

Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\vec{F} = P \hat{i} + Q \hat{j}$$

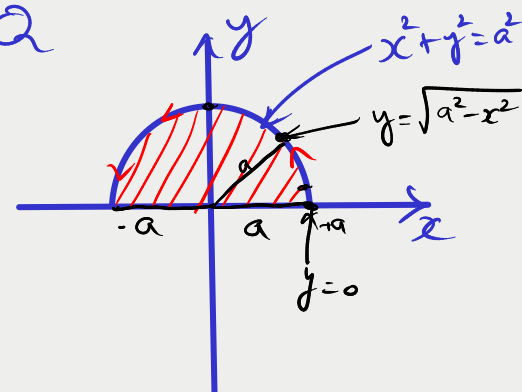
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (P \hat{i} + Q \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy$$

**Example 82.** Apply Green's Theorem to evaluate  $\int_C [(2x^2 - y^2) dx + (x^2 + y^2) dy]$ , where  $C$  is the boundary of the area enclosed by the  $x$ -axis and the upper half of circle  $x^2 + y^2 = a^2$ .  
(M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004)

$$I = \oint_C \underbrace{(2x^2 - y^2)}_P dx + \underbrace{(x^2 + y^2)}_Q dy$$

$$C : x^2 + y^2 = a^2$$



$$P = 2x^2 - y^2$$

$$Q = x^2 + y^2$$

$$\frac{\partial P}{\partial y} = -2y$$

$$\frac{\partial Q}{\partial x} = 2x$$

$$\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (-2y) \Rightarrow 2(x+y)$$

$$= 2 \iint_R (x+y) dx dy$$

$$= 2 \int_{-a}^a dx \int_0^{\sqrt{a^2-x^2}} (x+y) dy$$

$$= 2 \int_{-a}^a dx \left[ \int_0^{\sqrt{a^2-x^2}} x dy + \int_0^{\sqrt{a^2-x^2}} y \cdot dy \right]$$

$$= 2 \int_{-a}^a dx \left[ x \cdot y \Big|_0^{\sqrt{a^2-x^2}} + \frac{y^2}{2} \Big|_0^{\sqrt{a^2-x^2}} \right]$$

$$= 2 \int_{-a}^a dx \cdot \left[ x \cdot \sqrt{a^2-x^2} + \frac{(\sqrt{a^2-x^2})^2}{2} \right]$$

$$= 2 \int_{-a}^a dx \left[ x \cdot \sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right]$$

$$= 2 \cdot \int_{-a}^a \underbrace{x(a^2-x^2)^{1/2}}_{f_1(x)} dx + 2 \int_{-a}^a \underbrace{(a^2-x^2)}_{f_2(x)} dx$$

$\vdots f_1(x) \text{ odd}$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(x) \text{ is even fn}$$

$$= 0 \quad \text{if } f(x) \text{ is odd fn}$$

If  $f(-x) = f(x)$  then  $f(x)$  is even

If  $f(-x) = -f(x)$  then  $f(x)$  is odd

$f_1(x)$  is even or odd?

$$f_1(x) = x\sqrt{a^2-x^2}$$

$$x = -x$$

$$f_1(-x) = -\underbrace{x \cdot \sqrt{a^2-x^2}}_{f_1(x)}$$

$$f_1(-x) = -f_1(x) \leftarrow \text{odd}$$

$$f_2(x) = a^2 - x^2$$

$$f_2(-x) = a^2 - (-x)^2$$

$$= \underbrace{a^2 - x^2}_{f_2(x)}$$

$$\underline{f_2(-x) = f_2(x)}$$

$\nwarrow$  even

$$= 2 \cdot \int_0^a (a^2 - x^2) dx$$

$$= 2 \cdot \left[ a^2 \cdot x \Big|_0^a - \frac{x^3}{3} \Big|_0^a \right]$$

$$= 2 \cdot \left[ a^2 \cdot a - \frac{a^3}{3} \right] \Rightarrow 2 \cdot \left[ a^3 - \frac{a^3}{3} \right]$$

$$= 2 \cdot \frac{2a^3}{3} \Rightarrow 4 \frac{a^3}{3} //$$

**Example 83.** Evaluate  $\oint_C -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ , where  $C = C_1 \cup C_2$  with  $C_1 : x^2 + y^2 = 1$  and  $C_2 : x = \pm 2, y = \pm 2$ .  
(Gujarat, I Semester, Jan 2009)

$$\oint_C -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

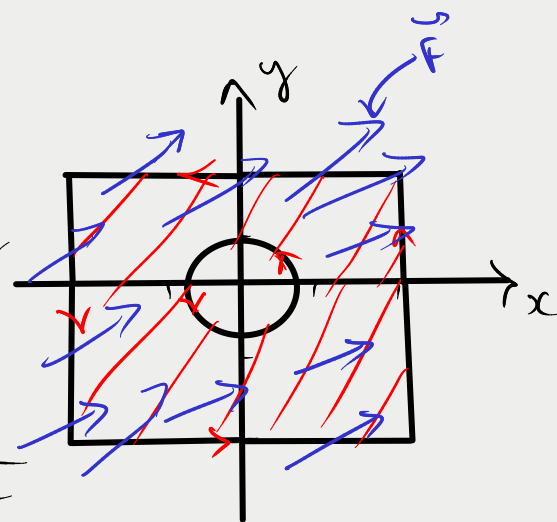
$$C_1 : x^2 + y^2 = 1$$

$$C_2 : x = \pm 2 ; y = \pm 2$$

$$\vec{F} = \underbrace{-\frac{y}{x^2+y^2}}_P \hat{i} + \underbrace{\frac{x}{x^2+y^2}}_Q \hat{j}$$

$$\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$P = \frac{-y}{x^2+y^2} ; \quad Q = \frac{x}{x^2+y^2}$$



$$P = -\underbrace{y}_u \cdot \underbrace{(x^2+y^2)^{-1}}_v$$

$$\frac{\partial P}{\partial y} = +y \cdot (x^2+y^2)^{-1-1} \cdot 2y + (x^2+y^2)^{-1} (-1)$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} \cdot \frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{2y^2 - x^2 - y^2}{(x^2+y^2)^2}$$

$$Q = \underbrace{x}_u \cdot \underbrace{(x^2+y^2)^{-1}}_v$$

$$\frac{\partial Q}{\partial x} = x \cdot -1 (x^2+y^2)^{-2} \cdot 2x + (x^2+y^2)^{-1} \cdot 1$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

