Lecture 16: Scalar and Vector Integration *(RHB chapter 9)*

16. 1. Scalar Integration

You should already be familar with integration in $I\!R^1, I\!R^2, I\!R^3$. Here we review integration of a scalar field with an example.

Consider a hemisphere of radius a centered on the $\underline{e}_3$ axis and with bottom face at $z = 0$. If the mass density (a scalar field) is $\rho(r) = r^{-n}$ where $n < 3$, then what is the total mass?

It is most convenient to use spherical polars (see lecture 15). Then

$$M = \int_{\text{hemisphere}} \rho(\underline{r})dV = \int_0^a r^2\rho(r)dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi \int_0^a r^{2-n}dr = \frac{2\pi a^{3-n}}{3-n}$$

Now consider the centre of mass vector

$$M\underline{R} = \int_V \underline{r}\rho(\underline{r})dV$$

This is our first example of integrating a vector field (here $\underline{r}\rho(\underline{r})$). To do so simply integrate each component using $\underline{r} = r\sin\theta\cos\phi\underline{e}_1 + r\sin\theta\sin\phi\underline{e}_2 + r\ \cos\theta\underline{e}_3$

$$\begin{aligned}
MX &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \cos\phi\ d\phi = 0 \quad \text{since } \phi \text{ integral gives } 0\\
MY &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi\ d\phi = 0 \quad \text{since } \phi \text{ integral gives } 0\\
MZ &= \int_0^a r^3\rho(r)dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = \int_0^a r^{3-n}dr \int_0^{\pi/2} \frac{\sin 2\theta}{2} 2\pi\\
&= \frac{\pi a^{4-n}}{4-n} \qquad \therefore\ \underline{R} = \frac{a(3-n)}{2(4-n)}\underline{e}_3
\end{aligned}$$

16. 2. Line Integrals (RHB 8.3, 9.1)

As an example recall that the work done by a force in moving a particle a distance ds: $dW = Fds$. Now consider a particle constrained to move on a wire say. Only the component of the force along the wire does any work. Therefore the work done in moving the particle from $\underline{r}$ to $\underline{r} + d\underline{r}$ is

$$dW = \underline{F} \cdot d\underline{r}\ .$$

The total work done in moving particle along a wire which follows some curve C between two points P, Q is

$$W_C = \int_P^Q dW = \int_C \underline{F}(\underline{r}) \cdot d\underline{r}\ .$$

This is a line integral along the curve C.

More generally let $\underline{A}(\underline{r})$ be a vector field defined in the region R, and let C be a curve in R joining two points P and Q. $\underline{r}$ is the position vector at some point on the curve; $d\underline{r}$ is an infinitesimal length *along* the curve at $\underline{r}$. $\underline{t}$ is the **unit-vector** tangent to the curve at $\underline{r}$ (points in the direction of $\underline{dr}$)

$$\underline{t} \;=\; \frac{\underline{dr}}{ds}$$

where ds is the infinitesimal **arc-length**: $\boxed{ds = \sqrt{\underline{dr}\cdot\underline{dr}}.}$ Clearly, $\underline{t}\cdot\underline{t} = 1$.

NB In general, $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ **depends on the path** joining P and Q.

In Cartesian coordinates, we have

$$\int_C \underline{A}\cdot\underline{dr} \;=\; \int_C A_i dx_i \;=\; \int_C (A_1dx_1 \,+\, A_2dx_2 \,+\, A_3dx_3)$$

16. 3. Parametric Representation of a line integral

Often a curve in 3d can be parameterised by a single parameter e.g. if the curve were the trajectory of a particle then time would be the parameter e.g. sometimes the parameter of a line integral is chosen to be the arc-length s along the curve C.

Generally for parameterisation by λ (varying from λ_P to λ_Q)

$$x_i \;=\; x_i(\lambda), \quad \text{with } \lambda_P \le \lambda \le \lambda_Q$$

then

$$\int_C \underline{A}\cdot\underline{dr} \;=\; \int_{\lambda_P}^{\lambda_Q}\left(\underline{A}\cdot\frac{\underline{dr}}{d\lambda}\right)d\lambda = \int_{\lambda_P}^{\lambda_Q}\left(A_1\,\frac{dx_1}{d\lambda} \,+\, A_2\,\frac{dx_2}{d\lambda} \,+\, A_3\,\frac{dx_3}{d\lambda}\right)\,d\lambda$$

If necessary, the curve C may be subdivided into sections, each with a different parameterisation (piecewise smooth curve).

Example: $\underline{A} = (3x^2 + 6y)\,\underline{e}_1 - 14yz\underline{e}_2 + 20xz^2\underline{e}_3$. Evaluate $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ between the points with Cartesian coordinates $(0,0,0)$ and $(1,1,1)$, along the paths C:

1. $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$ (straight lines).

2. $x = t$, $y = t^2$ $z = t^3$; from $t = 0$ to $t = 1$.

1. a Along line from $(0, 0, 0)$ to $(1, 0, 0)$, we have $y = z = 0$, so $dy = dz = 0$, hence $\underline{dr} = \underline{e}_1\, dx$ and $\underline{A} = 3x^2\, \underline{e}_1$, (here parameter is x):

$$\int_{(0,0,0)}^{(1,0,0)} \underline{A} \cdot \underline{dr} \;=\; \int_{x=0}^{x=1} 3x^2\, dx \;=\; \left[x^3\right]_0^1 \;=\; 1$$

b Along line from $(1, 0, 0)$ to $(1, 1, 0)$, we have $x = 1$, $dx = 0$, $z = dz = 0$, so $\underline{dr} = \underline{e}_2\, dy$ (here parameter is y) and

$$\underline{A} \;=\; \left(3x^2 + 6y\right)\Big|_{x=1} \underline{e}_1 \;=\; (3 + 6y)\, \underline{e}_1.$$

$$\int_{(1,0,0)}^{(1,1,0)} \underline{A} \cdot \underline{dr} \;=\; \int_{y=0}^{y=1} (3 + 6y)\; \underline{e}_1 \cdot \underline{e}_2\, dy \;=\; 0.$$

c Along line from $(1, 1, 0)$ to $(1, 1, 1)$, we have $x = y = 1$, $dx = dy = 0$, and hence $\underline{dr} = \underline{e}_3\, dz$ and $\underline{A} = 9\, \underline{e}_1 - 14z\, \underline{e}_2 + 20z^2\, \underline{e}_3$, therefore

$$\int_{(1,1,0)}^{(1,1,1)} \underline{A} \cdot \underline{dr} \;=\; \int_{z=0}^{z=1} 20z^2\, dz \;=\; \left[\frac{20}{3} z^3\right]_0^1 \;=\; \frac{20}{3}$$

Adding up the 3 contributions we get

$$\int_C \underline{A} \cdot \underline{dr} \;=\; 1 \,+\, 0 \,+\, \frac{20}{3} \;=\; \frac{23}{3} \qquad \text{along path (1)}$$

2. To integrate $\underline{A} = (3x^2 + 6y)\, \underline{e}_1 - 14yz\underline{e}_2 + 20xz^2\underline{e}_3$ along path (2) (where the parameter is t), we write

$$\underline{r} \;=\; t\, \underline{e}_1 + t^2\, \underline{e}_2 + t^3\, \underline{e}_3 \qquad \frac{\underline{dr}}{dt} = \underline{e}_1 \,+\, 2t\, \underline{e}_2 \,+\, 3t^2\, \underline{e}_3$$

$$\underline{A} \;=\; \left(3t^2 + 6t^2\right)\, \underline{e}_1 \;-\; 14t^5\, \underline{e}_2 \;+\; 20t^7\, \underline{e}_3$$

therefore
$$\int_C \left(\underline{A} \cdot \frac{\underline{dr}}{dt}\right) dt \;=\; \int_{t=0}^{t=1} \left(9t^2 \;-\; 28t^6 \;+\; 60t^9\right)\, dt$$

$$=\; \left[3t^3 \;-\; 4t^7 + 6t^{10}\right]_0^1 \;=\; 5$$

Hence
$$\int_C \underline{A} \cdot \underline{dr} \;=\; 5 \qquad \text{along path (2)}$$

In this case, the integral of $\underline{A}$ from $(0,0,0)$ to $(1,1,1)$ depends on the path taken.

The line integral $\displaystyle\int_C \underline{A}\cdot\underline{dr}$ is a **scalar** quantity. Another **scalar** line integral is $\displaystyle\int_C f\,ds$ where $f(\underline{r})$ is a scalar field and ds is the infinitesimal arc-length introduced earlier.

Line integrals around a **simple** (doesn't intersect itself) **closed** curve C are denoted by $\displaystyle\oint_C$

$$\text{e.g.} \qquad \oint_C \underline{A}\cdot\underline{dr} \qquad \equiv \text{the } \textbf{circulation} \text{ of } \underline{A} \text{ around } C$$

Example : Let $f(\underline{r}) = ax^2 + by^2$. Evaluate $\displaystyle\oint_C f\,ds$ around the unit circle C:

$$x = \cos\phi,\ y = \sin\phi,\ z = 0;\ 0 \le \phi \le 2\pi.$$

$$\begin{array}{rrcl} \text{We have} & f(\underline{r}) & = & ax^2 + by^2 \ = \ a\cos^2\phi \,+\, b\sin^2\phi \\ & \underline{r} & = & \cos\phi\,\underline{e}_1 \,+\, \sin\phi\,\underline{e}_2 \\ & \underline{dr} & = & (-\sin\phi\,\underline{e}_1 \,+\, \cos\phi\,\underline{e}_2)\ d\phi \\ \text{so} & ds & = & \sqrt{\underline{dr}\cdot\underline{dr}} \ = \ \left(\cos^2\phi \,+\, \sin^2\phi\right)^{1/2}\, d\phi \ = \ d\phi \end{array}$$

Therefore, for this example,

$$\oint_C f\,ds \ = \ \int_0^{2\pi} \left(a\cos^2\phi \,+\, b\sin^2\phi\right) d\phi \ = \ \pi\,(a \,+\, b)$$

The **length** s of a curve C is given by $s = \displaystyle\int_C ds$. In this example $s = 2\pi$.

We can also define **vector** line integrals e.g.:-

1. $\displaystyle\int_C \underline{A}\,ds \ = \ \underline{e}_i \int_C A_i\,ds$ in Cartesian coordinates *e.g.* centre of mass of 1 d object

2. $\displaystyle\int_C \underline{A}\times\underline{dr} \ = \ \underline{e}_i\,\epsilon_{ijk}\int_C A_j\,dx_k$ in Cartesians.

Example : Consider a current of magnitude I flowing along a wire following a closed path C. The magnetic force on an element $\underline{dr}$ of the wire is proportional to $I\underline{dr}\times\underline{B}$ where $\underline{B}$ is the magnetic field at $\underline{r}$. Let $\underline{B}(\underline{r}) = x\,\underline{e}_1 + y\,\underline{e}_2$. Here we evaluate $\displaystyle\oint_C \underline{B}\times\underline{dr}$ around a circle of radius a.

$$\begin{array}{rcl} \underline{B} & = & a\cos\phi\,\underline{e}_1 \,+\, a\sin\phi\,\underline{e}_2 \\ \underline{dr} & = & (-a\sin\phi\,\underline{e}_1 \,+\, a\cos\phi\,\underline{e}_2)\ d\phi \end{array}$$

$$\text{Hence} \qquad \oint_C \underline{B}\times\underline{dr} \ = \ \oint\int_0^{2\pi} \left(a^2\cos^2\phi \,+\, a^2\sin^2\phi\right)\underline{e}_3\,d\phi \ = \ \underline{e}_3\,a^2\int_0^{2\pi} d\phi \ = \ 2\pi a^2\,\underline{e}_3$$