Lecture 12: Fields *(RHB 8.6, 8.7.1; D chapter 4; BK chapter 4)*

In Physics we often have to consider properties that vary in some region of space e.g. temperature of a body. To do this we require the concept of fields.

If to each point $\underline{r}$ in some region of in ordinary 3-d space there corresponds a **scalar** $\phi(x_1, x_2, x_3)$, then $\phi(\underline{r})$ is a **scalar field**.

Scalar Fields: Temperature distribution in a body $T(\underline{r})$ or pressure in the atmosphere $P(\underline{r})$;
Electric charge density or mass density $\rho(\underline{r})$;
Electrostatic potential $\phi(\underline{r})$.

Similarly a **vector field** assigns a vector $\underline{V}(x_1, x_2, x_3)$ to each point $\underline{r}$ of some region.

Vector Fields: Velocity in a fluid $\underline{v}(\underline{r})$;
Electric current density $\underline{J}(\underline{r})$;
Electric field $\underline{E}(\underline{r})$; Magnetic field $\underline{B}(\underline{r})$

A vector field in 2-d can be represented graphically, at a carefully selected set of points $\underline{r}$, by an arrow whose length and direction is proportional to $\underline{V}(\underline{r})$ *e.g.* wind velocitymap on the weather forecast.

12. 1. Level Surfaces of a Scalar Field

If $\phi(\underline{r})$ is a non-constant scalar field, then the equation

$$\phi(\underline{r}) = c$$

where c is a constant, defines a **level surface** (or equipotential) of the field. Level surfaces do not intersect (else ϕ would be multi-valued at the point of intersection).

Familiar examples in two dimensions (level curves) are the contours of constant height on a geographical map, $h(x_1, x_2) = c$. Also isobars on a weather map are level curves of pressure $P(x_1, x_2) = c$.

Examples in three dimensions:

(i) Let $\phi(\underline{r})$ be $$\phi(\underline{r}) = x^2 + y^2 + z^2$$

The level surface $\phi(\underline{r}) = c$ is a sphere of radius $\sqrt{c}$ centred on the origin. As c is varied, we obtain a family of level surfaces.

(ii) Electrostatic potential due to a point charge q situated at the point $\underline{a}$ is

$$\phi(\underline{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\underline{r} - \underline{a}|}$$

The level surfaces are concentric spheres centred on the point $\underline{a}$.

(iii) Let $\phi(\underline{r}) = \underline{k} \cdot \underline{r}$. The level surfaces are planes $\underline{k} \cdot \underline{r} =$ constant with normal $\underline{k}$.

(iv) Let $\phi(\underline{r}) = \exp(i\underline{k} \cdot \underline{r})$. Note that this a complex scalar field. Since $\underline{k} \cdot \underline{r} =$ constant is the equation for a plane, the level surfaces are planes.

The function ϕ goes through phase 2π when we go from $\underline{r}$ to $\underline{r} + \Delta\underline{r}$ where $\Delta\underline{r} \cdot \underline{k} = 2\pi$ *i.e.* we move distance $2\pi/|\underline{k}|$ along the unit normal $\hat{\underline{k}}$.

Therefore $\exp(i\underline{k} \cdot \underline{r})$ is referred to as a 'plane wave' of wavelength $2\pi/|\underline{k}|$ in direction $\hat{\underline{k}}$.

12. 2. Gradient of a Scalar Field

How does a scalar field change as we change position?

As an example think of a 2-d contour map of the height h=h(x,y) of a hill say. The height is a scalar field. If we are on the hill and move in the x-y plane then the change in height will depend on the direction in which we move (unless the hill is completely flat!). For example there will be a direction in which the height increases most steeply ('straight up the hill') We now introduce a formalism to describe how a scalar field $\phi(\underline{r})$ changes as a function of $\underline{r}$.

Mathematical Note: A scalar field $\phi(\underline{r}) = \phi(x_1, x_2, x_3)$ is said to be *continuously differentiable* in a region R if its first order partial derivatives

$$\frac{\partial\phi(\underline{r})}{\partial x_1}, \qquad \frac{\partial\phi(\underline{r})}{\partial x_2} \quad \text{and} \quad \frac{\partial\phi(\underline{r})}{\partial x_3}$$

exist and are *continuous* at every point $\underline{r} \in R$. We will generally assume scalar fields are continuously differentiable.

Let $\phi(\underline{r})$ be a scalar field. Consider 2 nearby points: P (position vector $\underline{r}$) and Q (position vector $\underline{r} + \underline{\delta r}$). Assume P and Q lie on *different* level surfaces as shown:

Now use Taylor's theorem for a function of 3 variables to evaluate the change in ϕ as we move from P to Q

$$\begin{aligned}
\delta\phi &\equiv \phi(\underline{r} + \underline{\delta r}) - \phi(\underline{r}) \\
&= \phi(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3) - \phi(x_1, x_2, x_3) \\
&= \frac{\partial\phi(\underline{r})}{\partial x_1}\,\delta x_1 + \frac{\partial\phi(\underline{r})}{\partial x_2}\,\delta x_2 + \frac{\partial\phi(\underline{r})}{\partial x_3}\,\delta x_3 + O(\,\delta x_i^2\,)
\end{aligned}$$

where we have assumed that the higher order partial derivatives exist, and that terms of order $(\,\delta x_i^2\,)$ can always be neglected. Thus we can write

$$\boxed{\delta\phi = \underline{\nabla}\;\phi(\underline{r})\cdot\underline{\delta r}}$$

where the 3 quantities

$$\left(\underline{\nabla}\;\phi(\underline{r})\right)_i = \frac{\partial\phi(\underline{r})}{\partial x_i}$$

form the Cartesian components of a **vector field**. We write

$$\boxed{\underline{\nabla}\;\phi(\underline{r}) \equiv \underline{e}_i\,\frac{\partial\phi(\underline{r})}{\partial x_i} = \underline{e}_1\,\frac{\partial\phi(\underline{r})}{\partial x_1} + \underline{e}_2\,\frac{\partial\phi(\underline{r})}{\partial x_2} + \underline{e}_3\,\frac{\partial\phi(\underline{r})}{\partial x_3}}$$

or in the old 'x, y, z' notation (where $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$ and $\underline{e}_3 = \underline{k}$)

$$\underline{\nabla}\;\phi(\underline{r}) = \underline{e}_1\,\frac{\partial\phi(\underline{r})}{\partial x} + \underline{e}_2\,\frac{\partial\phi(\underline{r})}{\partial y} + \underline{e}_3\,\frac{\partial\phi(\underline{r})}{\partial z}$$

The **vector field** $\underline{\nabla}\,\phi(\underline{r})$, pronounced "grad phi", is called the **gradient** of $\phi(\underline{r})$.

Example: calculate the gradient of $\phi = r^2 = x^2 + y^2 + z^2$

$$\begin{aligned}
\underline{\nabla}\phi(\underline{r}) &= \left(\underline{e}_1\,\frac{\partial}{\partial x} + \underline{e}_2\,\frac{\partial}{\partial y} + \underline{e}_3\,\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) \\
&= 2x\,\underline{e}_1 + 2y\,\underline{e}_2 + 2z\,\underline{e}_3 = 2\underline{r}
\end{aligned}$$

12. 3. Interpretation of the gradient

In deriving the expression for $\delta\phi$ above, we assumed that the points P and Q lie on *different* level surfaces. Now consider the situation where P and Q are nearby points on the *same* level surface. In that case $\delta\phi = 0$ and so

$$\delta\phi = \underline{\nabla}\phi(\underline{r})\cdot\underline{\delta r} = 0$$

The infinitesimal vector $\underline{\delta r}$ vector lies in the level surface at $\hat{\underline{r}}$, and the above equation holds

for all such $\underline{\delta r}$, hence

$$\boxed{\underline{\nabla}\phi(\underline{r}) \text{ is } \textbf{normal to the level surface at } \underline{r}.}$$

To construct a **unit normal** $\underline{n}(\underline{r})$ to the level surface at $\underline{r}$, we divide $\underline{\nabla}\phi(\underline{r})$ by its length

$$\underline{n}(\underline{r}) \;=\; \frac{\underline{\nabla}\phi(\underline{r})}{|\underline{\nabla}\phi(\underline{r})|} \qquad \big(\text{valid for} \quad |\underline{\nabla}\phi(\underline{r})| \neq 0\big)$$

12. 4. Directional Derivative

Now consider the change in $\delta\phi$ poduced by moving distance δs in some direction say $\underline{\hat{s}}$. Then $\underline{\delta r} = \underline{\hat{s}}\delta s$ and

$$\delta\phi = (\underline{\nabla}\phi(\underline{r}) \cdot \underline{\hat{s}})\ \delta s$$

As $\delta s \to 0$, the rate of change of ϕ as we move in the direction of $\underline{\hat{s}}$ is

$$\frac{d\phi(\underline{r})}{ds} = \underline{\hat{s}} \cdot \underline{\nabla}\phi(\underline{r}) = |\underline{\nabla}\phi(\underline{r})|\ \cos\theta \tag{1}$$

where θ is the angle between $\underline{\hat{s}}$ and the normal to the level surface at $\underline{r}$.

$$\boxed{\underline{\hat{s}} \cdot \underline{\nabla}\phi(\underline{r}) \text{ is the } \textbf{directional derivative} \text{ of the scalar field } \phi \text{ in the direction of } \underline{\hat{s}}.}$$

Note that the directional derivative has its *maximum* value when $\underline{s}$ is parallel to $\underline{\nabla}\phi(\underline{r})$, and is *zero* when $\underline{s}$ lies in the level surface. Therefore

$$\boxed{\underline{\nabla}\phi \text{ points in the direction of } \textbf{maximum} \text{ increase in } \phi}$$

Also recall that this direction is normal to the level surface. For a familiar example think of the contour lines on a map. The steepest direction is perpendicular to the contour lines.

Example: Find the directional derivative of $\phi = xy(x+z)$ at point $(1, 2, -1)$ in the $(\underline{e}_1 + \underline{e}_2)/\sqrt{2}$ direction.

$$\underline{\nabla}\phi = (2xy + yz)\underline{e}_1 + x(x+z)\underline{e}_2 + xy\underline{e}_3 = 2\underline{e}_1 + 2\underline{e}_3$$

at (1,2,-1). Thus at this point

$$\frac{1}{\sqrt{2}}\,(\underline{e}_1 + \underline{e}_2) \cdot \underline{\nabla}\phi = \sqrt{2}$$

Physical example: Let $T(\underline{r})$ be the temperature of the atmosphere at the point $\underline{r}$. An object flies through the atmosphere with velocity $\underline{v}$. Obtain an expression for the rate of change of temperature experienced by the object.

As the object moves from $\underline{r}$ to $\underline{r} + \underline{\delta r}$ in time δt, it sees a change in temperature

$$\delta T(\underline{r}) \;=\; \underline{\nabla}T(\underline{r}) \cdot \underline{\delta r} \;=\; \left(\underline{\nabla}T(\underline{r}) \cdot \frac{\underline{\delta r}}{\delta t}\right)\ \delta t$$

Taking the limit $\delta t \to 0$ we obtain

$$\frac{dT(\underline{r})}{dt} = \underline{v} \cdot \underline{\nabla}T(\underline{r})$$