From Green's theorem w.k.T

$$\oint_C P\,dx + Q\,dy = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=1} dx\,dy$$

$$\left.\begin{array}{ll} P = -y & ; \quad Q = x \\ \dfrac{\partial P}{\partial y} = -1 & ; \quad \dfrac{\partial Q}{\partial x} = 1 \end{array}\right\}$$

$$\oint_C -y\,dx + x\,dy = \iint_R (1 - (-1))\,dx\,dy$$

$ds = dx\,dy$

$$\oint_C x\,dy - y\,dx = 2\iint_R dx\,dy$$

Area

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \underbrace{\iint_R dx\,dy}$$

$$\boxed{\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Example 84. *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$ (U.P. I, Semester, Dec. 2008)

$C_1 \Rightarrow y = x$ $\quad C_3 \Rightarrow y = \frac{x}{4}$

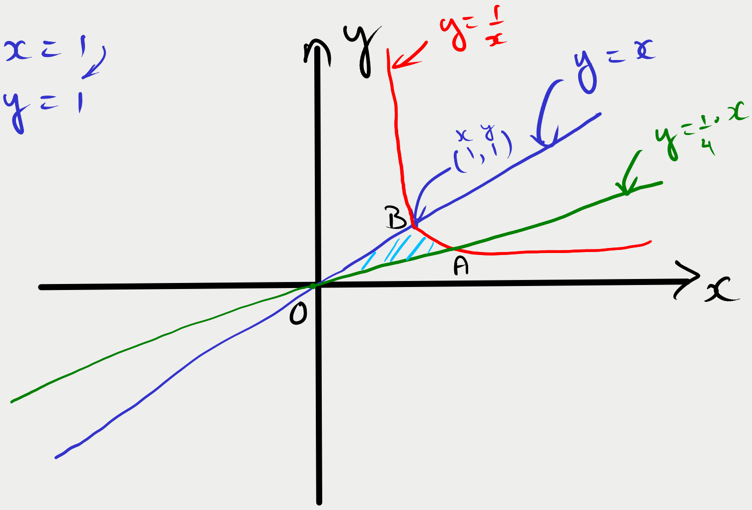
$C_2 \Rightarrow y = \frac{1}{x}$

C_1: $y = x \Rightarrow$ Slope $= 1$

when $x = 1$, $y = 1$

$y = \underset{\substack{\uparrow \\ \text{slope}}}{m}x + c \rightarrow$ intercept

$\left.\begin{array}{l} c = 0 \\ m = 1 \end{array}\right\} \Rightarrow y = x$



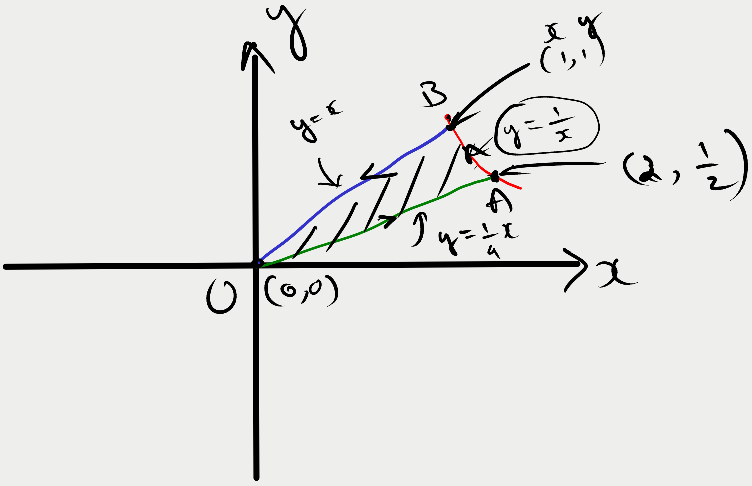
C_2: $y = \frac{1}{x}$

when $x = 0$; $y = \infty$

$x = 1$; $y = 1$

C_3: $y = \frac{1}{4}x$

Slope $m = \frac{1}{4}$



For what value of x; $y = x$ & $y = \frac{1}{x}$ are equal?

$x = 1$ $\quad \left.\begin{array}{l} \overset{C_1}{y} = x \\ \& \underset{C_2}{y} = \frac{1}{x} \end{array}\right\}$ are equal

For what value of x; $\underbrace{y = \frac{1}{4}x}_{C_3}$ & $\underbrace{y = \frac{1}{x}}_{C_2}$ are equal?

$\boxed{x = 2}$ $\quad y = \frac{1}{4} \cdot 2 = \frac{1}{2}$ $\qquad x = 2$

$\qquad y = \frac{1}{2} = \frac{1}{2}$ $\qquad y = \frac{1}{2}$

$OA \Rightarrow (0, 0), (2, \frac{1}{2})$; $AB \Rightarrow (2, \frac{1}{2}) (1, 1)$

$BO \Rightarrow (1,1),\ (0,0)$

$$\boxed{\text{Area} = \frac{1}{2}\oint_{OAB} x\,dy - y\,dx}$$

This is from Green's theorem formula

$$\oint_{OAB} x\,dy - y\,dx = \int_{OA} x\,dy - y\,dx + \int_{AB} x\,dy - y\,dx + \int_{BO} x\,dy - y\,dx$$

For OA $\quad C_3:\ y = \frac{1}{4}x \qquad x_i = 0 \qquad y_i = 0$

$$dy = \frac{1}{4}\,dx \qquad x_f = 2 \qquad y_f = \frac{1}{2}$$

$$\int_{OA} x\,dy - y\,dx = \int_{OA} x\cdot\left(\frac{dx}{4}\right) - \left(\frac{x}{4}\right)\cdot dx$$

$$= \int_0^2 \cancel{\frac{x\,dx}{4} - \frac{x\,dx}{4}} \rightarrow 0$$

For AB $\quad C_2:\ y = \frac{1}{x} \qquad x_i = 2\ ;\ y_i = \frac{1}{2}$

$$dy = \frac{-dx}{x^2} \qquad x_f = 1\ ;\ y_f = 1$$

$$\int_{AB} x\,dy - y\,dx = \int_{AB} \cancel{x}\cdot\left(\frac{-dx}{x^{\cancel{2}}}\right) - \left(\frac{1}{x}\right)\cdot dx$$

$$= \int_2^1 -\frac{dx}{x} - \frac{dx}{x}$$

$$= -2\int_{2}^{1} \frac{dx}{x}$$

$$= -2 \cdot \log x \Big|_{2}^{1} \Rightarrow -2 \cdot -\log x \Big|_{1}^{2}$$

$$= 2\left[\log(2) - \cancel{\log(1)}^{\,0}\right]$$

$$\int_{AB} x\,dx - y\,dy = 2 \cdot \log 2$$

for BO c_1: $y = x$ $\quad x_i = 1 \;;\; y_i = 1$

$dy = dx$ $\quad x_f = 0 \;;\; y_f = 0$

$$\int_{BO} x\,dy - y\,dx = \int_{BO} x(dx) - (x) \cdot dx$$

$$= \int_{1}^{0} \cancel{x\,dx - x\,dx}^{\,0}$$

line integrals.

$$\oint_{OAB} x\,dy - y\,dx = \underbrace{\int_{OA}}_{=0} + \underbrace{\int_{AB}}_{=2 \cdot \log 2} + \underbrace{\int_{BO}}_{=0}$$

$$\oint_{OAB} x\,dy - y\,dx = 0 + 2\log 2 + 0$$

$$\text{Area} = \frac{1}{2}\oint_{OAB} x\,dy - y\,dx \Rightarrow \text{Area} = \frac{1}{2} \cdot 2\log 2$$

$$\boxed{\text{Area} = \log 2}$$ //.

Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

The circulation of vector F around a closed curve C is equal to the flux of the curve of the vector through the surface S bounded by the curve C.

$$\oint_c \overline{F}\cdot d\overline{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\,d\vec{s} = \iint_S curl\,\vec{F}\cdot d\vec{S}$$

Example 85. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1$, $z = y^2$. (M.D.U., Dec 2009)

By Stoke's theorem w.r.t

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \underline{\vec{\nabla}\times\vec{F}}\cdot\hat{n}\cdot\frac{dx\,dy}{\underline{|\hat{n}\cdot\hat{k}|}}$$

If our $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

and our $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (yz\,dx + zx\,dy + xy\,dz)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\,[x - x] - \hat{j}\,[y - y] + \hat{k}\,[z - z]$$

$\vec{\nabla} \times \vec{F} = 0$

$\therefore$ By Stoke's theorem

$\oint_c \vec{F} \cdot d\vec{r} = 0 \quad \because \quad \vec{\nabla} \times \vec{F} = 0$

$$\oint_c \vec{f} \cdot d\vec{r} = \iint_S \cancel{\vec{\nabla} \times \vec{f}}^{\,0} \cdot \hat{n} \, \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

Example 86. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

where c is the circle $x^2 + y^2 = 1$, *corresponding to the surface of sphere of unit radius.* $x^2+y^2+z^2=1$

(U.P., I Semester, Winter 2001)

$x^2+y^2+z^2=1$

$\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$ $\qquad \vec{\nabla} \times \vec{F}$

$\hat{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

$\phi = x^2+y^2+z^2-1$

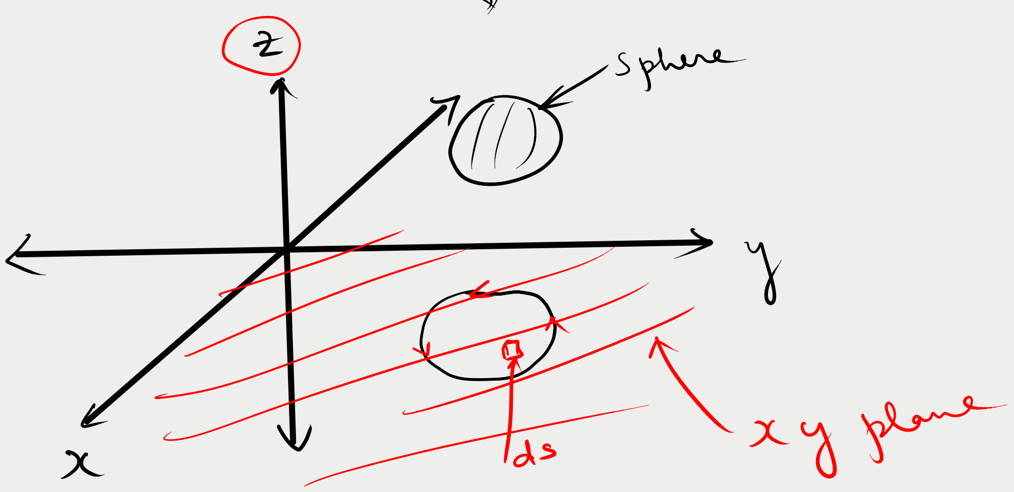
By Stoke's theorem:

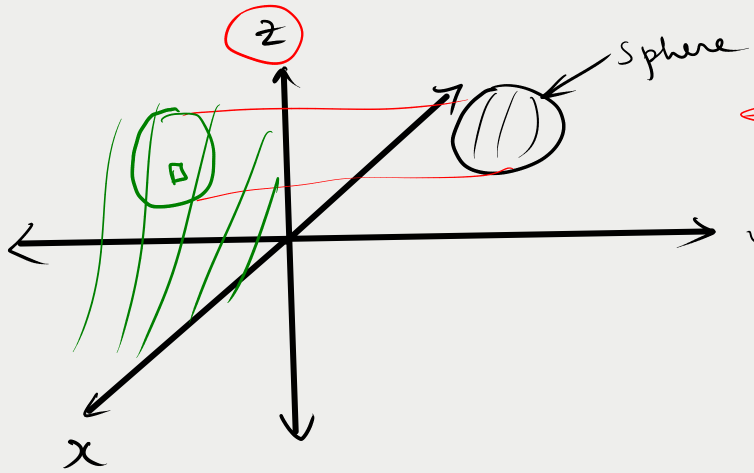
$\hat{n} \cdot \hat{k}$

$$\oint_c \vec{f} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \, \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

ds

$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$





$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\ \frac{dx\cdot dz}{|\hat{n}\cdot\hat{j}|}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$0-(-1)$

$$= \hat{i}\left[\cancel{\quad}^{\,0}\right] - \hat{j}\left[\cancel{\quad}^{\,0}\right] + \hat{k}\left[0+1\right]$$

$$\boxed{\vec{\nabla}\times\vec{F} = \hat{k}}$$

$$\phi = x^2+y^2+z^2-1$$

$$\vec{\nabla}\phi = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}}, \quad x^2+y^2+z^2 \to 1$$

$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

Given:

$x^2+y^2+z^2=1$

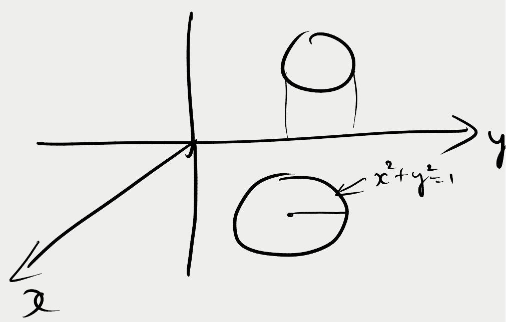
$$\boxed{\hat{n}\cdot\hat{k} = z}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

$$= \iint_S \cancel{\hat{k}\cdot\hat{n}}^{\,z} \cdot \frac{dx\,dy}{z}$$

$z^2=0$

$$= \iint_S \cancel{z} \cdot \frac{dx\,dy}{\cancel{z}}$$



$$\oint_C \tilde{f}\cdot d\tilde{r} = \iint_S dx\,dy$$

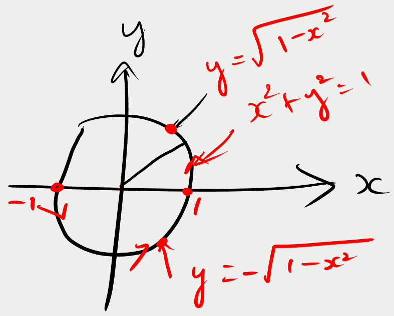
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$y=\pm\sqrt{1-x^2}$

$$= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$



$$= \int_{-1}^{1} dx \cdot y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1}\left(2\cdot\sqrt{1-x^2}\right)dx$$

$$= 2\int_{-1}^{1} \underbrace{\sqrt{1-x^2}}_{f(x)}\,dx$$

$f(x) = \sqrt{1-x^2}$ and its a even fn $\therefore$

$$\therefore \quad = 2 \cdot 2 \int_0^1 \left(\sqrt{1-x^2}\right) dx$$

~~Formula:~~ ← Pls chk this formula

$$\int_0^1 (a^2-x^2)^{1/2}\, dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \cancel{4} \cdot \frac{1}{\cancel{2}}\left[\underbrace{x(1-x^2)^{1/2}}_{\to 0 \text{ (for both limits)}} + 1 \sin^{-1}(x)\right]_0^1$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= \cancel{2}\left[\frac{\pi}{\cancel{2}} - 0\right]$$

$$\oint_c \vec{F} \cdot d\vec{r} = \pi$$

//.