Lecture 11: Diagonalisation of a Real, Symmetric Tensor *(RHB 7.15; BK 9.2)*

In the $\underline{e}_i$ basis T_{ij} is, in general, non-diagonal. i.e. T_{ij} is non-zero for $i \neq j$.
However if we transform to a basis constructed from the normalised eigenvectors—the 'principal axes'—we find that the tensor becomes diagonal.

Transform to the $\underline{e}_i{}'$ basis, chosen such that

$$\boxed{\underline{e}_i{}' \;=\; \underline{n}(i)}\,,$$

where $\underline{n}(i)$ are the three *normalized*, and *orthogonal*, eigenvectors of T_{ij} with eigenvalues $t(i)$ respectively.
e.g. in the notation of last lecture

$$\underline{n}(1) \;=\; \underline{\hat{A}} \quad ; \quad \underline{n}(2) \;=\; \underline{\hat{B}} \quad ; \quad \underline{n}(3) \;=\; \underline{\hat{C}}\,.$$
$$t(1) \;=\; t_A \quad ; \quad t(2) \;=\; t_B \quad ; \quad t(3) \;=\; t_C\,.$$

Now

$$\lambda_{ij} \;=\; \underline{e}_i{}' \cdot \underline{e}_j \;=\; \underline{n}(i) \cdot \underline{e}_j \;=\; n_j(i)\,.$$

i.e. the **rows** of λ are the components of the **normalised eigenvectors** of T.

In the $\underline{e}_i{}'$ basis

$$T'_{ij} = (\lambda T \lambda^T)_{ij}$$

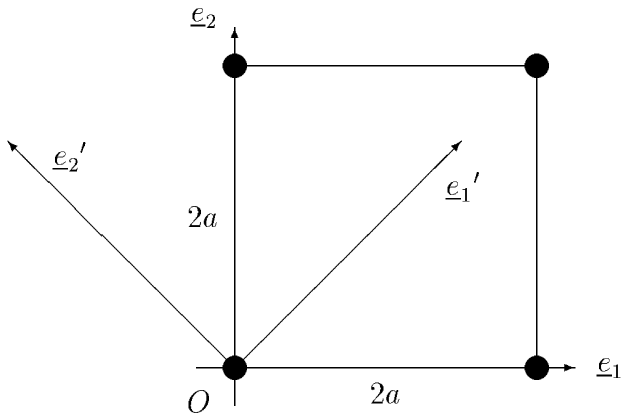
Now since the columns of λ^T are the eigenvectors of T we see

$$\lambda T \lambda^T \;=\; \lambda \begin{pmatrix} t(1)\lambda_{11} & t(2)\lambda_{21} & t(3)\lambda_{31} \\ t(1)\lambda_{12} & t(2)\lambda_{22} & t(3)\lambda_{32} \\ t(1)\lambda_{13} & t(2)\lambda_{23} & t(3)\lambda_{33} \end{pmatrix} = \begin{pmatrix} t(1) & 0 & 0 \\ 0 & t(2) & 0 \\ 0 & 0 & t(3) \end{pmatrix}$$

from the orthonormality of the $\underline{n}(i)$ (rows of λ; columns of λ^T). Thus, with respect to a basis defined by the eigenvectors or principal axes of the tensor, the tensor has *diagonal form*. [i.e. $T'_{ij} = \mathrm{diag}\{t(1),\, t(2),\, t(3)\}$.] The diagonal basis is often referred to as the '**principal axes basis**'.

Note: In the diagonal basis the trace of a tensor is the sum of the eigenvalues; the determinant of the tensor is the product of the eigenvalues. Since the trace and determinant are invariants this means that in any basis the trace and determinant are the sum and products of the eigenvalues respectively.

Example: Diagonalisation of Inertia Tensor. Consider the Inertia tensor for four masses arranged in a square with the origin at the left hand corner (see lecture 9 p 36). It is easy to check (exercise) that the eigenvectors (or principal axes of inertia) are $(\underline{e}_1 + \underline{e}_2)$ (eigenvalue $4ma^2$), $(\underline{e}_1 - \underline{e}_2)$ (eigenvalue $12ma^2$) and $\underline{e}_3$ (eigenvalue $16ma^2$).



Defining the $\underline{e}_i{}'$ basis as normalised eigenvectors: $\underline{e}_1{}' = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)$; $\underline{e}_2{}' = \frac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_2)$; $\underline{e}_3{}' = \underline{e}_3$, one obtains

$$\lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\sim \text{rotation of } \pi/4 \text{ about } \underline{e}_3 \text{ axis})$$

and the Inertia Tensor in the $\underline{e}_i{}'$ basis is

$$\begin{aligned} I'_{ij}(O) &= \left(\lambda\ I(O)\lambda^T\right)_{ij} \\ &= 4ma^2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underline{4ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}}\ . \end{aligned}$$

We see the tensor is diagonal with diagonal elements the eigenvalues (principal moments of inertia).

Remark: Diagonalisability is a very special and useful property of real, symmetric tensors. It is a property also shared by the more general class of Hermitean operators which you will meet in quantum mechanics in third year. A general tensor does not share the property. For example a real non-symmetric tensor cannot be diagonalised.

11. 1. Symmetry and Eigenvectors of the Inertia Tensor

In the previous example the eigenvectors had some physical significance: in the original basis $\underline{e}_3$ is perpendicular to the plane where the masses lie; $\underline{e}_1 + \underline{e}_2$ is along the diagonal of the square.

By using the transformation law for the inertia tensor we can see how the symmetry of the mass arrangement is related to the eigenvectors of the tensor. First we need to define symmetry axes and planes.

A **Symmetry Plane** is a plane under reflection in which the distribution of mass remains unchanged *e.g.* for a lamina with normal $\underline{e}_3$ the $\underline{e}_1 - \underline{e}_2$ plane is a reflection symmetry plane.

Claim: A normal to a symmetry plane is an eigenvector

Proof: Choose $\underset{\sim}{e}_3$ as the normal. Now since the mass distribution is invariant under reflection in the symmetry plane, the representation of the tensor must be unchanged when the axes are reflected in the plane *i.e.* the tensor should look exactly the same when the axes have been transformed in such a way that the mass distribution with repect to the new axes is the same as the mass distribution with respect to the old axes.

$$\therefore \quad I' = \lambda I \lambda^T = I \quad \text{for } \lambda \text{ a reflection in the } \underset{\sim}{e}_1\text{–}\underset{\sim}{e}_2 \text{ plane} \quad \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Calculating $I' = \lambda I \lambda^T$ gives

$$I' = \begin{pmatrix} I_{11} & I_{12} & -I_{13} \\ I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix} = I \quad \Rightarrow \quad I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

and $\underset{\sim}{e}_3$ is an eigenvector, eigenvalue I_{33}.

An **m-fold Symmetry Axis** is an axis about which rotation of the system by $2\pi/m$ leaves the mass distribution unchanged *e.g.* for the example of the previous subsection the diagonal of the square is 2-fold symmetry axis.

Claim: A 2-fold symmetry axis is an eigenvector

Proof: Choose $\underset{\sim}{e}_3$ as the symmetry axis. Now since the mass distribution is invariant under rotation of π about this axis, the representation of the tensor must be unchanged when the axes are rotated by π about $\underset{\sim}{e}_3$

$$\therefore \quad I' = \lambda I \lambda^T = I \quad \text{for } \lambda \text{ a rotation of } \pi \text{ about } \underset{\sim}{e}_3 \quad \lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculating $I' = \lambda I \lambda^T$ gives

$$I' = \begin{pmatrix} I_{11} & I_{12} & -I_{13} \\ I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix} = I \quad \Rightarrow \quad I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

and $\underset{\sim}{e}_3$ is an eigenvector, eigenvalue I_{33}.

Claim: An m-fold symmetry axis is an eigenvector and for $m > 2$ the orthogonal plane is a degenerate eigenspace *i.e.* if $\underset{\sim}{e}_3$ is chosen as the symmetry axis then I is of the form

$$I = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}$$

See *e.g.* the example in lecture 9 p.34/35 which has $\underset{\sim}{e}_3$ as a 4-fold symmetry axis.

Proof: The idea is the same as the $m = 2$ case above. Because it is a bit more complicated we do not include it here.

Note: The limit $m \to \infty$ yields a continuous symmetry axis. *e.g.* a cylinder, a cone ..

11. 2. Summary of Relation of Eigenvectors of Inertia Tensor to Symmetry

a Most general body has no symmetry.
3 orthogonal eigenvectors have
to be found the hard way!

b A body with a symmetry plane.
Normal to symmetry plane is eigenvector

c A body with 2-fold symmetry axis
Symmetry axis is eigenvector.

d A body with m-fold symmetry axis ($m > 2$)
Symmetry axis is eigenvector;
degenerate eigenvectors normal to symmetry axis

e A body with spherical symmetry
Any vector is an eigenvector
with the same eigenvalue! (triple degeneracy)