Stoke's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \, ds$$

(or)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (M.D.U., Dec 2009)

$$\phi = x^2 + y^2 - 1$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{\cancel{2}(x\hat{i} + y\hat{j})}{\cancel{2}\sqrt{x^2 + y^2}}$$

$$\hat{n} = \frac{(x\hat{i} + y\hat{j})}{\sqrt{x^2 + y^2}}$$

$$\vec{F} \cdot d\vec{r} = yz\,dx + zx\,dy + xy\,dz$$

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}[x - x] - \hat{j}[y - y] + \hat{k}[z - z]$$

$$\vec{\nabla} \times \vec{F} = 0$$

$\therefore$ By Stoke's theorem $\oint_C \sim\sim dx + \sim dy + \sim dz = 0 \quad \because \vec{\nabla} \times \vec{F} = 0$

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.
(*U.P., I Semester, Winter 2001*)

W.K.T By Stoke's theorem

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds \quad \text{or} \quad \iint_S \underbrace{\vec{\nabla} \times \vec{F}}\cdot d\vec{s}$$

$$\vec{F} = (2x - y)\hat{i} - yz^2 \hat{j} - y^2 z \hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \hat{i}\left[\cancel{-2yz - (-2yz)}^{\,0}\right] - \hat{j}\left[\cancel{0 - 0}^{\,0}\right] + \hat{k}\left[0 - (-1)\right]$$

$$\vec{\nabla} \times \vec{F} = \hat{k} \;;$$

$$\phi = x^2 + y^2 + z^2$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2} = \phi}$$

$$\hat{n} \cdot \hat{k} = \frac{z}{1} \qquad \therefore\; ds = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$ds = dx\,dy$$

$$\iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} \Rightarrow \iint \cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

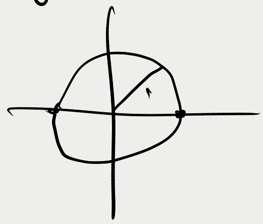
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy \Rightarrow$$

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$-1\le x\le 1$

$-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$



$$\int_{-1}^{1} dx\cdot y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)$$

$$2\sqrt{1-x^2}\;dx$$

f(x) is a even fn.

$$2\int_{-1}^{1}(1-x^2)\,dx = 2\cdot 2\int_{0}^{1}(1-x^2)\,dx$$

$$= 4\cdot\int_{0}^{1}(1-x^2)dx$$

$$=4\cdot\frac{1}{2}\left(\left(1^2-x^2\right)^{1/2}-a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^1$$

$$=\frac{1}{2}\left[x\cdot\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=2\cdot\left[x\cdot\cancel{\sqrt{1-x^2}}\to 0+1\sin^{-1}(x)\right|_0^1$$

$$=2\cdot\left[\sin^{-1}(1)-\cancel{\sin^{-1}(0)}\to 0\right]\Rightarrow \cancel{2}\cdot\frac{\pi}{\cancel{2}}\;\pi.$$

Example 87. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)

By Stoke's theorem w.k.T

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

So we have

$$\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

So given surface is

$$\phi = y + z - 2$$

$$\vec{\nabla}\phi = \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\hat{j}+\hat{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[1+2y]$$

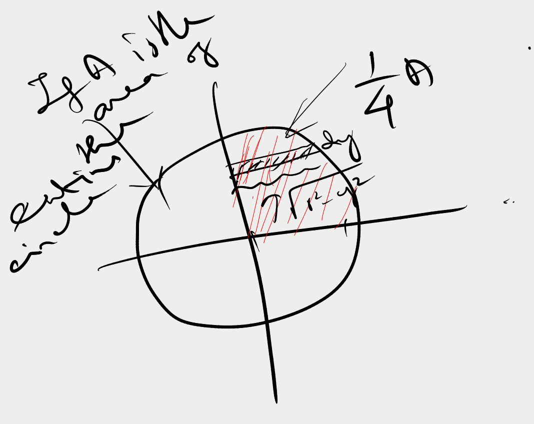
$$\vec{\nabla}\times\vec{F} = \hat{k}[1+2y]$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \hat{k}(1+2y)\cdot\frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1+2y}{\sqrt{2}}$$

By Stoke's theorem

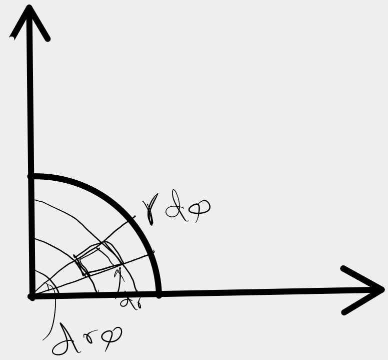
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1+2y}{\sqrt{2}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint_S (1+2y)\,dx\,dy$$



This integral is easy to solve in Spherical polar coordinates instead of Cartesian

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

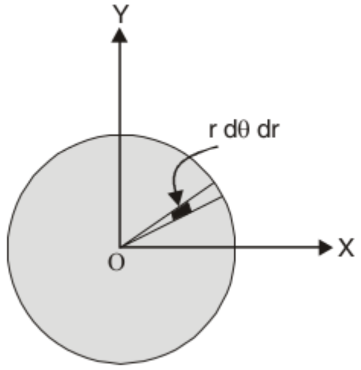


$$= \iint \frac{1+2y}{\sqrt{2}}\,\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

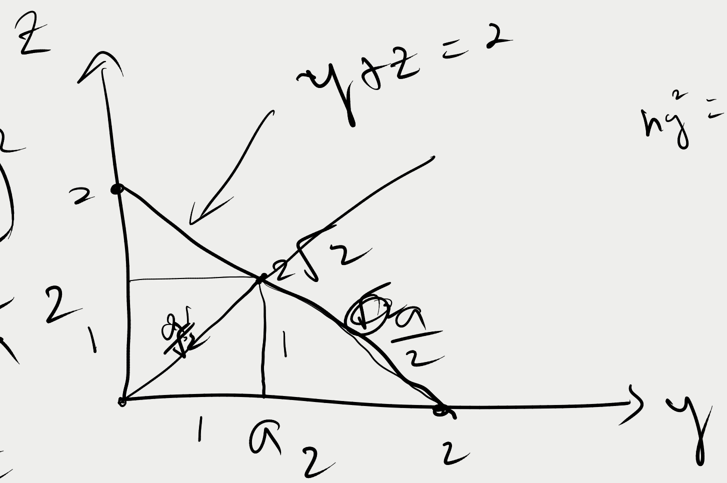
$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi$$ **Ans.**



mple 88. *Apply Stoke's Theorem to find the value of*

$$\left.\frac{\theta}{2}-\frac{2}{3}\cos\theta\right|_0^{2\pi} = \frac{\cancel{2\pi}^{\,\pi}}{\cancel{2}} - \frac{2}{3}\cancel{\cos(2\pi)}^{\,1} - \left(0-\frac{2}{3}\cancel{\cos 0}^{\,1}\right)$$

$= \pi$



$hy^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2$

$= \frac{a^2}{4} + \frac{a^2}{4}$

$hy^2 = \frac{2a^2}{2\times 2}$

$hy = \frac{a}{\sqrt{2}}$ ~~= 2-z~~

$hy^2 = ad^2 + op^2$

$= 2\sqrt{2}$

$y+z=2$

when $y=0$ $z=0$

when $z=0$ $y=2$

~~y=x~~ $y = z$

$y+z=2$

$y = \cancel{2}\ 2-z$

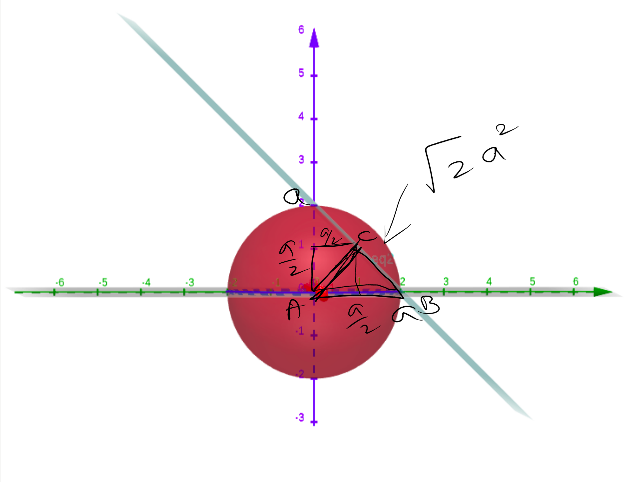
$y = z$

$AB^2 = BC^2 - AC^2$

r of circ.

radius of sphere

distance from centre



$p = a$

$op = a$

$ad = a$

$hy^2 = op^2 + ad^2$

$hy^2 = a^2 + a^2$

$hy^2 = 2a^2$

$hy = \sqrt{2}\,a^2$

$hy^2 = \frac{a^2}{4} + \frac{a^2}{4}$

$\sqrt{2}\,\frac{a^2}{2}$

$hy^2 = \frac{\cancel{2}a^2}{\cancel{4}2}$

$\frac{\cancel{2}a^2}{\cancel{4}2}$

$hy = \frac{a}{\sqrt{2}}$

$\frac{a}{2}$

$R^2 = r^2 + p^2$

$r^2 = R^2 - p^2$

$= a^2 -$

Stoke's Theorem
and
Application

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F}\cdot d\overline{r} = \iint_S curl\ \vec{F}\cdot\hat{n}\, d\vec{s} = \iint_S curl\ \vec{F}\cdot d\vec{S}$$

Problems:

✓ **Example 85.** *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (M.D.U., Dec 2009)

✓ **Example 86.** *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

where c is the circle $x^2 + y^2 = 1$*, corresponding to the surface of sphere of unit radius.* (U.P., I Semester, Winter 2001)

Example 87. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)

Example 88. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ *and* $x + z = a$. (Nagpur, Summer 2001)

Example 89. *Directly or by Stoke's Theorem, evaluate* $\iint_s curl\ \vec{v}\cdot\hat{n}\,ds$, $\vec{v} = \hat{i}y + \hat{j}z + \hat{k}x$, *s is the surface of the paraboloid* $z = 1 - x^2 - y^2$, $z^3 \geq 0$ *and* $\hat{n}$ *is the unit vector normal to s.*

Example 90. *Use Stoke's Theorem to evaluate* $\int_c \vec{v}\cdot d\vec{r}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

Example 91. *Evaluate the surface integral* $\iint_S \text{curl } \vec{F}\,.\,\hat{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$ and $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ · *(K. University, Dec. 2008)*

Example 92. *Evaluate* $\oint_C \vec{F}\cdot\vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$ *and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*
(U.P., I Semester, Winter 2000)

Example 93. *Evaluate* $\oint_C \vec{F}\cdot\vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = (x^2 + y^2)\,\hat{i} - 2\,xy\,\hat{j}$ *and C is the boundary of the rectangle* $x = \pm\, a$, $y = 0$ *and* $y = b$. *(U.P., I Semester, Winter 2002)*

Example 94. *Apply Stoke's Theorem to calculate* $\int_c 4\,y\,dx + 2\,z\,dy + 6\,y\,dz$ *where c is the curve of intersection of* $x^2 + y^2 + z^2 = 6\,z$ *and* $z = x + 3$.

Example 95. *Verify Stoke's Theorem for the function* $\overline{F} = z\hat{i} + x\,\hat{j} + y\,\hat{k}$, *where C is the unit circle in xy-plane bounding the hemisphere* $z = \sqrt{(1 - x^2 - y^2)}$. *(U.P., I Semester Comp. 2002)*

Example 96. *Verify Stoke's theorem for the vector field* $\overline{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2\,z\hat{k}$ *over the upper half of the surface* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on xy- plane.*
(Nagpur University, Summer 2001)

Example 97. *Verify Stoke's Theorem for* $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

Example 98. *Verify Stoke's theorem for a vector field defined by* $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ *in the rectangular in xy-plane bounded by lines* $x = 0$, $x = a$, $y = 0$, $y = b$.
(Nagpur University, Summer 2000)

Example 99. *Verify Stoke's Theorem for the function*

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z = 0$ *and bounded by the lines* $x = 0$, $y = 0$, $x = a$, $y = a$.

Y

Example 100. *Verify Stoke's Theorem for* $\vec{F} = (x + y)\,\hat{i} + (2x - z)\,\hat{j} + (y + z)\,\hat{k}$ *for the surface of a triangular lamina with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*
(Nagpur University 2004, K. U. Dec. 2009, 2008, A.M.I.E.T.E., Summer 2000)

Example 101. *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2)\ \hat{i} + (yz + 4)\ \hat{j} - (xz)\ \hat{k}$$

over the surface of a cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the XOY plane (open the bottom).

Gaussian Theorem :

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Example 102. *State Gauss's Divergence theorem* $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint Div\, \vec{F}\, dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

Example 103. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *where* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(U.P., Ist Semester, 2009, Nagpur University, Winter 2003)

Example 104. *Find* $\iint \vec{F} \cdot \hat{n} \cdot ds$, *where* $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ *and S is the surface of the sphere having centre (3, – 1, 2) and radius 3.*

(AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000)

Example 105. *Use Divergence Theorem to evaluate* $\iint_S \vec{A} \cdot \vec{ds}$,

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(AMIETE, Dec. 2009)

Example 106. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2)\, d\vec{s} = 6\, V$$

where S is any closed surface enclosing volume V. *(U.P., I Semester, Winter 2002)*

Example 107. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Example 108. *Use Divergence Theorem to evaluate* $\iint_S \vec{F} \cdot d\vec{S}$ *where* $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.
(*A.M.I.E.T.E., Summer 2003, 2001*)

Example 109. *Apply the Divergence Theorem to compute* $\iint \vec{u} \cdot \hat{n}\, ds$, *where s is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = \hat{i}x - \hat{j}y + \hat{k}z$.

Example 110. *Apply Divergence Theorem to evaluate* $\iiint_V \vec{F} \cdot \hat{n}\, ds$, *where* $\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$ *and* $z = b$.
(*U.P. Ist Semester, Dec. 2006*)

Example 111. *Evaluate surface integral* $\iint \vec{F} \cdot \hat{n}\, ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\ (\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron* $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ *and n is the unit normal in the outward direction to the closed surface S.*

Example 112. *Use the Divergence Theorem to evaluate*

$$\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

where S is the portion of the plane $x + 2y + 3z = 6$ *which lies in the first Octant.*
(*U.P., I Semester, Winter 2003*)

Example 113. *Use Divergence Theorem to evaluate :* $\iint (x\,dy\, dz + y\, dz\, dx + z\, dx\, dy)$ *over the surface of a sphere radius a.* (*K. University, Dec. 2009*)

Example 114. *Using the divergence theorem, evaluate the surface integral* $\iint\limits_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ *where* $S : x^2 + y^2 + z^2 = 4$.
(*AMIETE, Dec. 2010, UP, I Sem., Dec 2008*)

Example 115. *Evaluate* $\iint_S xz^2\, dy\, dz + (x^2y - z^3)\, dz\, dx + (2xy + y^2z)\, dx\, dy$ *where S is the surface of hemispherical region bounded by*

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = 0.$$

Example 116. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $F = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Example 117. *The vector field* $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ *is defined over the volume of the cuboid given by* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, *enclosing the surface S. Evaluate the surface integral*

$$\iint_S \vec{F} \cdot \vec{ds}$$

(*U.P., I Semester, Winter 2001*)

Example 118. *Verify the divergence Theorem for the function* $\overline{F} = 2x^2yi - y^2j + 4xz^2k$ *taken over the region in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$.

Example 119. *Verify the Gauss divergence Theorem for*

$\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. *(U.P., I Semester, Compartment 2002)*

Example 120. *Verify Divergence Theorem, given that* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by the planes* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.