Lecture 6: Change of Basis *(BK 1.5 RHB 7.14)*

6. 1. Linear Transformation of Basis

Suppose $\underline{e}_i$ and $\underline{e}_i{}'$ are two different orthonormal bases, how do we relate them?

Clearly $\underline{e}_1{}'$ can be written as a linear combination of the vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$. Let us write the linear combination as

$$\underline{e}_1{}' = \lambda_{11}\underline{e}_1 + \lambda_{12}\underline{e}_2 + \lambda_{13}\underline{e}_3$$

Similarly we may write

$$\boxed{\underline{e}_i{}' \; = \lambda_{ij}\,\underline{e}_j} \; ,$$

(assuming summation convention) where λ_{ij} $(i = 1, 2, 3$ and $j = 1, 2, 3)$ are the 9 numbers relating the basis vectors $\underline{e}_1{}'$, $\underline{e}_2{}'$ and $\underline{e}_3{}'$ to the basis vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$.

Notes

(*i*) λ_{ij} are nine numbers defining the change of basis (abbreviated to 'c.o.b') or 'linear transformation'. They are sometimes known as 'direction cosines'. [Here a linear transformation is 'passive' and only the basis changes. In your maths courses you may have also met 'active transformations' which are mappings between vector spaces]

(*ii*) Since $\underline{e}_i{}'$ are orthonormal

$$\underline{e}_i{}' \cdot \underline{e}_j{}' \;=\; \delta_{ij} \; .$$

Now the l.h.s. of this equation may be written as

$$(\lambda_{ik}\,\underline{e}_k) \cdot (\lambda_{jl}\,\underline{e}_l) = \lambda_{ik}\lambda_{jl}\delta_{kl} = \lambda_{ik}\,\lambda_{jk}$$

(in the final step we have used the sifting property of δ_{kl}) and we deduce

$$\boxed{\lambda_{ik}\lambda_{jk} \;=\; \delta_{ij}}$$

Since there are 6 distinct relations, only 3 of the 9 numbers λ_{ij} are independent.

(*iii*) In order to determine λ_{ij} from the two bases consider

$$\underline{e}_i{}' \cdot \underline{e}_j \;=\; (\lambda_{ik}\,\underline{e}_k) \cdot \underline{e}_j = \lambda_{ik}\,\delta_{kj} = \lambda_{ij} \, .$$

Thus

$$\boxed{\underline{e}_i{}' \cdot \underline{e}_j = \lambda_{ij}} \; .$$

6. 2. Inverse Relations

Consider expressing the unprimed basis in terms of the primed basis and suppose that

$$\underline{e}_i = \mu_{ij}\, \underline{e}_j{}'.$$

Then $\quad \lambda_{si} \;=\; \underline{e}_s{}' \cdot \underline{e}_i \;=\mu_{ij}\,(\underline{e}_s{}' \cdot \underline{e}_j{}') \;=\mu_{ij}\,\delta_{sj} = \mu_{is}\;.$
Therefore

$$\boxed{\mu_{ij} = \lambda_{ji} = \left(\lambda^T\right)_{ij}}$$

Note that $\underline{e}_i \cdot \underline{e}_j \;=\; \delta_{ij} \;=\lambda_{si}\,(\underline{e}_s{}' \cdot \underline{e}_j) \;=\lambda_{si}\lambda_{sj}$ and so

$$\lambda_{si}\lambda_{sj} \;=\; \delta_{ij}\;.$$

6. 3. The Transformation Matrix

The numbers λ_{ij} may be arranged in a square matrix, denoted by λ.

$$\lambda \;=\; \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \qquad \Leftarrow \text{ Transformation Matrix.}$$

Recall that in matrix form δ_{ij} is the identity matrix $\mathbb{1}$. The relations $\lambda_{ki}\lambda_{kj} = \lambda_{ik}\lambda_{jk} = \delta_{ij}$ can now be written as:-

$$\boxed{\lambda\lambda^T \;=\; \lambda^T\lambda \;=\; \mathbb{1}}\;, \; i.e. \; \boxed{\lambda^{-1} \;=\; \lambda^T}\;.$$

This is the condition for an **orthogonal matrix** and the transformation (from the $\underline{e}_i$ basis to the $\underline{e}_i{}'$ basis) is called an **orthogonal transformation**.

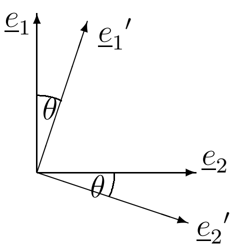
Now since $|\lambda\lambda^T| = |\mathbb{1}| = 1 = |\lambda|\,|\lambda^T|$ and $|\lambda^T| = |\lambda|$, we have that $|\lambda|^2 = 1$ hence

$$|\lambda| \;=\; \pm 1\;.$$

If $\lvert\lambda\rvert = +1$ the orthogonal transformation is said to be 'proper'
If $\lvert\lambda\rvert = -1$ the orthogonal transformation is said to be 'improper'

6. 4. Examples of Orthogonal Transformations

(a) Rotation about the $\underline{e}_3$ axis. For a rotation of θ, we have:–



$$\begin{aligned} \underline{e}_3{}' &= \underline{e}_3 \quad \Rightarrow \quad \underline{e}_3{}' \cdot \underline{e}_1 = \underline{e}_3{}' \cdot \underline{e}_2 = 0 \\ \underline{e}_1{}' \cdot \underline{e}_1 &= \cos\theta \\ \underline{e}_1{}' \cdot \underline{e}_2 &= \cos(\pi/2 - \theta) \;=\; \sin\theta \\ \underline{e}_2{}' \cdot \underline{e}_2 &= \cos\theta \\ \underline{e}_2{}' \cdot \underline{e}_1 &= \cos(\pi/2 + \theta) \;=\; -\sin\theta \end{aligned}$$

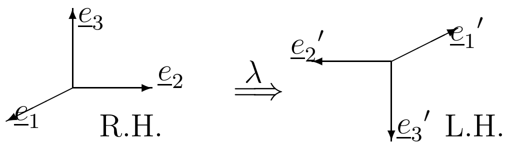
Thus
$$\lambda \;=\; \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $\lambda\lambda^T = \mathbb{1}$. Since $|\lambda| = \cos^2\theta + \sin^2\theta = 1$, this is a *proper* transformation. Note that rotations cannot change the handedness of the basis vectors.

(b) Inversion or Parity transformation. This is defined such that $\underline{e}_i{}' = -\underline{e}_i$.

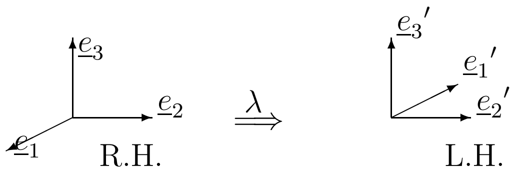
$$\text{i.e. } \lambda_{ij} \;=\; -\delta_{ij} \qquad \text{or} \qquad \lambda \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;=\; -\mathbb{1}\,.$$

Clearly $\lambda\lambda^T = \mathbb{1}$. Since $|\lambda| = -1$, this is an *improper* transformation. Note that the handedness of the basis is reversed.



(c) Reflection. Consider reflection of the axes in $\underline{e}_2$–$\underline{e}_3$ plane so that $\underline{e}_1{}' = -\underline{e}_1$, $\underline{e}_2{}' = \underline{e}_2$ and $\underline{e}_3{}' = \underline{e}_3$. The transformation matrix is:–
$$\lambda \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $|\lambda| = -1$, this is an *improper* transformation. Again the handedness of the basis changes.



6. 5. Products of Transformations

Consider a transformation λ to the $\underline{e}_i{}'$ basis followed by a transformation μ to another basis the $\underline{e}_i{}''$ basis
$$\underline{e}_i \overset{\lambda}{\Longrightarrow} \underline{e}_i{}' \overset{\mu}{\Longrightarrow} \underline{e}_i{}''$$
Clearly there must be an orthogonal transformation
$$\underline{e}_i \overset{\xi}{\Longrightarrow} \underline{e}_i{}''$$

Now
$$\underline{e}_i{}'' = \mu_{ij}\underline{e}_j{}' = \mu_{ij}\lambda_{jk}\underline{e}_k = (\mu\lambda)_{ik}\underline{e}_k$$
so
$$\boxed{\xi = \mu\lambda} \qquad \text{Note order of the product!}$$

Notes

(i) In general transformations do not commute *e.g.* rotation of θ about $\underline{e}_3$ then reflection in $\underline{e}_2 - \underline{e}_3$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whereas

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) The inversion and the identity transformations commute with all transformations.

6. 6. Improper Transformations

We may write any improper transformation λ (for which $|\lambda| = -1$) as

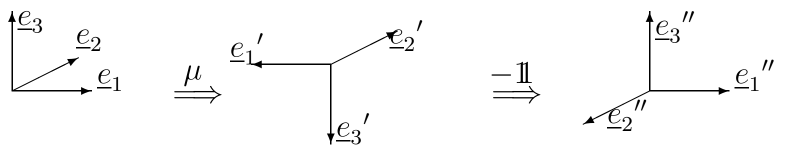
$$\lambda = (-\mathbb{1})\,\mu \quad \text{where} \quad \mu = -\lambda \quad \text{and} \quad |\mu| = +1$$

Thus an improper transformation can always be expressed as a proper transformation followed by an inversion.

e.g. consider λ for a reflection in the $1-3$ plane which may be written as

$$\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Identifying μ from $\lambda = (-\mathbb{1})\,\mu$ we see μ is a rotation of π about $\underline{e}_2$.



6. 7. Summary

If $|\lambda| = +1$ we have a **proper** orthogonal transformation which is equivalent to rotation of axes. It can be proven that any rotation is a proper orthogonal transformation and vice-versa.

If $|\lambda| = -1$ we have an **improper** orthogonal transformation which is equivalent to rotation of axes then inversion. This is known as an improper rotation since it *changes the handedness of the basis.*