Lecture 13: More on Gradient; the Operator 'Del'

13. 1. Examples of the Gradient in Physical Laws

Gravitational force due to Earth: Consider the potential energy of a particle of mass m, a height z above the Earth's surface $V = mgz$. Then the force due to gravity can be written as $F = -\underline{\nabla}\, V = -mg\, \underline{e}_3$.

Gravitational attraction: Now consider the gravitational force on a mass m at $\underline{r}$ due to a mass m_0 at the origin i.e. Newton's law of Gravitation: $\underline{F} = -(Gmm_0/r^2)\underline{\hat{r}}$. We can write this as

$$\underline{F} = -\underline{\nabla}\, V$$

where the potential energy $V = -Gmm_0/r$ (see p.51 for how to calculate $\underline{\nabla}(1/r)$).

In these two examples we see that force acts down the potential energy gradient.

Diffusion: In Physics 2 you may have encountered the idea of diffusion: for example in a gas the molecular motion effectively smoothes out the density. This can be described by the current of particles $\underline{j}(\underline{r})$ being proportional to the gradient of the density (Fick's Law)

$$\underline{j}(\underline{r}) = -D\underline{\nabla} n(\underline{r})$$

i.e. the diffusion current flows down the concentration gradient.

13. 2. Examples on gradient

Last lecture some examples using 'xyz' notation were given. Here we do some exercises with suffix notation (in the lecture we will repeat using 'xyz'). As usual suffix notation is most convenient for proving more complicated identities.

1. Let $\phi(\underline{r}) = r^2 = x_1^2 + x_2^2 + x_3^2$, then

$$\begin{aligned} \underline{\nabla}\phi(\underline{r}) &= \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right) (x_1^2 + x_2^2 + x_3^2) \\ &= 2x_1\, \underline{e}_1 + 2x_2\, \underline{e}_2 + 2x_3\, \underline{e}_3 = 2\underline{r} \end{aligned}$$

   In suffix notation

$$\underline{\nabla}\, r^2 = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) (x_j x_j) = \underline{e}_i\, (\delta_{ij} x_j + x_j \delta_{ij}) = \underline{e}_i\, 2x_i = 2\underline{r}$$

   In the above we have used the important fact

$$\boxed{\frac{\partial x_i}{\partial x_j} = \delta_{ij}}$$

   The level surfaces of r^2 are spheres centred on the origin, and the gradient of r^2 at $\underline{r}$ points radially outward with magnitude $2r$.

2. Let $\phi = \underline{a} \cdot \underline{r}$ where $\underline{a}$ is a *constant* vector.

$$\underline{\nabla}\,(\underline{a} \cdot \underline{r}) = \left(\underline{e}_i\,\frac{\partial}{\partial x_i}\right)\ (a_j x_j) = \underline{e}_i\,a_j \delta_{ij} = \underline{a}$$

   This is not surprising, since the level surfaces $\underline{a} \cdot \underline{r} = c$ are planes orthogonal to $\underline{a}$.

3. Let $\phi(\underline{r}) = r = \sqrt{x_1^2 + x_2^2 + x_3^2} = (x_j x_j)^{1/2}$

$$\begin{aligned}
\underline{\nabla}\,r &= \left(\underline{e}_i\,\frac{\partial}{\partial x_i}\right)\ (x_j x_j)^{1/2} \\
&= \underline{e}_i\,\frac{1}{2}\,(x_j x_j)^{-1/2}\,\frac{\partial}{\partial x_i}(x_k x_k) \qquad \text{(chain rule)} \\
&= \underline{e}_i\,\frac{1}{2r}\,2\,x_i \\
&= \frac{1}{r}\,\underline{r} = \hat{\underline{r}}
\end{aligned}$$

   The gradient of the length of the position vector is the unit vector pointing radially outwards from the origin. It is normal to the level surfaces which are spheres centered on the origin.

13. 3. Identities for gradients

If $\phi(\underline{r})$ and $\psi(\underline{r})$ are real scalar fields, then:

1. **Distributive law**

$$\boxed{\underline{\nabla}\ \big(\phi(\underline{r}) + \psi(\underline{r})\big)\ =\ \underline{\nabla}\,\phi(\underline{r}) + \underline{\nabla}\,\psi(\underline{r})}$$

   Proof:

$$\underline{\nabla}\ \big(\phi(\underline{r})\ +\ \psi(\underline{r})\big)\ =\ \underline{e}_i\,\frac{\partial}{\partial x_i}\ \big(\phi(\underline{r})\ +\ \psi(\underline{r})\big)\ =\ \underline{\nabla}\,\phi(\underline{r})\ +\ \underline{\nabla}\,\psi(\underline{r})$$

2. **Product rule**

$$\boxed{\underline{\nabla}\ \big(\phi(\underline{r})\ \psi(\underline{r})\big)\ =\ \psi(\underline{r})\ \ \underline{\nabla}\,\phi(\underline{r}) + \phi(\underline{r})\ \ \underline{\nabla}\,\psi(\underline{r})}$$

   Proof:

$$\begin{aligned}
\underline{\nabla}\ \big(\phi(\underline{r})\ \psi(\underline{r})\big) &= \underline{e}_i\,\frac{\partial}{\partial x_i}\ \big(\phi(\underline{r})\ \psi(\underline{r})\big) \\
&= \underline{e}_i\left(\psi(\underline{r})\ \frac{\partial \phi(\underline{r})}{\partial x_i}\ +\ \phi(\underline{r})\ \frac{\partial \psi(\underline{r})}{\partial x_i}\right) \\
&= \psi(\underline{r})\ \ \underline{\nabla}\,\phi(\underline{r})\ +\ \phi(\underline{r})\ \ \underline{\nabla}\,\psi(\underline{r})
\end{aligned}$$

3. **Chain rule:** If $F(\phi(\underline{r}))$ is a scalar field, then

$$\boxed{\underline{\nabla}\, F(\phi(\underline{r})) \;=\; \frac{\partial F(\phi)}{\partial \phi}\; \underline{\nabla}\, \phi(\underline{r})}$$

Proof:

$$\begin{aligned} \underline{\nabla}\, F(\phi(\underline{r})) &= \underline{e}_i \frac{\partial}{\partial x_i}\, F(\phi(\underline{r})) \\ &= \underline{e}_i \frac{\partial F(\phi)}{\partial \phi}\, \frac{\partial \phi(\underline{r})}{\partial x_i} \;=\; \frac{\partial F(\phi)}{\partial \phi}\; \underline{\nabla}\, \phi(\underline{r}) \end{aligned}$$

Example of Chain rule: If $\phi(\underline{r}) \;=\; r$

$$\underline{\nabla}\, F(r) \;=\; \frac{\partial F(r)}{\partial r}\, \underline{\nabla}\, r \;=\; \frac{\partial F(r)}{\partial r}\, \frac{1}{r}\, \underline{r}$$

where we used a result from 13.2. If $F(\phi(\underline{r})) = \phi(\underline{r})^n = r^n$ we find that

$$\underline{\nabla}\, (r^n) \;=\; (n\, r^{n-1})\, \frac{1}{r}\, \underline{r} \;=\; (n\, r^{n-2})\, \underline{r}$$

In particular

$$\underline{\nabla}\, \left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$$

13. 4. Transformation of the gradient

Here we prove the claim that the gradient actually is a vector (so far we assumed it was!).

Let the point P have coordinates x_i in the $\underline{e}_i$ basis and the *same* point P have coordinates x'_i in the $\underline{e}_i{}'$ basis i.e. we consider the vector transformation law $x_i \to x'_i = \lambda_{ij}\, x_j$.

$\phi(\underline{r})$ is a scalar if it depends only on the physical point P and not on the coordinates x_i or x'_i used to specify P. The *value* of ϕ at P is *invariant* under a change of basis λ (but the function may look different).

$$\phi(x_1,\, x_2,\, x_3) \to \phi'(x'_1,\, x'_2,\, x'_3) = \phi(x_1,\, x_2,\, x_3)$$

Now consider $\underline{\nabla}\phi$ in the new (primed) basis, its components are

$$\frac{\partial}{\partial x'_i}\, \phi'(x'_1,\, x'_2,\, x'_3)$$

Using the chain rule, we obtain

$$\frac{\partial}{\partial x'_i}\, \phi'(x'_1,\, x'_2,\, x'_3) = \frac{\partial x_j}{\partial x'_i}\, \frac{\partial}{\partial x_j}\, \phi(x_1,\, x_2,\, x_3)\ .$$

Since $x_j = \lambda_{kj}\, x'_k$ (inverse vector transformation law)

$$\frac{\partial x_j}{\partial x'_i} = \lambda_{kj}\, \frac{\partial x'_k}{\partial x'_i} = \lambda_{kj}\, \delta_{ik} = \lambda_{ij}\ .$$

Hence

$$\frac{\partial}{\partial x'_i}\,\phi(x'_1,\,x'_2,\,x'_3) = \lambda_{ij}\frac{\partial}{\partial x_j}\,\phi(x_1,\,x_2,\,x_3)\;.$$

which shows that the components of $\underline{\nabla}\phi$ respect the vector transformation law. Thus $\underline{\nabla}\phi(\underline{r})$ transforms as a **vector field** as claimed.

13. 5. The Operator 'Del'

We can think of the **vector operator** $\underline{\nabla}$ (confusingly pronounced "del") acting on the **scalar field** $\phi(\underline{r})$ to produce the **vector field** $\underline{\nabla}\,\phi(\underline{r})$.

In Cartesians:
$$\boxed{\underline{\nabla} = \underline{e}_i\,\frac{\partial}{\partial x_i} = \underline{e}_1\,\frac{\partial}{\partial x_1} + \underline{e}_2\,\frac{\partial}{\partial x_2} + \underline{e}_3\,\frac{\partial}{\partial x_3}}$$

We call $\underline{\nabla}$ an 'operator' since it operates on something to its *right*. It is a vector operator since it has vector transformation properties. (More precisely it is a linear differential vector operator!)

We have seen how $\underline{\nabla}$ acts on a scalar field to produce a vector field. We can make products of the vector operator $\underline{\nabla}$ with other vector quantities to produce new operators and fields in the same way as we could make scalar and vector products of two vectors.

For example, recall that the directional derivative of ϕ in direction $\underline{\hat{s}}$ was given by $\underline{\hat{s}}\cdot\underline{\nabla}\phi$. Generally, we can interpret $\underline{A}\cdot\underline{\nabla}$ as a **scalar operator**:

$$\underline{A}\cdot\underline{\nabla} = A_i\,\frac{\partial}{\partial x_i}$$

i.e. $\underline{A}\cdot\underline{\nabla}$ acts on a scalar field to its *right* to produce another scalar field

$$(\underline{A}\cdot\underline{\nabla})\;\phi(\underline{r}) \;=\; A_i\,\frac{\partial\phi(\underline{r})}{\partial x_i} \;=\; A_1\,\frac{\partial\phi(\underline{r})}{\partial x_1} \;+\; A_2\,\frac{\partial\phi(\underline{r})}{\partial x_2} \;+\; A_3\,\frac{\partial\phi(\underline{r})}{\partial x_3}$$

Actually we can also act with this operator on a vector field to get another vector field.

$$\begin{aligned}(\underline{A}\cdot\underline{\nabla})\;\underline{V}(\underline{r}) \;&=\; A_i\,\frac{\partial}{\partial x_i}\,\underline{V}(\underline{r}) \;=\; A_i\,\frac{\partial}{\partial x_i}\,\Big(V_j(\underline{r})\,\underline{e}_j\Big)\\ &=\; \underline{e}_1\,(\underline{A}\cdot\underline{\nabla})\,V_1(\underline{r}) \;+\; \underline{e}_2\,(\underline{A}\cdot\underline{\nabla})\,V_2(\underline{r}) \;+\; \underline{e}_3\,(\underline{A}\cdot\underline{\nabla})\,V_3(\underline{r})\end{aligned}$$

The alternative expression $\underline{A}\cdot\big(\underline{\nabla}\,\underline{V}(\underline{r})\big)$ is *undefined* because $\underline{\nabla}\,\underline{V}(\underline{r})$ doesn't make sense.

N.B. Great care is required with the order in products since, in general, products involving operators are not commutative. For example

$$\underline{\nabla}\cdot\underline{A} \;\neq\; \underline{A}\cdot\underline{\nabla}$$

$\underline{A}\cdot\underline{\nabla}$ is a scalar differential operator whereas

$$\underline{\nabla}\cdot\underline{A} = \frac{\partial A_i}{\partial x_i} \qquad \text{gives a scalar field called the } \textbf{divergence} \text{ of} \underline{A}$$