Lecture 4: Suffix Notation

4. 1. Free Indices and Summation Indices

Consider, for example, the vector equation

$$\underline{a} - (\underline{b} \cdot \underline{c})\, \underline{d} + 3\underline{n} = 0$$

Since the basis vectors are linearly independent the equation must hold for each component therefore

$$a_i - (\underline{b} \cdot \underline{c})\, d_i + 3n_i = 0 \quad \text{for} \quad i = 1, 2, 3$$

The free index i occurs once in each term of the equation. In general every term in the equation must be of the same kind *i.e.* have the same free indices. For example the following is forbidden

$$a_i - (\underline{b} \cdot \underline{c}) + 3n_i = 0 \quad \text{WRONG}$$

As we have seen summation indices are "dummy" indices and can be relabelled

$$\underline{A} \cdot \underline{B} = \sum_i A_i B_i = \sum_k A_k B_k$$

This freedom should *always* be used to avoid confusion with other indices in the equation. For example consider the following

$$\begin{aligned} (\underline{a} \cdot \underline{b})(\underline{c} \cdot \underline{d}) &= (\sum_{i=1}^{3} a_i b_i)(\sum_{j=1}^{3} c_j d_j) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_i c_j d_j \end{aligned}$$

If instead we write

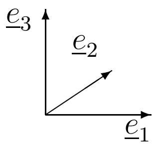
$$(\underline{a} \cdot \underline{b})(\underline{c} \cdot \underline{d}) = (\sum_{i=1}^{3} a_i b_i)(\sum_{i=1}^{3} c_i d_i)$$

then great confusion, inevitably leading to mistakes, arises when the brackets are removed!

4. 2. Handedness of Basis

Consider $\underline{e}_1 \times \underline{e}_2 = 1 \cdot 1 \cdot \sin(\pi/2)\ \hat{\underline{n}}$, where $\hat{\underline{n}} = \pm \underline{e}_3$. If we choose $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ to form a *right-handed* basis (sometimes referred to as right-handed system or right-handed triad), then $\underline{e}_1 \times \underline{e}_2 = +\underline{e}_3$. Similarly $\underline{e}_2 \times \underline{e}_3 = +\underline{e}_1$ and $\underline{e}_3 \times \underline{e}_1 = +\underline{e}_2$.

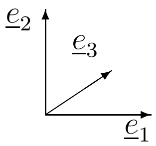
right handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_1 \times \underline{e}_2 \\ \underline{e}_1 &= \underline{e}_2 \times \underline{e}_3 \\ \underline{e}_2 &= \underline{e}_3 \times \underline{e}_1 \end{aligned}$$

$$\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = 1}$$

left handed



$$\begin{aligned} \underline{e}_3 &= \underline{e}_2 \times \underline{e}_1 \\ \underline{e}_1 &= \underline{e}_3 \times \underline{e}_2 \\ \underline{e}_2 &= \underline{e}_1 \times \underline{e}_3 \end{aligned}$$

$$\boxed{(\underline{e}_1, \underline{e}_2, \underline{e}_3) = -1}$$

4. 3. The Vector Product in r.h. basis

$$\begin{aligned} \underline{A} \times \underline{B} &= \left(\sum_{i=1}^{3} A_i \, \underline{e}_i \right) \times \left(\sum_{j=1}^{3} B_j \, \underline{e}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i \, B_j \, (\underline{e}_i \times \underline{e}_j) \, . \end{aligned}$$

Since $\underline{e}_1 \times \underline{e}_1 = \underline{e}_2 \times \underline{e}_2 = \underline{e}_3 \times \underline{e}_3 = 0$, and $\underline{e}_1 \times \underline{e}_2 = -\underline{e}_2 \times \underline{e}_1 = \underline{e}_3$, etc. we have

$$\begin{aligned} \underline{A} \times \underline{B} &= \underline{e}_1(A_2B_3 - A_3B_2) + \underline{e}_2(A_3B_1 - A_1B_3) + \underline{e}_3(A_1B_2 - A_2B_1) \\ &\equiv \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} . \end{aligned}$$

Here the vertical lines indicate the determinant expanded along the first row *i.e.*

$$(\underline{A} \times \underline{B})_1 = (A_2B_3 - A_3B_2) \text{ , etc.}$$

It is now easy to write down an expression for the scalar triple product

$$\begin{aligned} \underline{A} \cdot (\underline{B} \times \underline{C}) &= \sum_{i=1}^{3} A_i (\underline{B} \times \underline{C})_i \\ &= A_1(B_2C_3 - C_2B_3) - A_2(B_1C_3 - C_1B_3) + A_3(B_1C_2 - C_1B_2) \\ &= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} . \end{aligned}$$

Note that the cyclic symmetry of the scalar triple product corresponds to the property of a determinant that interchanging two rows (or columns) changes the value by a factor -1.

4. 4. Summary of algebraic approach to vectors

We are now able to define vectors and the various products of vectors in an algebraic way (as opposed to the geometrical approach of lectures one and two).

A **vector** is *represented* (in some orthonormal basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$) by an ordered set of 3 numbers with certain laws of addition.

$$\begin{aligned} &\textit{e.g.}\ \underline{A} &&\text{is represented by} && (A_1, A_2, A_3)\,; \\ &\underline{A}+\underline{B} &&\text{is represented by} && (A_1+B_1, A_2+B_2, A_3+B_3)\,. \end{aligned}$$

The various 'products' of vectors are defined as follows:–

The Scalar Product

This is denoted by $\underline{A}\cdot\underline{B}$ and **defined** as:–

$$\underline{A}\cdot\underline{B} \stackrel{\text{def}}{=} \sum_i A_i B_i\ .$$

$$\underline{A}\cdot\underline{A} = A^2 \text{ defines the magnitude } A \text{ of the vector.}$$

The Vector Product

This is denoted by $\underline{A}\times\underline{B}$, and is **defined** in a right-handed basis as:–

$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.$$

The Scalar Triple Product

$$\begin{aligned} (\underline{A},\underline{B},\underline{C}) &\stackrel{\text{def}}{=} \sum_i A_i(\underline{B}\times\underline{C})_i \\ &= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}. \end{aligned}$$

In all the above formula the summations imply sums over each index taking values $1, 2, 3$.

4. 5. The Kronecker Delta Symbol δ_{ij} RHB 19.8

We define the symbol δ_{ij} (pronounced "delta i j"), where i and j can take on the values 1 to 3, such that

$$\boxed{\begin{aligned} \delta_{ij} &= 1 \ \text{ if } i=j \\ &= 0 \ \text{ if } i\neq j \end{aligned}}$$

i.e. $\delta_{11}=\delta_{22}=\delta_{33}=1$ and $\delta_{12}=\delta_{13}=\delta_{23}=\cdots=0$.

The equations satisfied by the **orthonormal basis vectors** $\underline{e}_i$ can all now be written as:–

$$\boxed{\underline{e}_i\cdot\underline{e}_j \ = \ \delta_{ij}}\ .$$

eg $\quad \underline{e}_1\cdot\underline{e}_2 \ = \ \delta_{12}=0 \quad ; \quad \underline{e}_1\cdot\underline{e}_1 \ = \ \delta_{11}=1$

Notes

(i) Since there are two free indices i and j, $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ is equivalent to 9 equations

(ii) $\delta_{ij} = \delta_{ji}$ [*i.e.* δ_{ij} is *symmetric* in its indices.]

(iii) $\displaystyle\sum_{i=1}^{3} \delta_{ii} = 3 \quad (= \delta_{11} + \delta_{22} + \delta_{33})$

(iv) $\displaystyle\sum_{j=1}^{3} A_j \delta_{jk} = A_1 \delta_{1k} + A_2 \delta_{2k} + A_3 \delta_{3k}$

If $k = 1$ then only the first term on the rhs contributes and rhs $= A_1$, similarly if $k = 2$ then rhs $= A_2$ and if $k = 2$ then rhs $= A_3$. Thus $\displaystyle\sum_{j=1}^{3} A_j \delta_{jk} = A_k$

Generalising the reasoning in (iv) implies that $\boxed{\displaystyle\sum_{j=1}^{3} \cdots_j \, \delta_{jk} = \cdots_k}$ which is known as the sifting property of δ_{ij}.

Examples of the use of this symbol are:–

1.
$$\begin{aligned}
\underline{A} \cdot \underline{e}_j &= \left(\sum_{i=1}^{3} A_i \underline{e}_i \right) \cdot \underline{e}_j = \sum_{i=1}^{3} A_i \, (\underline{e}_i \cdot \underline{e}_j) \\
&= \sum_{i=1}^{3} A_i \delta_{ij} = A_j, \quad \text{since terms with } i \neq j \text{ vanish.}
\end{aligned}$$

2.
$$\begin{aligned}
\underline{A} \cdot \underline{B} &= \left(\sum_{i=1}^{3} A_i \underline{e}_i \right) \cdot \left(\sum_{j=1}^{3} B_j \underline{e}_j \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \, (\underline{e}_i \cdot \underline{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij} \\
&= \sum_{i=1}^{3} A_i B_i \left(\text{or } \sum_{j=1}^{3} A_j B_j \right).
\end{aligned}$$

4. 6. Matrix representation of δ_{ij}

We may label the elements of a (3×3) matrix M as M_{ij},

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

Thus we see that if we write δ_{ij} as a matrix we find that it is the identity matrix $\mathbb{1}$.

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$