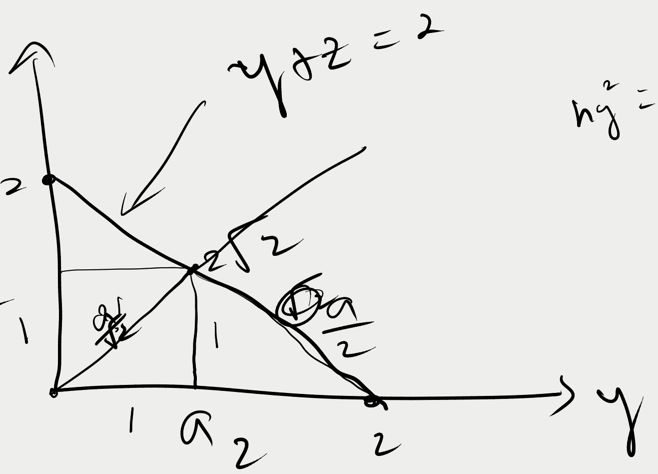
$$= \left[\frac{\theta}{2} - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi - \frac{2}{3} - 0 + \frac{2}{3}\right) = \pi \qquad \textbf{Ans.}$$



mple 88. *Apply Stoke's Theorem to find the value of*

$$\frac{\theta}{2} - \frac{2}{3}\cos\theta \Big|_0^{2\pi} = \frac{\cancel{2\pi}^{\pi}}{\cancel{2}} - \frac{2}{3}\cos(2\pi) - \left(0 - \frac{2}{3}\cos 0\right)$$

$= \pi$



$hy^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2$

$= \frac{a^2}{4} + \frac{a^2}{4}$

$hy^2 = \frac{a^2}{2 \times 2}$

$hy = \frac{a}{\sqrt{2}} = \cancel{2-z}$

$hy^2 = ad^2 + op^2$

$= 2\sqrt{2}$

$y + z = 2$

when $y = 0 \quad z = 0$

when $z = 0 \quad y = 2$

$\cancel{y = x} \quad y = z$

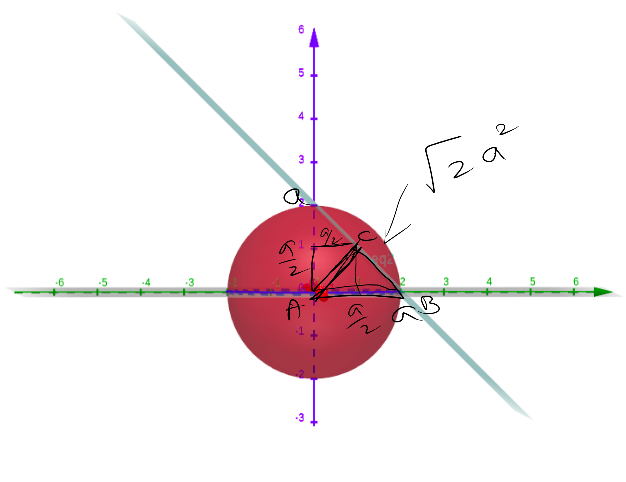
$y + z = 2$

$y = 2 - z$

$y = z$

$AB^2 = BC^2 - AC^2$

ray circ. radius of sphere distance from centre



$op = a$

$ad = a$

$hy^2 = op^2 + ad^2$

$hy^2 = a^2 + a^2$

$hy^2 = 2a^2$

$hy = \sqrt{2}\,a^2$

$p = a$

$hy^2 = \frac{a^2}{4} + \frac{a^2}{4}$

$\frac{\sqrt{2}\,a^2}{2}$

$hy^2 = \frac{\cancel{2}a^2}{\cancel{4}2}$

$R^2 = r^2 + p^2$ $\frac{a}{2}$

$\frac{\cancel{2}\,a^2}{\cancel{4}\,2}$

$hy = \frac{a}{\sqrt{2}}$

$r^2 = R^2 - p^2$

$= a^2 -$

Stoke's Theorem and Application

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F}\cdot d\overline{r} = \iint_S curl\ \vec{F}\cdot\hat{n}\, d\ \vec{s} = \iint_S curl\ \vec{F}\cdot d\ \vec{S}$$

Problems:

✓ **Example 85.** *Evaluate by Strokes theorem* $\oint_C (yz\, dx + zx\, dy + xy\, dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (*M.D.U., Dec 2009*)

✓ **Example 86.** *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\, dx - yz^2 dy - y^2 z\, dz]$$

where c is the circle $x^2 + y^2 = 1$, *corresponding to the surface of sphere of unit radius.* (*U.P., I Semester, Winter 2001*)

Example 87. *Evaluate* $\int_C \vec{F}\,.\,d\,\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (*Gujarat, I sem. Jan. 2009*)

Example 88. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\, dx + z\, dy + x\, dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ *and* $x + z = a$. (*Nagpur, Summer 2001*)

Example 89. *Directly or by Stoke's Theorem, evaluate* $\iint_s curl\ \vec{v}\cdot\hat{n}\, ds$, $\vec{v} = \hat{i}y + \hat{j}z + \hat{k}x$, *s is the surface of the paraboloid* $z = 1 - x^2 - y^2$, $z^3 \geq 0$ *and* $\hat{n}$ *is the unit vector normal to s.*

Example 90. *Use Stoke's Theorem to evaluate* $\int_c \vec{v}\cdot d\vec{r}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

Example 91. *Evaluate the surface integral* $\iint_S \text{curl}\,\vec{F}\,.\,\hat{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$ and $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}\cdot$ *(K. University, Dec. 2008)*

Example 92. *Evaluate* $\oint_C \vec{F}\cdot\vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$ *and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*
(U.P., I Semester, Winter 2000)

Example 93. *Evaluate* $\oint_C \vec{F}\cdot\vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = (x^2 + y^2)\,\hat{i} - 2\,xy\,\hat{j}$ *and C is the boundary of the rectangle* $x = \pm\, a$, $y = 0$ *and* $y = b$. *(U.P., I Semester, Winter 2002)*

Example 94. *Apply Stoke's Theorem to calculate* $\int_c 4\,y\,dx + 2\,z\,dy + 6\,y\,dz$ *where c is the curve of intersection of* $x^2 + y^2 + z^2 = 6\,z$ *and* $z = x + 3$.

Example 95. *Verify Stoke's Theorem for the function* $\overline{F} = z\hat{i} + x\,\hat{j} + y\,\hat{k}$, *where C is the unit circle in xy-plane bounding the hemisphere* $z = \sqrt{(1 - x^2 - y^2)}$. *(U.P., I Semester Comp. 2002)*

Example 96. *Verify Stoke's theorem for the vector field* $\overline{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2\,z\,\hat{k}$ *over the upper half of the surface* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on xy- plane.*
(Nagpur University, Summer 2001)

Example 97. *Verify Stoke's Theorem for* $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

Example 98. *Verify Stoke's theorem for a vector field defined by* $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ *in the rectangular in xy-plane bounded by lines* $x = 0$, $x = a$, $y = 0$, $y = b$.
(Nagpur University, Summer 2000)

Example 99. *Verify Stoke's Theorem for the function*

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z = 0$ *and bounded by the lines*

$$x = 0,\ y = 0,\ x = a,\ y = a.$$

Y

Example 100. *Verify Stoke's Theorem for* $\vec{F} = (x + y)\,\hat{i} + (2x - z)\,\hat{j} + (y + z)\,\hat{k}$ *for the surface of a triangular lamina with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*
(Nagpur University 2004, K. U. Dec. 2009, 2008, A.M.I.E.T.E., Summer 2000)

Example 101. *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - (xz)\hat{k}$$

over the surface of a cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the XOY plane (open the bottom).

Gaussian Theorem :

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Example 102. *State Gauss's Divergence theorem* $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint Div\, \vec{F}\, dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

Example 103. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *where* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(U.P., Ist Semester, 2009, Nagpur University, Winter 2003)

Example 104. *Find* $\iint \vec{F} \cdot \hat{n} \cdot ds$, *where* $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ *and S is the surface of the sphere having centre (3, – 1, 2) and radius 3.*

(AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000)

Example 105. *Use Divergence Theorem to evaluate* $\iint_S \vec{A} \cdot \vec{ds}$,

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(AMIETE, Dec. 2009)

Example 106. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2)\, d\vec{s} = 6V$$

where S is any closed surface enclosing volume V. *(U.P., I Semester, Winter 2002)*

Example 107. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Example 108. *Use Divergence Theorem to evaluate* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F} = 4\,x\hat{i} - 2\,y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.
(*A.M.I.E.T.E., Summer 2003, 2001*)

Example 109. *Apply the Divergence Theorem to compute* $\iint \vec{u}\cdot\hat{n}\,ds$, *where s is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = \hat{i}x - \hat{j}y + \hat{k}z$.

Example 110. *Apply Divergence Theorem to evaluate* $\iiint_V \vec{F}\,.\,\hat{n}\,ds$, *where* $\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$ *and* $z = b$.
(*U.P. Ist Semester, Dec. 2006*)

Example 111. *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\,ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\,(\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron* $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ *and n is the unit normal in the outward direction to the closed surface S.*

Example 112. *Use the Divergence Theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where S is the portion of the plane $x + 2\,y + 3\,z = 6$ *which lies in the first Octant.*
(*U.P., I Semester, Winter 2003*)

Example 113. *Use Divergence Theorem to evaluate* : $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *over the surface of a sphere radius a.* (*K. University, Dec. 2009*)

Example 114. *Using the divergence theorem, evaluate the surface integral* $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ *where* $S : x^2 + y^2 + z^2 = 4$.
(*AMIETE, Dec. 2010, UP, I Sem., Dec 2008*)

Example 115. *Evaluate* $\iint_S xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dx\,dy$ *where S is the surface of hemispherical region bounded by*

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = 0.$$

Example 116. *Evaluate* $\iint_S \vec{F}\,.\,\hat{n}\,ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $F = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.

Example 117. *The vector field* $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ *is defined over the volume of the cuboid given by* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, *enclosing the surface S. Evaluate the surface integral*

$$\iint_S \vec{F}\,.\,d\vec{s}$$

(*U.P., I Semester, Winter 2001*)

Example 118. *Verify the divergence Theorem for the function* $\overline{F} = 2x^2yi - y^2j + 4xz^2k$ *taken over the region in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$.

Example 119. *Verify the Gauss divergence Theorem for*

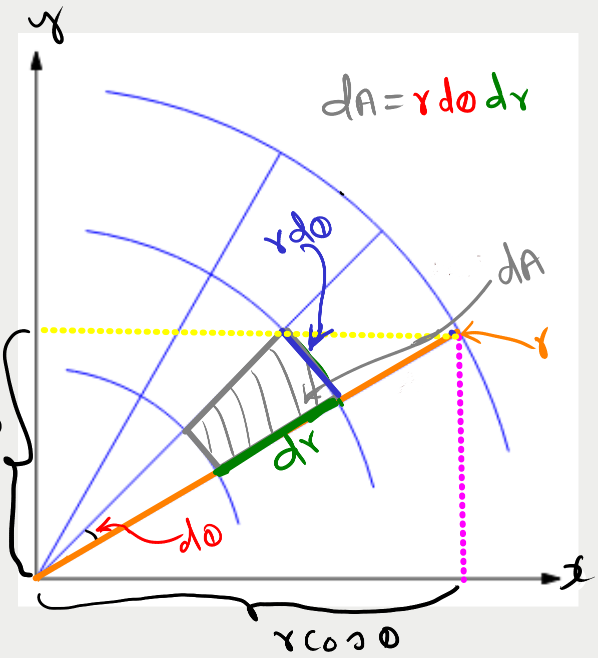
$\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$ *(U.P., I Semester, Compartment 2002)*

Example 120. *Verify Divergence Theorem, given that* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by the planes* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

In Cartesian co-ordinates:

$dA = dx\,dy$

Elemental Area



In cylindrical coordinate:

$dA = r\,d\theta\,dr$

Elemental area

Sector

$l = r\theta$

Transformation from Cartesian to Cylinderical coordinate

$x \longrightarrow r\cos\theta$

$y \longrightarrow r\sin\theta$

Elemental area

$dA = dx\,dy \longrightarrow r\,d\theta\,dr$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

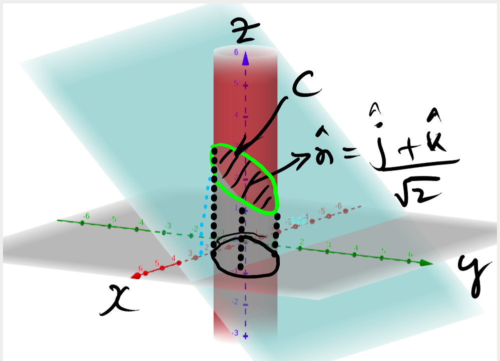
Date : **02-Dec-20**

Attendance : **9**

Example 87. *Evaluate* $\int_C \vec{F} \cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)

Stoke's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds$$



$$\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = (1+2y)\hat{k} \quad ;$$

$$\phi = y + z - 2$$

$$\vec{\nabla}\phi = \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{j}+\hat{k}}{\sqrt{1^2+1^2}} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$\hat{n} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$\hat{n} \cdot \hat{k} = \frac{1}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = (\hat{x} + 2y\hat{k})\cdot\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$$

$$= \frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k} + 2y\cdot\frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k}$$

$$\boxed{(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{2}}(1+2y)}$$

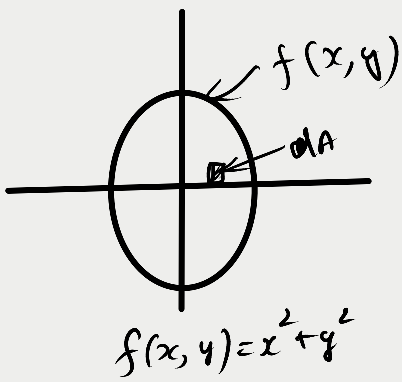
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,dx\,dy$$

$$\iint_S (x^2+y^2)\underbrace{dx\,dy}_{dA}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1+2y}{\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\,dx\,dy$$

$$= \iint_S (1+2y)\,\underbrace{dx\,dy}_{dA}$$



In cylinderical coordinate

$$x \to r\cos\theta \; ; \; y \to r\sin\theta$$

$$dx\,dy \to r\,d\theta\,dr$$

$$= \int_0^{2\pi} \int_0^{r=1} (1 + 2(r\sin\phi)) \cdot r \, d\phi \, dr$$

$$= \int_0^{2\pi} d\phi \int_0^{1} \left[r\,dr + 2 \cdot r^2 \sin\phi \, dr \right]$$

$$= \int_0^{2\pi} d\phi \left[\frac{r^2}{2} \Big|_0^1 + 2 \cdot \frac{r^3}{3} \sin\phi \Big|_0^1 \right]$$

$$= \int_0^{2\pi} d\phi \left[\frac{1}{2} + \frac{2}{3} \sin\phi \right]$$

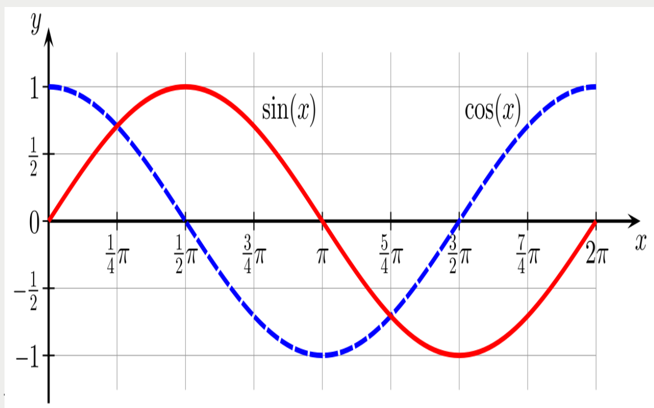
$$= \frac{1}{2} \phi \Big|_0^{2\pi} + \frac{2}{3} (-\cos\phi) \Big|_0^{2\pi}$$

$$= \left[\pi - 0 - \left(\frac{2}{3} \cancel{\cos 2\pi}^{\,1} - \frac{2}{3} \cancel{\cos 0}^{\,1} \right) \right]$$

$$= \pi - \cancel{\frac{2}{3}(1)} + \cancel{\frac{2}{3}(1)}$$

$$= \pi$$

$$\oint_C (-y^2 \hat{i} + x \hat{j} + z^2 \hat{k}) \cdot d\vec{r} = \pi$$

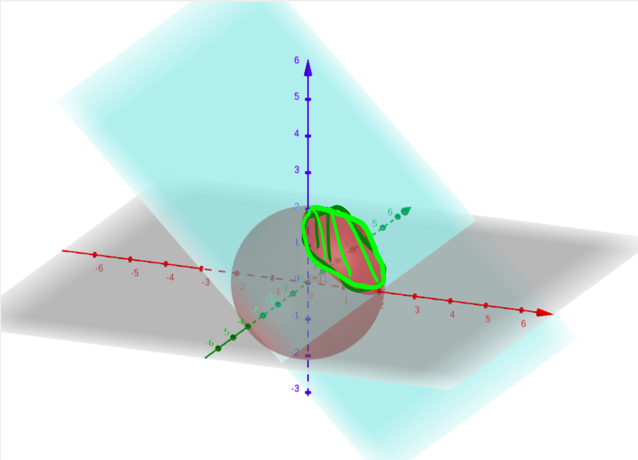


Example 88. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ *and* $x + z = a$. (*Nagpur, Summer 2001*)

$$\oint_c \vec{F}\cdot d\vec{r} = \oint_c (y\,dx + z\,dy + x\,dz)$$



$$\oint_c \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$$

$$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\vec{\nabla}\times\vec{F} = -(\hat{i}+\hat{j}+\hat{k}) \quad ; \quad \phi = x + z - a$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\hat{n} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, ds = \iint_S \frac{-2}{\sqrt{2}} \cdot ds$$

$$= \frac{-2}{\sqrt{2}} \underbrace{\iint_S ds}$$

area of the disc formed by the intersection the plane $x+z=a$, and $x^2+y^2+z^2=a^2$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, ds = \frac{-2}{\sqrt{2}} A_d$$

W.K.T area of a disc $= \pi r^2$

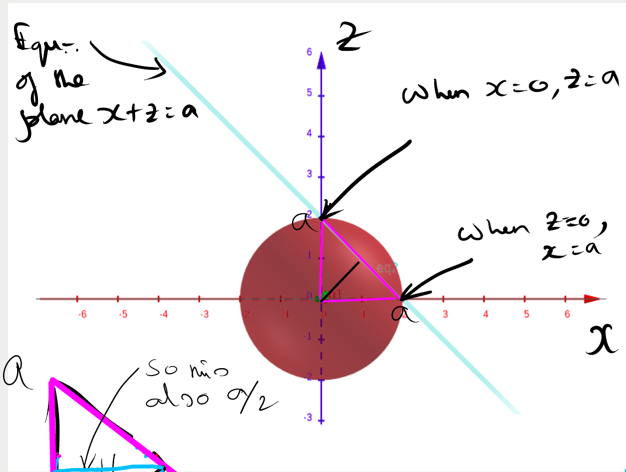
If "r_d" is the radius of the disc

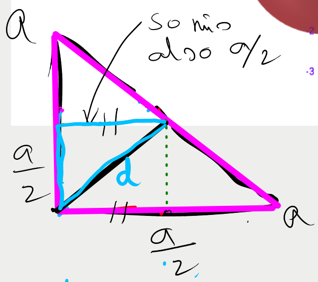
$$\boxed{a^2 = r_d^2 + d^2}$$

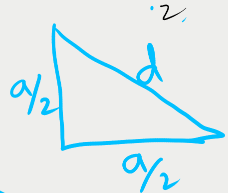
radius of the sphere

radius of the disc

distance bt the center of the disc and the center of the sphere



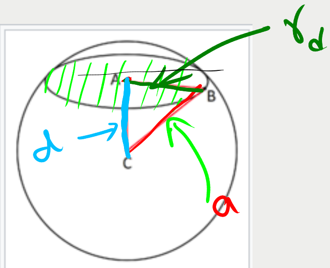




$$d^2 = \frac{a^2}{4} + \frac{a^2}{4}$$

$$d^2 = \frac{2a^2}{4}$$

$$\boxed{\therefore d^2 = \frac{a^2}{2}}$$



$BC^2 = AB^2 + AC^2$, where C is the center of the sphere, A is the center of the small circle, and B is a point in the boundary of the small circle. Therefore, knowing the radius of the sphere, and the distance from the plane of the small circle to C, the radius of the small circle can be determined using the Pythagorean theorem.

$$\therefore r_d^2 = a^2 - d^2$$

$$= a^2 - \frac{a^2}{2}$$

$$\boxed{\therefore r_d^2 = \frac{a^2}{2}}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = -\frac{2}{\sqrt{2}} \cdot A_d$$

$$A_d = \pi r_d^2$$

$$r_d^2 = \frac{a^2}{2}$$

$$\oint_c \vec{F} \cdot dr = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = -\frac{2}{\sqrt{2}} \pi \cdot \frac{a^2}{2}$$

$$= -\frac{\cancel{2}}{\sqrt{2}} \pi \cdot \frac{a^2}{\cancel{2}}$$

$$= -\frac{\pi a^2}{\sqrt{2}}$$

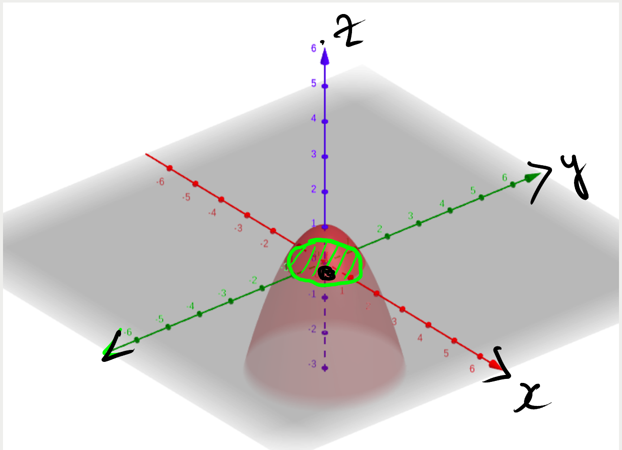
Example 89. *Directly or by Stoke's Theorem, evaluate* $\iint_s$ *curl* $\vec{v} \cdot \hat{n}\, ds$, $\vec{v} = \hat{i}y + \hat{j}z + \hat{k}x$, *s is the surface of the paraboloid* $z = 1 - x^2 - y^2$, $z^3 \geq 0$ *and* $\hat{n}$ *is the unit vector normal to s.*

0

$0 = 1 - x^2 - y^2$

$x^2 + y^2 = 1$

$$\vec{v} = y\hat{i} + z\hat{j} + x\hat{k}$$



Since this surface is on the xy plane $\hat{n} = \hat{k}$

$$\vec{\nabla} \times \vec{v} = -(\hat{i} + \hat{j} + \hat{k})$$

$$(\vec{\nabla} \times \vec{v}) \cdot \hat{n} = -1$$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dx\,dy}{1}$$

$$\iint_S \underbrace{(\vec{\nabla}\times\vec{v})\cdot\hat{n}}\, ds = \iint_S (-1)\cdot \frac{dx\,dy}{1}$$

$$= -1 \iint_S dx\,dy$$

$$= -1\cdot\pi(1)^2$$

$$= -\pi$$ //.

Example 90. *Use Stoke's Theorem to evaluate* $\int_c \vec{v}\cdot\vec{dr}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

$$\vec{v} = y^2\hat{i} + xy\hat{j} + xz\,\hat{k}$$

From Stoke's theorem

$$\oint_c \vec{v}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{v})\cdot\hat{n}\, ds$$

On the xy plane

$z = 0$

$x^2+y^2+\cancel{z^2}^{\to 0} = 9$

$\underline{x^2+y^2 = 3^2}$

$\hookrightarrow$ radius of the circle

$$\vec{\nabla}\times\vec{v} = -z\hat{j} - y\hat{k} \;;$$

$$\phi = x^2+y^2+z^2-9$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4x^2+4y^2+4z^2}$$

$$= 2\cdot\sqrt{x^2+y^2+z^2}$$

$$= 2\cdot\sqrt{9}$$

$$|\vec{\nabla}\phi| = 6$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{6}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

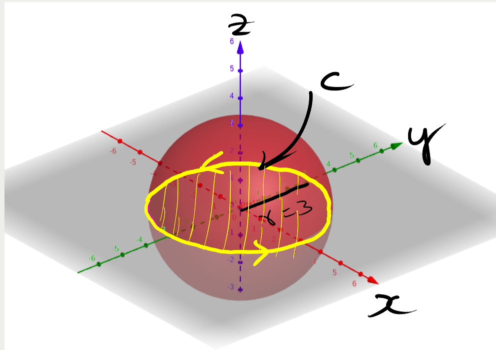
$$(\vec{\nabla}\times\vec{v})\cdot\hat{n} = -\frac{2}{3}zy$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\frac{z}{3}} = \frac{3}{z}\,dx\,dy$$

$$\iint_S \frac{\underbrace{(\vec{\nabla}\times\vec{V})\cdot\hat{n}}\,dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint_S -\frac{2}{\cancel{3}}\cancel{z}y\cdot\frac{\cancel{3}}{\cancel{z}}\,dx\,dy$$

$$= -2\iint_S y\cdot dx\,dy$$

In cylindrical coordinate



$$y \rightarrow r\sin\theta$$

$$dx\,dy \rightarrow r\,d\theta\,dr$$

$$= -2\int_0^{2\pi}\int_0^{3} r\sin\theta\, r\,d\theta\,dr$$

$$= -2\int_0^{2\pi} d\theta \int_0^{3} r^2\sin\theta\,dr$$

$$= -2\int_0^{2\pi}\sin\theta\,d\theta\cdot\left[\frac{r^3}{3}\right]_0^3$$

$$= -2\int_0^{2\pi}\sin\theta\,d\theta\cdot\frac{3^{\cancel{3}\,2}}{\cancel{3}}$$

$$= -18(-\cos\theta)_0^{2\pi}$$

$$= 18\left[\underset{1}{\cos 2\pi} - \cos 0\right]$$

$$= 18\{1-1\}$$

$$= 0$$

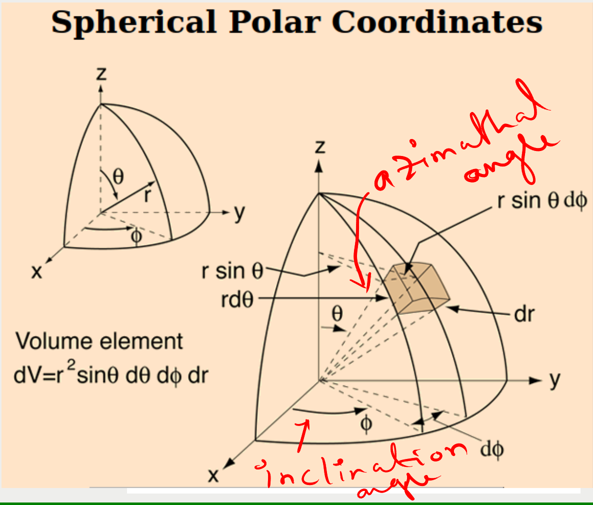
Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **09-Dec-2020**

Attendance : **9**

Spherical Polar Coordinates



$r \in [0, \infty)$ radial vector

$\theta \in [0, \pi]$ azimuth angle

$\phi \in [0, 2\pi]$ inclination angle

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$$\boxed{dv = dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

Gauss' Divergence Theorem:

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \vec{\nabla} \cdot \vec{F}\, dv$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Example 102. *State Gauss's Divergence theorem* $\iint_S \vec{F}.\hat{n}\,ds = \iiint Div\,\vec{F}\,dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

$$\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$$

$$x^2 + y^2 + z^2 = r^2$$

$$r = 4$$

$$\phi = x^2 + y^2 + z^2 - 16 \Rightarrow$$

Gauss' theorem w.k.T

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(3x\hat{i} + 4y\hat{j} + 5z\hat{k}\right)$$

$$= 3 + 4 + 5$$

$$\vec{\nabla}\cdot\vec{F} = 12$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \underbrace{12}\,dv$$

$$= 12 \iiint_V dv$$

$$= 12 \cdot \frac{4}{3}\pi\, 4^3$$

$$= \cancel{12}^{4} \cdot \frac{4}{\cancel{3}}\pi \cdot 64$$

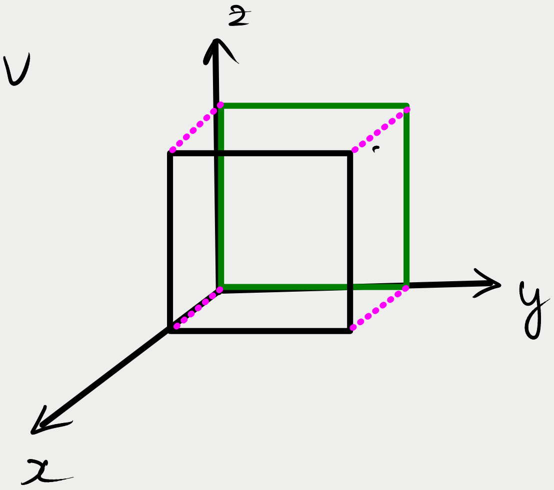
$$= 16 \times 64 \cdot \pi$$

$$= 1024\pi$$

Example 103. *Evaluate* $\iint_S \vec{F}\cdot\hat{n}\,ds$ *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

(*U.P., Ist Semester, 2009, Nagpur University, Winter 2003*)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$



$$\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = 4z - 2y + y$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 4z - y}$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)\,\overbrace{dx\,dy\,dz}^{dv}$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y)\,dz$$

$$= \int_0^1 dx \int_0^1 dy \left[\cancel{4}^{2}\cdot\frac{z^2}{\cancel{2}}\Big|_0^1 - y\cdot z\Big|_0^1\right]$$

$$= \int_0^1 dx \int_0^1 dy\,[2 - y]$$

complete the remaining steps.

$$\vdots$$

$$= \frac{3}{2}$$ //

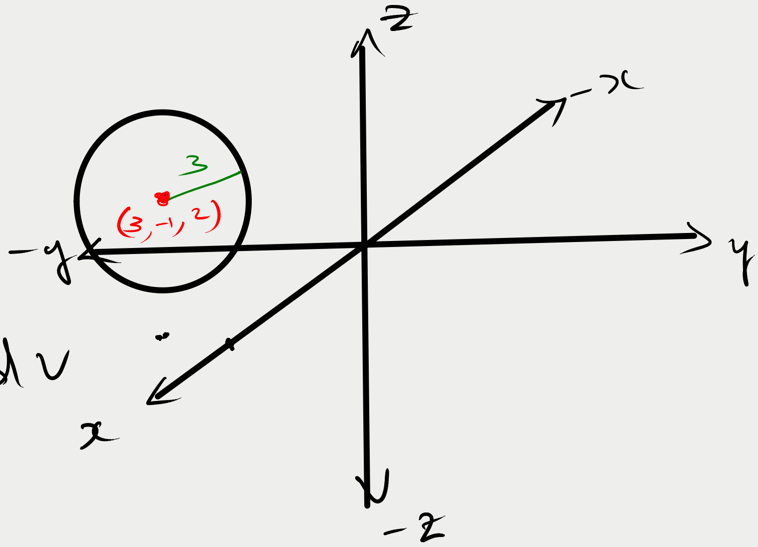
Example 104. *Find* $\iint \vec{F}\cdot\hat{n}\cdot ds$, *where* $\vec{F}=(2x+3z)\,\hat{i}-(xz+y)\,\hat{j}+(y^2+2z)\,\hat{k}$ *and S is the surface of the sphere having centre* (3, – 1, 2) *and radius* 3.

(AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000)

$x = 3$

$y = -1$

$z = 2$



$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dv$$

$$\vec{F} = (2x+\cancel{3z})\,\hat{i} - (xz+y)\,\hat{j} + (y^2+2z)\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = 2 - 1 + 2$$

$$\underline{\vec{\nabla}\cdot\vec{F} = 3}$$

$$= \iiint_V 3\cdot dv$$

$$= 3\iiint_V dv$$

$$= 3\cdot\frac{4}{\cancel{3}}\pi\, 3^{\cancel{3}}$$

$$= 12\cdot 3^2\pi$$

$$= 108\pi\ \text{A.}$$

Example 105. *Use Divergence Theorem to evaluate* $\iint_S \vec{A}\cdot\vec{ds}$,

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(AMIETE, Dec. 2009)

By Gauss theorem

$$\vec{ds} = \hat{n}\, ds$$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{A}\, dv$$

$$\vec{\nabla}\cdot\vec{A} = 3x^2 + 3y^2 + 3z^2$$

$$\vec{\nabla}\cdot\vec{A} = 3(x^2+y^2+z^2)$$

$$= 3\iiint_V \overbrace{(x^2+y^2+z^2)}^{a^2}\, dv$$

X you should not sub. a^2 here itself

$x \to r\sin\theta\cos\phi$

$y \to r\sin\theta\sin\phi$

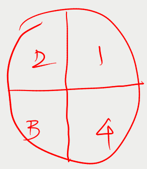
$z \to r\cos\theta$

$$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\iiint_V (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\iiint_V \left[r^2\sin^2\theta\,\overbrace{(\cos^2\phi + \sin^2\phi)}^{1} + r^2\cos^2\theta\right] r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\iiint_V r^2\,\overbrace{(\sin^2\theta + \cos^2\theta)}^{1}\cdot \underbrace{r^2\sin\theta\, dr\, d\theta\, d\phi}_{dv}$$



$$= 3\iiint_V r^2\cdot r^2\sin\theta\, dr\, \underbrace{d\theta}_{\to\, 0,\pi}\, d\phi \longleftrightarrow 0, 2\pi$$

$$3\times 8\int_0^{\pi/2} d\phi \int_0^{\pi/2}\sin\theta\, d\theta \cdot \int_0^a r^4\, dr$$

In the book

$$= 3\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= 3\cdot\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \cdot \left.\frac{r^5}{5}\right|_0^a$$

$$= 3 \cdot \frac{a^5}{5} \left(-\cos\theta\right)_0^{\pi} \cdot (2\pi)$$

$$= 3 \cdot \frac{a^5}{5} \cdot (1+1) \cdot 2\pi$$

$$= \frac{12\pi a^5}{5}$$

Example 106. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2) \cdot d\vec{s} = 6\, V$$

where S is any closed surface enclosing volume V. (*U.P., I Semester, Winter 2002*)

$$\iint_S \vec{F} \cdot d\vec{r} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dv$$

$$\vec{\nabla}(x^2+y^2+z^2) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$\vec{F} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$d\vec{s} = \hat{n} \cdot ds$$

$$\iint_S \vec{\nabla}(x^2+y^2+z^2) \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} \, ds$$

$$\iint_S 2(x\hat{i}+y\hat{j}+z\hat{k}) \cdot \hat{n} \, ds = 2\iiint_V \vec{\nabla} \cdot (x\hat{i}+y\hat{j}+z\hat{k}) \, dv$$

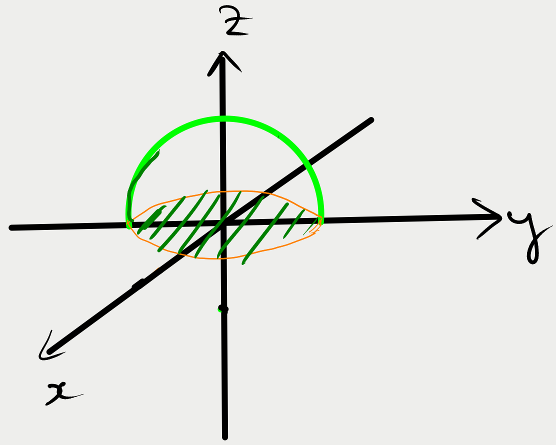
$$= 2 \cdot \iiint_V 3 \, dv$$

$$= 2 \cdot 3 \iiint_V dv \quad V$$

$$= 6V$$

Example 107. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,\hat{n}\,dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

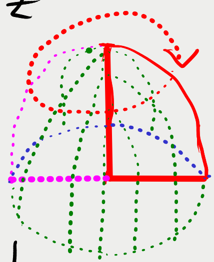
$$\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}$$

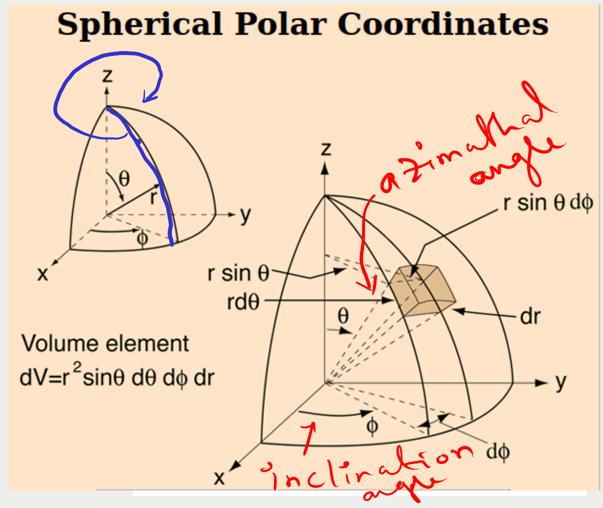


$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$\vec{\nabla}\cdot\vec{F} = 0 + 0 + y^2\cdot 2z$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 2y^2z}$$





$$x \to r\sin\theta\cos\phi$$

$$y \to r\sin\theta\sin\phi$$

$$z \to r\cos\theta$$

$$dv = r^2\sin\theta\,d\theta\,dr\,d\phi$$

For a hemisphere $\theta = 0$ to $\frac{\pi}{2}$; $\phi = 0$ to 2π

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dv$$

$$= \iiint_V 2y^2z\cdot dx\,dy\,dz$$

In spherical polar coordinates we have

$$= 2\iiint_V (r^2\sin^2\theta\sin^2\phi\, r\cos\theta)\cdot r^2\sin\theta\,d\theta\,dr\,d\phi$$

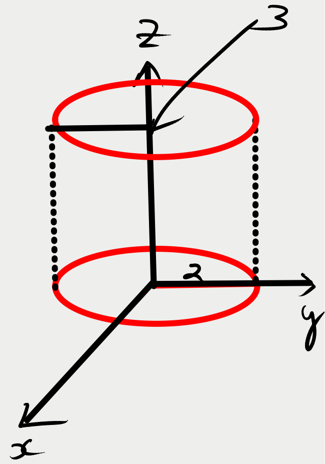
$$= 2\iiint_V r^5 dr\sin^3\theta\cos\theta\,d\theta\sin^2\phi\,d\phi$$

$$= 2\cdot\int_0^{2\pi}\sin^2\phi\,d\phi\cdot\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\cdot\int_0 r^5\,dr$$

Complete the remaining steps.

Example 108. *Use Divergence Theorem to evaluate* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.
(A.M.I.E.T.E., Summer 2003, 2001)

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dv$$



$$\vec{\nabla}\cdot\vec{F} = 4 - 4y + 2z$$

$$= \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int dx \int dy \int_0^3 (4 - 4y + 2z)\, dz$$

$$= \int dx \int dy \cdot \left[4\cdot z\Big|_0^3 - 4y\cdot z\Big|_0^3 + \cancel{2}\frac{z^2}{\cancel{2}}\Big|_0^3 \right]$$

$$= \int dx \int dy \left[12 - 12y + 9 \right]$$

$$= \int dx \int dy \left[21 - 12y \right]$$

$$= 3 \iint (7 - 4y)\, dx\, dy$$

$x^2 + y^2 = 4$

$y = \sqrt{4 - x^2}$

$x^2 = 4$

$x = \pm 2$

$$= 3 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4+x^2}} (7 - 4y)\, dx\, dy$$

We can use cylindrical coor:

$x \to r\cos\theta$

$y \to r\sin\theta$

$dx\,dy \to r\, d\theta\, dr$

$$= 3 \int_0^{2\pi}\int_0^{2} (7 - 4r\sin\theta)\, r\, d\theta\, dr$$

Complete the integral //.

Example 111. *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\,ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\,(\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron* $x = 0$, $y = 0$, $z = 0$, $\underline{x + y + z = 2}$ *and n is the unit normal in the outward direction to the closed surface S.*

Eqn of plane

$$\vec{F} = (x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$$

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})\right)$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)+\frac{\partial}{\partial y}(x^2+y^2+z^2)+\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= 2\cdot x + 2y + 2z$$

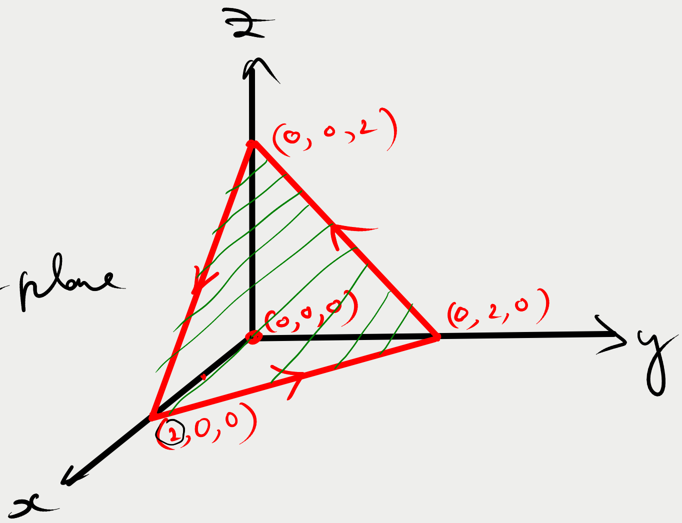
$$\vec{\nabla}\cdot\vec{F} = 2(x+y+z)$$

From Gauss' theorem w.k.T

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$= 2\iiint_V (x+y+z)\,dx\,dy\,dz$$

Before proceed further let find the volume.



Eqn of plane:

$$x+y+z=2$$

General form for Eqn of plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=1$$

$$\overset{x}{(0},\overset{y}{0},\overset{z}{0)} \quad \overset{x}{(a},\overset{y}{b},\overset{z}{c)}$$

$(2,2,2)$

$$= 2 \cdot \int_0^2 dx \int_0^{2-x} dy \int_0^{2-x-y} dz\,(x+y+z)$$

$x+y+z = 2$

Line on the xy plane

$z = 2-x-y$

Point on the xy plane $z=0$

$0 = 2-x-y$

$\therefore\ y = 2-x$

Point on the x axis $y=0$

$0 = 2-x$

or $\boxed{x = 2}$

Complete the integral 4.

Example 112. *Use the Divergence Theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where S is the portion of the plane x + 2 y + 3 z = 6 which lies in the first Octant.

(U.P., I Semester, Winter 2003)

Given

$$\iint_S x\,dy\,dz + \iint_S y\,dz\,dx + \iint_S z\,dx\,dy \qquad \text{——— (1)}$$

In order to use Gauss' Theorem, we need $\vec{F}$.

Let assume $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

From Gauss's Theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV \qquad \text{——— (2)}$$

Lets consider L.H.S of (2)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot\hat{n}\,ds \quad \text{--- (3)}$$

In xy plane w.r.t $\hat{n} = \hat{k}$
↳ $dx\,dy$

Lets rewrite Eqn (3) for xy plane

$$\iint_S \vec{F}\cdot\hat{k}\,dx\,dy = \iint_S (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot\hat{k}\,dx\,dy$$

$$\iint_S \vec{F}\cdot\hat{k}\,dx\,dy = \iint_S f_3\,dx\,dy \quad \text{--- (4a)}$$

$$\iint_S \vec{F}\cdot\hat{i}\,dy\,dz = \iint_S f_1\,dy\,dz \quad \text{--- (4b)}$$

$$\iint_S \vec{F}\cdot\hat{j}\,dz\,d\cancel{x} = \iint_S f_2\,\underline{dz\,dx} \quad \text{--- (4c)}$$

Comparing Eqns (4a), (4b) & (4c) with Eqn. (1)

w.r.t

$$\iint_S y\,\underline{dx\,dz} = \iint_S f_2\,dx\,dz$$

$$\boxed{f_2 = y}$$

$$\boxed{f_1 = x}$$

$$\boxed{f_3 = z}$$

Sub values of f_1, f_2 & f_3 in Eqn for $\vec{F}$ we have

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

From Gauss theorem w.k.T

$$\underbrace{\iint_S \vec{F}\cdot\hat{n}\,ds}_{} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$\iint_S \vec{F}\cdot\hat{i}\,dydz + \iint_S \vec{F}\cdot\hat{j}\,dxdz + \iint_S \vec{F}\cdot\hat{k}\,dxdy =$$

$$\boxed{\iint_S (x\,dydz + y\,dxdz + z\,dxdy)}$$

$$= \iiint_V \vec{\nabla}\cdot(x\hat{i} + y\hat{j} + z\hat{k})\cdot dv$$

$$= 3\cdot\iiint_V dv$$

Given Eqn :-

$$x + 2y + 3z = \underline{\underline{6}}\ 1$$

$$\frac{x}{6} + \frac{2y}{6} + \frac{3z}{6} = 1$$

comparing this Eqn

$$\frac{1}{6}(x) + \frac{1}{3}(y) + \frac{1}{2}(z) = 1 \longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

We have $a = 6$; $b = 3$; $c = 2$

$$(0,0,0)\ (a, b, c)$$

for our case we have

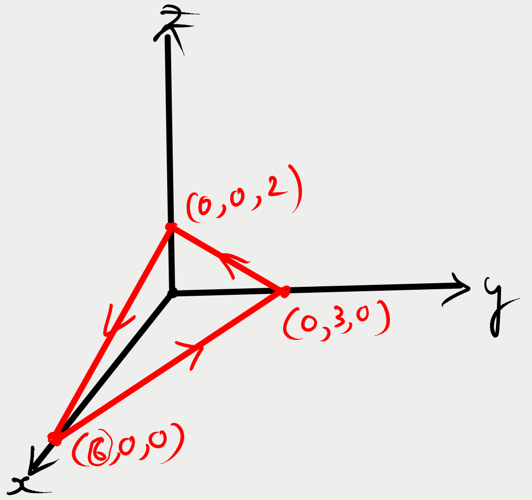
$$(0, 0, 0)\ \ (6, 3, 2)$$

$x_i \quad y_i \quad z_i \qquad x_f \quad y_f \quad z_f$

In the xy plane Eqn of line

$$3z = 6 - 2y - x$$

$$z = 2 - \frac{2}{3}y - \frac{1}{3}x$$



Eqn of y, for that we have to put z = 0 in the above eqn.

$$0 = 2 - \frac{2}{3}y - \frac{1}{3}x$$

$$\frac{2}{3}y = 2 - \frac{1}{3}x \Rightarrow y = \frac{2}{\frac{2}{3}} - \frac{\frac{x}{3}}{\frac{2}{3}}$$

$$y = 3 - \frac{x}{2}$$

$$= \frac{3 \times \cancel{2}}{2} - \frac{\cancel{3}x}{\cancel{3}\cdot 2}$$

For the value x we have put y = 0

$$3 - \frac{x}{2} = 0$$

$$\iiint_V 3 \cdot dV = 3 \cdot \iiint dV$$

$$\iiint dx\,dy\,dz$$

or $x = 6$ //

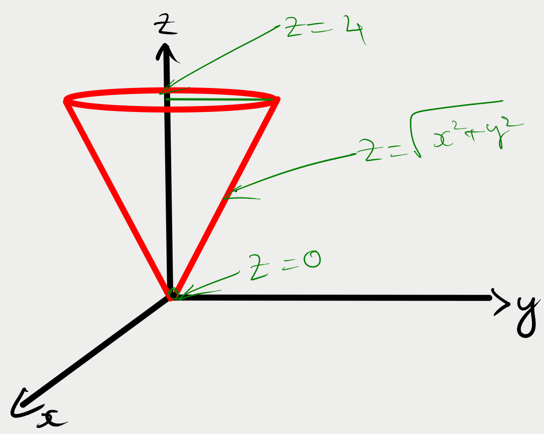
$$= 3 \cdot \int_0^6 dx \int_0^{3-\frac{x}{2}} dy \int_0^{2-\frac{2}{3}y-\frac{1}{3}x} dz$$

Complete the Integral.

Example 116. *Evaluate* $\iint_S \vec{F}.\hat{n}\, ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.

$$\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$$

$$\vec{\nabla} \cdot \vec{F} = 4z + xz^2 + 3$$



From Gauss' theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$$

$$= \iiint (4z + xz^2 + 3)\, dx\, dy\, dz$$

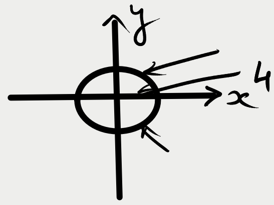
$$= \int dx \int dy \int_{\sqrt{x^2+y^2}}^{4} (4z + xz^2 + 3)\, dz$$

$$= \iint \Big[44 + \frac{64}{3} x - 2(x^2+y^2) - x(x^2+y^2)^{3/2} - 3\sqrt{x^2+y^2} \Big]\, dx\, dy$$

$x \to r\cos\theta$ $\qquad r \to 0, 4$

$y \to r\sin\theta$ $\qquad \theta \to 0, 2\pi$

$dx\, dy \to r\, d\theta\, dr$



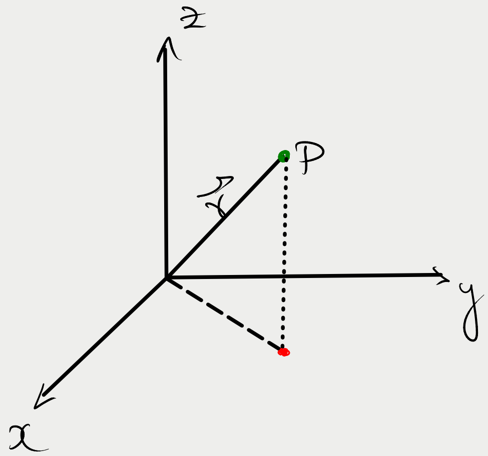
Replace $x \to r\cos\theta$; $y \to r\sin\theta$; $dx\,dy \to r\,d\theta\,d\phi$ and do the integral for $r \to 0, 4$

$\theta \to 0, 2\pi$

Complete this integral

Vector Operators in Curvilinear Coordinate System

Grad, Div, Curl and Laplacian.



$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Position vector of P be $\vec{r}$

$$\vec{r} = \vec{r}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{—(1)}$$

$$d\vec{r} = \underbrace{\frac{\partial \vec{r}}{\partial x}}_{\text{vector}}\cdot dx + \underbrace{\frac{\partial \vec{r}}{\partial y}}_{\text{vector}}\cdot dy + \underbrace{\frac{\partial \vec{r}}{\partial z}}_{\text{vector}}\cdot dz \quad \text{—(2)}$$

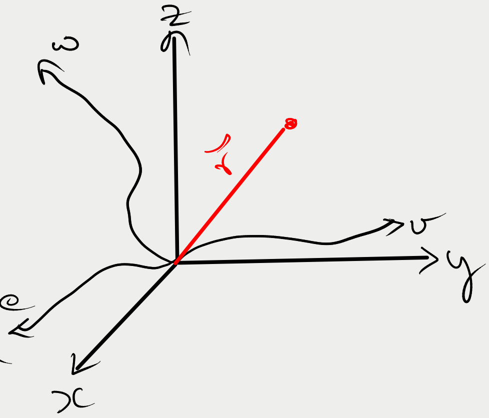
from (1) we can write

$$\frac{\partial \vec{r}}{\partial x} = \hat{i}\,; \quad \frac{\partial \vec{r}}{\partial y} = \hat{j}\,; \quad \frac{\partial \vec{r}}{\partial z} = \hat{k}$$

Curvilinear coordinates

In cartesian coordinate

$\vec{r}(x, y, z)$



In curvilinear coordinate

$\vec{r}(u, v, w)$

$$d\vec{r} = \underbrace{\frac{\partial \vec{r}}{\partial u}}_{\text{vector}} \cdot du + \underbrace{\frac{\partial \vec{r}}{\partial v}}_{\text{vector}} \cdot dv + \underbrace{\frac{\partial \vec{r}}{\partial w}}_{\text{vector}} \cdot dw$$

What the way to make a vector to an unit vector.

$$\hat{e}_u = \frac{\partial \vec{r}/\partial u}{|\partial \vec{r}/\partial u|} \quad ; \quad \hat{e}_v = \frac{\partial \vec{r}/\partial v}{|\partial \vec{r}/\partial v|}$$

$$\hat{e}_w = \frac{\partial \vec{r}/\partial w}{|\partial \vec{r}/\partial w|}$$

$\hat{e}_u = \frac{1}{h_u} \cdot \frac{\partial \vec{r}}{\partial u}$ where $h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|$

$\hat{e}_v = \frac{1}{h_v} \cdot \frac{\partial \vec{r}}{\partial v}$ " $h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|$

$\hat{e}_w = \frac{1}{h_w} \cdot \frac{\partial \vec{r}}{\partial w}$ $h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$

$\Big\}$ Scaling factor.

Curvilinear $\longrightarrow$ $d\vec{r} = h_u \underbrace{\hat{e}_u}_{\text{unit vector}} du + h_v \cdot \underbrace{\hat{e}_v}_{\text{unit vector}} dv + h_w \cdot \underbrace{\hat{e}_w}_{\text{unit vector}} dw$

Cartesian $\longrightarrow$ $d\vec{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$

$$\hat{e}_u = \frac{1}{h_u} \cdot \frac{\partial \vec{r}}{\partial u}$$

In Cartesian system these u, v, w are nothing but x, y, z

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{e}_x = \frac{1}{h_x} \cdot \frac{\partial \vec{r}}{\partial x}$$

w.r.t $\frac{\partial \vec{r}}{\partial x} = \hat{i}$

$h_x = \left|\frac{\partial \vec{r}}{\partial x}\right| = 1$

$$\hat{e}_x = \frac{\hat{i}}{1} = \hat{i}$$

Spherical coordinate ($u = r$; $v = \theta$; $w = \phi$)

Cylindrical coordinate ($u = r$; $v = \theta$; $w = z$)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ —— Cartesian}$$

$x \to r\sin\theta\cos\phi$

$y \to r\sin\theta\sin\phi$

$z \to r\cos\theta$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

—— Spherical coordinate

From Curvilinear coordinate w.r.t

$$d\vec{r} = h_r\hat{e}_r\,dr + h_\theta\hat{e}_\theta\,d\theta + h_\phi\hat{e}_\phi\,d\phi$$

$$\hat{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r};\quad \hat{e}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta};\quad \hat{e}_\phi = \frac{1}{h_\phi}\frac{\partial \vec{r}}{\partial \phi}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\frac{\partial \vec{r}}{\partial r} \cdot \frac{\partial \vec{r}}{\partial r}}$$

dot product

$$= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(1) + \cos^2\theta}$$

$$h_r = \sqrt{1} = 1 \,//$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{\frac{\partial \vec{r}}{\partial \theta} \cdot \frac{\partial \vec{r}}{\partial \theta}}$$

$$= \left[r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\right]^{1/2}$$

$$= \left[r^2\cos^2\theta\,\cancelto{1}{(\cos^2\phi + \sin^2\phi)} + r^2\sin^2\theta\right]^{1/2}$$

$$= \left[r^2\,\cancelto{1}{[\cos^2\theta + \sin^2\theta]}\right]^{1/2}$$

$$h_\theta = \sqrt{r^2} = r$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\,\hat{k}$$

$$= -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

$$h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{\frac{\partial \vec{r}}{\partial \phi}\cdot\frac{\partial \vec{r}}{\partial \phi}}$$

$$= \left[r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi\right]^{1/2}$$

$$= \left[r^2 \sin^2\theta \, (\cancelto{1}{\sin^2\phi + \cos^2\phi})\right]^{1/2}$$

$$h_\phi = r\sin\theta$$

$$\hat{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} = \frac{1}{1}\cdot \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \frac{1}{h_\theta}\cdot\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}\left(r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}\right)$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{1}{h_\phi}\cdot\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}\cdot\left(-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}\right)$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Let find the Gradient of a Scalar fn = ϕ in curvilinear system.

$$\phi = \phi(u, v, \omega)$$

$$d\phi = \frac{\partial \phi}{\partial u}\cdot du + \frac{\partial \phi}{\partial v}\cdot dv + \frac{\partial \phi}{\partial \omega}\cdot d\omega$$

Orthogonality property

$\hat{i}\cdot\hat{i} = 1$; $\hat{j}\cdot\hat{j} = 1$; $\hat{k}\cdot\hat{k} = 1$ $\longleftarrow$ Cartesian

$\hat{e}_u\cdot\hat{e}_u = 1$; $\hat{e}_v\cdot\hat{e}_v = 1$; $\hat{e}_\omega\cdot\hat{e}_\omega = 1$ $\longleftarrow$ Curvilinear

$$d\phi = \left(\frac{\partial \phi}{\partial u}\cdot\hat{e}_u + \frac{\partial \phi}{\partial v}\cdot\hat{e}_v + \frac{\partial \phi}{\partial w}\cdot\hat{e}_w\right)\cdot\left(du\cdot\hat{e}_u + dv\cdot\hat{e}_v + dw\cdot\hat{e}_w\right)$$

$$d\vec{r} = h_u\hat{e}_u du + h_v\hat{e}_v dv + h_w\hat{e}_w dw$$

$$= \left(\frac{h_u}{h_u}\cdot\frac{\partial \phi}{\partial u}\hat{e}_u + \frac{h_v}{h_v}\frac{\partial \phi}{\partial v}\cdot\hat{e}_v + \frac{h_w}{h_w}\frac{\partial \phi}{\partial w}\hat{e}_w\right)$$

$$\cdot\left(du\hat{e}_u + dv\hat{e}_v + dw\hat{e}_w\right)$$

$$= \left(\frac{1}{h_u}\frac{\partial \phi}{\partial u}\cdot\hat{e}_u + \frac{1}{h_v}\cdot\frac{\partial \phi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial \phi}{\partial w}\hat{e}_w\right)\cdot$$

dot product

$$\left(h_u\hat{e}_u du + h_v\hat{e}_v dv + h_w\hat{e}_w dw\right)$$

$$d\phi = \underbrace{\left(\frac{1}{h_u}\frac{\partial}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial}{\partial v}\cdot\hat{e}_v + \frac{1}{h_w}\frac{\partial}{\partial w}\hat{e}_w\right)}_{\hookrightarrow \vec{\nabla} \text{ in curvilinear.}}\phi\cdot d\vec{r}$$

In Cartesian system

$$d\phi = \vec{\nabla}\phi\cdot d\vec{r}$$

The vector differential operator " $\vec{\nabla}$ " in Curvilinear system

$$\vec{\nabla} = \left(\frac{\hat{e}_u}{h_u}\frac{\partial}{\partial u} + \frac{\hat{e}_v}{h_v}\frac{\partial}{\partial v} + \frac{\hat{e}_w}{h_w}\cdot\frac{\partial}{\partial w}\right)$$

Del operator in curvilinear system.

Spherical coordinate (r, θ, ϕ) (u, v, ω)

$h_r = 1$; $h_\theta = r$; $h_\phi = r \sin\theta$

$$\vec{\nabla} f(r,\theta,\phi) = \frac{\hat{e}_r}{1} \cdot \frac{\partial f}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{\hat{e}_\phi}{r \sin\theta} \frac{\partial f}{\partial \phi}$$

Let $\vec{F}$ be a vector field in a curvilinear system:

$$\vec{F} = \vec{F}(u,v,w) = F_1 \hat{e}_u + F_2 \hat{e}_v + F_3 \cdot \hat{e}_\omega \longleftarrow \text{curvilinear}$$

$$\vec{F} = \vec{F}(x,y,z) = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \longleftarrow \text{Cartesian}$$

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\hat{e}_u}{h_u} \frac{\partial}{\partial u} + \frac{\hat{e}_v}{h_v} \cdot \frac{\partial}{\partial v} + \frac{\hat{e}_\omega}{h_\omega} \frac{\partial}{\partial \omega} \right) \cdot \left(F_1 \hat{e}_u + F_2 \hat{e}_v + F_3 \hat{e}_\omega \right)$$

$$\hat{e}_u \cdot \hat{e}_u = 1 \quad \ldots \quad \hat{e}_\omega \cdot \hat{e}_\omega = 1$$

$$= \left(\frac{1}{h_u} \cdot \frac{\partial F_1}{\partial u} + \frac{1}{h_v} \frac{\partial F_2}{\partial v} + \frac{1}{h_\omega} \frac{\partial F_3}{\partial \omega} \right)$$

$$\underbrace{\vec{\nabla} \cdot \vec{F}}_{\text{Curvilinear coordinate..}} = \frac{1}{h_u h_v h_\omega} \left[\frac{\partial F_1}{\partial u} \cdot h_v \cdot h_\omega + \frac{\partial F_2}{\partial v} \cdot h_u \cdot h_\omega + \frac{\partial F_3}{\partial \omega} \cdot h_u \cdot h_v \right]$$

Example spherical coordinate $h_r = 1$; $h_\theta = r$; $h_\phi = r \sin\theta$

$(u = r ; v = \theta ; \omega = \phi)$

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{h_r h_\theta h_\phi} \cdot \left[\frac{\partial F_1}{\partial r} h_\theta h_\phi + \frac{\partial F_2}{\partial \theta} h_r \cdot h_\phi + \frac{\partial F_3}{\partial \phi} h_r \cdot h_\theta \right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial f_1}{\partial r}\cdot r^2 \sin\theta + \frac{\partial f_2}{\partial \phi}\cdot r \sin\theta + \frac{\partial f_3}{\partial \phi}\cdot r\right]$$

$$= \frac{1}{r^2 \cancel{\sin\theta}}\frac{\partial f_1}{\partial r}\cdot r^2 \cancel{\sin\theta} + \frac{1}{r^{\cancel{2}} \sin\theta}\frac{\partial f_2}{\partial \phi}\cancel{r}\sin\theta$$

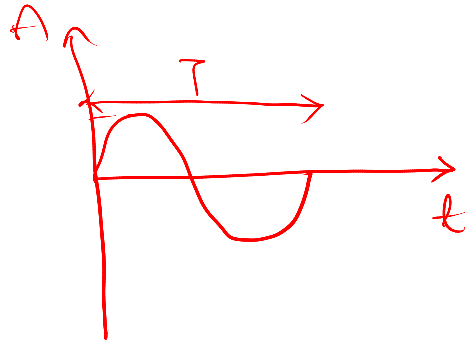
$$\left. + \frac{1}{r^{\cancel{2}} \sin\phi}\frac{\partial f_3}{\partial \phi}\cdot \cancel{r}\right]$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial f_1}{\partial r}r^2 + \frac{1}{r}\frac{\partial f_2}{\partial \phi}\cdot \sin\theta + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial \phi}$$

Similarly the Curl of $\vec{F}$ in curvilinear system

$$\vec{\nabla}\times\vec{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u f_1 & h_v F_2 & h_w F_3 \end{vmatrix}$$

Fourier Series:



i) finite extrema

ii) absolute integrable over a period.

iii) and has only finite discontinuity

Fourier Series of a fn = $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

D.C part

a_1 and b_1 are the fundamental frequency

$a_2, a_3 \dots$ and $b_2, b_3 \dots$ are called the harmonics.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

Some Important Integral:

(i) $\int_0^{2\pi} \sin nx\, dx = 0$ ii) $\int_0^{2\pi} \cos nx\, dx = 0$

iii) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$ iv) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

v) $\int_0^{2\pi} \sin nx \sin mx\, dx = 0$ vi) $\int_0^{2\pi} \cos nx \cos mx\, dx = 0$

vii) $\int_0^{2\pi} \sin nx \cos mx\, dx = 0$ viii) $\int_0^{2\pi} \sin nx \cos nx\, dx = 0$

$$\int u\, dv = uv - \int v\, du$$

Shortcut for choosing "u" for the integral

$\int u\, dv$

L I A T E

Log, Inverse, Algebric, Trignometric, Exponential

Ex: 1

Find the F.S for the fn: $f(x) = x$

$$0 < x < 2\pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots$$

$$+ b_1 \sin x + b_2 \sin 2x + \cdots$$

a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x\, dx \Rightarrow \frac{1}{\pi}\left.\frac{x^2}{2}\right|_0^{2\pi}$$

$$a_0 = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} \overset{u}{x}\, \overset{dv}{\cos nx\, dx}$$

$$u = x \quad ; \quad dv = \cos nx\, dx$$

$$du = dx \quad ; \quad v = \int \cos nx\, dx = \frac{\sin nx}{n}$$

$$\int u\, dv = uv - \int v\, du$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} x \cos nx\, dx = \frac{1}{\pi}\left[\left. x \cdot \frac{\sin nx}{n}\right|_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n}\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x \sin nx}{n}\Big|_0^{2\pi} - \frac{1}{n}\left(-\frac{\cos nx}{n}\right)\Big|_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n^2}\left[\cos 2\pi n - 1\right]\right]$$

$$= \frac{1}{n^2\pi}\left[\cos(2n)\pi - 1\right]$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 2n\pi = (-1)^{2n} = 1$

$$= \frac{1}{n^2\pi}\left[1-1\right] = 0 \,//$$ $\boxed{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x}\,\Big|\,\underset{dv}{\sin nx\,dx}$$

$u = x$	$dv = \sin nx\,dx$
$du = dx$	$v = \int \sin nx\,dx = -\frac{\cos nx}{n}$

$$\frac{1}{\pi}\int_0^{2\pi} x \sin nx\,dx = \frac{1}{\pi}\left[x\,\frac{(-\cos nx)}{n} + \frac{1}{n^2}\cancel{\sin nx}^{\,0}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n}(-1)\right] \Rightarrow -\frac{2}{n} \,//$$

$\boxed{b_n = -\frac{2}{n}}$

$a_0 = 2\pi$; $a_n = 0$; $b_n = -\dfrac{2}{n}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} \cancel{a_n \cos nx}^{\,0} + \sum_{n=1}^{p} b_n \sin nx$$

$$\boxed{f(x) = \frac{2\pi}{2} - 2\sum_{n=1}^{p} \frac{\sin nx}{n}}$$

Ex: 2 $\quad f(x) = x + x^2 \qquad -\pi < x < \pi$

using this

$-\pi \to 0 \to$ length π } 2π

$0 \to \pi \to$ " π

P.T $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (x + x^2)\, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x + x^2) \cos nx\, dx = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (x + x^2) \sin nx\, dx = -\frac{2}{n}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{p} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{p} \frac{(-1)^n}{n} \sin nx \quad \text{—— (1)}$$

$$x+x^2=\frac{\pi^2}{3}+4\left[-\frac{1}{1}\cos 1x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\cdots\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right] \quad \text{—} \ (2)$$

$x=\pi$ in (2) $\qquad \cos n\pi=(-1)^n$

$$\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad \text{—} \ (3)$$

$x=-\pi$ in (2)

$$-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad \text{—} \ (4)$$

(3) + (4) $\Rightarrow$

$$2\pi^2=\frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$2\pi^2-\frac{2\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\pi^2\left[2-\frac{2}{3}\right]= \quad ''$$

$$\frac{4}{3}\pi^2=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \;//$$

Ex: 3 F.S

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} (-1)\,dx + \int_{-\pi/2}^{\pi/2} (0)\,dx + \int_{\pi/2}^{\pi} 1\,dx\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\,dx + \int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} \cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\,dx + \int_{-\pi/2}^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} \sin nx\,d\right]$$

Odd & Even fn: Fourier Series

$$-\pi < x < \pi$$

If $f(x)$ is an odd fn: then

$$a_0 = 0 \; ; \; a_n = 0 \; ; \; b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

If $f(x)$ is even fn. then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$b_n = 0$ //

Ex 4: $f(x) = x^2$ $\quad -\pi < x < \pi$ ✓

$\cancel{0 < x < 2\pi}$ ✗

a_0

a_n

Ex 5: $f(x) = x^3$ $\quad -\pi < x < \pi$

$a_0 = 0 \quad ; \quad a_n = 0$

$b_n = 2/\pi \int_0^{\pi} f(x)\sin nx\,dx$

If $f(x)$ 0 to 2π $\quad a_0, a_n$ & b_n

If $f(x)$ $-\pi$ to π

If $f(x)$ is odd find only b_n

If $f(x)$ is even find only a_0 & a_n

Fourier Half range Series: $2T$

Fourier full range series. $\rightarrow \begin{cases} -\pi < x < \pi \\ 0 < x < 2\pi \end{cases}$

$f(x)$ defined in the $0 < x < \pi$

$-\pi/2 < x < \pi/2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \checkmark$$

T or π

Fourier cosine series $\quad b_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \checkmark \quad ; \quad a_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

Eg: Find the Fourier sine series for the fn: $f(x) = e^{ax}$ for $0 < x < \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\pi - n\cos n\pi) - \left[\frac{1}{a^2+n^2}\cdot -n(\cos(0))\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n(-1)^n) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\left\{\frac{n}{a^2+n^2}\left[e^{a\pi} - (-1)^n + 1\right]\right\}$$

$$b_n = \frac{2n}{\pi(a^2+n^2)}\left[1 - e^{a\pi}(-1)^n\right]$$

$$b_1 = \frac{2}{\pi(a^2+1)}\left[1+e^{a\pi}\right] \; ; \; b_2 = \frac{2\cdot 2}{\pi(a^2+4)}\left[1-e^{a\pi}\right]$$

$$f(x) = \sum_{n=1}^{p} b_n \sin nx$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2}{a^2+2^2}\sin 2x + \cdots\right]$$

Change of Intervals :

$-c \leq x \leq c \qquad f(x) = x \;;$

Full range | Half range
$-\pi < x < \pi$ | $-\frac{\pi}{2} < x < \frac{\pi}{2}$ $\qquad -2 < x < 2$
$0 < x < 2\pi$ | $0 < x < \pi$

$-c \qquad c$

Full range Series

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Half Range Series

$$a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx$$

$-\frac{1}{2} < x < \frac{1}{2}$

$0 < x < 1$

$0 < x < l$

c

$$a_n = \frac{2}{c}\int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{2}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Full Range Series

Eg: find the Fourier Series $f(x) = |x|$ $\quad -2 < x < 2$

$c \qquad c$

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx \;;\; a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$f(x) = \begin{cases} -x & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$$

$c \qquad c$

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0} -x \, dx + \int_{0}^{2} x \, dx\right]$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} -x \cos\frac{n\pi x}{2} \, dx + \int_{0}^{2} x \cos\frac{n\pi x}{2} \, dx\right]$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} -x \sin\frac{n\pi x}{2} \, dx + \int_{0}^{2} x \sin\frac{n\pi x}{2} \, dx\right]$$

$|x|$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

Example for Half Range Series.

Find the F.S $f(x) = \left(\frac{-x}{l} + 1\right) \quad 0 \le x \le l$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx \; ; \quad a_n = \frac{2}{c}\int_0^c f(x) \cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{2}{c}\int_0^c f(x) \sin\frac{n\pi x}{c}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l \left(\frac{-x}{l} + 1\right) dx \; ; \quad b_n = \frac{2}{l}\int_0^l \left(\frac{-x}{l} + 1\right) \sin\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{2}{l}\int_0^l \left(\frac{-x}{l} + 1\right) \cos\frac{n\pi x}{l}\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$\int e^{ax} \sin nx \, dx = \frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx) + c$$

$$\int e^{ax} \cos nx \, dx = \frac{e^{ax}}{a^2+n^2}(a \cos nx + n \sin nx) + c$$

Fourier transform of a fn = f(x)

$$F(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

Properties of F.T :

i) Linear Property

$$F[a f_1(x) + b f_2(x)] = a F_1(s) + b F_2(s)$$

W.K.T

$$F_1(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f_1(x)\, e^{isx}\, dx$$

$$F_2(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f_2(x)\, e^{isx}\, dx$$

$$F\{a f_1(x) + b f_2(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (a f_1(x) + b f_2(x))\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{a \int_{-\infty}^{\infty} f_1(x)\, e^{isx}\, dx}_{F_1(s)} + \underbrace{b \int_{-\infty}^{\infty} f_2(x)\, dx}_{F_2(s)} \right]$$

$$F\{a f_1(x) + b f_2(x)\} = a F_1(s) + b F_2(s)$$ //

2) Change of Scale property

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

W.K.T

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

$x \to \infty$, $t \to \infty$

$x \to -\infty$, $t \to -\infty$

$$ax = t$$

$$x = \frac{t}{a} \qquad dx = \frac{1}{a} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right) \cdot t}\, \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right) t}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \qquad \left[\text{because } F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

3) Shifting property

$$F\{f(x-a)\} = e^{isa} F(s)$$

W.K.T

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$t = x - a$

$x = t + a \qquad dx = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ist} dt \cdot e^{isa}$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{isa} F(s) \;//$$

4) $F\{e^{iax} f(x)\} = F(s+a)$

W.K.T $\quad F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i(s+a)x} dx$$

$$= F(s+a) \;//$$

5) $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

a)

W.K.T

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx$$

$$\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}$$

$$\cos ax = \frac{e^{iax}+e^{-iax}}{2}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\cdot\frac{e^{iax}+e^{-iax}}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right] //$$

5b)

$F\{f(x)\sin ax\} \longrightarrow \sin ax = \frac{e^{iax}-e^{-iax}}{2i}$

6) $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$

W.K.T $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

$$\frac{d}{ds} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot \frac{d}{ds} e^{isx} \, dx$$

$$\frac{d}{ds} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot ix \cdot e^{isx} \, dx$$

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot (ix)^n e^{isx} \, dx$$

$$= i^n \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, x^n e^{isx} \, dx$$

$$\frac{d^n}{ds^n} F(s) = i^n \, F\{x^n f(x)\}$$

$$\frac{1}{(i)^n} \frac{d^n}{ds^n} F(s) = F\{x^n f(x)\}$$

$$(-i)^n \frac{d^n}{ds^n} F(s) = F\{x^n f(x)\}$$

7) $$F\{f'(x)\} = (-is) F(s)$$

W.K.T

$$F\{f(x)\} \; F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$$

$$F\{d f(x)\} = F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{d f(x) \, dx}_{dv} \, \underbrace{e^{isx}}_{u}$$

$$u = e^{isx} \qquad dv = d f(x) \, dx$$

$$du = is \cdot e^{isx} \, dx \qquad v = \int d f(x) \, dx = f(x)$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{e^{isx} f(x)\Big|_{-\infty}^{\infty}}^{\,0} - is\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-is\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \right]$$

$$F\{f'(x)\} = -is\,\underbrace{F(s)}_{F\{f(x)\}} \;//$$

---

8) $$F\left\{\int_a^x f(x)\,dx\right\} = \frac{-1}{is}\,F\{f(x)\} \;//$$

W.K.T

$$F\{f'(x)\} = F\{df(x)\} = -is\,F\{f(x)\}$$

$$f_1(x) = \int_a^x f(x)\,dx$$

$$df_1(x) = d\int_a^x f(x)\,dx$$

$$df_1(x) = f(x) \;//$$

$$F\{df_1(x)\} = -is\,F\{f_1(x)\} = -is\,F\left\{\int_a^x f(x)\,dx\right\}$$

$\Downarrow$

$$F\{f(x)\} = (-is)\,F\left\{\int_a^x f(x)\,dx\right\}$$

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{-1}{is} F\{f(x)\}.$$

Similarly we can derive the properties for $F_c(s)$ and $F_s(s)$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

Eg! Find F.T of

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

W.K.T

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\, \frac{e^{isa} - e^{-isa}}{2\, is} \right]$$

$\because \sin\phi = \dfrac{e^{i\phi} - e^{-i\phi}}{2i}$

$$= \frac{2}{\sqrt{2\pi}\, s} \sin sa$$

$$= \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\cdot\sqrt{\pi}} \cdot \frac{\sin sa}{s} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin sa}{s}$$

Eg:7 Find F.S.T and F.C.T of

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a 1 \cdot \sin sx \, dx + \cancel{\int_a^\infty 0 \cdot \sin sx \, dx}^{\,0} \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sa}{s} + \frac{\cos 0}{s} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos sa}{s} \right]$$ //

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \, dx \Rightarrow \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - \cancel{\frac{\sin 0}{s}}^{\,0} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} \right]$$ //

Eg 2: Find the F.T of

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

W.K.T

$$F(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Choice "u"

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$$= \sqrt{\frac{1}{2\pi}} \int_{-1}^{1} \underbrace{(1-x^2)}_{u}\, \underbrace{e^{isx}\, dx}_{dv}$$

$$= \sqrt{\frac{1}{2\pi}} \left[(1-x^2) \frac{e^{isx}}{is} \Big|_{-1}^{1} - \int_{-1}^{1} \underbrace{\frac{e^{isx}}{is}\, dx}_{dv}\, \underbrace{(-2x)}_{u} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[(1-x^2) \frac{e^{isx}}{is} \Big|_{-1}^{1} - \left[(-2x) \frac{e^{isx}}{(is)^2} \Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{isx}}{(is)^2} (-2)\, dx \right] \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\cancel{(1-x^2) \frac{e^{isx}}{is} \Big|_{-1}^{1}} \to 0 + 2x \cdot \frac{e^{isx}}{(is)^2} \Big|_{-1}^{1} - 2 \frac{e^{isx}}{(is)^3} \Big|_{-1}^{1} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{2e^{is}}{(is)^2} - \frac{2(-1)e^{-is}}{(is)^2} - 2\frac{e^{is}}{(is)^3} + 2\frac{e^{-is}}{(is)^3} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2e^{is}}{s^2} - \frac{2e^{-is}}{s^2} + \frac{2e^{is}}{is^3} - \frac{2e^{-is}}{is^3} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2}{s^2} \left[2\underbrace{\frac{e^{is} + e^{-is}}{2}}_{\to \cos s} \right] + \frac{2}{s^3} \left[2\underbrace{\frac{e^{is} - e^{-is}}{2i}}_{\to \sin s} \right] \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2}{s^2} \cdot 2 \cdot \cos s + \frac{2}{s^3} \cdot 2 \sin s \right]$$

$= \sqrt{\frac{1}{2\pi}} \frac{4}{s^3}\left[-s\cos s + \sin s\right]$ //

Eg: 3 find F.S.T and F.C.T of $f(x) = e^{-ax}$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}\left[-a\sin sx - s\cos sx\right]\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^{0} \to 1}{a^2+s^2}\left[-a\cancel{\sin 0}^{\to 0} - s\cancel{\cos 0}^{\to 1}\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{a^2+s^2}\left[-s\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2}\right]$$ //

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cancel{e^{0}}^{\to 1}}{a^2+s^2}\left(-a\cancel{\cos 0}^{\to 1} + s\cancel{\sin 0}^{\to 0}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-1}{a^2+s^2}(-a)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right] //$$

Eg: 5 find the F.C.T of $f(x) = e^{-2x} + 4e^{-3x}$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(e^{-2x}+4e^{-3x}\right)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty e^{-2x}\cos sx + 4\int_0^\infty e^{-3x}\cos sx\, dx\right]$$

W.K.T $\displaystyle\int_0^\infty e^{-ax}\cos sx = \frac{a}{a^2+s^2}$ //

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2}{2^2+s^2} + 4\cdot\frac{3}{3^2+s^2}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{1}{4+s^2} + \frac{6}{9+s^2}\right] //$$

Eg. 4 find F.S.T of $f(x) = \frac{1}{x}$

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\, dx$$

Putting $sx = \phi$ $\qquad dx = d\phi/s$

$x = \frac{\phi}{s}$ $\qquad x \to 0,\ \phi \to 0 \ \big|\ x \to \infty,\ \phi \to \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\phi \cdot \cancel{s}}{\phi} \cdot \frac{d\phi}{\cancel{s}}$$

$$= \sqrt{\frac{2}{\pi}} \underbrace{\int_0^\infty \frac{\sin\phi}{\phi}\, d\phi}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

$$= \frac{\sqrt{2}}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}} \cdot \sqrt{\pi}}{\cancel{\sqrt{2}} \cdot \sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2}} \text{ or } \sqrt{\frac{\pi}{2}}$$

Eg 5: find F.S.T of $f(x) = \frac{e^{-ax}}{x}$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx \, dx \quad \text{—— (1)}$$

Diff (1) w.r.t ds on both sides we have

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} dx \cdot \frac{d}{ds}(\sin sx)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} dx \cdot \cos sx \cdot \cancel{x}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2} \right]$$

Integrating both sides w.r.t s

$$\int \cancel{d} F_s(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2} \cdot ds$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

$\therefore$ @ $s=0$ $\quad F_s(0) = \sqrt{\frac{2}{\pi}} \cancel{\tan^{-1}(0)}^{\,0} + c$

$$\therefore \quad c = 0$$

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) //$$

Eg 8: Find the F.T of Dirac Delta fn =

$$\delta(x-a)$$

$$\delta(x-a) = \underset{h \to 0}{Lt}\; f(x)$$

$$f(x) = \begin{cases} \frac{1}{h} & \text{for } a < x < a+h \\ 0 & \text{for } x < a \\ & \phantom{\text{for }} x > a+h \end{cases}$$

$$F(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \sqrt{\frac{1}{2\pi}} \underset{h \to 0}{Lt} \int_a^{a+h} \frac{1}{h} e^{isx} \, dx$$

$$= \sqrt{\frac{1}{2\pi}} \underset{h \to 0}{Lt} \frac{1}{h} \left[\frac{e^{isx}}{is} \right]_a^{a+h}$$

$$= \sqrt{\frac{1}{2\pi}} \underset{h \to 0}{Lt} \frac{1}{h} \left[\frac{e^{is(a+h)}}{is} - \frac{e^{isa}}{is} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \underset{h \to 0}{Lt} e^{isa} \underbrace{\left[\frac{e^{ish} - 1}{ish} \right]}$$

$$\underset{\theta \to 0}{Lt} \frac{e^{\theta} - 1}{\theta} = 1$$

$$= \sqrt{\frac{1}{2\pi}} e^{isa} \cdot \underset{h \to 0}{Lt} \left[\frac{e^{ish} - 1}{ish} \right] \to 1$$

$$= \sqrt{\frac{1}{2\pi}} \cdot e^{isa} \, //$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$e^{\theta} - 1 = \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$\frac{e^{\theta} - 1}{\theta} = 1 + \cancel{\frac{\theta}{2!} + \frac{\theta^2}{3!} + \frac{\theta^3}{4!} + \cdots} \to 0$$

$$\underset{\theta \to 0}{Lt} \frac{e^{\theta} - 1}{\theta} = 1 \, //$$

Eg 9: S.T $F_s\{xf(x)\} = -\frac{d}{ds}F_c(s)$

$$F_c\{xf(x)\} = \frac{d}{ds}F_s(s)$$ ✓

hence find the F.C.T and F.S.T of

$f(x) = xe^{-ax}$

W.K.T

$$F_c\{f(x)\} = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$\frac{d}{ds}F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot dx\cdot\frac{d}{ds}\cos sx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\cdot(-\sin sx)\cdot x$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin sx\,dx$$

$$-\frac{d}{ds}F_c(s) = F_s\{xf(x)\}$$

next proof

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$\frac{d}{ds}F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\cdot\frac{d}{ds}(\sin sx)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\cdot\cos sx\cdot x$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$\frac{d}{ds} \underbrace{F_s(s)}_{F_s\{f(x)\}} = F_c\{x f(x)\}$$

W.K.T

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_c(x \cdot \underbrace{e^{-ax}}_{f(x)}) = \frac{d}{ds} F_s\{e^{-ax}\}$$

$$= \frac{d}{ds}\left[\frac{s}{a^2+s^2}\right]$$

$$\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$= \frac{(a^2+s^2)\cdot 1 - s(2s)}{(a^2+s^2)^2}$$

$$= \frac{a^2+s^2-2s^2}{(a^2+s^2)^2}$$

$$\boxed{F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(a^2+s^2)^2}}$$

$$F_s\{x e^{-ax}\} = -\frac{d}{ds} F_c\{e^{-ax}\}$$

$$= -\frac{d}{ds}\left[\frac{a}{a^2+s^2}\right]$$

$$= -\frac{\cancel{(a^2+s^2)0}^{\to 0} - a(2s)}{(a^2+s^2)^2}$$

$$F_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}$$

Eg 10: Find F.C.T of $e^{-a^2x^2}$ and hence find F.S.T of $x \cdot e^{-a^2x^2}$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot\cos sx\,dx$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot(\cos sx + i\sin sx)\,dx$$

$$= R.P\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot e^{isx}\,dx$$

Standard Integral formula

$$\int_0^\infty e^{-ax^2}\cdot e^{bx}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{\frac{b^2}{4a}}$$

$$= R.P\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a^2}}\,e^{(is)^2/4a^2}$$

$$= R.P\sqrt{2/\pi}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a^2}}\cdot e^{\frac{-s^2}{4a^2}}$$

$$= R.P \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{\cancel{\sqrt{2}}\sqrt{2}} \cdot \frac{\sqrt{\pi}}{\sqrt{a^2}} \cdot e^{-\frac{s^2}{4a^2}}$$

$$= R.P \frac{1}{\sqrt{2}\cdot a} e^{-\frac{s^2}{4a^2}}$$

$$F_c\{e^{-a^2x^2}\} = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2}$$

Find the $F_s\{x \cdot \underbrace{e^{-a^2x^2}}\}$

W.K.T

$$F_s\{x \cdot f(x)\} = -\frac{d}{ds} F_c\{f(x)\}$$

$$F_s\{x \cdot e^{-a^2x^2}\} = -\frac{d}{ds} F_c\{e^{-a^2x^2}\}$$

$$= -\frac{d}{ds}\left[\frac{1}{\sqrt{2}\,a} \cdot e^{-\frac{s^2}{4a^2}}\right]$$

$$= -\frac{1}{\sqrt{2}\,a} e^{-\frac{s^2}{4a^2}} \cdot \left(-\frac{2s}{4a^2}\right)$$

$$= \frac{\cancel{\sqrt{2}} \cdot \sqrt{2} \cdot s}{\cancel{\sqrt{2}} \cdot a \cdot 4a^2} \cdot e^{-s^2/4a^2}$$

$$= \frac{\sqrt{2}s}{4a^3} \cdot e^{-s^2/4a^2}$$

$$F_s\{x \cdot e^{-a^2x^2}\} = \frac{\cancel{\sqrt{2}}s}{\cancel{\sqrt{2}}\cdot\sqrt{2}\cdot 2a^3} e^{-s^2/4a^2} = \boxed{\frac{s}{2\sqrt{2}a^3} e^{-s^2/4a^2}}$$

Solving Partial Differential Equation Using F.T

$$F\left\{\frac{d}{dx}f(x)\right\} = -is\,F\{f(x)\}$$

$$F\left\{\frac{d^n}{dx^n}f(x)\right\} = (-is)^n F\{f(x)\}$$

If our fn is a fn of 2 variables then $u(x,t)$

$$F\left\{\frac{\partial^n}{\partial x^n}u(x,t)\right\} = (-is)^n F\{u(x,t)\}$$

$$\boxed{F\left\{\frac{\partial^2}{\partial x^2}u(x,t)\right\} = -s^2 F\{u(x,t)\}}$$

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = s\cdot\underbrace{\left(u(x,t)\right)\Big|_{x=0}}_{u(0,t)} - s^2 F_s\{u(x,t)\}$$

$$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\underbrace{\left(\frac{\partial u}{\partial x}\right)\Big|_{x=0}}_{u'(0,t)} - s^2 F_c\{u(x,t)\}$$

$$\frac{\partial u}{\partial x} \rightarrow u'(0,t)$$

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Solution of heat conduction problems by Fourier sine Transforms

Example 31. *Solve the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x > 0, t > 0$$

subject to the conditions

(i) $u = 0$ when $x = 0, t > 0$

(ii) $u = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$ when $t = 0$

(iii) $u(x, t)$ is bounded.

$$u(\overset{x}{0}, t) = 0$$

$$\left.\begin{array}{l} u(x,t) = 1 \quad \text{when } x < 1 \\ u(x,t) = 0 \quad \text{when } x \geq 1 \end{array}\right\} \text{when } t = 0$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

Let take F_s on both sides in (1)

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{\partial}{\partial t}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin sx \, dx\right) = s \cdot u(0,t) - s^2 \cdot F_s\{u(x,t)\}$$

$$\frac{\partial}{\partial t}\underbrace{F_s\{u(x,t)\}}_{\bar{u}} = s \cdot u(0,t) - s^2 \cdot \underbrace{F_s\{u(x,t)\}}_{\bar{u}}$$

$$\frac{\partial \bar{u}}{\partial t} = s \cdot \cancel{u(0,t)}^{\,0} - s^2 \bar{u}$$

$$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad (\text{or})$$

$$\frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \leftarrow \text{Linear first order Differential Equation}$$

$$\frac{\partial x}{\partial t} + kx = 0$$

general solution

$$x = A \cdot e^{-kt}$$

$$\bar{u} = A e^{-s^2 t} \quad \text{——— (2)}$$

We also know $\Big\} \rightarrow \bar{u} = F_s\{u(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx$

$$\therefore \bar{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx$$

$$\bar{u}(x,t) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 u(x,t) \sin sx \, dx + \cancel{\int_1^\infty u(x,t) \sin sx \, dx} \right]$$

When $t = 0$ $\begin{cases} u(x,0) = 1 & \text{for } x < 1 \\ u(x,0) = 0 & \text{for } x \geq 1 \end{cases}$ by this condition

$$\bar{u}(x,0) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 u(x,0) \sin sx \, dx + \cancel{\int_1^\infty u(x,0) \sin sx \, dx} \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx \Rightarrow \sqrt{\frac{2}{\pi}} \left(-\frac{\cos sx}{s} \right)_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{-\cos s}{s} + \frac{1}{s} \right)$$

$$\bar{u}(x,0) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \quad \text{---} \ (3)$$

From (2) we k.T

$$\bar{u}(x,t) = Ae^{-s^2 t}$$

$$\bar{u}(x,0) = A \cdot e^{-s^2 \cdot 0}$$

$$\bar{u}(x,0) = A. \quad \text{---} \ (4)$$

Comparing (3) & (4) we have

$$A = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \quad \text{---} \ (5)$$

Sub. value of A in (2) we have

$$\bar{u}(x,t) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \cdot e^{-s^2 t}$$

$$F_s\left\{\underline{u(x,t)}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \cdot e^{-s^2 t}$$

Taking Inverse F_s Transform

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1-\cos s}{s}\right) \cdot e^{-s^2 t} \cdot \sin xs \, ds$$

Solution.

Example 33. *Solve* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for* $0 \le x < \infty, t > 0$ *given the conditions*

(i) $u(x, 0) = 0$ *for* $x \ge 0$ *(ii)* $\frac{\partial u}{\partial x}(0,t) = -a$ *(constant)*

(iii) $u(x, t)$ *is bounded.*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

$u(\overset{x}{0}, t) \leftarrow F_s$

$u'(0, t) \leftarrow F_c$

@ $t = 0$

$$u(x, 0) = 0$$

& for $t > 0$

$$\frac{\partial u}{\partial x}(0, t) = -a$$

Lets take F_c transform on both side of (1)

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = k F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

using the formula

$$\frac{\partial}{\partial t} F_c\{u(x,t)\} = k\left(-\underbrace{\frac{\partial u}{\partial x}\Big|_{x=0}}_{=-a} - s^2 F_c\{u(x,t)\}\right)$$

$$\underbrace{\frac{\partial}{\partial t} F_c\{u(x,t)\}}_{\bar{u}} = ka - s^2 k \underbrace{F_c\{u(x,t)\}}_{\bar{u}}$$

$$\frac{\partial}{\partial t} \underbrace{\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\, dx}_{\bar{u}} = ka - s^2 k \underbrace{\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\, dx}_{\bar{u}}$$

$$\frac{\partial \bar{u}}{\partial t} = ka - s^2 k \bar{u}$$

$$\underbrace{\frac{\partial \bar{u}^{\,\swarrow y}}{\partial t}}_{\uparrow x} + \underbrace{s^2 k}_{\uparrow P}\bar{u} = \underbrace{ka}_{\uparrow Q} \quad \text{—— (2)}$$

General solution

$$\boxed{\begin{gathered}\frac{\partial y}{\partial x} + Py = Q \\ y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} + c\end{gathered}}$$

For our case $P = s^2 k$; $dx = dt$

$$\therefore \int P dx = \int s^2 k \, dt$$

$$= s^2 k \cdot \int dt$$

$$\underline{\int P dx = s^2 k t}$$

$$\bar{u} \cdot e^{s^2 k t} = \int ka \cdot e^{s^2 k t} dt + c$$

$$= ka \cdot \int e^{s^2 k t} dt + c$$

$$= \cancel{k} a \cdot \frac{e^{s^2 k t}}{s^2 \cancel{k}} + c$$

$$\bar{u} \cdot e^{s^2 k t} = \frac{a}{s^2} e^{s^2 k t} + c$$

$\times \; e^{-s^2 k t}$

$$\bar{u}(x,t) = \frac{a}{s^2} + c \cdot e^{-s^2 k t} \quad \text{—— (3)}$$

@ $t = 0$

$$\bar{u}(x,0) = \frac{a}{s^2} + c \quad \text{—— (4)}$$

W.K.T

$$\bar{u}(x,0) = F_c\{u(x,0)\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cancel{u(x,0)}^{\,0} \cos sx \, dx$$

$$\bar{u}(x,0) = 0 \longleftarrow$$ Defin of $\bar{u}$

$$\text{———} \quad (5)$$

Comparing (5) & (4) we have

$$0 = \frac{a}{s^2} + c$$

$$c = -\frac{a}{s^2} \quad \text{———} \quad (6)$$

Sub (6) in (3) we have

$$\bar{u}(x,t) = \frac{a}{s^2} - \frac{a}{s^2} \cdot e^{-s^2 kt}$$

$$\bar{u}(x,t) = \frac{a}{s^2}\left(1 - e^{-s^2 kt}\right)$$

$$F_c\{\underline{u(x,t)}\} = \frac{a}{s^2}\left(1 - e^{-s^2 kt}\right)$$

Taking Inverse $\underline{\underline{F_c}}$ transform we have

$$u(x,t) = \dot{a} \int_0^\infty \frac{1 - e^{-s^2 kt}}{s^2} \cdot \cos sx \, \underline{\underline{ds}}$$