Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **28-NOV-20**

Attendance : **10**

Green's theorem :

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$\vec{F} = P\hat{i} + Q\hat{j}$$

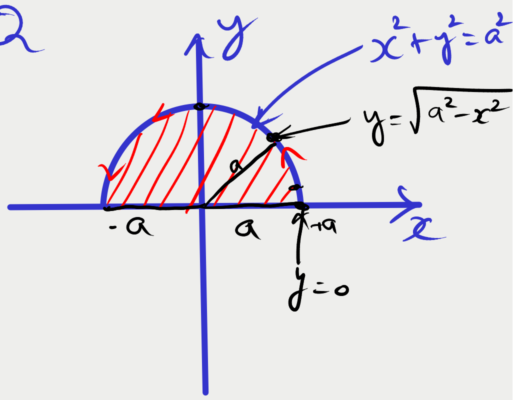
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (P\hat{i} + Q\hat{j})\cdot(dx\hat{i} + dy\hat{i})$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

Example 82. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.
(*M.D.U. Dec. 2009, U.P., I Sem., Dec. 2004*)

$$I = \oint_C \underbrace{(2x^2 - y^2)}_{P}\,dx + \underbrace{(x^2+y^2)}_{Q}\,dy$$

$$C : x^2 + y^2 = a^2$$



$$P = 2x^2 - y^2 \quad ; \quad Q = x^2 + y^2$$

$$\frac{\partial P}{\partial y} = -2y \quad ; \quad \frac{\partial Q}{\partial x} = 2x$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (-2y) \Rightarrow 2(x+y)$$

$$= 2\iint_R (x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} (x+y)\,dy$$

$$= 2\int_{-a}^{a} dx \left[\int_0^{\sqrt{a^2-x^2}} x\,dy + \int_0^{\sqrt{a^2-x^2}} y \cdot dy \right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot y \Big|_0^{\sqrt{a^2-x^2}} + \frac{y^2}{2} \Big|_0^{\sqrt{a^2-x^2}} \right]$$

$$= 2\int_{-a}^{a} dx \cdot \left[x \cdot \sqrt{a^2-x^2} + \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} \right]$$

$$= 2\int_{-a}^{a} dx \left[x \cdot \sqrt{a^2-x^2} + \frac{a^2-x^2}{2} \right]$$

$$= 2 \cdot \int_{-a}^{a} \underbrace{\overbrace{x(a^2-x^2)^{1/2}}^{f_1(x)}}_{f(x)} \cancel{dx} \rightarrow 0 \quad \because f_1(x) \text{ is odd} \quad + \; 2 \int_{-a}^{a} \underbrace{\overbrace{(a^2-x^2)}^{f_2(x)}}_{f(x)} dx$$

$$\boxed{\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even fn.} \\ &= 0 \quad \text{if } f(x) \text{ is odd fn.} \end{aligned}}$$

If $f(-x) = f(x)$ then $f(x)$ is even

If $f(-x) = -f(x)$ then $f(x)$ is odd

$f_1(x)$ is even or odd ?

$$f_1(x) = x\sqrt{a^2-x^2}$$

$$x = -x$$

$$f_1(-x) = -\overbrace{x \cdot \sqrt{a^2-x^2}}^{f_1(x)}$$

$$f_1(-x) = -f_1(x) \leftarrow \text{odd}$$

$$f_2(x) = a^2 - x^2$$

$$f_2(-x) = a^2 - (-x)^2$$

$$= \underbrace{a^2 - x^2}_{f_2(x)}$$

$$\underline{f_2(-x) = f_2(x)}$$

$\nwarrow$ even

$$= 2 \cdot \int_{0}^{a} (a^2 - x^2)\,dx$$

$$= 2 \cdot \left[a^2 \cdot x \Big|_0^a - \frac{x^3}{3} \Big|_0^a \right]$$

$$= 2 \cdot \left[a^2 \cdot a - \frac{a^3}{3} \right] \Rightarrow 2 \cdot \left[a^3 - \frac{a^3}{3} \right]$$

$$= 2 \cdot \frac{2a^3}{3} \Rightarrow 4\frac{a^3}{3} \; //.$$

Example 83. *Evaluate* $\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2+y^2=1$ *and* $C_2 : x = \pm 2,\ y = \pm 2.$ *(Gujarat, I Semester, Jan 2009)*

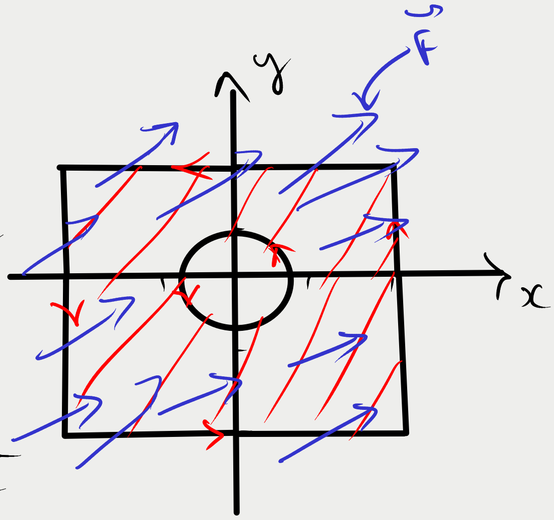
$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$C_1 : x^2+y^2=1$

$C_2 : x=\pm 2 \ ;\ y=\pm 2$

$$\vec{F} = \underbrace{\frac{-y}{x^2+y}}_{P}\,\hat{i} + \underbrace{\frac{x}{x^2+y^2}}_{Q}\,\hat{j}$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$P = \frac{-y}{x^2+y^2} \quad ; \quad Q = \frac{x}{x^2+y^2}$$

$$P = \underbrace{-y}_{u}\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial P}{\partial y} = +y\cdot(x^2+y^2)^{-1-1}\cdot 2y + (x^2+y^2)^{-1}(-1)$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}\cdot\frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = \frac{2y^2 - x^2 - y^2}{(x^2+y^2)^2}$$

$$Q = \underbrace{x}_{u}\cdot\underbrace{(x^2+y^2)^{-1}}_{v}$$

$$\frac{\partial Q}{\partial x} = x\cdot -1\,(x^2+y^2)^{-2}\cdot 2x + (x^2+y^2)^{-1}\cdot 1$$

$$= \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2} - \frac{(2y^2 - x^2 - y^2)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2 + x^2 + y^2 - 2y^2 + x^2 + y^2}{(x^2+y^2)^2}$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{0}{(x^2+y^2)^2} = 0$$

$\therefore$ from Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \underbrace{\oint_C P\,dx + Q\,dy}_{} = \iint_R \cancelto{0}{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dx\,dy$$

$$\underbrace{\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 0}_{}$$

$\uparrow$ By Green's theorem.