Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **4-Nov-2020**

Attendance : **10**

EXERCISE 5.8

1. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\dfrac{\vec{r}}{|\vec{r}|^3}\right) = 0$,

   (*ii*) div (grad r^n) $= n\,(n+1)\,r^{n-2}$ (*AMIETE, June 2010*) (*iii*) div $(\vec{r}\,\phi) = 3\phi + r$ grad ϕ.

2. Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

   (*R.G.P.V., Bhopal, Dec. 2003*)

3. Show that $\nabla.(\phi\, A) = \nabla\phi.A + \phi(\nabla.A)$
4. If ρ, ϕ, z are cylindrical coordinates, show that grad (log ρ) and grad ϕ are solenoidal vectors.
5. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their corresponding expression in orthogonal curvilinear coordinates.

Prove the following:

6. $\vec{\nabla}.(\phi\vec{F}) = (\vec{\nabla}\phi).\vec{F} + \phi(\vec{\nabla}.\vec{F})$

7. (*a*) $\nabla.(\nabla\phi) = \nabla^2\phi$ (*b*) $\vec{\nabla} \times \dfrac{(\vec{A}\times\vec{R})}{r^n} = \dfrac{(2-n)\vec{A}}{r^n} + \dfrac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$

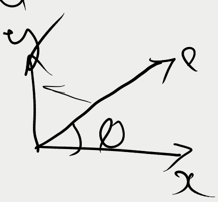
8. div $(f\,\nabla\, g)$ – div $(g\,\nabla\, f) = f\,\nabla^2 g - g\,\nabla^2 f$

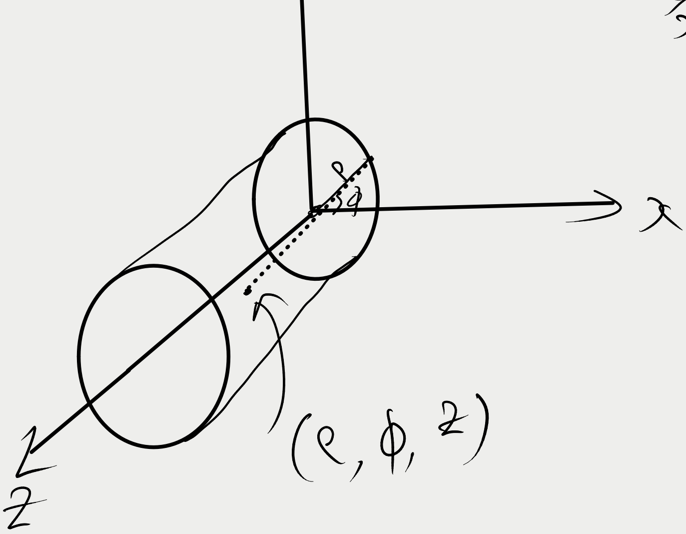
4, 7b, 8

Ex 5.8, 4th problem :

$\vec{\nabla}\cdot\vec{F} = 0$ then, $\vec{F}$ is solenoidal

$\vec{\nabla}(\log \rho)$; $\vec{\nabla}(\phi)$ are Solenoidal





$\rho = \sqrt{x^2+y^2}$

$x = \rho\cos\phi$
$y = \rho\sin\phi$

$\dfrac{y}{x} = \dfrac{\rho\sin\phi}{\rho\cos\phi} = \tan\phi$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\underbrace{\vec{\nabla}(\log \rho)}_{\vec{\nabla}\cdot} = ? \quad ; \quad \underbrace{\vec{\nabla}\phi}_{\vec{\nabla}\cdot} = ?$$

Ex 5.8

(b) $\vec{\nabla} \times \frac{(\vec{A} \times \vec{R})}{r^n} = \frac{(2-n)\vec{A}}{r^n} + \frac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{A} \times \vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$\vec{A} \times \vec{R} = \hat{i}(A_2 z - A_3 y) - \hat{j}\,(A_1 z - A_3 x) + \hat{k}\,(A_1 y - A_2 x)$$

$$\frac{\vec{A} \times \vec{R}}{r^n} = \frac{\hat{i}\,(A_2 z - A_3 y) + \hat{j}\,(A_3 x - A_1 z) + \hat{k}\,(A_1 y - A_2 x)}{(x^2 + y^2 + z^2)^{n/2}}$$

$$\frac{\vec{A} \times \vec{R}}{r^n} = \left\{ \frac{A_2 z - A_3 y}{(x^2+y^2+z^2)^{n/2}}\,\hat{i} + \frac{A_3 x - A_1 z}{(x^2+y^2+z^2)^{n/2}}\,\hat{j} + \frac{A_1 y - A_2 x}{(x^2+y^2+z^2)^{n/2}}\,\hat{k} \right.$$

$$\vec{\nabla} \times \frac{\vec{A} \times \vec{R}}{r^n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{A_2 z - A_3 y}{(x^2+y^2+z^2)^{n/2}} & \frac{A_3 x - A_1 z}{(x^2+y^2+z^2)^{n/2}} & \frac{A_1 y - A_2 x}{(x^2+y^2+z^2)^{n/2}} \end{vmatrix}$$

Ex 5.8 , 8 problem

$$\vec{\nabla}\cdot(f\vec{\nabla} g) - \vec{\nabla}\cdot(g\vec{\nabla} f) = f\nabla^2 g - g\nabla^2 f$$

$$\vec{\nabla} g = \vec{g} = g_1\hat{i} + g_2\hat{j} + g_3\hat{k}$$

$$f\vec{\nabla} g = f\vec{g} = fg_1\hat{i} + fg_2\hat{j} + fg_3\hat{k}$$

$$\vec{\nabla}\cdot(f\vec{\nabla}g) = \vec{\nabla}\cdot(f\vec{g}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot$$

$$\left(fg_1\hat{i} + fg_2\hat{j} + fg_3\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}fg_1 + \frac{\partial}{\partial y}fg_2 + \frac{\partial}{\partial z}fg_3\right)$$

$f(x,y,z)$

$$= f\frac{\partial g_1}{\partial x} + g_1\frac{\partial f}{\partial x} + f\frac{\partial g_2}{\partial y} + g_2\frac{\partial f}{\partial y}$$

$$+ f\frac{\partial g_3}{\partial z} + g_3\cdot\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right) + g_1\frac{\partial f}{\partial x} + g_2\frac{\partial f}{\partial y}$$

$$+ g_3\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{g}$$

$$+ \vec{g}\cdot\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\vec{\nabla}f}$$

$$= f\vec{\nabla}\cdot\vec{g} + \vec{g}\cdot\vec{\nabla}f$$

$$\vec{g} = \vec{\nabla}g$$

$$= f\vec{\nabla}\cdot\vec{\nabla}g + \vec{g}\cdot\vec{\nabla}f$$

$$\boxed{\vec{\nabla}\cdot(f\vec{\nabla}g) = f\nabla^2 g + \cancel{\vec{\nabla}g\cdot\vec{\nabla}f}} \quad \text{— } (1)$$

$$\boxed{\vec{\nabla}\cdot(g\vec{\nabla}f) = g\nabla^2 f + \cancel{\vec{\nabla}f\cdot\vec{\nabla}g}} \quad \text{— } (2)$$

① − ②

$$\boxed{\vec{\nabla}\cdot(f\vec{\nabla}g) - \vec{\nabla}\cdot(g\vec{\nabla}f) = f\nabla^2 g - g\nabla^2 f}$$

3. Prove that:

(*i*) $\nabla.(\phi\vec{A}) = \nabla\phi.\vec{A} + \phi(\nabla.\vec{A})$ ✓

(*ii*) $\nabla(A.B) = (A.\nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$ (*R.G.P.V. Bhopal, June 2004*)

(*iii*) $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$

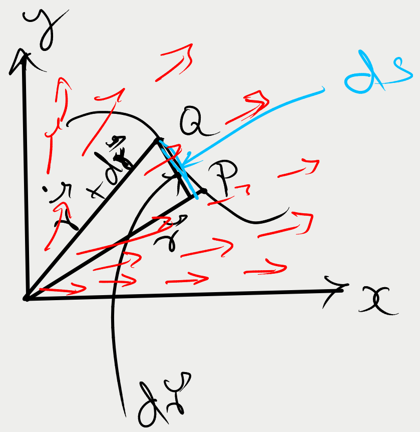
$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{A}\cdot\vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y} + \frac{\partial b_3}{\partial z}$$

Line Integral :

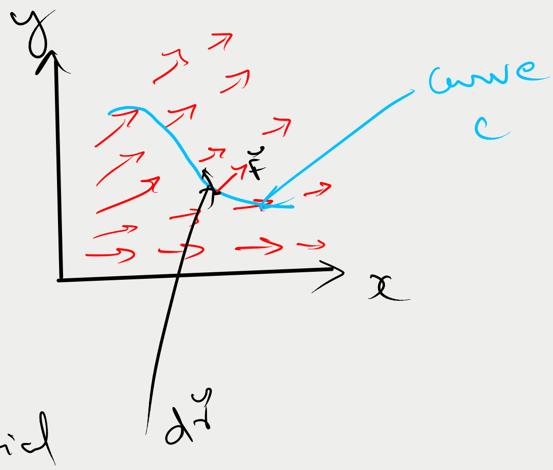


$$ds = |d\vec{r}|$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r}}$$

$$\left.\begin{array}{c}\text{Line Integral}\\ \text{of a}\\ \text{Vector field}\\ \vec{F} \text{ along a}\\ \text{Curve "C"}\end{array}\right\} = \int \vec{F}\cdot d\vec{r}$$

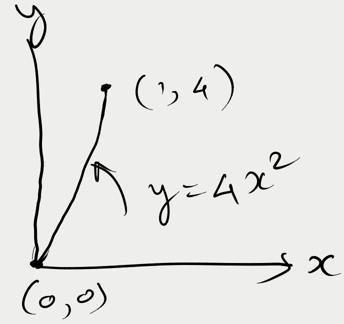
tangential vector to the curve C



Eg. 65

$$\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$$

$C \Rightarrow y = 4x^2$



$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\vec{F}\cdot d\vec{r} = 2x^2y\,dx + 3xy\,dy$$

$$W.d = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{\substack{(0,0)\\ x_i\ y_i}}^{\substack{x_f\ y_f\\ (1,4)}} 2x^2y\,dx + 3xy\,dy$$

$$= \int_0^1 2x^2 y\,dx + \int_0^4 3xy\,dy$$

$y = 4x^2$

$$y = 4x^2$$

$$dy = 4\cdot 2x$$

$$= 8x\,dx$$

$$= \int_0^1 2x^2\cdot 4x^2\,dx + 3x\cdot 4x^2\cdot 8x\,dx$$

$$= \frac{104}{5}$$ // check it.

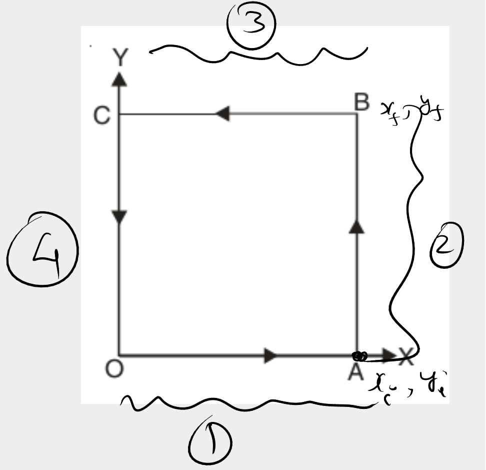
Example 66. *Evaluate $\int_C \vec{F}\,.\,d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

$C = OABC$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$$

$$+ \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$



$\vec{F} = x^2\hat{i} + xy\hat{j}$ $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

@ ① i.e OA $\quad x_i = 0 \quad y_i = 0$

$\underset{\sim}{x_f = a} \quad y_f = 0$

$dy = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^a$$

$$= \frac{a^3}{3}$$

@ ② i.e along AB

$x_i = a \quad y_i = 0$

$x_f = a \quad y_f = a$

$dx = 0$

$\vec{F}\cdot d\vec{r} = \underset{\sim}{x^2 dx} + xy\,dy$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a xy\,dy = \int_0^a ay\,dy$$

$$= a\int_0^a y\,dy \Rightarrow a\cdot\left[\frac{y^2}{2}\right]_0^a$$

$$= a\cdot\frac{a^2}{2}$$

$$= \frac{a^3}{2} \quad a.$$

@ ③ i.e along BC $x_i = a$ $y_i = a$

$x_f = 0$ $y_f = a$

$dy = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$

$$\int_{BC}\vec{F}\cdot d\vec{r} = \int_a^0 x^2\,dx = \left.\frac{x^3}{3}\right|_a^0$$

$$(\text{or}) \quad = \left.-\frac{x^3}{3}\right|_0^a$$

$$= -\frac{a^3}{3} \quad //.$$

@ ④ i.e along CO $x_i = 0$ $y_i = a$

$x_f = 0$ $y_f = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$

$dx = 0$

$x = 0$

$$\int_{CO}\vec{F}\cdot d\vec{r} = \int_0^a \cancel{0}^2 dx + 0\cdot y\,dy$$

$$\int_{CO}\vec{F}\cdot d\vec{r} = 0$$

$$\int_{OABC} \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \cancel{\frac{a^3}{3}} + \frac{a^3}{2} + \cancel{\left(\frac{-a^3}{3}\right)} + 0$$

$$\boxed{\int_{OABC} \vec{F}\cdot d\vec{r} = \frac{a^3}{2}}$$

Example 68. *The acceleration of a particle at time t is given by*

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity $\vec{v}$ and displacement $\vec{r}$ be zero at t = 0, find $\vec{v}$ and $\vec{r}$ at any point t.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \quad ; \quad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt - \hat{j}\int 8\sin 2t\,dt + 6\hat{k}\int t\,dt$$

$$= \hat{i}\,18\cdot\left(\frac{\sin 3t}{3}\right) - 8\hat{j}\left(\frac{-\cos 2t}{2}\right) + 6\hat{k}\left(\frac{t^2}{2}\right) + C$$

$$\vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\hat{k} + \underline{c}$$

At $t=0$; $\vec{v}=0$

$$0 = 6\sin 3(0)\,\hat{i} + 4\cos 2(0)\,\hat{j} + 0 + \underline{c}$$

$$0 = 0\hat{i} + 4\hat{j} + 0 + c$$

$$\therefore\ c = -4\hat{j}$$

$$\vec{v} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

$$\vec{r} = \int \vec{v}\cdot dt = \hat{i}\int 6\sin 3t\,dt + 4\hat{j}\int(\cos 2t - 1)\,dt$$
$$+ 3\hat{k}\int t^2\,dt$$

$$= 6\hat{i}\left(-\frac{\cos 3t}{3}\right) + 4\hat{j}\left(\frac{\sin 2t}{2}\right) - 4\hat{j}(t)$$
$$+ 3\hat{k}\left(\frac{t^3}{3}\right) + c$$

$$\vec{r} = -2\hat{i}\cos 3t + 2\hat{j}\sin 2t - 4t\hat{j}$$
$$+ t^3\hat{k} + c$$

@ $t=0$; $\vec{r}=0$

$$0 = -2\hat{i}\cos 0 + 2\hat{j}\sin 0 - 0\hat{j}$$
$$+ 0\hat{k} + c$$

$$0 = -2\hat{i} + c$$

$$c = 2\hat{i}$$

$$\vec{r} = -2\hat{i}\cos 3t + 2\hat{j}\sin 2t - 4t\hat{j} + t^3\hat{k} + 2\hat{i}$$

$$\vec{r} = 2\hat{i}\left(1 - \cos 3t\right) + 2\left(\sin 2t - 2t\right)\hat{j} + t^3\hat{k}$$

Ex 5.10 1 , 2