In Cartesian co-ordinates:

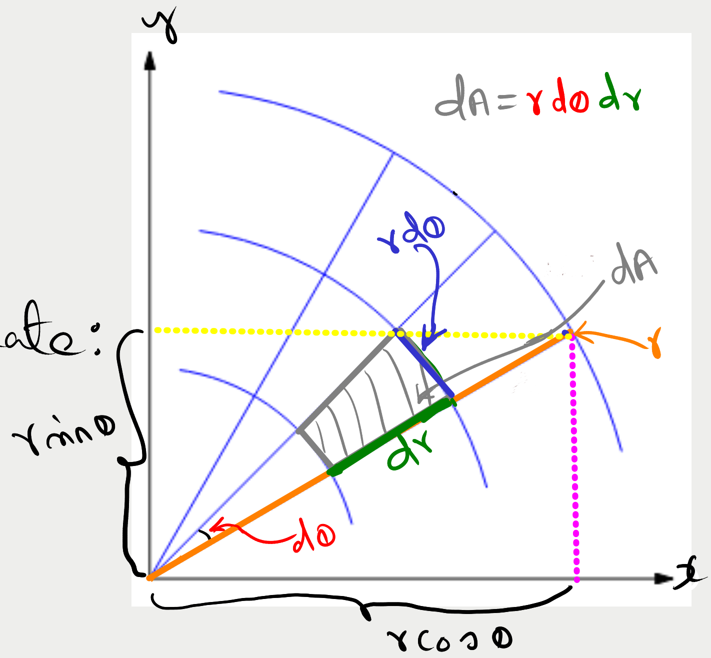
$$dA = dx\,dy$$

Elemental Area

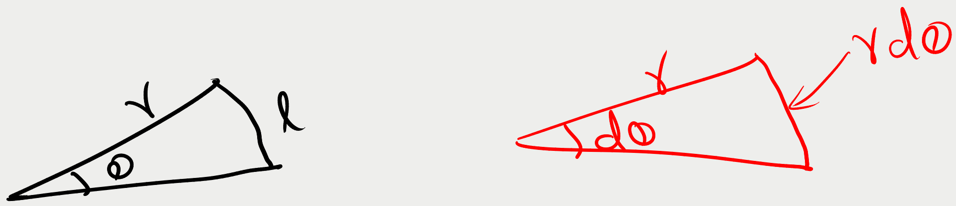
In cylindrical coordinate:

$$dA = r\,d\theta\,dr$$

Elemental area



Sector



$$l = r\theta$$

Transformation from Cartesian to Cylinderical coordinate

$$x \longrightarrow r\cos\theta$$

$$y \longrightarrow r\sin\theta$$

Elemental area

$$dA = dx\,dy \longrightarrow r\,d\theta\,dr$$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

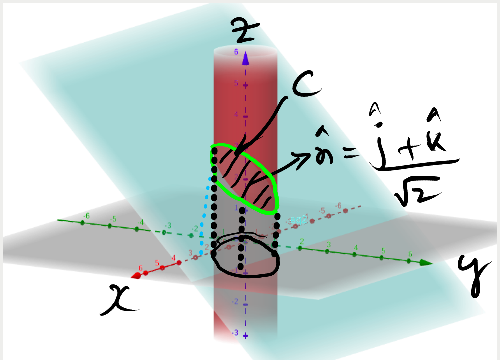
Date : **02-Dec-20**

Attendance : **9**

Example 87. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)

Stoke's theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,ds$$



$$\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{F} = (1+2y)\hat{k} \quad ; \quad \phi = y+z-2$$

$$\hat{n} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$\vec{\nabla}\phi = \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$= \frac{\hat{j}+\hat{k}}{\sqrt{1^2+1^2}} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = (\hat{\underset{\sim}{k}} + 2y\hat{k})\cdot\frac{1}{\sqrt{2}}(\hat{j}+\hat{\underset{\sim}{k}})$$

$$= \frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k} + 2y\cdot\frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k}$$

$$\boxed{(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{2}}(1+2y)}$$

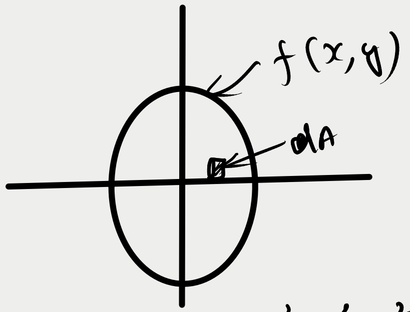
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,\underset{\sim}{ds}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,dx\,dy$$

$$\iint_S (x^2+y^2)\,\underset{\sim}{\overset{dA}{dx\,dy}}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1+2y}{\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\,dx\,dy$$

$$= \iint_S \underset{\sim}{(1+2y)}\,\underset{\sim}{\overset{dA}{dx\,dy}}$$



In cylinderical coordinate

$$x \rightarrow r\cos\phi \;;\quad y \rightarrow r\sin\phi$$

$$dx\,dy \rightarrow r\,d\phi\,dr$$

$$= \int_0^{2\pi}\int_0^{r=1} (1+2(r\sin\phi))\cdot r\, d\phi\, dr$$

$$= \int_0^{2\pi} d\phi \int_0^1 \left[r\,dr + 2\cdot r^2 \sin\phi\, dr\right]$$

$$= \int_0^{2\pi} d\phi \left[\frac{r^2}{2}\Big|_0^1 + 2\cdot\frac{r^3}{3}\sin\phi\Big|_0^1\right]$$

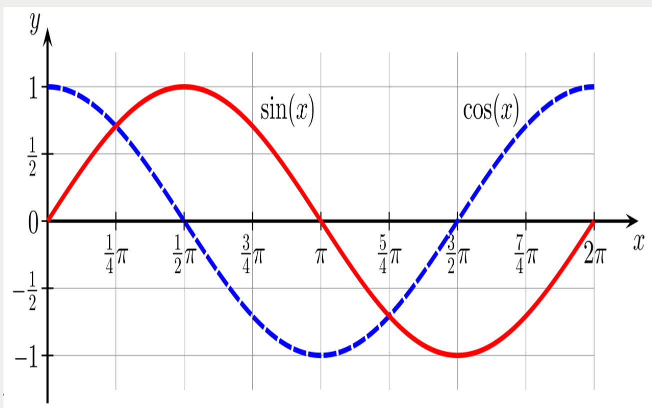
$$= \int_0^{2\pi} d\phi \left[\frac{1}{2} + \frac{2}{3}\sin\phi\right]$$

$$= \frac{1}{2}\phi\Big|_0^{2\pi} + \frac{2}{3}(-\cos\phi)\Big|_0^{2\pi}$$

$$= \left[\pi - 0 - \left(\frac{2}{3}\cancelto{1}{\cos 2\pi} - \frac{2}{3}\cancelto{1}{\cos 0}\right)\right]$$

$$= \pi - \cancel{\frac{2}{3}(1)} + \cancel{\frac{2}{3}(1)}$$

$$= \pi$$



$$\oint_C (-y^2\hat{i} + x\hat{j} + z^2\hat{k})\cdot d\vec{r} = \pi$$