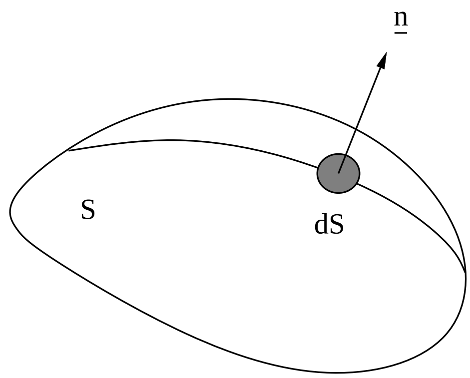
**Lecture 18: Surface Integrals**

RHB (9.5.1) does things a bit differently here. *(Bourne and Kendall 5.5)* follows our approach.

Let $S$ be a two-sided surface in ordinary three-dimensional space as shown. If an infinitesi-



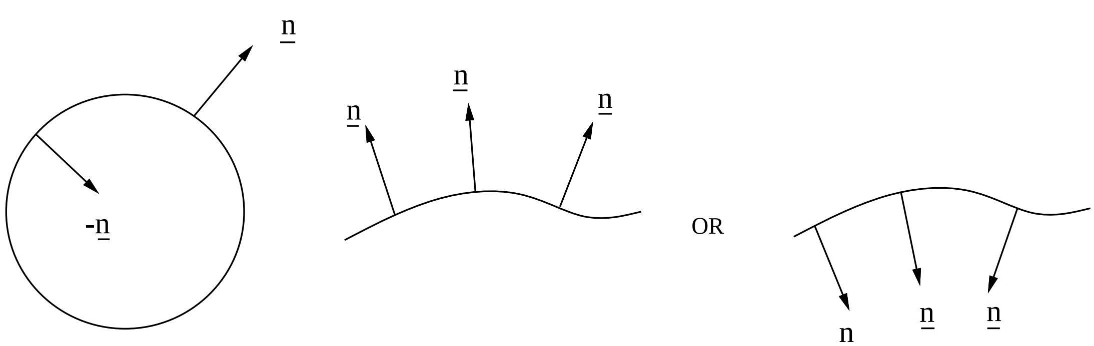

mal element of surface with (scalar) area $dS$ has unit normal $\underline{n}$, then the infinitesimal **vector element of area** is *defined* by:-

$$\boxed{\underline{dS} \;=\; \underline{n}\, dS}$$

**Example:** if $S$ lies in the $(x, y)$ plane, then $\underline{dS} = \underline{e}_3\, dx\, dy$ in Cartesian coordinates.

**Physical interpretation:** $\underline{dS} \cdot \underline{\hat{a}}$ gives the projected (scalar) element of area onto the plane with unit normal $\underline{\hat{a}}$.

For **closed** surfaces (eg, a sphere) we *choose* $\underline{n}$ to be the **outward normal**. For **open** surfaces, the sense of $\underline{n}$ is arbitrary — except that it is chosen in the same sense for all elements of the surface. See *Bourne & Kendall 5.5* for further discussion of surfaces.



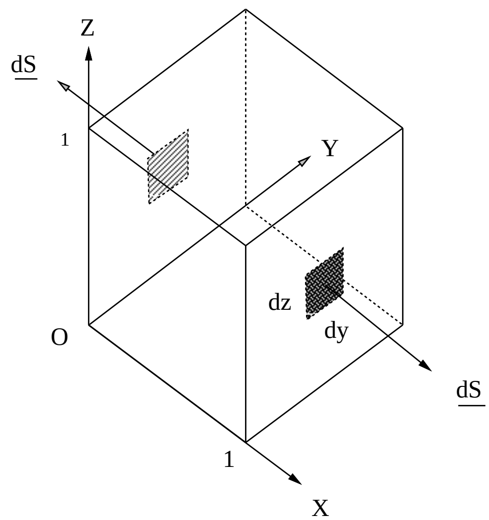

If $\underline{A}(\underline{r})$ is a vector field defined on $S$, we define the (normal) **surface integral**

$$\int_S \underline{A} \cdot \underline{dS} \;=\; \int_S \left(\underline{A} \cdot \underline{n}\right)\; dS \;=\; \lim_{\substack{n \to \infty \\ \delta S \to 0}} \sum_{i=0}^{n-1} \left(\underline{A}(\underline{r}^{\,i}) \cdot \underline{n}^{\,i}\right)\; \delta S^i$$

where we have formed the Riemann sum by dividing the surface $S$ into $n$ small areas, the $i$th area having vector area $\underline{\delta S}^{\,i}$. Clearly, the quantity $\underline{A}(\underline{r}^{\,i}) \cdot \underline{n}^{\,i}$ is the component of $\underline{A}$ *normal* to the surface at the point $\underline{r}^{\,i}$

- We use the notation $\displaystyle\int_S \underline{A}\cdot\underline{dS}$ for both *open* and *closed* surfaces. Sometimes the integral over a *closed* surface is denoted by $\displaystyle\oint_S \underline{A}\cdot\underline{dS}$ (*not* used here).
- Note that the integral over $S$ is a **double integral** in each case. Hence surface integrals are sometimes denoted by $\displaystyle\iint_S \underline{A}\cdot\underline{dS}$ (*not* used here).

**Example:** Let $S$ be the surface of a unit cube ($S$ = sum over **all six faces**). On the front




face, parallel to the $(y, z)$ plane, at $x = 1$, we have

$$\underline{dS} \;=\; \underline{n}\, dS \;=\; \underline{e}_1\, dy\, dz$$

On the back face at $x = 0$ in the $(y, z)$ plane, we have

$$\underline{dS} \;=\; \underline{n}\, dS \;=\; -\underline{e}_1\, dy\, dz$$

In each case, the unit normal $\underline{n}$ is an *outward* normal because $S$ is a *closed* surface.

If $\underline{A}(\underline{r})$ is a vector field, then the integral $\displaystyle\int_S \underline{A}\cdot\underline{dS}$ over the front face shown is

$$\int_{z=0}^{z=1}\int_{y=0}^{y=1} \underline{A}\cdot\underline{e}_1\, dy\, dz \;=\; \int_{z=0}^{z=1}\int_{y=0}^{y=1} A_1\Big|_{x=1}\, dy\, dz$$

The integral over $y$ and $z$ is an ordinary double integral over a square of side 1. The integral over the back face is

$$-\int_{z=0}^{z=1}\int_{y=0}^{y=1} \underline{A}\cdot\underline{e}_1\, dy\, dz \;=\; -\int_{z=0}^{z=1}\int_{y=0}^{y=1} A_1\Big|_{x=0}\, dy\, dz$$

The total integral is the sum of contributions from all 6 faces.

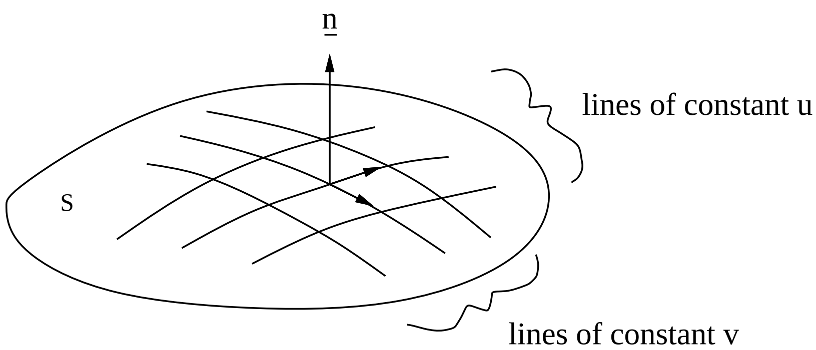
*18. 1. Parametric form of the surface integral*

Suppose the points on a surface $S$ are defined by **two** real parameters $u$ and $v$:-

$\underline{r} \;=\; \underline{r}(u, v) \;=\; (x(u, v),\, y(u, v),\, z(u, v))$ then

- the lines $\underline{r}(u,v)$ for fixed $u$, variable $v$, and
- the lines $\underline{r}(u,v)$ for fixed $v$, variable $u$

are **parametric lines** and form a **grid** on the surface $S$ as shown.




If we change $u$ and $v$ by $du$ and $dv$ respectively, then $\underline{r}$ changes by $\underline{dr}$:-

$$\underline{dr} = \frac{\partial \underline{r}}{\partial u}\, du + \frac{\partial \underline{r}}{\partial v}\, dv$$

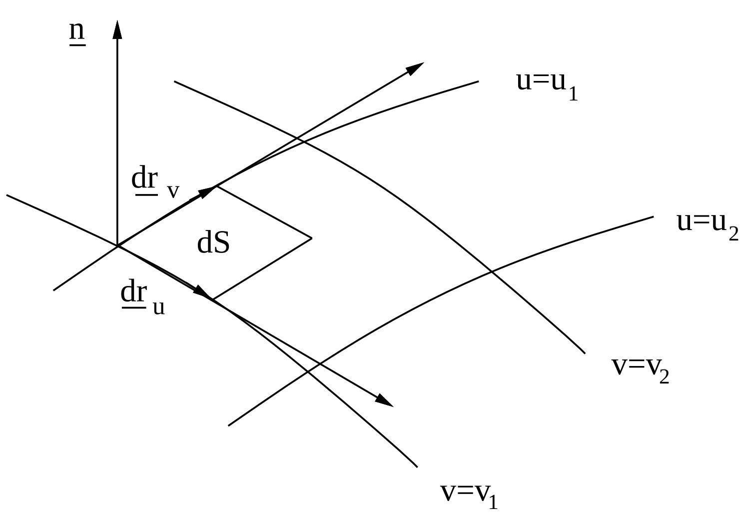
Along the curves $v = \text{constant}$, we have $dv = 0$, and so $\underline{dr}$ is simply:-

$$\underline{dr}_u = \frac{\partial \underline{r}}{\partial u}\, du$$

where $\dfrac{\partial \underline{r}}{\partial u}$ is a vector which is tangent to the surface, and tangent to the lines $v = \text{const}$. Similarly, for $u = \text{constant}$, we have

$$\underline{dr}_v = \frac{\partial \underline{r}}{\partial v}\, dv$$

so $\dfrac{\partial \underline{r}}{\partial v}$ is tangent to lines $u = \text{constant}$.



We can therefore construct a **unit vector** $\underline{n}$, **normal** to the surface at $\underline{r}$:-

$$\underline{n} \;=\; \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \bigg/ \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right|$$

The vector element of area, $\underline{dS}$, has magnitude equal to the area of the infinitesimal parallelogram shown, and points in the direction of $\underline{n}$, therefore we can write

$$\underline{dS} \;=\; \underline{dr}_u \times \underline{dr}_v = \left( \frac{\partial \underline{r}}{\partial u}\, du \right) \times \left( \frac{\partial \underline{r}}{\partial v}\, dv \right) = \left( \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right)\, du\, dv$$

$$\boxed{\underline{dS} = \left( \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right)\, du\, dv}$$

Finally, our integral is parameterised as

$$\boxed{\int_S \underline{A} \cdot \underline{dS} \;=\; \int_v \int_u \underline{A} \cdot \left( \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right)\, du\, dv}$$

Note: We use two integral signs when writing surface integrals in terms of **explicit** parameters $u$ and $v$. The limits for the integrals over $u$ and $v$ must be chosen appropriately for the surface.