$$\nabla \times \frac{(\vec{a} \times \vec{r})}{r^n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{a_2 z - a_3 y}{r^n} & \frac{a_3 x - a_1 z}{r^n} & \frac{a_1 y - a_2 x}{r^n} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^n}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{a_1 y - a_2 x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{a_3 x - a_1 z}{r^n}\right) - \frac{\partial}{\partial y}\left(\frac{a_2 z - a_3 y}{r^n}\right)\right]$$

Now, $\quad r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\quad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore\ \nabla \times \left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \hat{i}\left[\left\{-nr^{-n-1}.\left(\frac{y}{r}\right)(a_1 y - a_2 x) + \frac{1}{r^n} a_1\right\}\right.$$
$$\left.- \left\{-nr^{-n-1}\left(\frac{z}{r}\right)(a_3 x - a_1 z) + \frac{1}{r^n}(-a_1)\right\}\right] + \text{two similar terms}$$
$$= \hat{i}\left[-\frac{n}{r^{n+2}}(a_1 y^2 - a_2 xy) + \frac{a_1}{r^n} + \frac{n}{r^{n+2}}(a_3 xz - a_1 z^2) + \frac{a_1}{r^n}\right]$$
$$+ \text{two similar terms}$$
$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{n}{r^{n+2}} a_1(y^2 + z^2) + \frac{n}{r^{n+2}}(a_2 xy + a_3 xz)\right] + \text{two similar terms}$$

Adding and subtracting $\frac{n}{r^{n+2}} a_1 x^2$ to third and from second term, we get

$$\vec{\nabla} \times \left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}(x^2 + y^2 + z^2) + \frac{n}{r^{n+2}}(a_1 x^2 + a_2 xy + a_3 xz)\right]$$
$$+ \text{two similar terms}$$
$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}} r^2 + \frac{n}{r^{n+2}} x(a_1 x + a_2 y + a_3 z)\right] + \text{two similar terms}$$
$$= \hat{i}\left[\frac{2a_1}{r^n} - \frac{na_1}{r^n} + \frac{n}{r^{n+2}} x(a_1 x + a_2 y + a_3 z)\right] + \hat{j}\left[\frac{2a_2}{r^n} - \frac{na_2}{r^n} + \frac{n}{r^{n+2}} y(a_2 y + a_3 z + a_1 x)\right]$$
$$+ \hat{k}\left[\frac{2a_3}{r^n} - \frac{na_3}{r^n} + \frac{n}{r^{n+2}} z(a_3 z + a_1 x + a_2 y)\right]$$
$$= \frac{2}{r^n}(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) - \frac{n}{r^n}(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) + \frac{n}{r^{n+2}}(a_1 x + a_2 y + a_3 z)(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{2-n}{r^n}(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) + \frac{n}{r^{n+2}}(a_1 x + a_2 y + a_3 z)(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{2-n}{r^n}\vec{a} + \frac{n}{r^{n+2}}(\vec{a}.\vec{r})\vec{r}$$

**Proved.**

**Example 63.** *If f and g are two scalar point functions, prove that*

*div* $(f\,\nabla g) = f\,\nabla^2 g + \nabla f\,\nabla g.$ *(U.P., I Semester, compartment, Winter 2001)*

**Solution.** We have, $\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$

$$\Rightarrow \quad f\,\nabla g = f\frac{\partial g}{\partial x}\hat{i} + f\frac{\partial g}{\partial y}\hat{j} + f\frac{\partial g}{\partial z}\hat{k}$$

$$\Rightarrow \quad \text{div } (f\,\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)g + \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right).\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right)$$

$= f\,\nabla^2 g + \nabla f.\nabla g$ **Proved.**

**Example 64.** *For a solenoidol vector* $\vec{F}$*, show that curl curl curl curl* $\vec{F} = \nabla^4 \vec{F}$.

(*M.D.U., Dec. 2009*)

**Solution.** Since vector $\vec{F}$ is solenoidal, so div $\vec{F} = 0$ ... (1)

We know that curl curl $\vec{F}$ = grad div ($\vec{F} - \nabla^2 \vec{F}$) ... (2)

Using (1) in (2), grad div $\vec{F}$ = grad (0) = 0 ... (3)

On putting the value of grad div F in (2), we get

curl curl $\vec{F} = -\nabla^2 \vec{F}$ ... (4)

Now, curl curl curl curl $\vec{F}$ = curl curl ($-\nabla^2 \vec{F}$) [Using (4)]

$= -$ curl curl $(\nabla^2 \vec{F}) = -[\text{grad div } (\nabla^2 \vec{F}) - \nabla^2 (\nabla^2 \vec{F})]$ [Using (2)]

$= -\text{grad } (\nabla . \nabla^2 \vec{F}) + \nabla^2 (\nabla^2 \vec{F}) = -\text{grad } (\nabla^2 \nabla . \vec{F}) + \nabla^4 \vec{F}$ $[\nabla . \vec{F} = 0]$

$= 0 + \nabla^4 \vec{F} = \nabla^4 \vec{F}$ [Using (1)] **Proved.**

## EXERCISE 5.9

1. Find the divergence and curl of the vector field $V = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$.

   **Ans.** Divergence $= 4x$, Curl $= (2y - x)\hat{i} + y\hat{j} + 4y\hat{k}$

2. If $a$ is constant vector and $r$ is the radius vector, prove that

   (*i*) $\nabla(\vec{a}.\vec{r}) = \vec{a}$ (*ii*) $\text{div}(\vec{r} \times \vec{a}) = 0$ (*iii*) $\text{curl}(\vec{r} \times \vec{a}) = -2\vec{a}$

   where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

3. Prove that:

   (*i*) $\nabla.(\phi A) = \nabla\phi.A + \phi(\nabla.A)$

   (*ii*) $\nabla(A.B) = (A.\nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$ (*R.G.P.V. Bhopal, June 2004*)

   (*iii*) $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$

4. If $F = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $F$.curl $F = 0$.

   (*R.G.P.V. Bhopal, Feb. 2006, June 2004*)

**Prove that**

5. $\vec{\nabla} \times (\phi\vec{F}) = (\vec{\nabla}\phi) \times \vec{F} + \phi(\vec{\nabla} \times \vec{F})$

6. $\nabla.(\vec{F} \times \vec{G}) = \vec{G}.(\nabla \times \vec{F}) - \vec{F}.(\nabla \times \vec{G})$

7. Evaluate div $(\vec{A} \times \vec{r})$ if curl $\vec{A} = 0$.

8. Prove that curl $(\vec{a} \times \vec{r}) = 2a$

9. Find div $\vec{F}$ and curl $F$ where $F$ = grad $(x^3 + y^3 + z^3 - 3xyz)$. (*R.G.P.V. Bhopal Dec. 2003*)

**Ans.** div $\vec{F} = 6(x + y + z)$, curl $\vec{F} = 0$

10. Find out values of $a, b, c$ for which $\vec{v} = (x + y + az)\hat{i} + (bx + 3y - z)\hat{j} + (3x + cy + z)\hat{k}$ is irrotational.

**Ans.** $a = 3,\ b = 1,\ c = -1$

11. Determine the constants $a, b, c$, so that $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. Hence find the scalar potential $\phi$ such that $\vec{F}$ = grad $\phi$.

(*R.G.P.V. Bhopal, Feb. 2005*) **Ans.** $a = 4,\ b = 2,\ c = 1$

Potential $\phi = \left(\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx\right)$

**Choose the correct alternative:**

12. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is

(*i*) $\frac{2}{3}$ (*ii*) $\frac{3}{2}$ (*iii*) 3 (*iv*) 6 (*A.M.I.E.T.E., Summer 2000*) **Ans.** (*iv*)

13. If $u = x^2 - y^2 + z^2$ and $\bar{V} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\nabla\ (u\bar{V})$ is equal to

(*i*) $5u$ (*ii*) $5|\vec{V}|$ (*iii*) $5(u - |\vec{V}|)$ (*iv*) $5(u - |\vec{V}|)$ (*A.M.I.E.T.E., June 2007*)

14. A unit normal to $x^2 + y^2 + z^2 = 5$ at $(0, 1, 2)$ is equal to

(*i*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} + \hat{k})$ (*ii*) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} - \hat{k})$ (*iii*) $\frac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$ (*iv*) $\frac{1}{\sqrt{5}}(\hat{i} - \hat{j} + \hat{k})$

(*A.M.I.E.T.E., Dec. 2008*)

15. The directional derivative of $\phi = x\ y\ z$ at the point $(1, 1, 1)$ in the direction $\hat{i}$ is:

(*i*) $-1$ (*ii*) $-\frac{1}{3}$ (*iii*) 1 (*iv*) $\frac{1}{3}$ **Ans.** (*iii*)

(*R.G.P.V. Bhopal, II Sem., June 2007*)

16. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ then $\nabla\phi\ (r)$ is:

(*i*) $\phi'\ (r)\ \vec{r}$ (*ii*) $\frac{\phi(r)\,\vec{r}}{r}$ (*iii*) $\frac{\phi'(r)\,\vec{r}}{r}$ (*iv*) None of these **Ans.** (*iii*)

(*R.G.P.V. Bhopal, II Semester, Feb. 2006*)

17. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then value of $\nabla(\log r)$ is (*U.P., I Sem, Dec 2008*)

(*i*) $\frac{\vec{r}}{r}$ (*ii*) $\frac{\vec{r}}{r^2}$ (*iii*) $-\frac{\vec{r}}{r^3}$ (*iv*) none of the above. **Ans.** (*ii*)

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = r$, then div $\vec{r}$ is:

(*i*) 2 (*ii*) 3 (*iii*) $-3$ (*iv*) $-2$ **Ans.** (*ii*)

(*R.G.P.V. Bhopal, II Semester, Feb. 2006*)

19. If $\vec{V} = xy^2\hat{i} + 2yx^2z\hat{j} - 3yz^2\hat{k}$ then curl $\vec{V}$ at point $(1, -1, 1)$ is

(*i*) $-(\hat{j} + 2\hat{k})$ (*ii*) $(\hat{i} + 3\hat{k})$ (*iii*) $-(\hat{i} + 2\hat{k})$ (*iv*) $(\hat{i} + 2\hat{j} + \hat{k})$

(*R.G.P.V. Bhopal, II Semester, Feb 2006*) **Ans.** (*iii*)

20. If $\vec{A}$ is such that $\nabla \times \vec{A} = 0$ then $\vec{A}$ is called

(*i*) Irrotational (*ii*) Solenoidal (*iii*) Rotational (*iv*) None of these

(*A.M.I.E.T.E., Dec. 2008*)

21. If $\vec{F}$ is a conservative force field, then the value of curl $\vec{F}$ is

(*i*) 0 (*ii*) 1 (*iii*) $\overline{\nabla F}$ (*iv*) $-1$ (*A.M.I.E.T.E., June 2007*)

**22.** If $\nabla^2$ $[(1 - x)(1 - 2x)]$ is equal to

(*i*) 2 (*ii*) 3 (*iii*) 4 (*iv*) 6 (*A.M.I.E.T.E., Dec. 2009*) **Ans.** (*iii*)

**23.** If $\vec{R} = xi + yj + zk$ and $\vec{A}$ is a constant vector, curl $(\vec{A} \times \vec{R})$ is equal to

(*i*) $\vec{R}$ (*ii*) $2\vec{R}$ (*iii*) $\vec{A}$ (*iv*) $2\vec{A}$ (*A.M.I.E.T.E., Dec. 2009*) **Ans.** (*iv*)

**24.** If $r$ is the distance of a point (x, y, z) from the origin, the value of the expression $\hat{j} \times \text{grad}\,\frac{1}{2}$ equals

(*i*) $(x^2+y^2+z^2)^{-\frac{3}{2}}(\hat{j}\,z - \hat{k}\,x)$ (*ii*) $(x^2+y^2+z^2)^{-\frac{3}{2}}(\hat{j}\,z - \hat{i}\,z)$

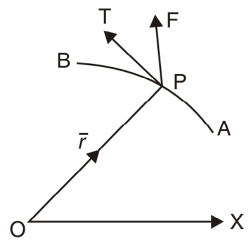
(*iii*) zero (*iv*) $(x^2+y^2+z^2)^{-\frac{3}{2}}(\hat{j}\,y - \hat{k}\,x)$

(*AMIETE, Dec. 2010*) **Ans.** (*ii*)

## 5.33 LINE INTEGRAL

Let $\vec{F}(x, y, z)$ be a vector function and a curve $AB$.

Line integral of a vector function $\vec{F}$ along the curve $AB$ is defined as integral of the component of $\vec{F}$ along the tangent to the curve $AB$.




Component of $\vec{F}$ along a tangent $PT$ at $P$

$$= \text{Dot product of } \vec{F} \text{ and unit vector along PT}$$

$$= \vec{F}\cdot\frac{d\vec{r}}{ds}\left(\frac{d\vec{r}}{ds}\text{ is a unit vector along tangent PT}\right)$$

$$\text{Line integral} = \sum \vec{F}\cdot\frac{d\vec{r}}{ds} \text{ from A to B along the curve}$$

$$\therefore \text{Line integral} = \int_c \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_c \vec{F}\cdot d\vec{r}$$

**Note (1) Work.** If $\overline{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F}\cdot d\vec{r}$

**(2) Circulation.** If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V}\cdot d\vec{r}$ is called the circulation of $V$ round the closed curve $c$.

If the circulation of V round every closed curve is zero then V is said to be irrotational there.

**(3)** When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

**Example 65.** *If a force $\vec{F} = 2x^2 y\hat{i} + 3\,xy\hat{j}$ displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve $y = 4\,x^2$. Find the work done.*

**Solution.** Work done $= \int_c \vec{F}.d\vec{r}$ $\left[\begin{array}{l}\vec{r} = x\hat{i} + y\hat{j}\\ d\vec{r} = dx\hat{i} + dy\,\hat{j}\end{array}\right]$

$$= \int_c (2\,x^2 y\,\hat{i} + 3\,xy\,\hat{j})\,.\,(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_c (2\,x^2 y\;dx + 3\,xy\;dy)$$

Putting the values of $y$ and $dy$, we get $\left(\begin{array}{l} y = 4x^2 \\ dy = 8x\,dx \end{array}\right)$

$$= \int_0^1 [2x^2 (4x^2)\,dx + 3x(4x^2)\,8x\,dx]$$

$$= 104\int_0^1 x^4\,dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$ **Ans.**

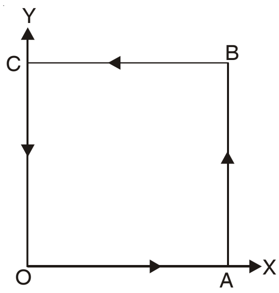
**Example 66.** *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *where* $\vec{F} = x^2\hat{i} + xy\hat{j}$ *and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

**Solution.** $\int_C \vec{F}\cdot\vec{d}r = \int_{OA}\vec{F}\cdot\vec{dr} + \int_{AB}\vec{F}\cdot\vec{dr} + \int_{BC}\vec{F}\cdot\vec{dr} + \int_{CO}\vec{F}\cdot\vec{dr}$

Here $\vec{r} = x\hat{i} + y\hat{j},\quad \vec{d}r = dx\hat{i} + dy\hat{j},\quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$

$$\vec{F}\cdot\vec{dr} = x^2dx + xydy \qquad ...(1)$$




On $OA$, $y = 0$ $\quad\therefore\ \vec{F}\cdot\vec{dr} = x^2dx$

$$\int_{OA}\vec{F}\cdot\vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ...(2)$$

On $AB$, $x = a$ $\quad\therefore\ dx = 0$

(1) becomes

$$\therefore\ \vec{F}\cdot\vec{dr} = aydy$$

$$\int_{Ab}\vec{F}\cdot\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \qquad ...(3)$$

On $BC$, $y = a$ $\quad\therefore\ dy = 0$

$\Rightarrow$ (1) becomes $\quad \vec{F}\cdot\vec{dr} = x^2dx$

$$\int_{BC}\vec{F}\cdot\vec{dr} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \qquad ...(4)$$

On $CO$, $x = 0$, $\quad\therefore\ \vec{F}\cdot\vec{dr} = 0$

(1) becomes

$$\int_{CO}\vec{F}\cdot\vec{dr} = 0 \qquad ...(5)$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F}\cdot\vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ **Ans.**

**Example 67.** *A vector field is given by*

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *along the path c is* $x = 2t$, $y = t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$.

(*Nagpur University, Winter 2003*)

**Solution.** $\int_C \vec{F}\cdot\vec{dr} = \int_C (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$

$$\left[\begin{array}{lll} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore\ \dfrac{dx}{dt} = 2 & \dfrac{dy}{dt} = 1 & \dfrac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t+3)(2\,dt) + (2t)(t^3)\,dt + (t^4 - 2t)(3t^2\,dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**

**Example 68.** *The acceleration of a particle at time t is given by*

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

*If the velocity $\vec{v}$ and displacement $\vec{r}$ be zero at t = 0, find $\vec{v}$ and $\vec{r}$ at any point t.*

**Solution.** Here, $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$

On integrating, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$$

$$\Rightarrow \quad \vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \vec{c} \qquad \text{...(1)}$$

At $t = 0$, $\vec{v} = \vec{0}$

Putting $t = 0$ and $\vec{v} = 0$ in (1), we get

$$\vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$$

$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

Again integrating, we have

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$$

$$\Rightarrow \quad \vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\,\hat{k} + \vec{c_1} \qquad \text{...(2)}$$

At, $t = 0$, $\vec{r} = 0$

Putting $t = 0$ and $\vec{r} = 0$ in (2), we get

$$\therefore \quad \vec{0} = -2\hat{i} + \vec{C_1} \quad \Rightarrow \quad \vec{C_1} = 2\hat{i}$$

Hence, $$\vec{r} = 2(1 - \cos 3t)\,\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}$$ **Ans.**

**Example 69.** *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\vec{dr}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*

$x = t,\ y = t^2,\ z = t^3.$ (*Uttarakhand, I Semester, Dec. 2006*)

**Solution.** We have,

$$\int_C \vec{A}.\vec{dr} = \int_C [(3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= \int_C [(3x^2 + 6y)\,dx - 14yzdy + 20xz^2dz]$$

If $x = t,\ y = t^2,\ z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

Now, $$\int_C \vec{A}.\vec{dr} = \int_{t=0}^{t=1} [(3t^2 + 6t^2)\,d(t) - 14t^2\,t^3\,d(t^2) + 20t\,(t^3)^2\,d(t^3)]$$

$$= \int_{t=0}^{t=1} [9t^2\,dt - 14t^5\,.\,2t\,dt + 20\,t^7\,.\,3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 \qquad = 3 - 4 + 6 = 5 \qquad \textbf{Ans.}$$

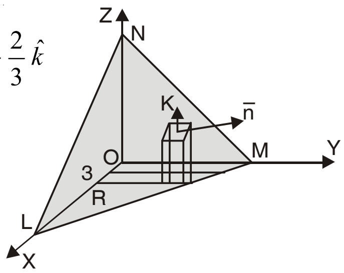
**Example 70.** *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\,ds$ *where* $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (*Nagpur University, Summer 2000*)

**Solution.** A vector normal to the surface "S" is given by

$$\nabla(2x + y + 2z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

And $\hat{n}$ = a unit vector normal to surface $S$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$




$$\hat{k}\cdot\hat{n} = \hat{k}\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \qquad \iint_S \overline{A}\cdot\hat{n}\,ds = \iint_R \overline{A}\cdot\hat{n}\,\frac{dx\,dy}{\hat{k}\cdot\overline{n}}$$

Where $R$ is the projection of $S$.

Now, $\vec{A}\cdot\hat{n} = [(x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}]\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

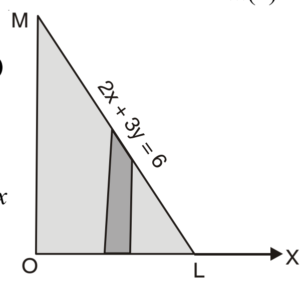
$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz \qquad ...(1)$$

Putting the value of $z$ in (1), we get

$$\vec{A}\cdot\hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right) \qquad \left(\because \text{ on the plane } 2x + y + 2z = 6,\; z = \frac{(6 - 2x - y)}{2}\right)$$

$$\vec{A}\cdot\hat{n} = \frac{2}{3}y\,(y + 6 - 2x - y) = \frac{4}{3}y\,(3 - x) \qquad ...(2)$$

Hence, $$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \overline{A}\cdot\overline{n}\,\frac{dx\,dy}{|\hat{k}\cdot\overline{n}|} \qquad ...(3)$$




Putting the value of $\vec{A}\cdot\hat{n}$ from (2) in (3), we get

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \frac{4}{3}y\,(3 - x)\cdot\frac{3}{2}\,dx\,dy = \int_0^3\int_0^{6-2x} 2y\,(3 - x)\,dy\,dx$$

$$= \int_0^3 2\,(3 - x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3 - x)\,(6 - 2x)^2\,dx = 4\int_0^3 (3 - x)^3\,dx$$

$$= 4.\left[\frac{(3 - x)^4}{4\,(-1)}\right]_0^3 = -(0 - 81) = 81 \qquad \textbf{Ans.}$$

**Example 71.** *Compute* $\int_c \vec{F}\cdot\vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*

**Solution.** $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z,\ d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\int_c \vec{F}\,.\,d\vec{r} = \int_c \frac{\hat{i}y - \hat{j}x}{x^2+y^2}\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \frac{ydx - xdy}{x^2+y^2} = \int_c (ydx - xdy) \qquad \ldots(1)\ [\because\ x^2+y^2=1]$$

Parametric equation of the circle are $x = \cos\theta$, $y = \sin\theta$.
Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\,\theta$, $dy = \cos\theta\, d\,\theta$ in (1), we get

$$\int_C \vec{F}\,d\,\vec{r} = \int_0^{2\pi} \sin\theta\,(-\sin\theta\, d\,\theta) - \cos\theta\,(\cos\theta\, d\,\theta)$$

$$= -\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\,d\,\theta = -\int_0^{2\pi} d\,\theta = -(\theta)_0^{2\pi} = -2\pi$$ **Ans.**

**Example 72.** *Show that the vector field* $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ *is conservative. Find its scalar potential and the work done in moving a particle from* (–1, 2, 1) to (2, 3, 4).
(*A.M.I.E.T.E. June 2010, 2009*)

**Solution.** Here, we have

$$\vec{F} = 2x(y^2+z^3)\,\hat{i} + 2x^2y\,\hat{j} + 3x^2z^2\hat{k}$$

Curl $\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix} = (0-0)i - (6xz^2 - 6xz^2)\hat{j} + (4xy - 4xy)\hat{k} = 0$$

Hence, vector field $\vec{F}$ is irrotational.
To find the scalar potential function $\phi$

$$\vec{F} = \vec{\nabla}\phi$$

$$d\,\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.\left(d\,\vec{r}\right) = \nabla\phi.d\,\vec{r} = \vec{F}\,.\,d\,\vec{r}$$

$$= \left[2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}\right](\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2x\,(y^2+z^3)\,dx + 2x^2y\,dy + 3x^2z^2dz$$

$$\phi = \int\left[2x(y^2+z^3)dx + 2x^2ydy + 3x^2z^2dz\right] + C$$

$$\int(2xy^2dx + 2x^2ydy) + (2xz^3dx + 3x^2z^2dz) + C = x^2y^2 + x^2z^3 + C$$

Hence, the scalar potential is $x^2y^2 + x^2z^3 + C$
Now, for conservative field

$$\text{Work done} = \int_{(-1,2,1)}^{(2,3,4)} \vec{F}\,.\,d\,\vec{r} = \int_{(-1,2,1)}^{(2,3,4)} d\phi = [\phi]_{(-1,2,1)}^{(2,3,4)} = \left[x^2y^2 + x^2z^3 + c\right]_{(-1,2,1)}^{(2,3,4)}$$

$$= (36 + 256) - (2 - 1) = 291$$ **Ans.**

**Example 73.** *A vector field is given by* $\vec{F} = (\sin y)\,\hat{i} + x(1+\cos y)\,\hat{j}$. *Evaluate the line integral over a circular path* $x^2 + y^2 = a^2$, $z = 0$. `(*Nagpur University, Winter 2001*)

**Solution.** We have,

$$\text{Work done} = \int_C \vec{F}\cdot\vec{dr}$$

$$= \int_C [(\sin y)\,\hat{i} + x(1+\cos y)\,\hat{j}]\cdot[dx\hat{i} + dy\hat{j}] \quad (\because\ z = 0 \text{ hence } dz = 0)$$

$$\Rightarrow \int_C \vec{F}\cdot\vec{dr} = \int_C \sin y\, dx + x(1+\cos y)\, dy = \int_C (\sin y\, dx + x\cos y\, dy + x\,dy)$$

$$= \int_C d(x\sin y) + \int_C x\, dy$$

(where $d$ is differential operator).

The parametric equations of given path

$x^2 + y^2 = a^2$ are $x = a\cos\theta$, $y = a\sin\theta$,

Where $\theta$ varies form 0 to $2\pi$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^{2\pi} d[\,a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a\cos\theta\,.\,a\cos\theta\, d\theta$$

$$= \int_0^{2\pi} d[a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a^2\cos^2\theta\,.\,d\theta$$

$$= [a\cos\theta\sin(a\sin\theta)\,]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta\, d\theta$$

$$= 0 + a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\,.\,2\pi = \pi a^2$$ **Ans.**

**Example 74.** *Determine whether the line integral* $\int (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$ *is independent of the path of integration ? If so, then evaluate it from (1, 0, 1) to* $\left(0, \frac{\pi}{2}, 1\right)$.

**Solution.** $\int_c (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

$$= \int_c [(2xyz^2\hat{i}) + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \vec{F}\cdot\vec{dr}$$

This integral is independent of path of integration if

$$\vec{F} = \nabla\phi \;\Rightarrow\; \nabla\times\vec{F} = 0$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz) = \hat{i} - (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$

$= 0$

Hence, the line integral is independent of path.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{(Total differentiation)}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \nabla\phi\cdot dr = \vec{F}\cdot\vec{d}\,r$$

$$= [(2xyz^2)\,\hat{i} + (x^2z^2 + z\cos y\,z)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2xyz^2\,dx + (x^2z^2 + z\cos y\,z)\,dy + (2x^2yz + y\cos yz)\,dz$$

$$= [(2x\,dx)\,yz^2 + x^2(dy)\,z^2 + x^2y\,(2z\,dz)] + [(\cos yz\,dy)\,z + (\cos yz\,dz)\,y]$$

$$= d\,(x^2yz^2) + d\,(\sin yz)$$

$$\phi = \int d\,(x^2yz^2) + \int d\,(\sin yz) = x^2yz^2 + \sin yz$$

$$[\phi]_A^B = \phi\,(B) - \phi\,(A)$$

$$= [x^2yz^2 + \sin yz]_{(0,\frac{\pi}{2},1)} - [x^2yz^2 + \sin yz]_{(1,0,1)} = \left[0 + \sin\left(\frac{\pi}{2}\times 1\right)\right] - [0+0]$$

$$= 1$$ **Ans.**

**Example 75.** *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\,d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

**Solution.** Here, $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $f(x, y, z) = 2x + 3y + 6z - 12$

Normal vector $= \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n}$ = unit normal vector at any point $(x, y, z)$ of $2x + 3y + 6z = 12$

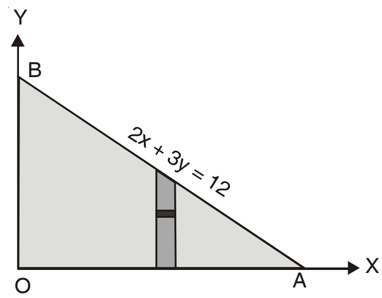
$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$dS = \frac{dx\,dy}{\hat{n}\,.\,\hat{k}} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\,.\,\hat{k}} = \frac{dxdy}{\frac{6}{7}} = \frac{7}{6}\,dx\,dy$$

Now, $\iint \vec{A}\,.\,\hat{n}\,dS = \iint (18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k})\,.\,\frac{1}{7}(2\,\hat{i} + 3\,\hat{j} + 6\,\hat{k})\,\frac{7}{6}\,dx\,dy$

$$= \iint (36z - 36 + 18y)\,\frac{dx\,dy}{6} \qquad = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get




$$= \int_0^6\int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$= \int_0^6\int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\,dx\,dy$$

$$= \int_0^6 (6 - 2x)\,dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\,dx\,(y)_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6 - 2x)\,\frac{1}{3}\,(12 - 2x)\,dx = \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx$$

$$= \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6 = \frac{1}{3}[4\times 36\times 2 - 18\times 36 + 72\times 6] = \frac{72}{3}[4 - 9 + 6] = 24$$ **Ans.**

## EXERCISE 5.10

1. Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $(\hat{i} + \hat{j})$. **Ans.** 1

2. Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$ moves a particle from origin to (1, 1) along a parabola $y^2 = x$. **Ans.** $\frac{2}{3}$

3. Show that $\vec{V} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find its scalar potential $\phi$ such that $\vec{V}$ = grad $\phi$. Find the work done by the force $\vec{V}$ in moving a particle from (1, – 2, 1) to (3, 1, 4). **Ans.** $x^2y + xz^3$, 202

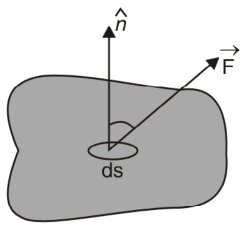
4. Show that the line integral $\int_c (2xy + 3)\,dx + (x^2 - 4z)\,dy - 4y\,dz$ where $c$ is any path joining (0, 0, 0) to (1, – 1, 3) does not depend on the path $c$ and evaluate the line integral. **Ans.** 14

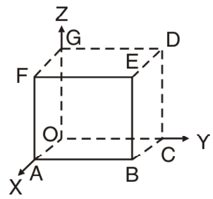
5. Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$, under the field of force given by $F = (2x - y + z)\ \hat{i}\ + (x + y - z^2)\ \hat{j}\ + (3x - 2y + 4z)\ \hat{k}$. Is the field of force conservative? (*A.M.I.E.T.E., Winter 2000*) **Ans.** 40 $\pi$

6. If $\vec{\nabla}\phi = (y^2 - 2xyz^3)\ \hat{i}\ + (3 + 2xy - x^2z^3)\ \hat{j}\ + (z^3 - 3x^2yz^2)\ \hat{k}$, find $\phi$. **Ans.** $3y + \frac{z^4}{4} + xy^2 - x^2yz^3$

7. $\int_C \vec{R}\,.\,d\vec{R}$ is independent of the path joining any two point if it is. (*A.M.I.E.T.E., June 2010*)
(*i*) irrotational field (*ii*) solenoidal field (*iii*) rotational field (*iv*) vector field. **Ans.** (*i*)

### 5.34 SURFACE INTEGRAL




A surface $r = f(u, v)$ is called smooth if $f\ (u, v)$ posses continous first order partial derivative.

Let $\vec{F}$ be a vector function and $S$ be the given surface.

Surface integral of a vector function $\vec{F}$ over the surface $S$ is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal

$= \vec{F}\,.\,\hat{n}$, where $n$ is the unit normal vector to an element $ds$ and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f\,|} \qquad ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

Surface integral of F over S

$$= \Sigma\,\vec{F}\cdot\hat{n} \qquad = \iint_S (\vec{F}\cdot\hat{n})\,ds$$

**Note.** (1) Flux = $\iint_S (\vec{F}\cdot\hat{n})\,ds$ where, $\overline{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F}\cdot\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a *solenoidal* vector point function.

**Example 76.** *Evaluate* $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\vec{ds}$ *where S is the surface of the sphere*

$x^2 + y^2 + z^2 = a^2$ *in the first octant.* (*U.P., I Semester, Dec. 2004*)

**Solution.** Here, $\phi = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Here, $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

$$\vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $$\iint_S F\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{k}\,.\,\hat{n}|} = \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xyz\,dx\,dy}{a\left(\dfrac{z}{a}\right)}$$

$$= 3\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\,(a^2 - x^2)\,dx = \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}. \qquad \textbf{Ans.}$$

**Example 77.** *Show that* $\iint_S \vec{F}\cdot\hat{n}\,ds = \dfrac{3}{2}$, *where* $\vec{F} = 4\,xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes,*

*$x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.*

| S.No. | Surface | Outward normal | $ds$ | |
|---|---|---|---|---|
| 1 | OABC | $-k$ | $dx\,dy$ | $z = 0$ |
| 2 | DEFG | $k$ | $dx\,dy$ | $z = 1$ |
| 3 | OAGF | $-j$ | $dx\,dz$ | $y = 0$ |
| 4 | BCED | $j$ | $dx\,dz$ | $y = 1$ |
| 5 | ABDG | $i$ | $dy\,dz$ | $x = 1$ |
| 6 | OCEF | $-i$ | $dy\,dz$ | $x = 0$ |

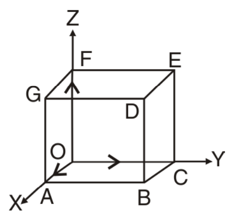
**Solution.** $$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC}\vec{F}\cdot\hat{n}\,ds + \iint_{DEFG}\vec{F}\cdot\hat{n}\,ds + \iint_{OAGF}\vec{F}\cdot\hat{n}\,ds + \iint_{BCED}\vec{F}\cdot\hat{n}\,ds + \iint_{ABDG}\vec{F}\cdot\hat{n}\,ds + \iint_{OCEF}\vec{F}\cdot\hat{n}\,ds \qquad ...(1)$$

Now, $\iint_{OABC}\vec{F}\cdot n\,ds = \iint_{OABC}(4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\,(-k)\,dx\,dy = \int_0^1\int_0^1 -yz\,dx\,dy = 0$ (as $z = 0$)

$$\iint_{DEFG}(4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot\hat{k}\,dx\,dy$$

$$= \iint_{DEFG} yz\,dx\,dy = \int_0^1\int_0^1 y\,(1)\,dx\,dy$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = [x]_0^1\,\frac{1}{2} = \frac{1}{2}$$




$\iint_{OAGF}(4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot(-j)\,dx\,dz = \iint_{OAGF} y^2\,dx\,dz = 0$ (as $y = 0$)

$$\iint_{BCED} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot\hat{j}\,dx\,dz = \iint_{BCED} (-y^2)\,dx\,dz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1\,(z)_0^1 = -1 \qquad \text{(as } y = 1)$$

$$\iint_{ABDG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}\,dy\,dz = \iint 4xz\,dy\,dz = \int_0^1\int_0^1 4\,(1)\,z\,dy\,dz$$

$$= 4\,(y)_0^1 \left(\frac{z^2}{2}\right)_0^1 = 4\,(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCEF} (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\,(-\hat{i})\,dy\,dz = \int_0^1\int_0^1 -4xz\,dy\,dz = 0 \qquad \text{(as } x = 0)$$

On putting these values in (1), we get

$$\iint_S \bar{F}\cdot\hat{n}\,ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$ **Proved.**

## EXERCISE 5.11

1. Evaluate $\iint_S \vec{A}.\hat{n}\,ds$, where $\vec{A} = (x+y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and $S$ is the surface of the plane $2x + y + 2z = 6$ in the first octant. **Ans.** 81

2. Evaluate $\iint_S \vec{A}.\hat{n}\,ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and $S$ is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. **Ans.** 90

3. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\,\hat{k}$ and $\vec{S} = 2t^2\hat{i} + 6t\hat{k}$, evaluate $\int_0^2 \vec{r}\cdot\vec{S}\,dt$. **Ans.** 12

4. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where, $\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\hat{k}$ and $S$ is the surface of the plane $2x + 3y + 6z = 12$ in the first octant. **Ans.** 24

5. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where, $F = 2yx\,\hat{i} - yz\hat{j} + x^2\hat{k}$ over the surface $S$ of the cube bounded by the coordinate planes and planes $x = a$, $y = a$ and $z = a$. **Ans.** $\frac{1}{2}a^4$

6. If $\vec{F} = 2y\hat{i} - 3\hat{j} + x^2\hat{k}$ and $S$ is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, and $z = 6$, then evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$. **Ans.** 132

## 5.35 VOLUME INTEGRAL

Let $\vec{F}$ be a vector point function and volume $V$ enclosed by a closed surface.

The volume integral $= \iiint_V \vec{F}\,dv$

**Example 78.** *If* $\vec{F} = 2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}$, *evaluate* $\iiint_V \vec{F}\,dv$ *where, v is the region bounded by the surfaces*

$$x = 0,\ \ y = 0,\ \ x = 2,\ \ y = 4,\ \ z = x^2,\ \ z = 2.$$

**Solution.** $\iiint_V \vec{F}\,dv = \iiint (2z\,\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz = \int_0^2 dx \int_0^4 dy\,[z^2\,\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\,[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k}]$$

$$= \int_0^2 dx \left[ 4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - \frac{x^2 y^2}{2}\hat{k} \right]_0^4$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\, dx$$

$$= \left[ 16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k} \right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$ **Ans.**

**EXERCISE 5.12**

**1.** If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \vec{F}\, dV$, where $V$ is bounded by the plane $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}$

**2.** Evaluate $\iiint_V \phi\, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0, y = 0, z = 0$ **Ans.** 128

**3.** If $\vec{F} = (2x^2 - 3z)\ \hat{i} - 2xy\ \hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \times \vec{F}\, dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}(\hat{j} - \hat{k})$

**4.** Evaluate $\iiint_V (2x + y)\, dV$, where $V$ is closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 2$ and $z = 0$. **Ans.** $\frac{80}{3}$

**5.** If $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, evaluate $\iiint \vec{F}\, dV$ over the region bounded by the surfaces $x = 0, y = 0, y = 6$ and $z = x^2, z = 4$. **Ans.** $(16\hat{i} - 3\hat{j} + 48\hat{k})$

## 5.36 GREEN'S THEOREM (For a plane)

**Statement.** *If* $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ *be continuous functions over a region R bounded by simple closed curve C in x – y plane, then*

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left( \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$ (*AMIETE, June 2010, U.P., I Semester, Dec. 2007*)

**Proof.** Let the curve $C$ be divided into two curves $C_1$ ($ABC$) and $C_1$ ($CDA$).
Let the equation of the curve $C_1$ ($ABC$) be $y = y_1(x)$ and equation of the curve $C_2$ ($CDA$) be $y = y_2(x)$.
Let us see the value of

$$\iint_R \frac{\partial \phi}{\partial y}\, dx\, dy = \int_{x=a}^{x=c} \left[ \int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial \phi}{\partial y}\, dy \right] dx = \int_a^c [\phi(x, y)]_{y=y_1(x)}^{y=y_2(x)}\, dx$$

$$= \int_a^c [\phi(x, y_2) - \phi(x, y_1)]\, dx = -\int_c^a \phi(x, y_2)\, dx - \int_a^c \phi(x, y_1)\, dx$$

$$= -\left[ \int_c^a \phi(x, y_2)\, dx + \int_a^c \phi(x, y_1)\, dx \right]$$

$$= -\left[ \int_{c2} \phi(x, y)\, dx + \int_{c1} \phi(x, y)\, dx \right] = -\oint_c \phi(x, y)\, dx$$

Thus, $$\oint_c \phi\, dx = -\iint_R \frac{\partial \phi}{\partial y}\, dx\, dy \qquad ...(1)$$

Similarly, it can be shown that

$$\oint_c \psi\, dy = \iint \frac{\partial \psi}{\partial x}\, dx\, dy \qquad ...(2)$$

On adding (1) and (2), we get

$$\oint (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy \quad \textbf{Proved.}$$

**Note.** Green's Theorem in vector form

$$\int_c \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dR$$

where, $\vec{F} = \phi\, \hat{i} + \psi\, \hat{j}, \bar{r} = x\hat{i} + y\hat{j}, \hat{k}$ is a unit vector along $z$-axis and $dR = dx\, dy$.

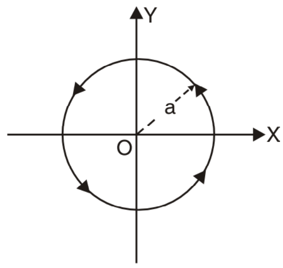
**Example 79.** *A vector field* $\vec{F}$ *is given by* $\vec{F} = \sin y\hat{i} + x\,(1 + \cos y)\,\hat{j}$.

*Evaluate the line integral* $\int_C \vec{F} \cdot \vec{dr}$ *where C is the circular path given by* $x^2 + y^2 = a^2$.

**Solution.** $\vec{F} = \sin y\hat{i} + x\,(1 + \cos y)\,\hat{j}$

$$\int_C \vec{F} \cdot \vec{dr} = \int_C [\sin y\hat{i} + x\,(1 + \cos y)\,\hat{j}] \cdot (\hat{i}dx + \hat{j}dy) = \int_C \sin y\, dx + x\,(1 + \cos y)\, dy$$

On applying Green's Theorem, we have




$$\oint_c (\phi\, dx + \psi\, dy) = \iint_s \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \iint_s [(1 + \cos y) - \cos y]\, dx\, dy$$

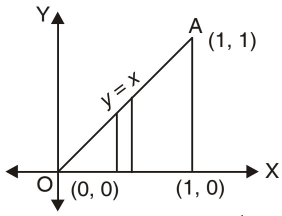
*where s is the circular plane surface of radius a.*

$$= \iint_s dx\, dy = \text{Area of circle} = \pi\, a^2. \quad \textbf{Ans.}$$

**Example 80.** *Using Green's Theorem, evaluate* $\int_c (x^2\ ydx + x^2\ dy)$, *where c is the boundary described counter clockwise of the triangle with vertices* (*0, 0*), (*1, 0*), (*1, 1*).

(*U.P., I Semester, Winter 2003*)

**Solution.** By Green's Theorem, we have




$$\int_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$\int_c (x^2 y\, dx + x^2\, dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy = \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$

$$= \int_0^1 (2x - x^2)\,(x)\, dx = \int_0^1 (2x^2 - x^3)\, dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12} \quad \textbf{Ans.}$$

**Example 81.** *State and verify Green's Theorem in the plane for* $\oint (3x^2 - 8y^2)\, dx + (4y - 6xy)$ *dy where C is the boundary of the region bounded by* $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$.

(*Uttarakhand, I Semester, Dec. 2006*)

**Solution. Statement:** See Article 24.4 on page 576.

Here the closed curve $C$ consists of straight lines $OB$, $BA$ and $AO$, where coordinates of $A$ and $B$ are (3, 0) and (0, – 2) respectively. Let $R$ be the region bounded by $C$.

Then by Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy \qquad ...(1)$$

$$= \iint_R (-6y + 16y)\, dx\, dy = \iint_R 10y\, dx\, dy$$

$$= 10\int_0^3 dx \int_{\frac{1}{3}(2x-6)}^{0} y\, dy = 10\int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^{0} = -\frac{5}{9}\int_0^3 dx\,(2x-6)^2$$

$$= -\frac{5}{9}\left[\frac{(2x-6)^3}{3\times 2}\right]_0^3 = -\frac{5}{54}(0+6)^3 = -\frac{5}{54}(216) = -20 \qquad ...(2)$$

Now we evaluate L.H.S. of (1) along $OB$, $BA$ and $AO$.

Along $OB$, $x = 0$, $dx = 0$ and $y$ varies form 0 to –2.

Along $BA$, $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2}\, dy$ and $y$ varies from –2 to 0.

and along $AO$, $y = 0$, $dy = 0$ and $x$ varies from 3 to 0.

L.H.S. of (1) $= \oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$

$$= \int_{OB} [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy] + \int_{BA} [(3x^2 - 8y^2)\, dx + (4x - 6xy)\, dy]$$
$$+ \int_{AO} [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \int_0^{-2} 4y\, dy + \int_{-2}^{0}\left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}\, dy\right) + [4y - 3(6+3y)\, y]\, dy + \int_3^0 3x^2\, dx$$

$$= [2y^2]_0^{-2} + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right] dy + (x^3)_3^0$$

$$= 2[4] + \int_{-2}^{0}\left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right] dy + (0 - 27)$$

$$= 8 + \left[\frac{9}{8}\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27 = -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2\right]$$

$$= -19 + 27 - 56 + 28 = -20 \qquad ...(3)$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

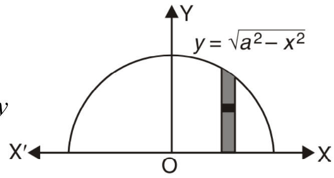
**Example 82.** *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.

(*M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004*)

**Solution.** $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$

By Green's Theorem, we've $\int_C (\phi\, dx + \psi\, dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$

$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}(2x^2-y^2)\right]dx\,dy$$




$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(2x+2y)\,dx\,dy \;=\; 2\int_{-a}^{a}dx\int_{0}^{\sqrt{a^2-x^2}}(x+y)\,dy$$

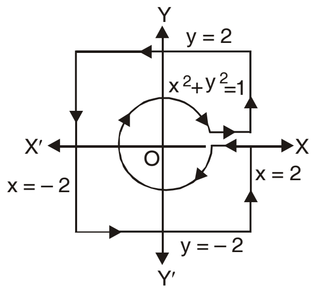
$$= 2\int_{-a}^{a}dx\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} = 2\int_{-a}^{a}\left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx$$

$$= 2\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx+\int_{-a}^{a}(a^2-x^2)\,dx \qquad \left[\begin{aligned}\int_{-a}^{a}f(x)\,dx &= 2\int_0^a f(x)\,dx, && f \text{ is even}\\ &=0, && f \text{ is odd}\end{aligned}\right]$$

$$= 0+2\int_0^a(a^2-x^2)\,dx \;=\; 2\left(a^2x-\frac{x^3}{3}\right)_0^a = 2\left(a^3-\frac{a^3}{3}\right) = \frac{4a^3}{3}$$ **Ans.**

**Example 83.** *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2=1$ *and* $C_2 : x = \pm 2, y = \pm 2.$ (*Gujarat, I Semester, Jan 2009*)

**Solution.** $\oint_C -\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$




$$= \iint\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2}+\frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)dx\,dy$$

$$= \iint\left[\frac{(x^2+y^2)1-2x\,(x)}{(x^2+y^2)^2}+\frac{(x^2+y^2)1-2y\,(y)}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}+\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint\left[\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}\right]dx\,dy \;=\; \iint\frac{0}{(x^2+y^2)^2}\,dx\,dy = 0$$ **Ans.**

## 5.37 AREA OF THE PLANE REGION BY GREEN'S THEOREM

**Proof.** We know that

$$\int_C M\,dx+N\,dy \;=\; \iint_A\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \qquad ...(1)$$

On putting $N = x\left(\frac{\partial N}{\partial x}=1\right)$ and $M = -y\left(\frac{\partial M}{\partial y}=1\right)$ in (1), we get

$$\int_C -y\,dx+x\,dy \;=\; \iint_A[1-(-1)]\,dx\,dy \;=\; 2\iint dx\,dy \;=2A$$

$$\text{Area} = \frac{1}{2}\int_C(x\,dy-y\,dx)$$

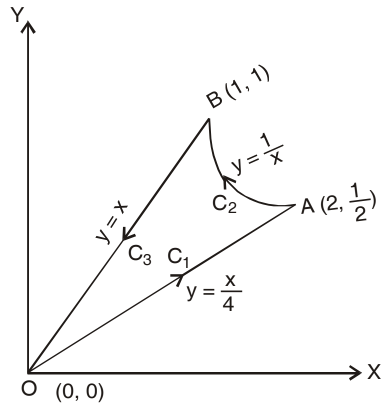
**Example 84.** *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\; y = \frac{1}{x},\; y = \frac{x}{4}$$ (*U.P. I, Semester, Dec. 2008*)

**Solution.** By Green's Theorem Area A of the region bounded by a closed curve $C$ is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

Here, $C$ consists of the curves $C_1 : y = \frac{x}{4}$, $C_2 : y = \frac{1}{x}$ and $C_3 : y = x$ So




$$\left[A = \frac{1}{2}\oint_C = \frac{1}{2}\left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] = \frac{1}{2}(I_1 + I_2 + I_3)\right]$$

Along $\quad C_1 : y = \frac{x}{4}, dy = \frac{1}{4}dx, x : 0 \text{ to } 2$

$$I_1 = \int_{C_1}(xdy - ydx) = \int_{C_1}\left(x\frac{1}{4}dx - \frac{x}{4}dx\right) = 0$$

Along $\quad C_2 : y = \frac{1}{x}, dy = -\frac{1}{x^2}dx, x : 2 \text{ to } 1$

$$I_2 = \int_{C_2}(xdy - ydx) = \int_2^1\left[x\left(-\frac{1}{x^2}\right)dx - \frac{1}{2}dx\right] = [-2\log x]_2^1 = 2\log 2$$

Along $\quad C_3 : y = x, dy = dx\ ;\ x : 1 \text{ to } 0\ ;$

$$I_3 = \int_{C_3}(xdy - ydx) = \int(xdx - xdx) = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{2}(0 + 2\log 2 + 0) = \log 2$$ **Ans.**

## EXERCISE 5.13

1. Evaluate $\int_c [(3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz]$ from (0, 0, 0) to (1, 1, 1) along the path $c$ given by the straight line from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (1, 1, 1).
2. Verify Green's Theorem in plane for $\int_C (x^2 + 2xy)\,dx + (y^2 + x^3y)\,dy$, where $c$ is a square with the vertices $P$ (0, 0), $Q$ (1, 0), $R$ (1, 1) and $S$ (0, 1). **Ans.** $-\frac{1}{2}$
3. Verify Green's Theorem for $\int_c (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$ around the boundary $c$ of the region $y^2 = 8x$ and $x = 2$.
4. Use Green's Theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where $c$ is the boundary in the $xy$-plane of the area enclosed by the $x$-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half $xy$-plane. **Ans.** $\frac{4}{3}$
5. Apply Green's Theoem to evaluate $\int_c [(y - \sin x)\,dy + \cos x\,dy]$, where $c$ is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$. **Ans.** $-\frac{\pi^2 + 8}{4\pi}$
6. Either directly or by Green's Therorem, evaluate the line integral $\int_c e^{-x}(\cos y\,dx - \sin y\,dy)$, where $c$ is the rectangle with vertices (0, 0), ($\pi$, 0,), $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. **Ans.** $2(1 - e^{-\pi})$
(*AMIETE II Sem June 2010*)
7. Verify the Green's Theorem to evaluate the line integral $\int_c (2y^2\,dx + 3x\,dy)$, where $c$ is the boundary of the closed region bounded by $y = x$ and $y = x^2$.
(*U.P., I Semester, Dec. 20005, AMIETE Summer 2004, Winter 2001*) **Ans.** $\frac{27}{4}$

**8.** Evaluate $\iint_s \bar{F}\,.\,\hat{n}.ds$, where $\vec{F} = x\,y\,\hat{i} - x^2\hat{j} + (x+z)\,\hat{k}$ and $s$ is the region of the plane $2x + 2y + z = 6$ in the first octant. (*A.M.I.E.T.E., Summer 2004, Winter 2001*) **Ans.** $\frac{27}{4}$

**9.** Verify Green's Theorem for $\int_C \left[(xy+y^2)\,dx + x^2dy\right]$ where $C$ is the boundary by $y = x$ and $y = x^2$. (*AMIETE, June 2010*)

## 5.38 STOKE'S THEOREM (Relation between Line Integral and Surface Integral)

(*Uttarakhand, I Sem. 2008, U.P., Ist Semester, Dec. 2006*)

**Statement.** Surface integral of the component of curl $\overline{F}$ along the normal to the surface $S$, taken over the surface $S$ bounded by curve $C$ is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve $C$.

Mathematically

$$\oint \vec{F}\,.\,d\,\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface $ds$,

**Proof.** Let

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$F = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

On putting the values of $\vec{F}, d\,\vec{r}$ in the statement of the theorem

$$\oint_c (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \iint_S \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (F_1\,\hat{i} + F_2\,\hat{j} + F3\,\hat{k}).\,(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\,ds$$

$$\oint (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right].$$
$$(\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma)\,ds$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]ds \quad ...(1)$$

Let us first prove

$$\oint_c F_1\,dx = \iint_S \left[\left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\right]ds \quad ...(2)$$

Let the equation of the surface $S$ be $z = g\,(x, y)$. The projection of the surface on $x - y$ plane is region $R$.

$$\oint_c F_1\,(x, y, z)\,dx = \oint_c F_1\,[x, y, g(x, y)]\,dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1\,(x, y, g)\,dx\,dy \qquad \text{[By Green's Theorem]}$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right)dx\,dy \quad ...(3)$$

The direction consines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{\frac{-\partial g}{\partial x}} = \frac{\cos\beta}{\frac{-\partial g}{\partial y}} = \frac{\cos\gamma}{1}$$

And $dx\,dy$ = projection of $ds$ on the $xy$-plane = $ds \cos\gamma$

Putting the values of $ds$ in R.H.S. of (2)

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds = \iint_R \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\frac{dx\,dy}{\cos\gamma}$$

$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\partial F_1}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right) dx\,dy \qquad \text{...(4)}$$

From (3) and (4), we get

$$\oint_c F_1\,dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) ds \qquad \text{...(5)}$$

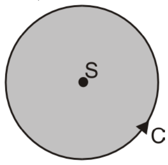
Similarly, $$\oint_c F_2\,dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right) ds \qquad \text{...(6)}$$

and $$\oint_c F_3\,dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds \qquad \text{...(7)}$$

On adding (5), (6) and (7), we get

$$\oint_c (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma + \frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha \right.$$
$$\left. + \frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right) ds$$ **Proved.**

## 5.39 ANOTHER METHOD OF PROVING STOKE'S THEOREM




*The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.*

$$\oint_c \overline{F}\cdot d\overline{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\,d\vec{s} = \iint_S curl\,\vec{F}\cdot d\vec{S}$$

**Proof :** The projection of any curved surface over $xy$-plane can be treated as kernal of the surface integral over actual surface

Now, $$\iint_S (\nabla\times\vec{F})\cdot\hat{k}\,d\vec{S} = \iint_S (\nabla\times\vec{F})\cdot(\vec{i}\times\vec{j})\,dx\,dy \qquad [\hat{k} = \hat{i}\times\hat{j}]$$

$$= \iint_S [(\nabla\cdot\hat{i})(\vec{F}\cdot\hat{j}) - (\nabla\cdot\hat{j})(\vec{F}\cdot\hat{i})]\,dx\,dy = \iint_S \left[\frac{\partial}{\partial x}(F_y) - \frac{\partial}{\partial y}(F_x)\right] dx\,dy$$

$$= \iint_S [F_x\,dx + F_y\,dy] \text{ [By Green's theorem]}$$

$$= \iint_S [\hat{i}\,F_x + \hat{j}\,F_y]\cdot(\hat{i}\,dx + \hat{j}\,dy) = \oint_c \vec{F}\,.\,\vec{dr}$$

$$\iint_S curl\,\vec{F}\cdot\hat{n}\,dS = \oint_c \vec{F}\,.\,\vec{d}\,r.$$

where, $\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$ and $\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

**Example 85.** *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$. (*M.D.U., Dec 2009*)

**Solution.** Here we have $\oint yz\,dx + zx\,dy + xy\,dz$

$$= \int (yz\hat{i} + zx\hat{j} + xy\hat{k}).(\hat{i}dx + \hat{j}dy + kdz)$$

$$= \oint F.dx$$

$$= \int curl\ F.\eta\ ds$$

$$\text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= (x - x)\ \hat{i}\ + (y - y)\ \hat{j}\ + (z - z)\ \hat{k}$$

$$= 0 \quad = 0$$ **Ans.**

**Example 86.** *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

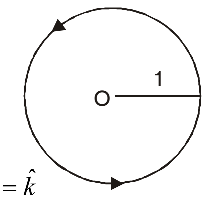
*where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.*
*(U.P., I Semester, Winter 2001)*

**Solution.** $\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$

$$= \int_c [(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

By Stoke's theorem $\oint \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\,ds$ ...(1)

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$




$$= (-2\,yz + 2\,yz)\,\hat{i} - (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$

Putting the value of curl $\vec{F}$ in (1), we get

$$= \iint \hat{k}\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint dx\,dy = \text{Area of the circle} = \pi \qquad \left[\because ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}\right]$$

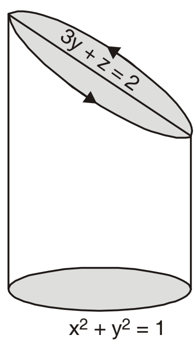
**Example 87.** *Evaluate* $\int_C \vec{F}\,.\,d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)*

**Solution.** $\oint_C \vec{F}\,.\,d\vec{r} = \iint_S \text{curl } \vec{F}\,.\,\hat{n}\,ds = \iint_S \text{curl } (-y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k})\,\hat{n}\,ds$ ...(1)

$$F(x, y, z) = -y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k}$$ *(By Stoke's Theorem)*

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - 0) + \hat{k}\,(1 + 2y) = (1 + 2y)\,\hat{k}$$




Normal vector $= \nabla\,\vec{F}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(y + z - 2) = \hat{j} + \hat{k}$$

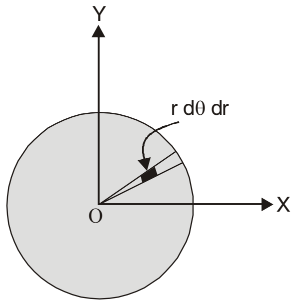
Unit normal vector $\hat{n} = \dfrac{\hat{j} + \hat{k}}{\sqrt{2}}$

$$ds = \frac{dx\,dy}{\hat{\eta}\,.\,\hat{k}}$$

On putting the values of curl $\vec{F}, \hat{n}$ and $ds$ in (1), we get

$$\int_C \vec{F}\,.\,\vec{dr} = \iint_S (1+2y)\,\hat{k}\,.\,\frac{\hat{j}+\hat{k}}{\sqrt{2}}\,\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right).\,\hat{k}}$$

$$= \iint \frac{1+2y}{\sqrt{2}}\,\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\,\theta\,d\,r$$




$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\,\theta\,d\,r$$

$$= \int_0^{2\pi} d\,\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\,\theta$$

$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi \qquad \textbf{Ans.}$$

**Example 88.** *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

*where c is the curve of intersection of* $x^2+y^2+z^2=a^2$ *and* $x+z=a$. (*Nagpur, Summer 2001*)

**Solution.** $\int_c (y\,dx + z\,dy + x\,dz)$

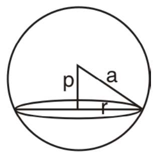
$$= \int_c (y\hat{i}+z\hat{j}+x\hat{k})\cdot(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz) = \int_C (y\hat{i}+z\hat{j}+x\hat{k})\cdot\vec{dr}$$

$$= \iint_S \text{curl}\,(y\hat{i}+z\hat{j}+x\hat{k})\cdot\hat{n}\,ds \qquad \text{(By Stoke's Theorem)}$$

$$= \iint_S \left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)\times(y\hat{i}+z\hat{j}+x\hat{k})\cdot\hat{n}\,ds = \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\hat{n}\,ds \quad ...(1)$$

*where S is the circle formed by the intersection of* $x^2+y^2+z^2=a^2$ and $x+z=a$.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)(x+z-a)}{|\nabla\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{1+1}}$$




$$\therefore \qquad \hat{n} = \frac{\hat{i}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}$$

Putting the vlaue of $\hat{n}$ in (1), we have

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}\right)ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds \qquad \left[\text{Use } r^2 = R^2-p^2 = a^2-\frac{a^2}{2}=\frac{a^2}{2}\right]$$

$$= \frac{-2}{\sqrt{2}}\iint_S ds = \frac{-2}{\sqrt{2}}\,\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}} \qquad \textbf{Ans.}$$

**Example 89.** *Directly or by Stoke's Theorem, evaluate* $\iint_s \text{curl}\,\vec{v}\cdot\hat{n}\,ds,\ \vec{v}=\hat{i}y+\hat{j}z+\hat{k}x,$ *s is the surface of the paraboloid* $z=1-x^2-y^2,\ z^3 \geq 0$ *and* $\hat{n}$ *is the unit vector normal to s.*

**Solution.**
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Obviously $\hat{n} = \hat{k}$.

Therefore $(\nabla \times \vec{v}) \cdot \hat{n} = (-\hat{i} - \hat{j} - \hat{k}).\, \hat{k} = -1$

Hence
$$\iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, ds = \iint_S (-1)\, dx\, dy = -\iint_S dx\, dy$$
$$= -\pi (1)^2 = -\pi. \qquad \text{(Area of circle} = \pi r^2\text{)} \ \textbf{Ans.}$$

**Example 90.** *Use Stoke's Theorem to evaluate* $\int_c \vec{v} \cdot \vec{dr}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

**Solution.** By Stoke's theorem

$$\int_c \vec{v} \cdot \vec{dr} = \iint_S (\text{curl } \vec{v}) \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, ds$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = (0-0)\,\hat{i} - (z-0)\,\hat{j} + (y - 2y)\,\hat{k} = -z\hat{j} - y\hat{k}$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)}{|\nabla \phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}$$

$$(\nabla \times \vec{v}) \cdot \hat{n} = (-z\hat{j} - y\hat{k}) \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3} = \frac{-yz - yz}{3} = \frac{-2yz}{3}$$

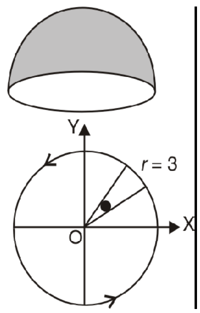
$$\hat{n} \cdot \hat{k}\, ds = dx\, dy \Rightarrow \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3} . \hat{k}\ dx = dx\, dy \Rightarrow \frac{z}{3} ds = dx\, dy$$

$$\therefore \qquad ds = \frac{3}{z}\, dx\, dy$$

$$\iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, ds = \iint \left(\frac{-2yz}{3}\right)\left(\frac{3}{z}\, dx\, dy\right) = -\iint 2y\, dx\, dy$$

$$= -\iint 2r \sin\theta\, r\, d\theta\, dr = -2 \int_0^{2\pi} \sin\theta\, d\theta \int_0^3 r^2\, dr$$

$$= -2\,(-\cos\theta)_0^{2\pi} \cdot \left[\frac{r^3}{3}\right]_0^3 = -2\,(-1+1)\, 9 = 0 \quad \textbf{Ans.}$$




**Example 91.** *Evaluate the surface integral* $\iint_S \text{curl } \vec{F} . \hat{n}\, dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$ and $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$. (*K. University, Dec. 2008*)

**Solution.** $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Obviously $\hat{n} = \hat{k}.$

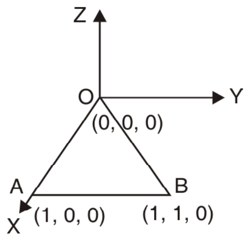
Therefore $(\nabla \times \vec{F}) \cdot \hat{n} = (-\hat{i} - \hat{j} - \hat{k}).\, \hat{k} = -1$

Hence $$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (-1)\, dx\, dy = -\iint_S dx\, dy$$
$$= -\pi\, (1)^2 = -\pi.$$ (Area of circle $= \pi r^2$) **Ans.**

**Example 92.** *Evaluate* $\oint_C \vec{F} \cdot \vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = y^2\, \hat{i} + x^2\, \hat{j} - (x+z)\, \hat{k}$ *and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*
(*U.P., I Semester, Winter 2000*)

**Solution.** We have, curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0.\, \hat{i} + \hat{j}\, 2\,(x-y)\, \hat{k}.$$




We observe that $z$ co-ordinate of each vertex of the triangle is zero. Therefore, the triangle lies in the $xy$-plane.

$\therefore \quad \hat{n} = \hat{k}$

$\therefore \quad$ curl $\vec{F} \cdot \hat{n} = [\hat{j} + 2\,(x-y)\hat{k}]\,.\,\hat{k} = 2\,(x-y).$

In the figure, only $xy$-plane is considered.
The equation of the line OB is $y = x$
By Stoke's theorem, we have

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\text{curl}\, \vec{F} \cdot \hat{n})\, ds$$
$$= \int_{x=0}^{1} \int_{y=0}^{x} 2\,(x-y)\, dx\, dy = 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$$
$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2}\right] dx = 2\int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$ **Ans.**

**Example 93.** *Evaluate* $\oint_C \vec{F} \cdot \vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = (x^2 + y^2)\, \hat{i} - 2\, xy\, \hat{j}$ *and C is the boundary of the rectangle* $x = \pm\, a,\ y = 0$ *and* $y = b$. (*U.P., I Semester, Winter 2002*)

**Solution.** Since the $z$ co-ordinate of each vertex of the given rectangle is zero, hence the given rectangle must lie in the $xy$-plane.

Here, the co-ordinates of $A, B, C$ and $D$ are $(a, 0), (a, b), (-a, b)$ and $(-a, 0)$ respectively.

$$\therefore \quad \text{Curl}\ \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = -4\, y\, k$$

Here, $\hat{n} = \hat{k}$, so by Stoke's theorem, we've

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\, \overline{F} \cdot \hat{n}\, ds$$

$$= \iint_S (-4y\hat{k}) \cdot (\hat{k})\, dx\, dy = -4 \int_{x=-a}^{a} \int_{y=0}^{b} y\, dx\, dy$$

$$= -4 \int_{-a}^{a} \left[\frac{y^2}{2}\right]_0^b dx = -2b^2 \int_{-a}^{a} dx = -4ab^2$$ **Ans.**

**Example 94.** *Apply Stoke's Theorem to calculate* $\int_c 4y\, dx + 2z\, dy + 6y\, dz$ *where c is the curve of intersection of* $x^2 + y^2 + z^2 = 6z$ *and* $z = x + 3$.

**Solution.**

$$\int_c \vec{F} \cdot \vec{dr} = \int_c 4y\, dx + 2z\, dy + 6y\, dz$$

$$= \int_c (4y\hat{i} + 2z\hat{j} + 6y\hat{k}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\vec{F} = 4y\hat{i} + 2z\hat{j} + 6y\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix} = (6-2)\,\hat{i} - (0-0)\,\hat{j} + (0-4)\,\hat{k} = 4\hat{i} - 4\hat{k}$$

S is the surface of the circle $x^2 + y^2 + z^2 = 6z$, $z = x + 3$, $\hat{n}$ is normal to the plane $x - z + 3 = 0$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x - z + 3)}{|\nabla \phi|} = \frac{\hat{i} - \hat{k}}{\sqrt{1+1}} = \frac{\hat{i} - \hat{k}}{\sqrt{2}}$$

$$(\nabla \times F) \cdot \hat{n} = (4\hat{i} - 4\hat{k}) \cdot \frac{\hat{i} - \hat{k}}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = 4\sqrt{2}$$

$$\int_c \vec{F} \cdot \vec{dr} = \iint_S (\text{curl } F) \cdot \hat{n}\, ds = \iint_S 4\sqrt{2}\,(dx\, dz) = 4\sqrt{2}\ (\text{area of circle})$$

Centre of the sphere $x^2 + y^2 + (z-3)^2 = 9$, $(0, 0, 3)$ lies on the plane $z = x + 3$. It means that the given circle is a great circle of sphere, where radius of the circle is equal to the radius of the sphere.

Radius of circle $= 3$, Area $= \pi\,(3)^2 = 9\pi$

$$\iint_S (\nabla \times F) \cdot \hat{n}\, ds = 4\sqrt{2}(9\pi) = 36\sqrt{2}\,\pi$$ **Ans.**

**Example 95.** *Verify Stoke's Theorem for the function* $\overline{F} = z\hat{i} + x\hat{j} + y\hat{k}$, *where C is the unit circle in xy-plane bounding the hemisphere* $z = \sqrt{(1 - x^2 - y^2)}$. *(U.P., I Semester Comp. 2002)*

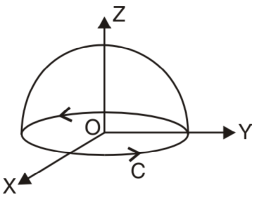
**Solution.** Here $\overline{F} = z\hat{i} + x\hat{j} + y\hat{k}$. ...(1)

Also, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \overline{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

$\therefore$ $\overline{F} \cdot \overline{dr} = z\, dx + x\, dy + y\, dz$.

$\therefore$ $\oint_C \overline{F} \cdot \overline{dr} = \oint_C (z\, dx + x\, dy + y\, dz)$. ...(2)

On the circle $C$, $x^2 + y^2 = 1$, $z = 0$ on the $xy$-plane. Hence on $C$, we have $z = 0$ so that $dz = 0$. Hence (2) reduces to




$$\oint_C \overline{F} \cdot \overline{dr} = \oint_C x\, dy. \qquad \text{...(3)}$$

Now the parametric equations of $C$, *i.e.*, $x^2 + y^2 = 1$ are

$$x = \cos\phi,\ y = \sin\phi. \qquad \text{...(4)}$$

Using (4), (3) reduces to $\oint_C \vec{F} \cdot \vec{dr} = \int_{\phi=0}^{2\pi} \cos\phi \cos\phi\, d\phi = \int_0^{2\pi} \frac{1+\cos 2\phi}{2}\, d\phi$

$$= \frac{1}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi} = \pi \qquad \text{...(5)}$$

Let $P(x, y, z)$ be any point on the surface of the hemisphere $x^2 + y^2 + z^2 = 1$, O origin is the centre of the sphere.

Radius $= \text{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ Normal $= x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = x\hat{i} + y\hat{j} + z\hat{k}$$

(Radius is $\perp$ to tangent *i.e.* Radius is normal)

$$x = \sin\theta\cos\phi,\quad y = \sin\theta\sin\phi,\quad z = \cos\theta \qquad \text{...(6)}$$

$$\hat{n} = \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k}$$

Also,
$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k} \qquad \text{...(7)}$$

$$\text{Curl } \vec{F} \cdot \hat{n} = (\hat{i} + \hat{j} + \hat{k}) \cdot (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \sin\theta\,\hat{k})$$
$$= \sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta$$

$$\therefore \qquad \iint_s \text{Curl } \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\hat{i} + \hat{j} + \hat{k})$$
$$.(\sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k}) \sin\theta\, d\theta\, d\phi$$
$$= \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\, d\phi$$

[$\because$ $dS$ = Elementary area on hemisphere = $\sin\theta\, d\theta\, d\phi$]

$$= \int_0^{\pi/2} \sin\theta\, d\theta\, [\sin\theta\sin\phi + \sin\theta(-\cos\phi) + \phi\cos\theta]_0^{2\pi} = \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \int_0^{\pi/2} (0 + 0 + 2\pi\sin\theta\cos\theta)\, d\theta = \pi\int_0^{\pi/2} \sin 2\theta\, d\theta = \pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -(\pi/2)\,[-1 - 1] = \pi.$$

From (5) and (8), $\oint_C \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{dr}} = \iint_S \text{curl } \overline{F} \cdot \hat{n}\, dS$, which verifies Stokes's theorem.

**Example 96.** *Verify Stoke's theorem for the vector field* $\overline{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$ *over the upper half of the surface* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on xy- plane.*

(*Nagpur University, Summer 2001*)

**Solution.** Let $S$ be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary $C$ or $S$ is a circle in the $xy$ plane of radius unity and centre $O$. The equation of C are $x^2 + y^2 = 1$, $z = 0$ whose parametric form is

$$x = \cos t,\ y = \sin t,\ z = 0,\ 0 < t < 2\pi$$

$$\int_C \vec{F} \cdot \vec{dr} = \int_C [(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}] . [\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz]$$

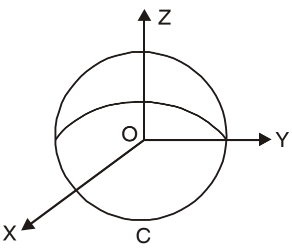
$$= \int_C [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz] = \int_C (2x - y)\,dx, \text{ since on } C,\ z = 0 \text{ and } 2z = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt$$

$$= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\,dt = \int_0^{2\pi}\left(-\sin 2t + \frac{1 - \cos 2t}{2}\right)dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi \qquad \text{...(1)}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\,\hat{i} + (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$




$$\text{Curl } \vec{F}.\hat{n} = \hat{k}.\hat{n} = \hat{n}.\hat{k}$$

$$\iint_S \text{Curl } \vec{F}.\hat{n}\,ds = \iint_S \hat{n}.\hat{k}\,ds = \iint_R \hat{n}.\hat{k}.\frac{dx\,dy}{\hat{n}.\hat{k}}$$

Where $R$ is the projection of $S$ on $xy$-plane.

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy = \int_{-1}^{1} 2\sqrt{1 - x^2}\,dx = 4\int_0^1 \sqrt{1 - x^2}\,dx$$

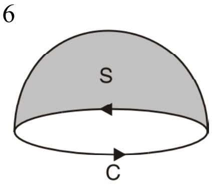
$$= 4\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = 4\left[\frac{1}{2}.\frac{\pi}{2}\right] = \pi \qquad \text{...(2)}$$

From (1) and (2), we have

$$\therefore \quad \int_C \vec{F}.\vec{dr} = \iint \text{Curl } \overline{F}.\hat{n}\,ds \text{ which is the Stoke's theorem.}$$ **Ans.**

**Example 97.** *Verify Stoke's Theorem for* $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

**Solution.** $\int_c \vec{F}\cdot\vec{dr}$, where $c$ is the boundary of the circle $x^2 + y^2 + z^2 = 16$ (bounding the hemispherical surface)




$$= \int_c [(x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}]\cdot(\hat{i}dx + \hat{j}dy)$$

$$= \int_c [(x^2 + y - 4)\,dx + 3\,xy\,dy)]$$

Putting $x = 4\cos\theta$, $y = 4\sin\theta$, $dx = -4\sin\theta\,d\theta$, $dy = 4\cos\theta\,d\theta$

$$= \int_0^{2\pi} [(16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\,d\theta) + (192\sin\theta\cos^2\theta\,d\theta)]$$

$$= 16\int_0^{2\pi} [-4\cos^2\theta\sin\theta - \sin^2\theta + \sin\theta + 12\sin\theta\cos^2\theta]\,d\theta$$

$$= 16\int_0^{2\pi} (8\sin\theta\cos^2\theta - \sin^2\theta + \sin\theta)\,d\theta$$

$$= -16\int_0^{2\pi} \sin^2\theta\,d\theta \qquad \left\{\begin{array}{l} \int_0^{2\pi} \sin^n\theta\cos\theta\,d\theta = 0 \\ \int_0^{2\pi} \cos^n\theta\sin\theta\,d\theta = 0 \end{array}\right\}$$

$$= -16 \times 4\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = -64\left(\frac{1}{2}\frac{\pi}{2}\right) = -16\,\pi.$$

To evaluate surface integral
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= (0-0)\,\hat{i} - (2z-0)\,\hat{j} + (3y-1)\,\hat{k} = -2z\hat{j} + (3y-1)\,\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(x^2+y^2+z^2-16)}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4}$$

$$(\nabla\times\vec{F})\cdot\hat{n} = [-2z\hat{j} + (3y-1)\,\hat{k}]\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{4} = \frac{-2yz+(3y-1)z}{4}$$

$$\hat{k}\cdot\hat{n}\cdot ds = dx\,dy \;\Rightarrow\; \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4}\,.\,k\,ds = dx\,dy \;\Rightarrow\; \frac{z}{4}ds = dx\,dy$$

$$\therefore \qquad ds = \frac{4}{z}\,dx\,dy$$

$$\iint(\nabla\times F)\cdot\hat{n}\,ds = \iint\frac{-2yz+(3y-1)z}{4}\left(\frac{4}{z}\,dx\,dy\right) = \iint[-2y+(3y-1)]\,dx\,dy = \iint(y-1)\,dx\,dy$$

On putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$, we get

$$= \iint(r\sin\theta - 1)\,r\,d\theta\,dr = \int d\theta\int(r^2\sin\theta - r)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{r^3}{3}\sin\theta - \frac{r^2}{2}\right)_0^4 = \int_0^{2\pi} d\theta\left(\frac{64}{3}\sin\theta - 8\right)$$

$$= \left(-\frac{64}{3}\cos\theta - 8\theta\right)_0^{2\pi} = \frac{-64}{3} - 16\pi + \frac{64}{3} = -16\pi$$

The line integral is equal to the surface integral, hence Stoke's Theorem is verified. **Proved.**

**Example 98.** *Verify Stoke's theorem for a vector field defined by* $\vec{F} = (x^2-y^2)\,\hat{i} + 2xy\,\hat{j}$ *in the rectangular in xy-plane bounded by lines* $x = 0$, $x = a$, $y = 0$, $y = b$.

(*Nagpur University, Summer 2000*)

**Solution.** Here we have to verify Stoke's theorem $\int_C \vec{F}\,.\,\vec{dr} = \iint_S (\nabla\times\bar{F})\,.\,\hat{n}\,ds$

Where 'C' be the boundary of rectangle (*ABCD*) and *S* be the surface enclosed by curve *C*.

$$\vec{F} = (x^2-y^2)\,\hat{i} + (2xy)\,\hat{j}$$

$$\vec{F}\,.\,\vec{dr} = [(x^2-y^2)\,\hat{i} + 2xy\,\hat{j}]\,.\,[\hat{i}\,dx + \hat{j}\,dy]$$

$$\Rightarrow \qquad \vec{F}\,.\,\vec{dr} = (x^2+y^2)\,dx + 2xy\,dy \qquad ...(1)$$

$$\text{Now,} \qquad \int_C \vec{F}\,.\,\vec{dr} = \int_{OA} F\,.\,\vec{dr} + \int_{AB}\vec{F}\,.\,\vec{dr} + \int_{BC}\vec{F}\,.\,\vec{dr} + \int_{CO}\vec{F}\,.\,\vec{dr} \qquad ...(2)$$

Along *OA*, put $y = 0$ so that $k\,dy = 0$ in (1) and $\vec{F}\,.\,\vec{d}r = x^2\,dx$,
Where $x$ is from 0 to $a$.

$$\therefore \qquad \int_{OA}\vec{F}\,.\,\vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ...(3)$$

Along *AB*, put $x = a$ so that $dx = 0$ in (1), we get $\vec{F}\,.\,\vec{d}r = 2ay\,dy$
Where $y$ is from 0 to $b$.

$$\therefore \qquad \int_{AB}\vec{F}\,.\,\vec{d}r = \int_0^b 2ay\,dy = [ay^2]_0^b = ab^2 \qquad ...(4)$$

Along $BC$, put $y = b$ and $dy = 0$ in (1) we get $\bar{F}\,.\,\bar{dr} = (x^2 - b^2)\,dx$, where $x$ is from $a$ to 0.

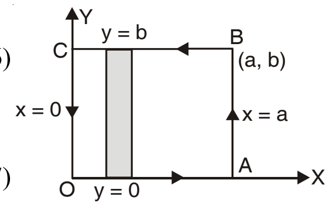
$$\therefore \qquad \int_{BC} \vec{F}\,.\,\vec{dr} = \int_a^0 (x^2 - b^2)\,dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = \frac{-a^3}{3} + b^2 a \qquad \text{...(5)}$$

Along CO, put $x = 0$ and $dx = 0$ in (1), we get $\vec{F}\,.\,\vec{dr} = 0$

$$\therefore \qquad \int_{CO} \vec{F}\,.\,\vec{dr} = 0 \qquad \text{...(6)}$$

Putting the values of integrals (3), (4), (5) and (6) in (2), we get

$$\int_C \vec{F}\,.\,\vec{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2 \qquad \text{...(7)}$$




Now we have to evaluate R.H.S. of Stoke's Theorem *i.e.* $\iint_S (\nabla \times \vec{F})\,.\,\hat{n}\,ds$

We have,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = (2y + 2y)\,\hat{k} = 4y\,\hat{k}$$

Also the unit vector normal to the surface $S$ in outward direction is $\hat{n} = k$

($\because$ $z$-axis is normal to surface $S$)

Also in $xy$-plane $ds = dx\,dy$

$$\therefore \qquad \iint_S (\nabla \times \vec{F})\,.\,\hat{n}.\,ds = \iint_R 4y\,\hat{k}\,.\,\hat{k}\,dx\,dy = \iint_R 4y\,dx\,dy.$$

Where $R$ be the region of the surface $S$.

Consider a strip parallel to $y$-axis. This strip starts on line y = 0 (*i.e.* $x$-axis) and end on the line $y = b$, We move this strip from $x = 0$ ($y$-axis) to $x = a$ to cover complete region $R$.

$$\therefore \qquad \iint_S (\nabla \times \vec{F})\,.\,\hat{n}.\,ds = \int_0^a \left[\int_0^b 4y\,dy\right] dx = \int_0^a [2y^2]_0^b\,dx$$

$$= \int_0^a 2b^2\,dx = 2b^2\,[x]_0^a = 2ab^2 \qquad \text{...(8)}$$

$\therefore$ From (7) and (8), we get

$\int_C \vec{F}\,.\,\vec{dr} = \iint_S (\nabla \times \bar{F})\,.\,\hat{n}\,ds$ and hence the Stoke's theorem is verified.

**Example 99.** *Verify Stoke's Theorem for the function*

$$\vec{F} = x^2\hat{i} - xy\hat{j}$$

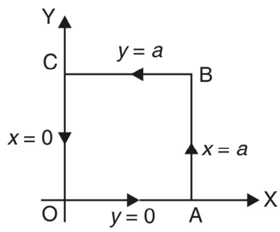
*integrated round the square in the plane z = 0 and bounded by the lines*

*$x = 0,\ y = 0,\ x = a,\ y = a.$*

**Solution.** We have, $\vec{F} = x^2\hat{i} - xy\hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= (0-0)\,\hat{i} - (0-0)\,\hat{j} + (-y-0)\,\hat{k} = -y\hat{k}$$




($\hat{n} \perp$ to $xy$ plane *i.e.* $\hat{k}$)

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (-yk) \cdot k\, dx\, dy$$

$$= \int_0^a dx \int_0^a -y dy = \int_0^a dx \left[ -\frac{y^2}{2} \right]_0^a = -\frac{a^2}{2} (x)_0^a = -\frac{a^3}{2} \qquad ...(1)$$

To obtain line integral

$$\int_C \vec{F} \cdot \vec{dr} = \int_C (x^2 \hat{i} - xy\hat{j}) \cdot (\hat{i}\, dx + \hat{j}\, dy) = \int_C (x^2\, dx - xy\, dy)$$

where $c$ is the path $OABCO$ as shown in the figure.

Also, $$\int_C \vec{F} \cdot \vec{dr} = \int_{OABCO} \vec{F} \cdot dr = \int_{OA} \vec{F} \cdot dr + \int_{AB} \vec{F} \cdot dr + \int_{BC} \vec{F} \cdot dr + \int_{CO} \vec{F} \cdot dr \qquad ...(2)$$

Along $OA$, $y = 0$, $dy = 0$

| line | Eq. of line | | Lower limit | Upper limit |
|---|---|---|---|---|
| $OA$ | $y = 0$ | $dy = 0$ | $x = 0$ | $x = a$ |
| $AB$ | $x = a$ | $dx = 0$ | $y = 0$ | $y = a$ |
| $BC$ | $y = a$ | $dy = 0$ | $x = a$ | $x = 0$ |
| $CO$ | $x = 0$ | $dx = 0$ | $y = a$ | $y = 0$ |

$$\int_{OA} \vec{F} \cdot \vec{dr} = \int_{OA} (x^2 dx - xy dy)$$

$$= \int_0^a x^2 dx = \left[ \frac{x^3}{3} \right]_0^a = \frac{a^3}{3}$$

Along $AB$, $x = a$, $dx = 0$

$$\int_{AB} \vec{F} \cdot \vec{dr} = \int_{AB} (x^2\, dx - x\, y\, d\, y)$$

$$= \int_0^a - a\, y\, d\, y = -a \left[ \frac{y^2}{2} \right]_0^a = -\frac{a^3}{2}$$

Along $BC$, $y = a$, $dy = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_{BC} (x^2 dx - xy\, dy) = \int_a^0 x^2 dx = \left[ \frac{x^3}{3} \right]_a^0 = -\frac{a^3}{3}$$

Along $CO$, $x = 0$, $dx = 0$

$$\int_{CO} \vec{F} \cdot \vec{dr} = \int_{CO} (x^2 dx - xy\, dy) = 0$$

Putting the values of these integrals in (2), we have

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \qquad ...(3)$$

From (1) and (3), $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \int_C \vec{F} \cdot \vec{dr}$

Hence, Stoke's Theorem is verified. **Ans.**

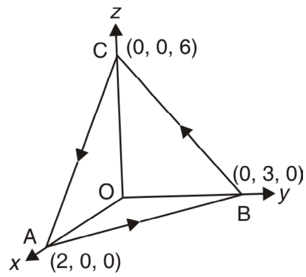
**Example 100.** *Verify Stoke's Theorem for* $\vec{F} = (x + y)\, \hat{i} + (2x - z)\, \hat{j} + (y + z)\, \hat{k}$ *for the surface of a triangular lamina with vertices* (*2, 0, 0*), (*0, 3, 0*) *and* (*0, 0, 6*).

(*Nagpur University 2004, K. U. Dec. 2009, 2008, A.M.I.E.T.E., Summer 2000*)

**Solution.** Here the path of integration $c$ consists of the straight lines AB, BC, CA where the co-ordinates of A, B, C and (2, 0, 0), (0, 3, 0) and (0, 0, 6) respectively. Let $S$ be the plane surface of triangle ABC bounded by $C$. Let $\hat{n}$ be unit normal vector to surface $S$. Then by Stoke's Theorem, we must have

$$\oint_c \vec{F} \cdot \vec{dr} = \iint_s \text{curl}\, \vec{F} \cdot \hat{n}\, ds \qquad ...(1)$$

L.H.S. of (1)$= \int_{ABC}^{c} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CA} \vec{F}\cdot\vec{dr}$




Along line $AB$, $z = 0$, equation of $AB$ is $\frac{x}{2} + \frac{y}{3} = 1$

$\Rightarrow \qquad y = \frac{3}{2}(2 - x),\ dy = -\frac{3}{2}dx$

At $A$, $x = 2$, At $B$, $x = 0$, $\bar{r} = x\hat{i} + y\hat{j}$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{AB} [(x+y)\hat{i} + 2x\hat{j} + y\hat{k}]\cdot(\hat{i}dx + \hat{j}dy)$$
$$= \int_{AB} (x+y)\,dx + 2xdy$$
$$= \int_{AB} \left(x + 3 - \frac{3x}{2}\right)dx + 2x\left(-\frac{3}{2}\,dx\right)$$
$$= \int_2^0 \left(-\frac{7x}{2} + 3\right)dx = \left(-\frac{7x^2}{4} + 3x\right)_2^0$$
$$= (7 - 6) = +1$$

| line | Eq. of line | | Lower limit | Upper limit |
|---|---|---|---|---|
| $AB$ | $\frac{x}{2} + \frac{y}{3} = 1$, $z = 0$ | $dy = -\frac{3}{2}\,dx$ | At $A$ $x = 2$ | At $B$ $x = 0$ |
| $BC$ | $\frac{y}{3} + \frac{z}{6} = 1$, $x = 0$ | $dz = -2dy$ | At $B$ $y = 3$ | At $C$ $y = 0$ |
| $CA$ | $\frac{x}{2} + \frac{z}{6} = 1$, $y = 0$ | $dz = -3dx$ | At $C$ $x = 0$ | At $A$ $x = 2$ |

Along line $BC$, $x = 0$, Equation of $BC$ is $\frac{y}{3} + \frac{z}{6} = 1$ or $z = 6 - 2y$, $dz = -2dy$

At $B$, $y = 3$, At $C$, $y = 0$, $\vec{r} = y\hat{j} + z\hat{k}$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{BC} [yi + zj + (y+z)k]\cdot(jdy + kdz) = \int_{BC} -zdy + (y+z)\,dz$$
$$= \int_3^0 (-6 + 2y)\,dy + (y + 6 - 2y)\,(-2dy)$$
$$= \int_3^0 (4y - 18)\,dy = (2y^2 - 18y)_3^0 = 36$$

Along line CA, $y = 0$, Eq. of CA, $\frac{x}{2} + \frac{z}{6} = 1$ or $z = 6 - 3x$, $dz = -3dx$

At C, $x = 0$, at A, $x = 2$, $\vec{r} = x\hat{i} + z\hat{k}$

$$\int_{CA} \vec{F}\cdot\vec{dr} = \int_{CA} [x\hat{i} + (2x - z)\,\hat{j} + z\hat{k}]\cdot[dx\hat{i} + dz\hat{k}] = \int_{CA} (xdx + zdz)$$
$$= \int_0^2 xdx + (6 - 3x)\,(-3dx) = \int_0^2 (10x - 18)\,dx = [5x^2 - 18x]_0^2 = -16$$

$$\text{L.H.S. of (1)} = \int_{ABC} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CA} \vec{F}\cdot\vec{dr} = 1 + 36 - 16 = 21 \quad ...(2)$$

$$\text{Curl } \overline{F} = \nabla\times\overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times[(x+y)\,\hat{i} + (2x-z)\,\hat{j} + (y+z)\,k]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = (1+1)\,\hat{i} - (0-0)\,\hat{j} + (2-1)\,\hat{k} = 2\hat{i} + \hat{k}$$

Equation of the plane of ABC is $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right) = \frac{\hat{i}}{2} + \frac{\hat{j}}{3} + \frac{\hat{k}}{6}$$

$$\text{Unit Normal Vector} = \frac{\dfrac{\hat{i}}{2} + \dfrac{\hat{j}}{3} + \dfrac{\hat{k}}{6}}{\sqrt{\dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{36}}}$$

$$\hat{n} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$$

$$\text{R.H.S. of (1)} = \iint_s \text{curl } \vec{F}\cdot\hat{n}\, ds = \iint_s (2\hat{i} + \hat{k})\cdot\frac{1}{\sqrt{4}}(3\hat{i} + 2\hat{j} + \hat{k})\frac{dx\,dy}{\dfrac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).\hat{k}}$$

$$= \iint_s \frac{(6+1)}{\sqrt{14}}\frac{dx\,dy}{\dfrac{1}{\sqrt{14}}} = 7\iint dx\,dy = 7 \text{ Area of } \Delta \text{ OAB}$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right) = 21 \quad ...(3)$$

with the help of (2) and (3) we find (1) is true and so Stoke's *Theorem* is verified.

**Example 101.** *Verify Stoke's Theorem for*

$$\vec{F} = (y - z + 2)\ \hat{i} + (yz + 4)\ \hat{j} - (xz)\ \hat{k}$$

*over the surface of a cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the XOY plane* (*open the bottom*).

**Solution.** Consider the surface of the cube as shown in the figure. Bounding path is *OABCO* shown by arrows.

$$\int_c \vec{F}\cdot\vec{d}\,r = \int[(y-z+2)\,\hat{i} + (yz+4)\,\hat{j} - (xz)\,\hat{k}]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c (y-z+2)\,dx + (yz+4)\,dy - xzdz$$

$$\int_c \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr} \quad ...(1)$$

(1) Along *OA*, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

| | Line | Equ. of line | | Lower limit | Upper limit | $\bar{F}.\bar{dr}$ |
|---|---|---|---|---|---|---|
| 1 | $OA$ | $y = 0$<br>$z = 0$ | $dy = 0$<br>$dz = 0$ | $x = 0$ | $x = 2$ | $2\,dx$ |
| 2 | $AB$ | $x = 2$<br>$z = 0$ | $dx = 0$<br>$dz = 0$ | $y = 0$ | $y = 2$ | $4\,dy$ |
| 3 | $BC$ | $y = 2$<br>$z = 0$ | $dy = 0$<br>$dz = 0$ | $x = 2$ | $x = 0$ | $4\,dx$ |
| 4 | $CO$ | $x = 0$<br>$z = 0$ | $dx = 0$<br>$dz = 0$ | $y = 2$ | $y = 0$ | $4\,dy$ |

$$\int_{OA} \vec{F}\cdot\vec{dr} = \int_0^2 2\,dx = [2\,x]_0^2 = 4$$

(2) Along $AB$, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_0^2 4\,dy = 4\,(y)_0^2 = 8$$

(3) Along $BC$, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_0^2 (2 - 0 + 2)\,dx = (4x)_2^0 = -8$$

(4) Along $CO$, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$

$$\int_{CO} \vec{F}\cdot\vec{dr} = \int (y - 0 + 2) \times 0 + (0 + 4)\,dy - 0$$

$$= 4\int dy = 4\,(y)_2^0 = -8$$

On putting the values of these integrals in (1), we get

$$\int_c \vec{F}\cdot\vec{dr} = 4 + 8 - 8 - 8 = -4$$

To obtain surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= (0 - y)\,\hat{i} - (-z + 1)\,\hat{j} + (0 - 1)\,\hat{k} = -y\hat{i} + (z - 1)\,\hat{j} - \hat{k}$$

Here we have to integrate over the five surfaces, *ABDE, OCGF, BCGD, OAEF, DEFG.*

Over the surface $ABDE$ $(x = 2)$, $\hat{n} = i$, $ds = dy\,dz$

$$\iint (\nabla \times \vec{F})\cdot\hat{n}\,ds = \iint [-yi + (z-1)\,j - k]\cdot i\,dx\,dz = \iint -y\,dy\,dz$$

$$= \iint_R [F_3\,(x, y, z)]_{z = f_1(x, y)}^{z = f_2(x, y)}\,dx\,dy$$

| | Surface | Outward normal | ds | |
|---|---|---|---|---|
| 1 | *ABDE* | *i* | *dy dz* | $x = 2$ |
| 2 | *OCGF* | $-i$ | *dy dz* | $x = 0$ |
| 3 | *BCGD* | *j* | *dx dz* | $y = 2$ |
| 4 | *OAEF* | $-j$ | *dx dz* | $y = 0$ |
| 5 | *DEFG* | *k* | *dx dy* | $z = 2$ |

$$= -\int_0^2 y\,dy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$$

Over the surface OCGF ($x = 0$), $\hat{n} = -i$, $ds = dy\,dz$

$$\iint (\nabla \times \vec{F})\cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\,\hat{j} - \hat{k}]\cdot(-\hat{i})\,dy\,dz$$

$$= \iint y\,dy\,dz = \int_0^2 y\,dy \int_0^2 dz = 2\left[\frac{y^2}{2}\right]_0^2 = 4$$

(3) Over the surface *BCGD*, ($y = 2$), $\hat{n} = j$, $ds = dx\,dz$

$$\iint (\nabla \times \vec{F})\cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\,\hat{j} - \hat{k}]\cdot \hat{j}\,dx\,dz$$

$$= -\iint (z-1)\,dx\,dz = \int_0^2 dx \int_0^2 (z-1)\,dz = -(x)_0^2 \left(\frac{z^2}{2} - z\right)_0^2 = 0$$

(4) Over the surface *OAEF*, ($y = 0$), $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint (\nabla \times \vec{F})\cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\hat{j} - \hat{k}]\cdot(-\hat{j})\,dx\,dz$$

$$= -\iint (z-1)\,dx\,dz = -\int_0^2 dx \int_0^2 (z-1)\,dz = -(x)_0^2 \left(\frac{z^2}{2} - z\right)_0^2 = 0$$

(5) Over the surface *DEFG*, ($z = 2$), $\hat{n} = k$, $ds = dx\,dy$

$$\iint (\nabla \times \vec{F})\cdot \hat{n}\,ds = \iint [-y\hat{i} + (z-1)\hat{j} - \hat{k}]\cdot \hat{k}\,dx\,dy = -\iint dx\,dy$$

$$= -\int_0^2 dx \int_0^2 dy = -[x]_0^2\,[y]_0^2 = -4$$

Total surface integral $= -4 + 4 + 0 + 0 - 4 = -4$

Thus $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_c \vec{F}\cdot\vec{dr} = -4$

which verifies Stoke's Theorem. **Ans.**

## EXERCISE 5.14

1. Use the Stoke's Theorem to evaluate $\int_C y^2 dx + xy\, dy + xz\, dz$,

   where $C$ is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the positive direction. **Ans.** 0

2. Evaluate $\int_s (\text{curl } F) \cdot \hat{n}\, dA$, using the Stoke's Theorem, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and $s$ is the paraboloid $z = f(x, y) = 1 - x^2 - y^2$, $z \geq 0$. **Ans.** $\pi$

3. Evaluate the integral for $\int_C y^2 dx + z^2 dy + x^2 dz$, where C is the triangular closed path joining the points (0, 0, 0), (0, $a$, 0) and (0, 0, $a$) by transforming the integral to surface integral using Stoke's Theorem. **Ans.** $\frac{a^3}{3}$.

4. Verify Stoke's Theorem for $\vec{A} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$, where $S$ is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and $c$ is its boundary traversed in the clockwise direction. **Ans.** $-20\pi$

5. Evaluate $\int_C \vec{F} \cdot \vec{d}R$ where $\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$, $C$ is the circl $x^2 + y^2 = 4$, $z = 1.5$ **Ans.** $\frac{19}{2}\pi$

6. If $S$ is the surface of the sphere $x^2 + y^2 + z^2 = 9$. Prove that $\int_S \text{curl } \vec{F} \cdot ds = 0$.

7. Verify Stoke's Theorem for the vector field

   $$\vec{F} = (2y + z)\,\hat{i} + (x - z)\,\hat{j} + (y - x)\,\hat{k}$$

   over the portion of the plane $x + y + z = 1$ cut off by the co-ordinate planes.

8. Evaluate $\int_c \vec{F} \cdot dr$ by Stoke's Theorem for $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,k$ and $C$ is the curve of intersection of $x^2 + y^2 = 1$ and $y = z^2$. **Ans.** 0

9. If $\vec{F} = (x - z)\,\hat{i} + (x^3 + yz)\,\hat{j} + 3xy^2\,\hat{k}$ and $S$ is the surface of the cone $z = a - \sqrt{(x^2 + y^2)}$ above the $xy$-plane, show that $\iint_s \text{curl } \vec{F} \cdot dS = 3\pi a^4 / 4$.

10. If $\vec{F} = 3y\hat{i} - xy\hat{j} + yz^2\hat{k}$ and $S$ is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$, show by using Stoke's Theorem that $\iint_s (\nabla \times \vec{F}) \cdot dS = 20\pi$.

11. If $\vec{F} = (y^2 + z^2 - x^2)\,\hat{i} + (z^2 + x^2 - y^2)\,\hat{j} + (x^2 + y^2 - z^2)\,\hat{k}$, evaluate $\int \text{curl } \vec{F} \cdot \hat{n}\, ds$ integrated over the portion of the surface $x^2 + y^2 - 2ax + az = 0$ above the plane $z = 0$ and verify Stoke's Theorem; where $\hat{n}$ is unit vector normal to the surface. (*A.M.I.E.T.E., Winter 20002*) **Ans.** $2\pi a^3$

12. Evaluate by using Stoke's Theorem $\int_C [\sin z\, dx - \cos x\, dy + \sin y\, dz]$ where C is the boundary of rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $z = 3$. (*AMIETE, June 2010*)

## 5.40 GAUSS'S THEOREM OF DIVERGENCE

(Relation between surface integral and volume integral)

(*U.P., Ist Semester, Jan., 2011, Dec, 2006*)

**Statement.** The surface integral of the normal component of a vector function $F$ taken around a closed surface $S$ is equal to the integral of the divergence of $F$ taken over the volume $V$ enclosed by the surface $S$.

Mathematically

$$\iint_S \vec{F} . \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

**Proof.** Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Putting the values of $\vec{F}, \hat{n}$ in the statement of the divergence theorem, we have

$$\iint_S F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\cdot\hat{n}\,ds = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\,dx\,dy\,dz.$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \qquad ...(1)$$

We require to prove (1).

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy$$

$$= \iint_R [F_3\,(x, y, f_2) - F_3\,(x, y, f_1)]\,dx\,dy \qquad ...(2)$$

For the upper part of the surface *i.e.* $S_2$, we have

$$dx\,dy = ds_2 \cos r_2 = \hat{n}_2.\hat{k}\;ds_2$$

Again for the lower part of the surface *i.e.* $S_1$, we have,

$$dx\,dy = -\cos r_1,\,ds_1 = \hat{n}_1.\;\hat{k}\,ds_1$$

$$\iint_R F_3\,(x, y, f_2)\,dx\,dy = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,ds_2$$

and $$\iint_R F_3\,(x, y, f_1)\,dx\,dy = -\iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,ds_1$$

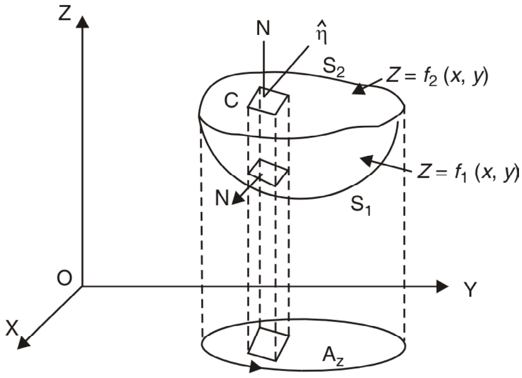
Putting these values in (2), we have

$$\iiint_V \frac{\partial F_3}{\partial z}\,dv = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,ds_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,ds_1 = \iint_S F_3\,\hat{n}\cdot\hat{k}\,ds \qquad ...(3)$$

Similarly, it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\,dv = \iint_S F_2\,\hat{n}\cdot\hat{j}\,ds \qquad ...(4)$$

$$\iiint_V \frac{\partial F_1}{\partial x}\,dv = \iint_S F_1\,\hat{n}\cdot\hat{i}\,ds \qquad ...(5)$$




Adding (3), (4) & (5), we have

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$$

$$= \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot\hat{n}\cdot ds$$

$$\Rightarrow \iiint_V (\nabla\cdot\overline{F})\,dv = \iint_S \overline{F}\cdot\hat{n}\cdot ds$$ **Proved.**

**Example 102.** *State Gauss's Divergence theorem* $\iint_S \vec{F}.\hat{n}\,ds = \iiint Div\,\vec{F}\,dv$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$.

(*Nagpur University, Winter 2004*)

**Solution.** Statement of Gauss's Divergence theorem is given in Art 24.8 on page 597.
Thus by Gauss's divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,ds = \iint_v\int \nabla.\vec{F}\,dv \qquad \text{Here } \vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$$

$$\nabla . \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(3x\hat{i} + 4y\hat{j} + 5z\hat{k})$$

$$\nabla . \vec{F} = 3 + 4 + 5 = 14$$

Putting the value of $\nabla$ . F, we get

$$\iint_S \vec{F} . \hat{n}\, ds = \iint_v\int 14 . dv \qquad \text{where } v \text{ is volume of a sphere}$$

$$= 14\, v$$

$$= 14\frac{4}{3}\pi(4)^3 = \frac{3584\pi}{3} \qquad \textbf{Ans.}$$

**Example 103.** *Evaluate* $\iint_S \vec{F} . \hat{n}\, ds$ *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

(*U.P., Ist Semester, 2009, Nagpur University, Winter 2003*)

**Solution.** By Divergence theorem,

$$\iint_S \vec{F} . \hat{n}\, ds = \iint_v\int (\nabla . \vec{F})\, dv$$

$$= \iiint_v \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\, dv$$

$$= \iint_v\int \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dx\, dy\, dz$$

$$= \iint_v\int (4z - 2y + y)\, dx\, dy\, dz$$

$$= \iint_v\int (4z - y)\, dx\, dy\, dz = \int_0^1\int_0^1\left(\frac{4z^2}{2} - yz\right)_0^1 dx\, dy$$

$$= \int_0^1\int_0^1 (2z^2 - yz)_0^1\, dx\, dy = \int_0^1\int_0^1 (2 - y)\, dx\, dy$$

$$= \int_0^1\left(2y - \frac{y^2}{2}\right)_0^1 dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}(1) = \frac{3}{2} \quad \textbf{Ans.}$$

**Note:** This question is directly solved as on example 14 on Page 574.

**Example 104.** *Find* $\iint \vec{F}\cdot\hat{n}\cdot ds$, *where* $\vec{F} = (2x + 3z)\,\hat{i} - (xz + y)\,\hat{j} + (y^2 + 2z)\,\hat{k}$ *and S is the surface of the sphere having centre (3, – 1, 2) and radius 3.*

(*AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000*)

**Solution.** Let $V$ be the volume enclosed by the surface $S$.
By Divergence theorem, we've

$$\iint_S \vec{F}\cdot\hat{n}\cdot ds = \iiint_V div\, \vec{F}\, dv.$$

Now, $div\, \vec{F} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}[-(xz + y)] + \frac{\partial}{\partial z}(y^2 + 2z) = 2 - 1 + 2 = 3$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\cdot ds = \iiint_V 3\, dv = 3\iiint_V dv = 3V.$$

Again V is the volume of a sphere of radius 3. Therefore

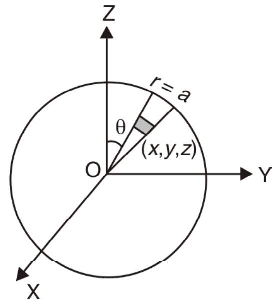
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 = 36\,\pi.$$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\cdot ds = 3V = 3 \times 36\,\pi = 108\,\pi \qquad \textbf{Ans.}$$

**Example 105.** *Use Divergence Theorem to evaluate* $\iint_S \vec{A}\cdot\vec{ds}$,

*where* $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*AMIETE, Dec. 2009*)




**Solution.** $\iint_S \vec{A}\cdot\vec{ds} = \iiint_v \text{div}\,\vec{A}\,dV$

$$= \iiint_v \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k})\,dV$$

$$= \iiint_v (3x^2 + 3y^2 + 3z^2)\,dV = 3\iiint_v (x^2 + y^2 + z^2)\,dV$$

On putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we get

$$= 3\iiint_V r^2\,(r^2\sin\theta\,dr\,d\theta\,d\phi) = 3\times 8\int_0^{\frac{\pi}{2}} d\phi\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^a r^4\,dr$$

$$= 24\,(\phi)_0^{\frac{\pi}{2}}\,(-\cos\theta)_0^{\frac{\pi}{2}}\left(\frac{r^5}{5}\right)_0^a = 24\left(\frac{\pi}{2}\right)(-0+1)\left(\frac{a^5}{5}\right) = \frac{12\pi a^5}{5}$$ **Ans.**

**Example 106.** *Use divergence Theorem to show that*

$$\iint_S \nabla\,(x^2 + y^2 + z^2)\,d\vec{s} = 6\,V$$

*where S is any closed surface enclosing volume V.* (*U.P., I Semester, Winter 2002*)

**Solution.** Here $\nabla\,(x^2 + y^2 + z^2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 + z^2)$

$$= 2\,x\,\hat{i} + 2\,y\,\hat{j} + 2\,z\,\hat{k} = 2\,(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

$$\therefore \quad \iint_S \nabla\,(x^2 + y^2 + z^2)\cdot ds = \iint_S \nabla\,(x^2 + y^2 + z^2)\cdot\hat{n}\,ds$$

$\hat{n}$ being outward drawn unit normal vector to $S$

$$= \iint_S 2\,(x\hat{i} + y\,\hat{j} + z\,\hat{k})\cdot\hat{n}\,ds$$

$$= 2\iiint_V div\,(x\hat{i} + y\,\hat{j} + z\,\hat{k})\,d\,v \qquad ...(1)$$

(By Divergence Theorem)
(V being volume enclosed by *S*)

Now, div. $(x\,\hat{i} + y\,\hat{j} + z\hat{k}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\,\hat{j} + z\,\hat{k})$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \qquad ...(2)$$

From (1) & (2), we have

$$\iint \nabla\,(x^2 + y^2 + z^2)\cdot dS = 2\iiint_V 3\,dv = 6\iiint_V dv = 6\,V$$ **Proved.**

**Example 107.** *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,\hat{n}\,dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

**Solution.** Let $V$ be the volume enclosed by the surface $S$. Then by divergence Theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,\hat{n}\,dS = \iiint_V div\,(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right]dV = \iint_V 2z\,y^2\,dV = 2\iint_V zy^2\,dV$$

Changing to spherical polar coordinates by putting

$x = r \sin\theta \cos\phi,\quad y = r \sin\theta \sin\phi,\quad z = r\cos\theta,\ dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

To cover $V$, the limits of $r$ will be 0 to 1, those of $\theta$ will be 0 to $\frac{\pi}{2}$ and those of $\phi$ will be 0 to $2\pi$.

$$\therefore \quad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta \cdot dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5 \sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

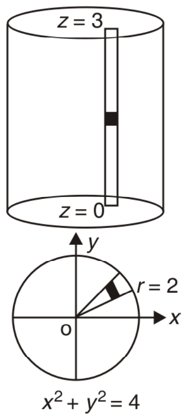
$$= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\sin^2\phi \left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi}\sin^2\phi \cdot \frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\pi}{12} \qquad \textbf{Ans.}$$

**Example 108.** *Use Divergence Theorem to evaluate* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.

(*A.M.I.E.T.E., Summer 2003, 2001*)

**Solution.** By Divergence Theorem,




$$\iint_S \vec{F}\cdot dS = \iiint_V div\,\vec{F}\, dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\, dV$$

$$= \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \iint dx\, dy \int_0^3 (4 - 4y + 2z)dz = \iint dx\, dy\,[4z - 4yz + z^2]_0^3$$

$$= \iint (12 - 12y + 9)\, dx\, dy = \iint (21 - 12y)\, dx\, dy$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$

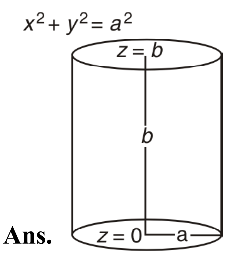
$$= \iint (21 - 12r\sin\theta)\, r\, d\theta\, dr = \int_0^{2\pi} d\theta \int_0^2 (21r - 12r^2\sin\theta)\, dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{21r^2}{2} - 4r^3\sin\theta\right]_0^2 = \int_0^{2\pi} d\theta\,(42 - 32\sin\theta) = (42\theta + 32\cos\theta)_0^{2\pi}$$

$= 84\pi + 32 - 32 = 84\pi$ **Ans.**

**Example 109.** *Apply the Divergence Theorem to compute* $\iint \vec{u}\cdot\hat{n}\, ds$, *where s is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = \hat{i}x - \hat{j}y + \hat{k}z$.

**Solution.** By Gauss's Divergence Theorem




$$\iint \vec{u}\cdot\hat{n}ds = \iiint_V (\nabla\cdot\vec{u})\, dv$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x - \hat{j}y + \hat{k}z)dv$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dv = \iiint_V (1 - 1 + 1)\, dv$$

$$= \iiint_V dv = \iiint_V dx\, dy\, dz = \text{Volume of the cylinder} = \pi a^2 b \quad \textbf{Ans.}$$

**Example 110.** *Apply Divergence Theorem to evaluate* $\iiint_V \vec{F}\,.\,\hat{n}\,ds$, *where* $\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$ *and* $z = b$. (*U.P. Ist Semester, Dec. 2006*)

**Solution.** We have,

$$\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$$

$$\therefore \quad div\,\overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k})$$

$$= \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(-x^2y) + \frac{\partial}{\partial z}(x^2z) = 12x^2 - x^2 + x^2 = 12\,x^2$$

Now, $$\iiint_V div\,\vec{F}\,dV = 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\,dz\,dy\,dx$$

$$= 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,(z)_0^b\,dy\,dx = 12\,b\int_{-a}^{a} x^2(y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx$$

$$= 12\,b\int_{-a}^{a} x^2.2\sqrt{a^2-x^2}\,dx = 24\,b\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx$$

$$= 48\,b\int_0^a x^2\sqrt{a^2-x^2}\,dx \qquad [\text{Put } x = a\sin\theta,\ dx = a\cos\theta\,d\theta]$$

$$= 48\,b\int_0^{\pi/2} a^2\sin^2\theta\,a\cos\theta\,a\cos\theta\,d\theta$$

$$= 48\,ba^4\int_0^{\pi/2}\sin^2\theta\cdot\cos^2\theta\,d\theta = 48\,ba^4\,\frac{\left|\frac{3}{2}\right.\left|\frac{3}{2}\right.}{2\left|3\right.}$$

$$= 48\,ba^4\,\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 2} = 3\,b\,a^4\pi \qquad \textbf{Ans.}$$

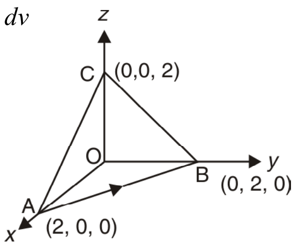
**Example 111.** *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\,ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\,(\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron* $x = 0, y = 0, z = 0, x + y + z = 2$ *and n is the unit normal in the outward direction to the closed surface S.*

**Solution.** By Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\cdot dv$$

where $S$ is the surface of tetrahedron $x = 0, y = 0, z = 0, x + y + z = 2$

$$= \iiint_V\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 + z^2)\,(\hat{i} + \hat{j} + \hat{k})\,dv$$




$$= \iiint_V (2x + 2y + 2z)\,dv$$

$$= 2\iiint_V (x + y + z)\,dx\,dy\,dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\int_0^{2-x-y}(x + y + z)\,dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\left(xz + yz + \frac{z^2}{2}\right)_0^{2-x-y}$$

$$= 2\int_0^2 dx \int_0^{2-x} dy \left( 2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2} \right]$$

$$= 2\int_0^2 dx \left[ 2xy - x^2 y - x\,y^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6} \right]_0^{2-x}$$

$$= 2\int_0^2 dx \left[ 2x\,(2-x) - x^2(2-x) - x\,(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6} \right]$$

$$= 2\int_0^2 \left( 4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6} \right]$$

$$= 2\left[ 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24} \right]_0^2$$

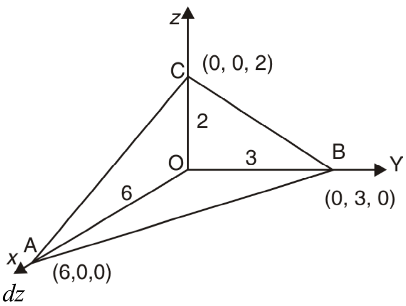
$$= 2\left[ -\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24} \right]_0^2 = 2\left[ \frac{8}{3} - \frac{16}{12} + \frac{16}{24} \right] = 4 \qquad \textbf{Ans.}$$

**Example 112.** *Use the Divergence Theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

*where S is the portion of the plane x + 2 y + 3 z = 6 which lies in the first Octant.*
(*U.P., I Semester, Winter 2003*)

**Solution.** $\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy)$




$$= \iiint_V \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) d\,x\,d\,y\,d\,z$$

where $S$ is a closed surface bounding a volume V.

$$\therefore \qquad \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$= \iiint_V \left[ \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right] dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz = 3\iiint_V dx\,dy\,dz$$

= 3 (Volume of tetrahedron $OABC$)

$$= 3\,[(\tfrac{1}{3}\text{ Area of the base } \Delta\, OAB) \times \text{height } OC]$$

$$= 3\left[ \frac{1}{3}\left( \frac{1}{2} \times 6 \times 3 \right) \times 2 \right] = 18 \qquad \textbf{Ans.}$$

**Example 113.** *Use Divergence Theorem to evaluate :* $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *over the surface of a sphere radius a.* (*K. University, Dec. 2009*)

**Solution.** Here, we have

$$\iint_S [x\,dy\,dz + y\,dx\,dz + z\,dx\,dy]$$

$$= \iiint_V \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx\,dy\,dz \quad = \iiint_V \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz = 3 \text{ (volume of the sphere)}$$

$$= 3\left( \frac{4}{3}\pi a^3 \right) = 4\,\pi\,a^3 \qquad \textbf{Ans.}$$

**Example 114.** *Using the divergence theorem, evaluate the surface integral* $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ *where* $S : x^2 + y^2 + z^2 = 4.$

(*AMIETE, Dec. 2010, UP, I Sem., Dec 2008*)

**Solution.** $\iint_S (f_1\, dy\, dz + f_2\, dx\, dz + f_3\, dx\, dy)$

$$= \iiint_v \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz$$

where $S$ is closed surface bounding a volume $V$.

$\therefore \quad \iint_S (yz\, dy\, dz + zx\, dx\, dz + xy\, dx\, dy)$

$$= \iiint_v \left(\frac{\partial (yz)}{\partial x} + \frac{\partial (zx)}{\partial y} + \frac{\partial (xy)}{\partial z}\right) dx\, dy\, dz = \iiint_v (0+0+0)\, dx\, dy\, dz$$

$$= 0$$ **Ans.**

**Example 115.** *Evaluate* $\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$ *where S is the surface of hemispherical region bounded by*

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = 0.$$

**Solution.** $\iint_S (f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy) = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz$

where S is a closed surface bounding a volume V.

$\therefore \quad \iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$

$$= \iiint_V \left[\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2 y - z^3) + \frac{\partial}{\partial z}(2xy + y^2 z)\right] dx\, dy\, dz$$

(Here $V$ is the volume of hemisphere)

$$= \iiint_V (z^2 + x^2 + y^2)\, dx\, dy\, dz$$

Let $x = r \sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$

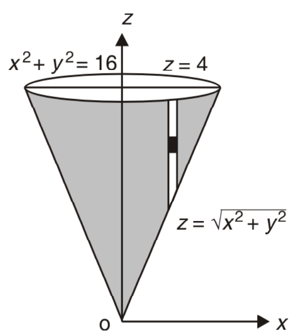
$$= \iiint r^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= (\phi)_0^{2\pi} (-\cos\theta)_0^{\pi/2} \left(\frac{r^5}{5}\right)_0^a = 2\pi(-0+1)\frac{a^5}{5} = \frac{2\pi a^5}{5}$$ **Ans.**

**Example 116.** *Evaluate* $\iint_S \vec{F}\,.\,\hat{n}\, ds$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $F = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}.$

**Solution.** If V is the volume enclosed by S, then V is bounded by the surfaces $z = 0$, $z = 4$, $z^2 = x^2 + y^2$.

By divergence theorem, we have




$$\iint_S \overline{F}\,.\,\hat{n}\, ds = \iiint_V div\,\vec{F}\, dx\, dy\, dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)\right] dx\, dy\, dz$$

$$= \iiint_V (4z + xz^2 + 3)\, dx\, dy\, dz$$

Limits of $z$ are $\sqrt{x^2 + y^2}$ and 4.

$$\iiint_{\sqrt{x^2+y^2}}^{4} (4z + xz^2 + 3)\, dz\, dy\, dx = \iint \left[2z^2 + \frac{xz^3}{3} + 3z\right]_{\sqrt{x^2+y^2}}^{4} dy\, dx$$

$$= \iint \left[\left(32 + \frac{64x}{3} + 12\right) - \{2(x^2+y^2) + x(x^2+y^2)^{3/2} + 3\sqrt{x^2+y^2}\}\right] dy\, dx$$

$$= \iint \left(44 + \frac{64x}{3} - 2(x^2+y^2) - x(x^2+y^2)^{3/2} - 3\sqrt{x^2+y^2}\right) dy\, dx$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$= \iint \left(44 + \frac{64\, r\cos\theta}{3} - 2r^2 - r\cos\theta\, r^3 - 3r\right) r\, d\theta\, dr$$

Limits of $r$ are 0 to 4.
and limits of $\theta$ are 0 to $2\pi$.

$$= \int_0^{2\pi}\int_0^4 \left(44r + \frac{64\, r^2\cos\theta}{3} - 2r^3 - r^5\cos\theta - 3r^2\right) d\theta\, dr$$

$$= \int_0^{2\pi} \left[22r^2 + \frac{64\times r^3\cos\theta}{9} - \frac{r^4}{2} - \frac{r^6}{6}\cos\theta - r^3\right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left[22(4)^2 + \frac{64\times(4)^3\cos\theta}{9} - \frac{(4)^4}{2} - \frac{(4)^6}{6}\cos\theta - (4)^3\right] d\theta$$

$$= \int_0^{2\pi} \left[352 + \frac{64\times 64}{9}\cos\theta - 128 - \frac{(4)^6}{6}\cos\theta - 64\right] d\theta$$

$$= \int_0^{2\pi} \left[160 + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\cos\theta\right] d\theta$$

$$= \left[160\,\theta + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\sin\theta\right]_0^{2\pi} = 160\,(2\pi) + \left(\frac{64\times 64}{9} - \frac{(4)^6}{6}\right)\sin 2\pi$$

$= 320\,\pi$ **Ans.**

**Example 117.** *The vector field* $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ *is defined over the volume of the cuboid given by* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, *enclosing the surface S. Evaluate the surface integral*

$$\iint_S \vec{F}\,.\,\vec{ds}$$ (*U.P., I Semester, Winter 2001*)

**Solution.** By Divergence Theorem, we have

$$\iint_S (x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,.ds = \iiint_v div\,(x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,dv,$$

where $V$ is the volume of the cuboid enclosing the surface $S$.

$$= \iiint_v \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right).(x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k})\,dv$$

$$= \iiint_v \left\{\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)\right\} dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (2x+y)\,dx\,dy\,dz = \int_0^a dx\int_0^b dy\int_0^c (2x+y)\,dz$$

$$= \int_0^a dx\int_0^b [2xz + yz]_0^c\,dy = \int_0^a dx\int_0^b (2xc + yc)\,dy$$

$$= c\int_0^a dx \int_0^b (2x+y)\,dy = c\int_0^a \left[2xy + \frac{y^2}{2}\right]_0^b dx = c\int_0^a \left(2bx + \frac{b^2}{2}\right) dx$$

$$= c\left[\frac{2bx^2}{2} + \frac{b^2 x}{2}\right]_0^a = c\left[a^2 b + \frac{ab^2}{2}\right] = abc\left(a + \frac{b}{2}\right) \qquad \textbf{Ans.}$$

**Example 118.** *Verify the divergence Theorem for the function* $\overline{F} = 2x^2yi - y^2j + 4xz^2k$ *taken over the region in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$.

**Solution.** $\iiint_V \nabla\cdot\vec{F}\,dV = \iiint\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x^2y\hat{i} - y^2\hat{j} + 4xz^2k)\,dV$

$$= \iiint (4xy - 2y + 8xz)\,dx\,dy\,dz = \int_0^2 dx \int_0^3 dy \int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dz$$

$$= \int_0^2 dx \int_0^3 dy\,(4xyz - 2yz + 4xz^2)_0^{\sqrt{9-y^2}}$$

$$= \int_0^2 dx \int_0^3 [4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2)]\,dy$$

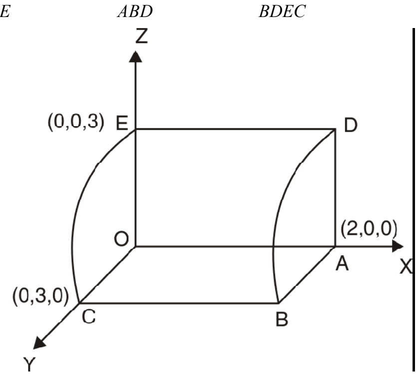
$$= \int_0^2 dx \left[-\frac{4x}{2}\,\frac{2}{3}(9-y^2)^{3/2} + \frac{2}{3}(9-y^2)^{3/2} + 36xy - \frac{4xy^3}{3}\right]_0^3$$

$$= \int_0^2 (0 + 0 + 108x - 36x + 36x - 18)\,dx = \int_0^2 (108x - 18)\,dx = \left[108\frac{x^2}{2} - 18x\right]_0^2$$

$$= 216 - 36 = 180 \qquad \ldots(1)$$

Here $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{OCE} \vec{F}\cdot\hat{n}\,ds + \iint_{OADE} \vec{F}\cdot\hat{n}\,ds + \iint_{ABD} \vec{F}\cdot\hat{n}\,ds + \iint_{BDEC} \vec{F}\cdot\hat{n}\,ds$

$$\iint_{BDEC} \overline{F}\cdot\hat{n}\,ds = \iint_{BDEC} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\,.\,\hat{n}\,ds$$




Normal vector

$$= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y^2 + z^2 - 9)$$

$$= 2y\hat{j} + 2z\hat{k}$$

Unit normal vector $= \hat{n} = \dfrac{2y\hat{j} + 2z\hat{k}}{\sqrt{4y^2 + 4z^2}} = \dfrac{y\hat{j} + z\hat{k}}{\sqrt{y^2 + z^2}}$

$$= \frac{y\hat{j} + z\hat{k}}{\sqrt{9}} = \frac{y\hat{j} + z\hat{k}}{3}$$

$$\iint_{BDEC} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2k)\cdot\frac{y\hat{j} + zk}{3}\,ds = \frac{1}{3}\iint_{BDEC} (-y^3 + 4xz^3)\,ds$$

$$\left[dx\,dy = ds\,(\hat{n}\cdot k) = ds\left(\frac{y\hat{j} + z\hat{k}}{3}\cdot\hat{k}\right) = ds\,\frac{z}{3} \text{ or } ds = \frac{dx\,dy}{\frac{z}{3}}\right]$$

$$= \frac{1}{3}\iint_{BDEC} (-y^3 + 4xz^3)\frac{dxdy}{\frac{z}{3}} = \int_0^2 dx \int_0^3 \left(-\frac{y^3}{z} + 4xz^2\right) dy \qquad \begin{pmatrix} y = 3\sin\theta, \\ z = 3\cos\theta \end{pmatrix}$$

$$= \int_0^2 dx \int_0^{\frac{\pi}{2}} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x(9\cos^2\theta)\right]$$

$$= \int_0^2 dx\left(-27\times\frac{2}{3}+108\,x\times\frac{2}{3}\right) = \int_0^2 (-18+72\,x)\,dx$$

$$= \left[-18x+36x^2\right]_0^2 = 108 \qquad ...(2)$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{k})\,ds$$

$$= \iint_{OABC} 4xz^2\,ds = 0$$ ...(3) because in $OABC$ $xy$-plane, $z = 0$

$$\iint_{OADE} \vec{F}\cdot\hat{n}\,ds = \iint_{OADE} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{j})\,ds = \iint_{OADE} y^2\,ds = 0 \qquad ...(4)$$

because in $OADE$ $xz$-plane, $y = 0$

$$\iint_{OCE} \vec{F}\cdot\hat{n}\,ds = \iint_{OCE} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{i})\,ds = \iint_{OCE} -2x^2y\,ds = 0 \qquad ...(5)$$

because in $OCE$ $yz$-plane, $x = 0$

$$\iint_{ABD} \vec{F}\cdot\hat{n}\,ds = \iint_{ABD} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(\hat{i})\,ds = \iint_{ABD} 2x^2y\,ds$$

$$= \iint 2x^2y\,dy\,dz = \int_0^3 dz \int_0^{\sqrt{9-z^2}} 2\,(2)^2\,y\,dy$$ because in $ABD$ plane, $x = 2$

$$= 8\int_0^3 dz\left[\frac{y^2}{2}\right]_0^{\sqrt{9-z^2}} = 4\int_0^3 dz\,(9-z^2) = 4\left[9z-\frac{z^3}{3}\right]_0^3 = 4\,[27-9] = 72 \qquad ...(6)$$

On adding (2), (3), (4), (5) and (6), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 108+0+0+0+72 = 180 \qquad ...(7)$$

From (1) and (7), we have $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$

Hence the theorem is verified.

**Example 119.** *Verify the Gauss divergence Theorem for*

$\vec{F} = (x^2-yz)\,\hat{i} + (y^2-zx)\,\hat{j} + (z^2-xy)\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. (*U.P., I Semester, Compartment 2002*)

**Solution.** We have

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot[(x^2-yz)\,\hat{i}+(y^2-zx)\,\hat{j}+(z^2-xy)\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy) = 2x+2y+2z$$

$\therefore$ Volume integral $= \iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 2\,(x+y+z)\,dV$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dx\,dy\,dz = 2\int_0^a dx\int_0^b dy\int_0^c (x+y+z)\,dz$$

$$= 2\int_0^a dx\int_0^b dy\left(xz+yz+\frac{z^2}{2}\right)_0^c = 2\int_0^a dx\int_0^b dy\left(cx+cy+\frac{c^2}{2}\right)$$

$$= 2\int_0^a dx\left(c\,x\,y+c\,\frac{y^2}{2}+\frac{c^2y}{2}\right)_0^b = 2\int_0^a dx\left(bcx+\frac{b^2c}{2}+\frac{bc^2}{2}\right)$$

$$= 2\left[\frac{bc\,x^2}{2} + \frac{b^2\,cx}{2} + \frac{bc^2 x}{2}\right]_0^a = [a^2bc + ab^2c + abc^2]$$

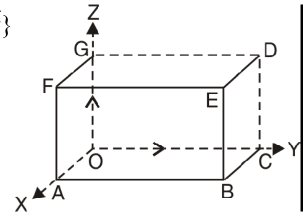
$$= abc\,(a + b + c) \qquad \text{...(A)}$$

To evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $S$ consists of six plane surfaces.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAFG} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{BCDE} \vec{F}\cdot\hat{n}\,ds + \iint_{ABEF} \vec{F}\cdot\hat{n}\,ds + \iint_{OCDG} \vec{F}\cdot\hat{n}\,ds$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \{(x^2 - yz)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\,\hat{k}\}$$




$$= -\int_0^a\int_0^b (z^2 - xy)\,dx\,dy$$

$$= -\int_0^a\int_0^b (0 - xy)\,dx\,dy = \frac{a^2\,b^2}{4} \qquad \text{...(1)}$$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFG} \{(x^2 - yz)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\,(\hat{k})\,dx\,dy$$

| S.No. | Surface | Outward normal | $ds$ | |
|---|---|---|---|---|
| 1 | OABC | $-k$ | $dx\,dy$ | $z = 0$ |
| 2 | DEFG | $k$ | $dx\,dy$ | $z = c$ |
| 3 | OAFG | $-j$ | $dx\,dz$ | $y = 0$ |
| 4 | BCDE | $j$ | $dx\,dz$ | $y = b$ |
| 5 | ABEF | $i$ | $dy\,dz$ | $x = a$ |
| 6 | OCDG | $-i$ | $dy\,dz$ | $x = 0$ |

$$= \int_0^a\int_0^b (z^2 - xy)\,dx\,dy = \int_0^a\int_0^b (c^2 - xy)\,dx\,dy$$

$$= \int_0^a \left[c^2 y - \frac{xy^2}{2}\right]_0^b dx = \int_0^a \left(c^2 b - \frac{x\,b^2}{2}\right) dx$$

$$= \left[c^2 bx - \frac{x^2\,b^2}{4}\right]_0^a = abc^2 - \frac{a^2\,b^2}{4} \qquad \text{...(2)}$$

$$\iint_{OAFG} \vec{F}\cdot\hat{n}\,ds = \iint_{OAFG} \{(x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\,(-\hat{j})\,dx\,dz$$

$$= -\iint_{OAFG} (y^2 - zx)\,dx\,dz$$

$$= -\int_0^a dx \int_0^c (0 - zx)\,dz = \int_0^a dx \left(\frac{x\,z^2}{2}\right)_0^c = \int_0^a \frac{x\,c^2}{2}\,dx = \left[\frac{x^2\,c^2}{4}\right]_0^a = \frac{a^2\,c^2}{4} \qquad \text{...(3)}$$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \iint \{(x^2 - y\,z)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}\}\cdot\hat{j}\,dx\,dz = \iint_{BCDE} (y^2 - xz)\,dx\,dz$$

$$= -\int_0^a dx \int_0^c (b^2 - x\,z)\,dz = \int_0^a \left(b^2\,z - \frac{x\,z^2}{2}\right)_0^c dx = \int_0^a \left(b^2\,c - \frac{x\,c^2}{2}\right) dx$$

$$= \left[b^2\,c\,x - \frac{x^2\,c^2}{4}\right]_0^a = a\,b^2\,c - \frac{a^2 c^2}{4} \qquad \text{...(4)}$$

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \iint_{ABEF} \{(x^2 - y\,z)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - x\,y)\,\hat{k}\}\cdot\hat{i}\,dy\,dz$$

$$= \iint_{ABEF} (x^2 - y\,z)\,dy\,dz = \int_0^b dy \int_0^c (a^2 - y\,z)\,d\,z = \int_0^b dy \left(a^2 z - \frac{y\,z^2}{2}\right)_0^c$$

$$= \int_0^b \left( a^2 c - \frac{y c^2}{2} \right) dy = \left[ a^2 c y - \frac{y^2 c^2}{4} \right]_0^b = a^2 b c - \frac{b^2 c^2}{4} \qquad ...(5)$$

$$\iint_{OCDG} \vec{F} \cdot \hat{n}\, ds = \iint_{OCDG} \{(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\} \cdot (-\hat{i})\, dy\, dz$$

$$= \int_0^b \int_0^c (x^2 - yz)\, dy\, dz = -\int_0^b dy \int_0^c (-yz)\, dz = -\int_0^b dy \left[ \frac{-y z^2}{2} \right]_0^c$$

$$= \int_0^b \frac{y c^2}{2}\, dy = \left[ \frac{y^2 c^2}{4} \right]_0^b = \frac{b^2 c^2}{4} \qquad ...(6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint \vec{F} \cdot \hat{n}\, ds = \left( \frac{a^2 b^2}{4} \right) + \left( abc^2 - \frac{a^2 b^2}{4} \right) + \left( \frac{a^2 c^2}{4} \right) + \left( a b^2 c - \frac{a^2 c^2}{4} \right) + \left( \frac{b^2 c^2}{4} \right) + \left( a^2 b c - \frac{b^2 c^2}{4} \right)$$

$$= abc^2 + ab^2c + a^2bc$$

$$= abc\,(a + b + c) \qquad ...(B)$$

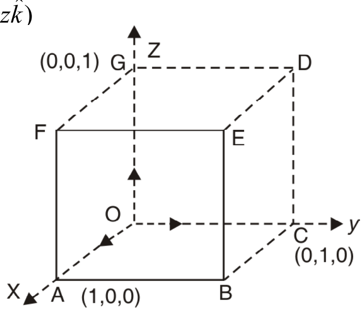
From (A) and (B), Gauss divergence Theorem is verified. **Verified.**

**Example 120.** *Verify Divergence Theorem, given that* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.*

**Solution.** $\nabla \cdot \vec{F} = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (4\,zx\hat{i} - y^2\hat{j} + yz\hat{k})$

$$= 4z - 2y + y$$

$$= 4z - y$$




$$\text{Volume Integral} = \iiint \nabla \cdot \vec{F}\, dv$$

$$= \iiint (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y)\, dz$$

$$= \int_0^1 dx \int_0^1 dy\, (2z^2 - yz)_0^1 = \int_0^1 dx \int_0^1 dy\, (2 - y)$$

$$= \int_0^1 dx \left( 2y - \frac{y^2}{2} \right)_0^1 = \int_0^1 dx \left( 2 - \frac{1}{2} \right) = \frac{3}{2} \int_0^1 dx = \frac{3}{2} (x)_0^1 = \frac{3}{2} \qquad ...(1)$$

To evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$, where $S$ consists of six plane surfaces.

Over the face $OABC$, $z = 0$, $dz = 0$, $\hat{n} = -\hat{k}$, $ds = dx\, dy$

$$\iint \vec{F} . \hat{n}\, ds = \int_0^1 \int_0^1 (-y^2\hat{j}) \cdot (-\hat{k})\, dx\, dy = 0$$

Over the face $BCDE$, $y = 1$, $dy = 0$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4\,xz\hat{i} - \hat{j} + z\hat{k})\cdot(\hat{j})\,dx\,dz$$

$$\hat{n} = \hat{j},\ ds = dx\,dz = \int_0^1\int_0^1 -\,dxdz$$

$$= -\int_0^1 dx\int_0^1 dz = -(x)_0^1\,(z)_0^1 = -(1)\,(1) = -1$$

Over the face $DEFG$, $z = 1$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 [4\,x\,(1) - y^2\hat{j} + y\,(1)\,\hat{k}]\cdot(\hat{k})\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 dx\int_0^1 y\,dy = (x)_0^1\left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Over the face $OCDG$, $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (0\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = 0$$

Over the face $AOGF$, $y = 0$, $dy = 0$, $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4\,xz\hat{i})\cdot(-\hat{j})\,dx\,dz = 0$$

Over the face $ABEF$, $x = 1$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 [(4\,z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})]\,dy\,dz = \int_0^1\int_0^1 4\,z\,dy\,dz$$

$$= \int_0^1 dy\int_0^1 4\,z\,dz = \int_0^1 dy\,(2\,z^2)_0^1 = 2\int_0^1 dy = 2\,(y)_0^1 = 2$$

On adding we see that over the whole surface

$$\iint \vec{F}\cdot\hat{n}\,ds = \left(0 - 1 + \frac{1}{2} + 0 + 0 + 2\right) = \frac{3}{2} \qquad ...(2)$$

From (1) and (2), we have $\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$ **Verified.**

## EXERCISE 5.15

**1.** Use Divergence Theorem to evaluate $\iint_s (y^2z^2\hat{i} + z^2x^2\hat{j} + x^2y^2\hat{k}).\overline{ds}$,

where $S$ is the upper part of the sphere $x^2 + y^2 + z^2 = 9$ above $xy$- plane. **Ans.** $\frac{243\pi}{8}$

**2.** Evaluate $\iint_S (\nabla\times\vec{F})\,.\,ds$, where $S$ is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the $xy$-plane and $\vec{F} = (x^2 + y - 4)\,\hat{i} + 3\,xy\hat{j} + (2\,xz + z^2)\,\hat{k}$. **Ans.** $-4\,\pi$

**3.** Evaluate $\iint_s [xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2\,xy + y^2z)\,dx\,dy]$, where $S$ is the surface enclosing a region bounded by hemisphere $x^2 + y^2 + z^2 = 4$ above $XY$-plane.

**4.** Verify Divergence Theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$, taken over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

**5.** Evaluate $\iint_S (2xy\,\hat{i} + yz^2\hat{j} + xz\,\hat{k})\cdot\vec{ds}$ over the surface of the region bounded by

$x = 0$, $y = 0$, $y = 3$, $z = 0$ and $x + 2\,z = 6$ **Ans.** $\frac{351}{2}$

**6.** Verify Divergence Theorem for $\vec{F} = (x + y^2)\,\hat{i} - 2\,x\hat{j} + 2\,yz\hat{k}$ and the volume of a tetrahedron bounded by co-ordinate planes and the plane $2\,x + y + 2\,z = 6$.

(*Nagpur, Winter 2000, A.M.I.E.T.E.. Winter 2000*)

**7.** Verify Divergence Theorem for the function $\vec{F} = y\hat{i} + x\hat{j} + z^2\,\hat{k}$ over the region bounded by

$x^2 + y^2 = 9, \quad z = 0$ and $z = 2$.

**8.** Use the Divergence Theorem to evaluate $\iint_S x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy$,

where $S$ is the surface of the region bounded by the closed cylinder

$x^2 + y^2 = a^2$, $(0 \le z \le b)$ and $z = 0$, $z = b$. **Ans.** $\dfrac{5\pi\,a^4 b}{4}$

**9.** Evaluate the integral $\iint_S (z^2 - x)\,dy\,dz - xy\,dx\,dz + 3z\,dx\,dy$, where S is the surface of closed region bounded by $z = 4 - y^2$ and planes $x = 0$, $x = 3$, $z = 0$ by transforming it with the help of Divergence Theorem to a triple integral. **Ans.** 16

**10.** Evaluate $\iint_S \dfrac{ds}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$ over the closed surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ by applying Divergence Theorem. **Ans.** $\dfrac{4\pi}{\sqrt{(a\,b\,c)}}$

**11.** Apply Divergence Theorem to evaluate $\iint (l\,x^2 + m\,y^2 + n\,z^2)\,ds$ taken over the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$, $l, m, n$ being the direction cosines of the external normal to the sphere. (*AMIETE June 2010, 2009*) **Ans.** $\dfrac{8\pi}{3}(a + b + c)\,r^3$

**12.** Show that $\iiint_V (u\,\nabla\cdot\vec{V} + \nabla u\cdot\vec{V})\,dv = \iint_S u\,\vec{V}\cdot ds$.

**13.** If $E = \text{grad}\ \phi$ and $\nabla^2\phi = 4\,\pi\,\rho$, prove that $\iint_S \vec{E}\cdot\vec{n}\,ds = -4\,\pi\iiint_V \rho\,dv$

where $\vec{n}$ is the outward unit normal vector, while $dS$ and $dV$ are respectively surface and volume elements.

**Pick up the correct option from the following:**

**14.** If $\vec{F}$ is the velocity of a fluid particle then $\int_C \vec{F}\cdot\vec{dr}$ represents.

(*a*) Work done (*b*) Circulation (*c*) Flux (*d*) Conservative field.

(*U.P. Ist Semester, Dec 2009*) **Ans.** (*b*)

**15.** If $\vec{f} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$, $a, b, c$, constants, then $\iint f.dS$ where $S$ is the surface of a unit sphere is

(*a*) $\dfrac{\pi}{3}(a + b + c)$ (*b*) $\dfrac{4}{3}\pi\,(a + b + c)$ (*c*) $2\,\pi\,(a + b + c)$ (*d*) $\pi\,(a + b + c)$

(*U.P., Ist Semester, 2009*) **Ans.** (*b*)

**16.** A force field $\vec{F}$ is said to be conservative if

(*a*) Curl $\vec{F} = 0$ (*b*) grad $\bar{F} = 0$ (*c*) Div $\bar{F} = 0$ (*d*) Curl (grad $\bar{F}$) = 0

(*AMIETE, Dec. 2006*) **Ans.** (*a*)

**17.** The line integral $\int_C x^2\,dx + y^2\,dy$, where $C$ is the boundary of the region $x^2 + y^2 < a^2$ equals

(*a*) 0, (*b*) $a$ (*c*) $\pi\,a^2$ (*d*) $\dfrac{1}{2}\pi\,a^2$

(*AMIETE, Dec. 2006*) **Ans.** (*b*)