Lecture 17: The Scalar Potential (*RHB 9.4*)

Consider again the work done by a force. If the force is *conservative*, i.e. total energy is conserved, then the work done is equal to the change in potential energy

$$dV = -dW = -\underline{F} \cdot \underline{dr} = -F_i dx_i$$

Now we can also write dV as

$$dV = \frac{\partial V}{\partial x_i} dx_i = (\underline{\nabla} V)_i dx_i$$

Therefore we can identify $\boxed{\underline{F} = -\underline{\nabla} V}$

Thus the force is minus the gradient of the (scalar) potential. The minus sign is conventional and chosen so that potential energy decreases as the force does work.

In this example we knew that a potential existed (we postulated conservation of energy). More generally one would like to know under what conditions can a vector field $\underline{A}(\underline{r})$ be written as the gradient of a scalar field ϕ, *i.e.* when does $\underline{A}(\underline{r}) \;=\; (\pm)\,\underline{\nabla}\,\phi(\underline{r})$ hold?

Aside: A **simply connected** region R is a region where every closed curve in R can be shrunk continuously to a point while remaining entirely in R. The inside of a sphere is simply connected while the region between two cylinders is **not** simply connected: it's doubly connected. See *RHB 9.2* for disussion. For this course we shall be concerned with simply connected regions

17. 1. Theorems on Scalar Potentials

For a vector field $\underline{A}(\underline{r})$ defined in a simply connected region R, the following three statements are equivalent, *i.e.*, **any one implies the other two**:-

1. $\underline{A}(\underline{r})$ can be written as the **gradient** of a **scalar potential** $\phi(\underline{r})$

   $$\boxed{\underline{A}(\underline{r}) \;=\; \underline{\nabla}\,\phi(\underline{r}) \quad \text{with} \quad \phi(\underline{r}) \;=\; \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'}$$

   where $\underline{r_0}$ is some arbitrary fixed point in R.

2. (a) $\displaystyle\oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; 0$, where C is any **closed** curve in R

   (b) $\displaystyle\phi(\underline{r}) \equiv \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$ does not depend on the path between $\underline{r_0}$ and $\underline{r}$.

3. $\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0$ for all points $\underline{r} \in R$

Proof that (2) implies (1)
Consider two neighbouring points $\underline{r}$ and $\underline{r} + \underline{dr}$, define the potential as before

$$\phi(\underline{r}) \;=\; \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

and define $d\phi$ as

$$\begin{aligned}
d\phi(\underline{r}) &= \phi(\underline{r}+\underline{dr}) - \phi(\underline{r}) = \left\{ \int_{\underline{r_0}}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}') \cdot \underline{dr}' - \int_{\underline{r_0}}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' \right\} \quad \text{(by definition)} \\
&= \left\{ \int_{\underline{r_0}}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}') \cdot \underline{dr}' + \int_{\underline{r}}^{\underline{r}_0} \underline{A}(\underline{r}') \cdot \underline{dr}' \right\} \quad \text{(swapped limits on 2nd } \textstyle\int \text{)} \\
&= \int_{\underline{r}}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}') \cdot \underline{dr}' \quad \text{(combined integrals using } \textbf{path independence}\text{)} \\
&= \underline{A}(\underline{r}) \cdot \underline{dr} \quad \text{(for infinitesimal } \underline{dr}\text{)}
\end{aligned}$$

But, by Taylor's theorem, we also have

$$d\phi(\underline{r}) = \frac{\partial \phi(\underline{r})}{\partial x_i}\, dx_i = \underline{\nabla}\, \phi(\underline{r}) \cdot \underline{dr}$$

Comparing the two different equations for $d\phi(\underline{r})$, which hold for all $\underline{dr}$, we deduce

$$\underline{A}(\underline{r}) = \underline{\nabla}\, \phi(\underline{r})$$

Thus we have shown that **path independence** implies the existence of a scalar potential ϕ for the vector field $\underline{A}$. (Also path independence implies 2)a).

Proof that (1) implies (3) (the easy bit!)

$$\underline{A} = \underline{\nabla}\, \phi \quad \Rightarrow \quad \underline{\nabla} \times \underline{A} = \underline{\nabla} \times \left(\underline{\nabla}\, \phi\right) \equiv 0$$

because curl (grad ϕ) is identically zero (ie it's zero for *any* scalar field ϕ).

Proof that (3) implies (2): (the hard bit!)
We defer the proof until we have met Stokes' theorem in a few lectures time.

Terminology: A vector field is

- **irrotational** if $\underline{\nabla} \times \underline{A}(\underline{r}) = 0$.
- **conservative** if $\underline{A}(\underline{r}) = \underline{\nabla}\phi$.
- For simply connected regions we have shown irrotational and conservative are synonymous. But note that for a multiply connected region this is not the case.

Note: $\phi(\underline{r})$ is only determined up to a **constant**: if $\psi = \phi + constant$ then $\underline{\nabla}\psi = \underline{\nabla}\phi$ and ψ can equally well serve as a potential. The freedom in the constant corresponds to the freedom in choosing $\underline{r}_0$ to calculate the potential. Equivalently the absolute value of a scalar potential has no meaning, only **potential differences** are significant.

17. 2. Methods for finding Scalar Potentials

Method (1): Integration along a straight line.

We have shown that the scalar potential $\phi(\underline{r})$ for a *conservative* vector field $\underline{A}(\underline{r})$ can be constructed via $\phi(\underline{r}) \;=\; \int_{\underline{r_0}}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$ for some suitably chosen $\underline{r_0}$ and any path which joins $\underline{r_0}$ and $\underline{r}$. Sensible choices for $\underline{r_0}$ are often $\underline{r_0} = 0$ or $\underline{r_0} = \infty$.

We have also shown that the line integral is *independent* of the path of integration between the endpoints. Therefore, a convenient way of evaluating such integrals is to integrate along a **straight line** between the points $\underline{r}_0$ and $\underline{r}$. Choosing $\underline{r}_0 = 0$, we can write this integral in parametric form as follows:

$$\begin{aligned} \underline{r}' &= \lambda\, \underline{r} \quad \text{where} \quad \{0 \;\leq\; \lambda \;\leq\; 1\} \\ \text{so} \quad \underline{dr'} &= d\lambda\, \underline{r} \qquad \text{and therefore} \\ \phi(\underline{r}) &= \int_{\lambda=0}^{\lambda=1} \underline{A}(\lambda\, \underline{r}) \cdot (d\lambda\, \underline{r}) \end{aligned}$$

Example 1: Let $\underline{A}(\underline{r}) \;=\; (\underline{a} \cdot \underline{r})\, \underline{a}$ where $\underline{a}$ is a constant vector.
It is easy to show that $\underline{\nabla} \times ((\underline{a} \cdot \underline{r})\, \underline{a}) = 0$ (tutorial). Thus

$$\begin{aligned} \phi(\underline{r}) &= \int_0^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' = \int_0^{\underline{r}} \Big((\underline{a} \cdot \underline{r}')\, \underline{a} \Big) \cdot \underline{dr}' \\ &= \int_0^1 \Big((\underline{a} \cdot \lambda\, \underline{r})\, \underline{a} \Big) \cdot (d\lambda\, \underline{r}) = (\underline{a} \cdot \underline{r})^2 \int_0^1 \lambda\, d\lambda \\ &= \frac{1}{2} (\underline{a} \cdot \underline{r})^2 \end{aligned}$$

Note: Always check that your $\phi(\underline{r})$ satisfies $\underline{A}(\underline{r}) \;=\; \underline{\nabla}\, \phi(\underline{r})$!

Example 2: Let $\underline{A}(\underline{r}) \;=\; 2\,(\underline{a} \cdot \underline{r})\, \underline{r} \,+\, r^2\, \underline{a}$ where $\underline{a}$ is a constant vector.
It is straightforward to show that $\underline{\nabla} \times \underline{A} = 0$. Thus

$$\begin{aligned} \phi(\underline{r}) &= \int_0^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; \int_0^1 \underline{A}(\lambda\, \underline{r}) \cdot (d\lambda\, \underline{r}) \\ &= \int_0^1 \Big[2\,(\underline{a} \cdot \lambda\, \underline{r})\, \lambda\, \underline{r} \,+\, \lambda^2 r^2\, \underline{a} \Big] \cdot (d\lambda\, \underline{r}) \\ &= \Big[2\,(\underline{a} \cdot \underline{r})\, \underline{r} \cdot \underline{r} \,+\, r^2\, (\underline{a} \cdot \underline{r}) \Big] \int_0^1 \lambda^2\, d\lambda \\ &= r^2\, (\underline{a} \cdot \underline{r}) \end{aligned}$$

Method (2): Direct Integration "by inspection".

Example 1 (revisited):
Let $\underline{A}(\underline{r}) \;=\; (\underline{a} \cdot \underline{r})\, \underline{a}$ where $\underline{a}$ is a constant vector. We can write

$$\underline{A}(\underline{r}) \;=\; (\underline{a} \cdot \underline{r})\, \underline{a} \;=\; (\underline{a} \cdot \underline{r})\, \underline{\nabla}(\underline{a} \cdot \underline{r}) \;=\; \underline{\nabla} \bigg(\frac{1}{2}\, (\underline{a} \cdot \underline{r})^2 \,+\, \text{const} \bigg)$$

in agreement with the result of Method 1, viz $\phi(\underline{r}) = \frac{1}{2}(\underline{a} \cdot \underline{r})^2$ if we *choose* the constant of integration to be zero.

Example 2 (revisited): Again, let

$$\underline{A}(\underline{r}) = 2\,(\underline{a} \cdot \underline{r})\,\underline{r} + r^2\,\underline{a}$$

where $\underline{a}$ is a constant vector. So

$$\underline{A}(\underline{r}) = 2\,(\underline{a} \cdot \underline{r})\,\underline{r} + r^2\,\underline{a} = (\underline{a} \cdot \underline{r})\underline{\nabla} r^2 + r^2\,\underline{\nabla}(\underline{a} \cdot \underline{r}) = \underline{\nabla}\Big((\underline{a} \cdot \underline{r})\,r^2 + \text{const}\Big)$$

in agreement with what we had before if we choose const $= 0$.
While this method is not as systematic as Method 1, it can be quicker if you spot the trick!

17. 3. Conservative forces: conservation of energy

Let us now see how the name *conservative field* arises. Consider a vector field $\underline{F}(\underline{r})$ corresponding to the only force acting on some test particle of mass m. We will show that for a conservative force (where we can write $\underline{F} = -\underline{\nabla} V$) the total energy is **constant** in time.

Proof: The particle moves under the influence of Newton's Second Law:

$$m\ddot{\underline{r}} = \underline{F}(\underline{r}).$$

Consider a small displacement $\underline{dr}$ along the path taking time dt. Then

$$m\ddot{\underline{r}} \cdot \underline{dr} = \underline{F}(\underline{r}) \cdot \underline{dr} = -\underline{\nabla} V(\underline{r}) \cdot \underline{dr}.$$

Integrating this expression along the path from $\underline{r_A}$ at time $t = t_A$ to $\underline{r_B}$ at time $t = t_B$ yields

$$m \int_{\underline{r_A}}^{\underline{r_B}} \ddot{\underline{r}} \cdot \underline{dr} = -\int_{\underline{r_A}}^{\underline{r_B}} \underline{\nabla} V(\underline{r}) \cdot \underline{dr}.$$

We can simplify the left-hand side of this equation to obtain

$$m \int_{\underline{r_A}}^{\underline{r_B}} \ddot{\underline{r}} \cdot \underline{dr} = m \int_{t_A}^{t_B} \ddot{\underline{r}} \cdot \dot{\underline{r}} dt = m \int_{t_A}^{t_B} \frac{1}{2}\frac{d}{dt}\dot{\underline{r}}^2 dt = \frac{1}{2} m[v_B^2 - v_A^2],$$

where v_A and v_B are the magnitudes of the velocities at points A and B respectively.

The right hand side simply gives

$$-\int_{\underline{r_A}}^{\underline{r_B}} \underline{\nabla} V(\underline{r}) \cdot \underline{dr} = -\int_{\underline{r_A}}^{\underline{r_B}} dV = V_A - V_B$$

where V_A and V_B are the values of the potential V at $\underline{r_A}$ and $\underline{r_B}$, respectively. Therefore

$$\frac{1}{2} m v_A^2 + V_A = \frac{1}{2} m v_B^2 + V_B$$

and the total energy $\quad E = \frac{1}{2} m v^2 + V \quad$ is **conserved**, i.e. *constant in time.*

Newtonian Gravity and the *electrostatic force* are both conservative. *Frictional forces* are not conservative; energy is dissipated and work is done in traversing a closed path. In general, time-dependent forces are not conservative.