Date : **22-Oct-2020**

Attendance : **9**

→ Assignment Doubts :

1) $\underline{\underline{11^{th}}}$

Gradient : Quick recap

$\vec{\nabla}\phi \rightarrow$ vector quantity

$\vec{\nabla}\phi \rightarrow$ "$\vec{n}$" normal vector

$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} \rightarrow \hat{n}$ unit normal vector

$|\vec{\nabla}\phi| \longrightarrow$ Greatest rate of change of ϕ

If $\vec{r}$ is some vector the $\vec{\nabla}\phi \cdot \underline{\underline{\hat{r}}}$ is the

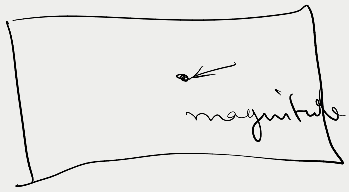
unit vector of $\vec{r}$

Directional derivative of "ϕ" along the $\vec{r}$.

→ Scalar point fn. Example: temperature
Mass

So today we will be examining vector point fn.

Scalar fn.

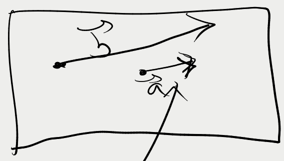


Some of the example for vector point fn.

$l \times l$

l

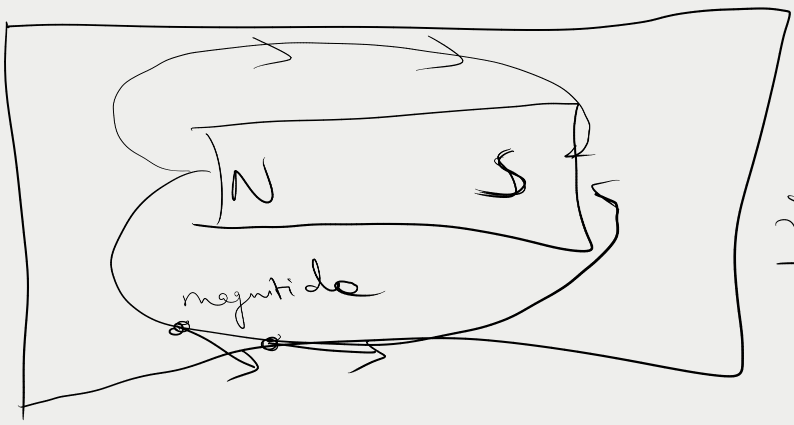
$[\text{area}] = l \times l$

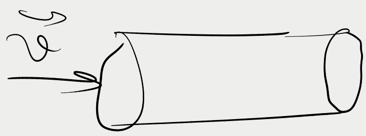


length of the arrow defines the magnitude and the head points the direction

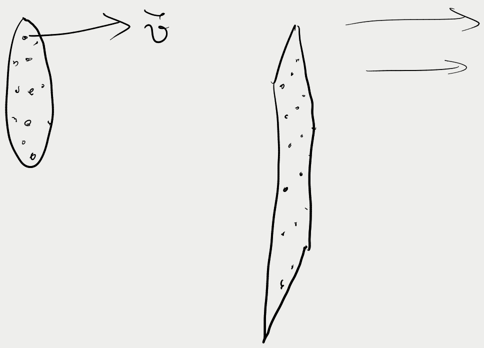
Velocity
Force
Acceleration
magnetization

} Vector
magnitude Direction





Speed - scalar
Velocity - vector



If ρ is the density of the fluid

ρ, M, V

$$\left.\rho = \frac{M}{V}\right\}$$ ← static case

Dynamic case :

$$\rho = \frac{M}{\frac{d\vec{V}}{dt}}$$ ← volume element

$$\rho = \frac{M}{\underbrace{dV}_{\text{decompose}}} \cdot dt$$

If we consider the flow only along the "x" direction

$$\rho = \frac{M}{\frac{dx}{dt}\,dy\,dz}$$

The mass flow along x direction :

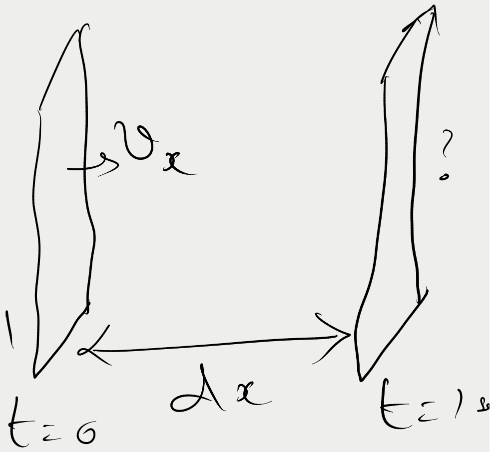
$$= \rho \cdot \overbrace{\frac{dx}{dt}}^{v_x}\,dy\,dz$$

$$= \rho \cdot v_x \cdot dy\,dz \qquad \text{--- (1)}$$

← velocity along x direction.

If there is a change in velocity along the length "dx"

$$v_x + \frac{\partial v_x}{\partial x} \cdot dx$$



The ~~change~~ in mass flow @ $t=1$

$$= \rho\left(v_x + \frac{\partial v_x}{\partial x} \cdot dx\right) dy\, dz \quad \text{—} \; (2)$$

Change in mass flow:

(1) − (2)

$$= \rho v_x \, dy\, dz - \rho\left(v_x + \frac{\partial v_x}{\partial x} dx\right) \cdot dy\, dz$$

$$= \cancel{\rho v_x \, dy\, dz} - \cancel{\rho v_x \, dy\, dz} - \rho \frac{\partial v_x}{\partial x} \cdot dx \, dy\, dz$$

$$\boxed{\Delta M_x = - \rho \frac{\partial v_x}{\partial x} \cdot dx\, dy\, dz}$$

$$\Delta M_y = - \rho \frac{\partial v_y}{\partial y} dx\, dy\, dz$$

$$\Delta M_z = - \rho \frac{\partial v_z}{\partial z} dx\, dy\, dz$$

If there is loss

The total change in mass due to flow $= \overbrace{\Delta M_x + \Delta M_y + \Delta M_z}^{\text{change}}$

$$= -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \underbrace{dx\,dy\,dz}$$

$$= -\rho\left(\underbrace{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}_{\text{scalar}}\right) \cdot dV$$

volume element

vector $\rightarrow \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

vector $\rightarrow \vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$

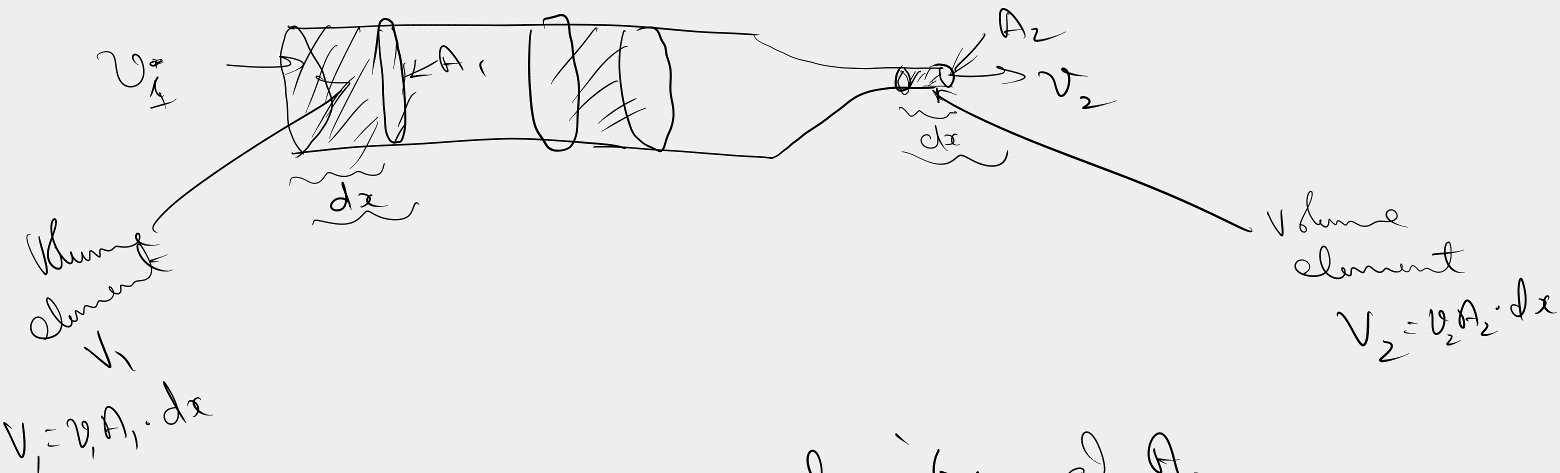
$$\vec{\nabla}\cdot\vec{v} = \text{Scalar}$$

Total change in "m" due to the flow $\Big\} = -\rho\, \underset{\text{volume}}{dV} \cdot \vec{\nabla}\cdot\underset{\text{velocity}}{\vec{v}}$

If we consider unit volume then $(dV = 1)$

$$= \underset{\text{loss}}{-}\rho\,\underbrace{\vec{\nabla}\cdot\vec{v}}$$

Loss of "m" due to flow $\Big\} = \underset{\text{constant}}{\rho}\, \vec{\nabla}\cdot\vec{v}$



$v_2 \leftarrow$ is the velocity at A_2

$v_1 \leftarrow$ " " at A_1

If $A_2 < A_1$, then $v_1 < v_2$, because the fluid is incompressible V_1 must be equal to V_2

$\therefore$ If $\vec{\nabla} \cdot \vec{v} = 0$, then the fluid is incompressible

$$\vec{\nabla} \cdot \vec{A} = 0$$

$\vec{A}$ is solenoidal.

Example 34. *If* $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of div* $\bar{v}$.

(U.P., I Semester, Winter 2000)

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{div } \vec{v} = \vec{\nabla} \cdot \vec{v}$$

$$\vec{\nabla} \cdot \vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \frac{\partial}{\partial x}\Big(\underset{u}{x} \cdot \underset{v}{(x^2+y^2+z^2)^{-1/2}}\Big) + \frac{\partial}{\partial y}\Big(y \cdot (x^2+y^2+z^2)^{-1/2}\Big)$$

$$+ \frac{\partial}{\partial z}\Big(z \cdot (x^2+y^2+z^2)^{-1/2}\Big)$$

$$\frac{\partial}{\partial x}\Big(\underset{u}{x}\underbrace{(x^2+y^2+z^2)^{-1/2}}_{v}\Big) = x \cdot \left(-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1} \cdot 2x\right) +$$

$$(x^2+y^2+z^2)^{-1/2} \cdot 1$$

$$\frac{\partial}{\partial x}(\quad '' \quad) = -x^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial y}(\ '' \ ) = -y^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial z}(\ '' \ ) = -z^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial}{\partial x}('') + \frac{\partial}{\partial y}('') + \frac{\partial}{\partial z}('') = 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{1-\frac{3}{2}}$$

$$= 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{-1/2}$$

$$= (x^2+y^2+z^2)^{-1/2}(3-1)$$

$$= \frac{2}{[x^2+y^2+z^2]^{1/2}}$$

Example 35. *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.*

(*A.M.I.E.T.E., Summer 2004*)

Scalar fn   Vector

Follow

$$u\vec{r} = (x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= (x^2+y^2+z^2)x\hat{i} + (x^2+y^2+z^2)y\hat{j} + (x^2+y^2+z^2)z\hat{k}$$

$$\vec{\nabla}\cdot(u\vec{r})$$

Example 36. *Find the value of n for which the vector* $r^n\vec{r}$ *is solenoidal, where* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$(x\hat{i}+y\hat{j}+z\hat{k})^n$$

$$r = |\vec{r}|$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$r^n\vec{r}$

$$\vec{\nabla}\cdot(r^n\vec{r}) = 0$$ since

Since solenoidal

v   u

$$r^n\vec{r} = (x^2+y^2+z^2)^{n/2}\cdot x\hat{i} + (x^2+y^2+z^2)^{n/2}\cdot y\hat{j} + (x^2+y^2+z^2)^{n/2} z\hat{k}$$

$$\vec{\nabla}\cdot(r^n\vec{r}) = nx^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$
$$+ny^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$
$$+nz^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + (x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (x^2+y^2+z^2)^1$$

$$+ 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1+1} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\boxed{\vec{\nabla}\cdot(r^n\vec{r}) = (n+3)(x^2+y^2+z^2)^{\frac{n}{2}}}$$

$r^n\vec{r}$ is solenoidal

$$\therefore \vec{\nabla}\cdot(r^n\vec{r}) = 0$$

$$\Rightarrow \underbrace{(n+3)}\underbrace{(x^2+y^2+z^2)^{\frac{n}{2}}} = 0$$

$$n+3 = 0 \Rightarrow \boxed{n=-3}$$

(or)

$$(x^2+y^2+z^2)^{\frac{n}{2}} = 0$$

Example 38. *Let* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$ *and* $\vec{a}$ *is a constant vector. Find the value of*

$$div\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$$

17.ω

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

① $$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

② $$\frac{\vec{a}\times\vec{r}}{r^n}$$

③ $$\vec{\nabla}\cdot\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$$

Example 39. *Find the directional derivative of div* $(\vec{u})$ *at the point (1, 2, 2) in the direction of the outer normal of the sphere* $x^2 + y^2 + z^2 = 9$ *for* $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.

$$\vec{\nabla}\cdot\vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}\right)$$

Let

$$\phi_1 = \vec{\nabla}\cdot\vec{u} = 4x^3 + 4y^3 + 4z^3$$

Direction Vector to ϕ_1

$$\vec{\nabla}\phi_1 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(4x^3 + 4y^3 + 4z^3\right)$$

$$= (4\cdot 3)\, x^2 \hat{i} + (4\cdot 3)\, y^2 \hat{j} + (4\cdot 3)\, z^2 \hat{k}$$

$$\vec{\nabla}\phi_1 = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$$

$$\vec{\nabla}\phi_1\Big|_{1,2,2} = 12(1)^2\hat{i} + 12(2)^2\hat{j} + 12(2)^2\hat{k}$$

$$\boxed{\vec{\nabla}\phi_1\Big|_{1,2,2} = 12\hat{i} + 48\hat{j} + 48\hat{k}}$$

$$\left.\begin{array}{l}\text{Outer normal to the}\\ \text{surface of spher}\end{array}\right\} = \vec{\nabla}\phi_2$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\therefore \vec{\nabla}\phi_2 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2-9\right)$$

$$\vec{\nabla}\phi_2 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{Normal @ } (1,2,2) = \vec{\nabla}\phi_2\Big|_{(1,2,2)}$$

$$= 2(1)\hat{i} + 2(2)\hat{j} + 2(2)\hat{k}$$

Let $\vec{d}$

$$\vec{d} = \vec{\nabla}\phi_2\Big|_p = 2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\therefore \hat{d} = \frac{\vec{\nabla}\phi_2}{|\vec{\nabla}\phi_2|} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{2^2+4^2+4^2}} \Rightarrow \frac{2(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{36}}$$

$$\hat{d} = \frac{1}{3}\,\hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore$ DD of ϕ_2 along $\vec{d} \Rightarrow \vec{\nabla}\phi_1 \cdot \hat{d}$

$$= (12\hat{i} + 48\hat{j} + 48\hat{k}) \cdot \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= \frac{12 + 96 + 96}{3}$$

$$= \frac{204}{3} \Rightarrow 68$$

Example 40. *Show that div (grad r^n) = $n(n+1)r^{n-2}$, where*

$$r = \sqrt{x^2 + y^2 + z^2}$$

Hence, show that $\Delta^2\left(\frac{1}{r}\right) = 0.$ (*U.P. I Semester, Dec. 2004, Winter 2002*)

∇^2

$$\vec{\nabla} r^n = n \cdot r^{n-2} \vec{r}.$$

$$\text{div}\cdot(\vec{\nabla} r^n) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(n r^{n-2} x \hat{i} + n r^{n-2} y \hat{j} + n r^{n-2} z \hat{k}\right)$$

$$= \underbrace{\frac{\partial}{\partial x}(\underset{u}{n r^{n-2}} \underset{v}{x})} + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z)$$

$r(x)$

$$= n r^{n-2} + x \cdot n(n-2) r^{(n-3)} \frac{\partial r}{\partial x} +$$

$$n r^{n-2} + y \cdot n(n-2) r^{n-3} \frac{\partial r}{\partial y} +$$

$$n r^{n-2} + z \cdot n(n-2) r^{n-3} \frac{\partial r}{\partial z}$$

$$|r| = (x^2 + y^2 + z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x$$

$$= \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{x}{r}$$

$$= n r^{n-2} + x\, n(n-2) r^{n-3} \cdot \frac{x}{r} +$$

$$n r^{n-2} + y(n-2)n \cdot r^{n-3} \cdot \frac{y}{r} +$$

$$n r^{n-2} + z\, n(n-2)\, r^{n-3} \frac{z}{r}$$

||| ^{ly} $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$$= 3n r^{n-2} + x^2 \cdot n(n-2) \cdot r^{n-4} + y^2 n(n-2) r^{n-4} + z^2 n(n-2) r^{n-4}$$

$$= 3n r^{n-2} + (n(n-2) r^{n-4})(x^2+y^2+z^2)$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} \cdot r^2$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= r^{n-2}(3n + n^2 - 2n)$$

$$= r^{n-2}(n^2 + n)$$

$$= r^{n-2} n(n+1)$$

$$\operatorname{div}(\operatorname{grad} r^n) = r^{n-2} n(n+1)$$

$$\operatorname{div}(\operatorname{grad} r^n) = \vec{\nabla} \cdot \vec{\nabla} r^n = \nabla^2 r^n$$

$$\boxed{\nabla^2 r^n = r^{n-2} \cdot n(n+1)} \quad \text{—— } (1a)$$

(ii) prove $\nabla^2\left[\frac{1}{r}\right] = 0$

$$\nabla^2\left[\frac{1}{r}\right] \Rightarrow \nabla^2 r^{-1} \quad \text{———} \quad (2a)$$

Comparing L.H.S of Eqn (1a) & (2a) we have

$$\nabla^2 r^n = \nabla^2 r^{-1}$$

$$\therefore \boxed{n = -1}$$

Sub:- $\boxed{n = -1}$ in eqn (1a) we have

$$\nabla^2 r^{-1} = \nabla^2\left[\frac{1}{r}\right] = r^{-1-2}(-1)(-1+1)$$

$$= 0$$

Hence proved.