Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **28-NOV-20**

Attendance : **10**

Green's theorem :

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$\vec{F} = P\hat{i} + Q\hat{j}$$

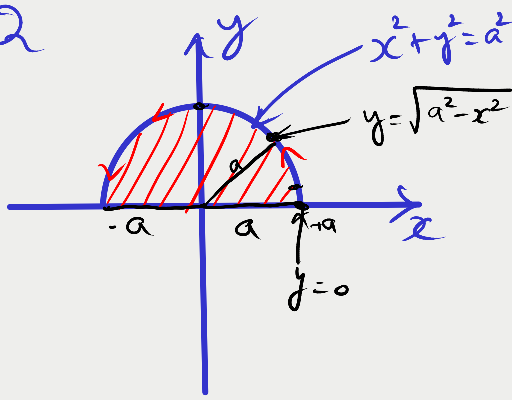
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (P\hat{i} + Q\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

Example 82. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.
(*M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004*)

$$I = \oint_C \underbrace{(2x^2 - y^2)}_{P}\,dx + \underbrace{(x^2 + y^2)}_{Q}\,dy$$

$$C : x^2 + y^2 = a^2$$



$P = 2x^2 - y^2 \quad ; \quad Q = x^2 + y^2$

$\frac{\partial P}{\partial y} = -2y \quad ; \quad \frac{\partial Q}{\partial x} = 2x$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (-2y) \Rightarrow 2(x+y)$$

$$= 2\iint_R (x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} (x+y)\,dy$$

$$= 2\int_{-a}^{a} dx \left[\int_0^{\sqrt{a^2-x^2}} x\,dy + \int_0^{\sqrt{a^2-x^2}} y \cdot dy \right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot y \Big|_0^{\sqrt{a^2-x^2}} + \frac{y^2}{2} \Big|_0^{\sqrt{a^2-x^2}} \right]$$

$$= 2\int_{-a}^{a} dx \cdot \left[x \cdot \sqrt{a^2-x^2} + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} \right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot \sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right]$$

$$= 2\cdot\int_{-a}^{a} \cancel{\underbrace{\overbrace{x(a^2-x^2)^{1/2}}^{f_1(x)}}_{f(x)}\,dx} \xrightarrow{} 0 + \cancel{2}\int_{-a}^{a} \underbrace{\overbrace{\frac{(a^2-x^2)}{\cancel{2}}}^{f_2(x)}}_{f(x)}\,dx$$

$\because f_1(x)$ is odd

$$\boxed{\begin{aligned}\int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even fn:}\\ &= 0 \quad\quad\quad\quad\quad\ \ \text{if } f(x) \text{ is odd fn:}\end{aligned}}$$

If $f(-x) = f(x)$ then $f(x)$ is even

If $f(-x) = -f(x)$ then $f(x)$ is odd

$f_1(x)$ is even or odd ?

$$f_1(x) = x\sqrt{a^2-x^2}$$

$$x = -x$$

$$f_1(-x) = -\overbrace{x\cdot\sqrt{a^2-x^2}}^{f_1(x)}$$

$$f_1(-x) = -f_1(x) \leftarrow \text{odd}$$

$$f_2(x) = a^2-x^2$$

$$f_2(-x) = a^2-(-x)^2$$

$$= \underbrace{a^2-x^2}_{f_2(x)}$$

$$\underline{f_2(-x) = f_2(x)}$$

even

$$= 2\cdot\int_{0}^{a}(a^2-x^2)\,dx$$

$$= 2\cdot\left[a^2\cdot x\Big|_0^a - \frac{x^3}{3}\Big|_0^a\right]$$

$$= 2\cdot\left[a^2\cdot a - \frac{a^3}{3}\right] \Rightarrow 2\cdot\left[a^3-\frac{a^3}{3}\right]$$

$$= 2\cdot\frac{2a^3}{3} \Rightarrow 4\frac{a^3}{3}\ //.$$

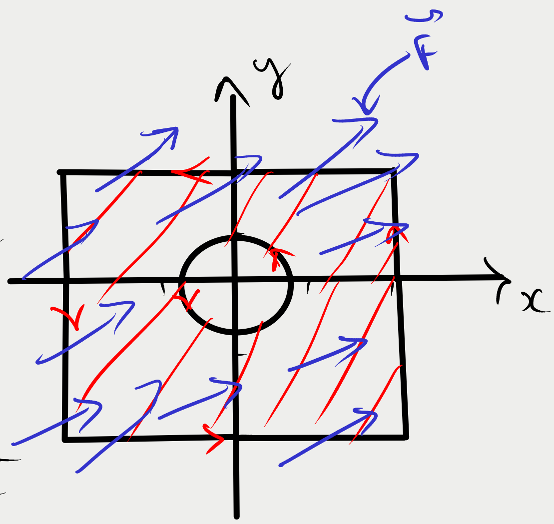
Example 83. *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2=1$ *and* $C_2 : x=\pm 2,\ y=\pm 2$. *(Gujarat, I Semester, Jan 2009)*

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$C_1 : x^2+y^2=1$

$C_2 : x=\pm 2\ ;\ y=\pm 2$

$$\vec{F} = \underbrace{\frac{-y}{x^2+y}}_{P}\,\hat{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q}\,\hat{j}$$



$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$P = \frac{-y}{x^2+y^2} \quad ; \quad Q = \frac{x}{x^2+y^2}$$

$$P = \underbrace{-y}_{u}\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial P}{\partial y} = +y\cdot(x^2+y^2)^{-1-1}\cdot 2y + (x^2+y^2)^{-1}(-1)$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\cdot\frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{2y^2-x^2-y^2}{(x^2+y^2)^2}$$

$$Q = \underbrace{x}_{u}\cdot\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial Q}{\partial x} = x\cdot -1(x^2+y^2)^{-2}\cdot 2x + (x^2+y^2)^{-1}\cdot 1$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$