Department of Medical Physics
Bharathidasan University

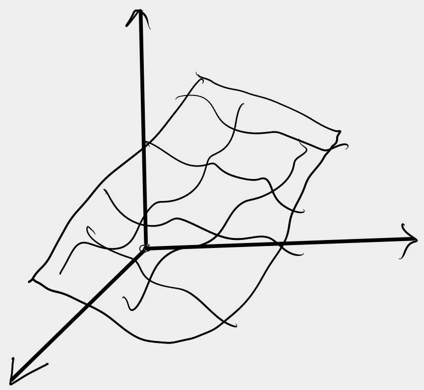
Mathematical Physics (MP101)

Date : **14-10-2020**

Attendance : **10**

→ <u>Gradient</u>, Divergence, Curl & Laplacian.

So our aim is to construct a surface $\phi_1(x, y, z) = C_1$



⇓

Example

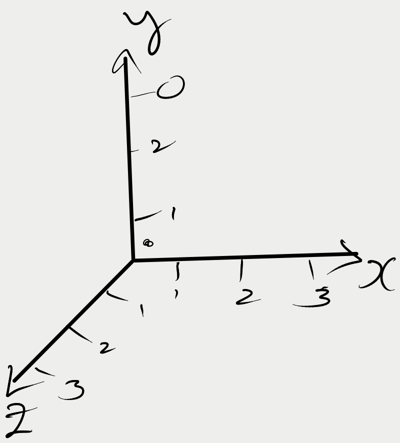
$$\phi_1(x, y, z) = x + y + z$$

Let the constant $C_1 = 3$ // example

$$\phi_1(x, y, z) = C_1$$

$$\Rightarrow \boxed{x + y + z = C_1}$$

$C_1 = 3$



x	y	z	C_1
1	1	1	
0	2	1	
2	0	1	
1	2	0	3
1.5	1.5	0	
1.75	0	1.25	
1.3	1.7	0	

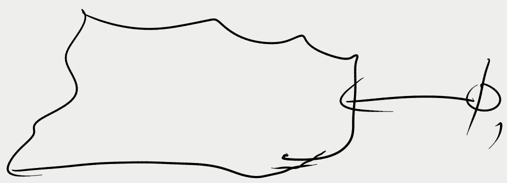
We use Gradient to study the properties of surface (s)

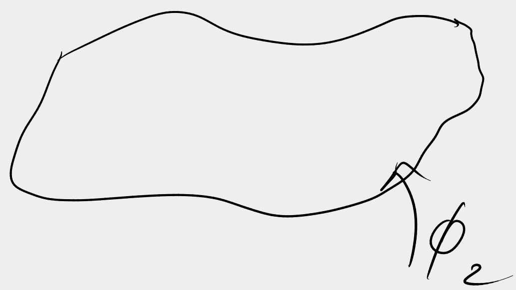
→ If we have two surface ϕ_1 & ϕ_2

→ ϕ_1 & ϕ_2 intersect?

→ Are ϕ_1 & ϕ_2 $||^{ell}$

→ Are $\phi_2 \perp^{lar} \phi_2$





$\phi_1 \Rightarrow \underbrace{\phi_1(x,y,z)} = C_1$

$\phi_2 \Rightarrow \underbrace{\phi_2(x,y,z)} = C_2$

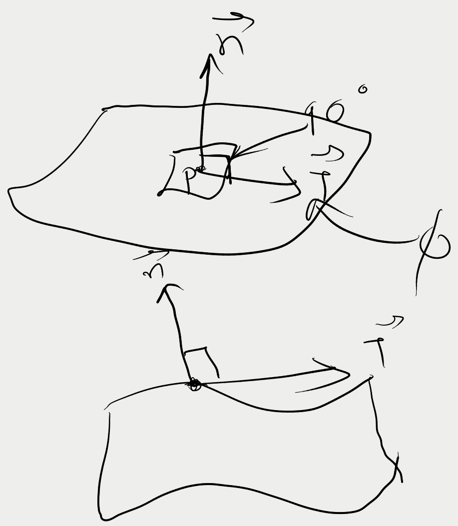
Concepts

(i) Tangent to surface @ somepoint $\underline{P}$

(ii) Normal to Surface " " P

$\phi_1 \Rightarrow \vec{n}_1$ & $\vec{T}_1$ ← Tangent

normal



$\phi_2 \Rightarrow \vec{n}_2$ & $\vec{T}_2$

$\phi_1 \perp \phi_2 \Rightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$

How to find normal to some surface ϕ?

This is where our "gradient" enters the picture.

Let $\phi(x,y,z)$ be some surface, then

W.K.T

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz$$

Can we relate

$$d\phi \rightsquigarrow \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Scalar

Vector

We want a scalar from a vector

dot product

$$\vec{\nabla}\phi \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$d\phi = \vec{\nabla}\phi \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

then we have

$$\underbrace{d\phi}_{\text{scalar}} = \underbrace{\vec{\nabla}\phi}_{\text{vector}} \cdot \underbrace{d\vec{r}}_{\text{vector}}$$

If $\phi(x, y, z) = C_1$, then $d\phi = 0$

for a surface w.r.t $\phi(x, y, z) = 0 \Rightarrow d\phi = 0$

$$\therefore \quad \vec{\nabla}\phi \cdot d\vec{r} = 0$$

$\vec{a} \cdot \vec{b} = 0$

dot

then

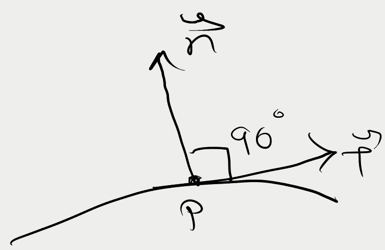
$\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$

So for our case

$$\vec{\nabla}\phi \perp d\vec{r} \in \text{surface}$$

Generally $d\vec{r} \longrightarrow$ tangent vector to the surface.

Any vector ⊥ar to the tangent vector is normal vector



So the conclusion is

$\vec{\nabla}\phi$ is a normal vector to the surface "ϕ" @ a point P.

$$\boxed{\vec{\nabla}\phi \text{ is the normal vector}}$$

If we have a surface $\phi(x, y, z) = c_1$, then

i.e. $\vec{\nabla}\phi$ is the normal to that surface.

Example 16. *If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point (1, –2, –1).*

$$\phi = 3x^2y - y^3z^2$$

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$\vec{\nabla}\phi = \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\frac{\partial\phi}{\partial x} = 6xy$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2$$

$$\frac{\partial\phi}{\partial z} = -2y^3z$$

$$\vec{\nabla}\phi\Big|_{\substack{(1,\,-2,\,-1)\\ x\quad y\quad z}} = \hat{i}(-12) + \hat{j}(-9) + \hat{k}(-16)$$

$$\boxed{\vec{\nabla}\phi_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k}}$$

Example 19. *Find the unit normal to the surface $xy^3z^2 = 4$ at (–1, –1, 2).*

point

$$\phi \Rightarrow xy^3z^2 = 4$$

$$\phi = xy^3z^2 - 4$$

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\underline{\underline{\vec{\nabla}\phi}} = y^3 z^2 \hat{i} + 3y^2 x z^2 \hat{j} + 2z x y^3 \hat{k}$$

$\uparrow$ normal vector

Normal @ $(-1, -1, 2)$

$$\vec{n} \rightarrow \boxed{\vec{\nabla}\phi\Big|_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}}$$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Unit vector of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

144
32

Unit Normal

unit normal $\rightarrow$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = \frac{4(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{16(1 + 9 + 1)}}$$

$$= \frac{4}{4}\frac{(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{11}}$$

$$= -\frac{1}{\sqrt{11}} \cdot (\hat{i} + 3\hat{j} - \hat{k})$$

$$\boxed{\hat{n} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})}$$

Directional Derivative : (DD)

DD of "ϕ" along any vector "$\vec{d}$" is the dot product of $\vec{\nabla}\phi$ and $\vec{d}$ in other words

$$DD \text{ of } \phi \Rightarrow \vec{\nabla}\phi \cdot \vec{d}$$

Example 18. *Find the directional derivative of $x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.*

Given $\phi = x^2y^2z^2$ $\qquad \vec{\nabla}\phi \cdot \vec{T} = DD$

$\vec{T} = \frac{d\vec{r}}{dt}$ $\qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{\nabla}\phi = 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

$$\underbrace{\vec{\nabla}\phi\Big|_{(1,1,-1)} = 2\hat{i} + 2\hat{j} - 2\hat{k}}$$

$\vec{r}\Big|_{(1,1,-1)}$ $\qquad x = e^t$

$y = \sin 2t + 1$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\qquad z = 1 - \cos t$

$\vec{r} = \underline{e^t\hat{i}} + (\sin 2t + 1)\hat{j} + (1 - \cos t)\hat{k}$

$\vec{T} = \frac{d\vec{r}}{dt} = e^t\hat{i} + 2\cos 2t\hat{j} + \sin t\hat{k}$ $\qquad \frac{de^t}{dt} = ?$

$\vec{T} = e^t\hat{i} + 2\cos 2t\hat{j} + \sin t\hat{k}$ $\qquad \frac{d}{dt}(e^t) = ?$

$$\vec{T}\Big|_{t=0} = e^{0}\hat{i} + 2\cos(0)\hat{j} + \sin(0)\hat{k}$$

$$\boxed{\vec{T}\Big|_{t=0} = \hat{i} + 2\hat{j}}$$

Directional Derivative of ϕ along $\vec{T}$

$$DD(\phi) \text{ along } \vec{T} = \vec{\nabla}\phi \cdot \vec{T}$$

$$= (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot (\hat{i} + 2\hat{j} + 0\hat{k})$$

$$= (2\cdot 1) + (2\cdot 2) - (2\cdot 0)$$

$$= 6$$

$$DD(\phi) \text{ along } \hat{T} = \vec{\nabla}\phi \cdot \hat{T} \qquad \hat{T} = \frac{\vec{T}}{|\vec{T}|}$$

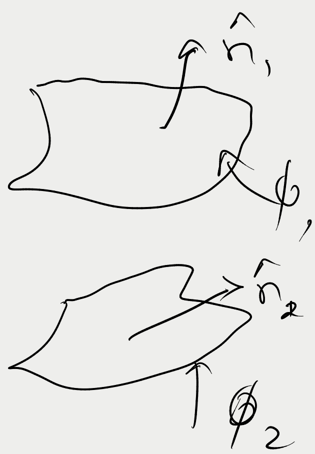
$$= 6/\sqrt{5}$$

Home work for (15-10-20)

Example 23. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (Nagpur University, Summer 2002)

$$\phi_1 = x^2 + y^2 + z^2 - 9$$

$$\phi_2 = x^2 + y^2 - z - 3$$



unit normal $\rightarrow$ $\hat{n}_1 \cdot \hat{n}_2 = n_1 n_2 \cos\theta$

$$\theta = \cos^{-1}\left(\frac{\hat{n}_1 \cdot \hat{n}_2}{n_1 n_2}\right)$$

$\phi_1 \xrightarrow[\text{vector}]{\text{normal}} \vec{\nabla}\phi_1 \xrightarrow{\text{unit normal}} \dfrac{\vec{\nabla}\phi_1}{|\vec{\nabla}\phi_1|}$

Answer: $\theta = \cos^{-1} \dfrac{8}{3\sqrt{21}}$