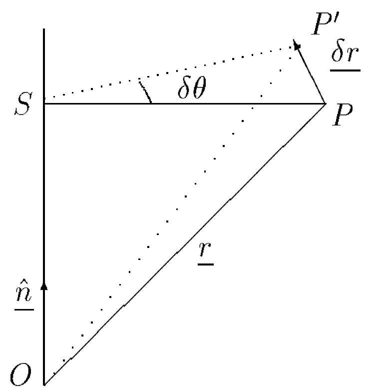
## Lecture 9: The Inertia Tensor

Consider a body rotating with angular velocity $\underline{\omega}$. As we shall see the angular momentum about the origin $\underline{L}(0)$ is NOT in general parallel to $\underline{\omega}$. Therefore one needs a tensor—the inertia tensor— to relate the two vectors $\underline{L}(O)$ and $\underline{\omega}$:

$$L(O)_i = I_{ij}(0)\,\omega_j$$

In Physics 2 you have met moments of inertia but not the inertia tensor. This is because in the situations considered there, it was tacitly assumed that $\underline{\omega}$ was parallel to $\underline{L}$ *i.e.* $\underline{\omega}$ was in fact assumed to be an *eigenvector* of the inertia tensor.

Consider a *rigid* body, rotating about an axis $\hat{\underline{n}}$ through a *fixed point* $O$.




In time $\delta t$, the body rotates through angle $\delta\theta$ and $P \to P'$ :

$$|\underline{PP'}| \simeq |\underline{SP}|\,\delta\theta\,.$$

Thus $\quad \delta\underline{r} \simeq \hat{\underline{n}} \times \underline{r}\,\delta\theta$

and so
$$\underline{v} = \dot{\theta}\,\hat{\underline{n}} \times \underline{r} = \underline{\omega} \times \underline{r}\,,$$

where $\underline{\omega} = \dot{\theta}\,\hat{\underline{n}}$ is the **angular velocity** of the body.

For a mass $m$ at $P$, its **angular momentum** $\underline{L}(O)$ about $O$ is

$$\begin{aligned}
\underline{L}(O) &\stackrel{\text{def}}{=} \underline{r} \times \underline{p} \\
&= \underline{r} \times \{m\,(\underline{\omega} \times \underline{r})\} \\
&= m\left\{(\underline{r}\cdot\underline{r})\,\underline{\omega} - (\underline{\omega}\cdot\underline{r})\,\underline{r}\right\}\,.
\end{aligned}$$

In the $\underline{e}_i$ basis the components of $\underline{L}(O)$ and $\underline{\omega}$ are related by

$$\begin{aligned}
L_i(O) &= m\left\{(\underline{r}\cdot\underline{r})\,\omega_i - (\underline{\omega}\cdot\underline{r})\,x_i\right\} \\
&= m\left\{r^2\,\omega_i - \omega_j x_j x_i\right\} \\
&= m\left\{r^2\,\delta_{ij} - x_i x_j\right\}\,\omega_j\,.
\end{aligned}$$

Thus

$$\boxed{L_i = I_{ij}(O)\,\omega_j \qquad \text{where} \qquad I_{ij}(O) = m\left\{r^2\,\delta_{ij} - x_i x_j\right\}}$$

$I_{ij}(O)$ is the **Inertia Tensor**, relative to $O$, in the $\underline{e}_i$ basis.

For a **collection of particles** of mass $m_\alpha$ at $\underline{r}^\alpha$,

$$\boxed{I_{ij}(O) = \sum_\alpha m_\alpha\left\{(\underline{r}^\alpha\cdot\underline{r}^\alpha)\,\delta_{ij} - x_i^\alpha x_j^\alpha\right\}}\,.$$

For a **continuous body**, the sums become integrals, giving

$$\boxed{I_{ij}(O) \;=\; \int_V \rho(\underline{r}) \left\{ (\underline{r} \cdot \underline{r})\, \delta_{ij} - x_i x_j \right\} \, dV} \,.$$

[ $\rho(\underline{r})\, dV$ = mass of volume element $dV$ at $\underline{r}$, where density is $\rho(\underline{r})$. ]

For laminae (flat objects) and solid bodies, these are 2 and 3-dimensional integrals.

If the basis is *fixed relative to the body*, the $I_{ij}(O)$ are **constants** in time.

Consider the **diagonal** term

$$\begin{aligned} I_{11}(O) &= \sum_\alpha m_\alpha \left\{ (\underline{r}^\alpha \cdot \underline{r}^\alpha) - (x_1^\alpha)^2 \right\} \\ &= \sum_\alpha m_\alpha \left\{ (x_2^\alpha)^2 + (x_3^\alpha)^2 \right\} \\ &= \sum_\alpha m_\alpha \, (r_\perp^\alpha)^2 \,, \end{aligned}$$

where $r_\perp^\alpha$ is the **perpendicular distance** of $m_\alpha$ from the $\underline{e}_1$ axis through $O$.
This term is called the **moment of inertia** about the $\underline{e}_1$ axis. Similarly the other diagonal terms are moments of inertia.

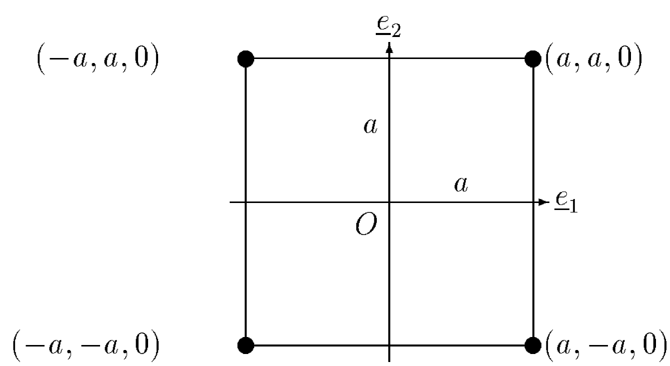
The **off-diagonal** terms are called the **products of inertia**, having the form, for example

$$I_{12}(O) \;=\; -\sum_\alpha m_\alpha \, x_1^\alpha \, x_2^\alpha \,.$$

**Example**

Consider 4 masses $m$ at the vertices of a square of side $2a$.

(i) $O$ **at centre of the square.**




For $m$ at $(a, a, 0)$, $r^2 = 2a^2$ and

$$I_{ij}(O) \;=\; m \left\{ 2a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \;=\; ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \,.$$

For $m$ at $(a,-a,0)$,

$$I_{ij}(O) \;=\; m\left\{2a^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - a^2\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\right\} \;=\; ma^2\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}.$$

Similarly for the masses at $(-a,a,0)$ and $(-a,-a,0)$, giving the inertia tensor for all 4 particles as:–

$$\underline{I_{ij}(O) \;=\; 4ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}.}$$

Note that the final inertia tensor is diagonal and in this basis we have no products of inertia. (Of course there are other bases where the tensor is not diagonal.) This implies the basis vectors are eigenvectors of the inertia tensor *e.g.* if If $\underline{\omega} = \omega(0,0,1)$ then $\underline{L}(O) = 8m\omega a^2(0,0,1)$. In general $\underline{L}(O)$ is not parallel to $\underline{\omega}$ *e.g.* If $\underline{\omega} = \omega(0,1,1)$ then $\underline{L}(O) = 4m\omega a^2(0,1,2)$. Note that the inertia tensors for the individual masses are not diagonal.

### *9. 1. Two Useful Theorems*

**Perpendicular Axis Theorem**

For a lamina, or collection of particles confined to a plane, (choose $\underline{e}_3$ as normal to the plane), with $O$ in the plane

$$\boxed{I_{11}(O) + I_{22}(O) \;=\; I_{33}(O)}\,.$$

This is simply checked by using the formula at the bottom of p.33 and noting $x_3^\alpha = 0$.

**Parallel Axis Theorem**

If $G$ is the **centre of mass** of the body its position vector $\underline{R}$ is given by

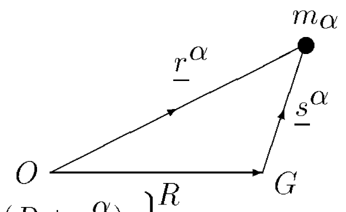
$$\underline{R} = \sum_\alpha m_\alpha \underline{r}^\alpha \Big/ M,$$

where $\underline{r}^\alpha$ are the position vectors relative to $O$ and $M = \sum_\alpha m_\alpha$, is the **total mass** of the system.

The parallel axis theorem states that

$$\boxed{I_{ij}(O) - I_{ij}(G) \;=\; M\left\{(\underline{R}\cdot\underline{R})\,\delta_{ij} - R_iR_j\right\}}\,,$$

**Proof** Let $\underline{s}^\alpha$ be the position of $m_\alpha$ **with respect to** $G$, then



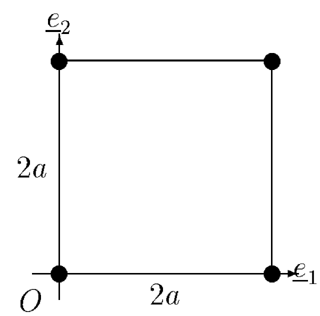

$$
\begin{aligned}
I_{ij}(G) &= \sum_\alpha m_\alpha \left\{ (\underline{s}^\alpha \cdot \underline{s}^\alpha)\, \delta_{ij} - s_i^\alpha s_j^\alpha \right\} ; \\
I_{ij}(O) &= \sum_\alpha m_\alpha \left\{ (\underline{r}^\alpha \cdot \underline{r}^\alpha)\, \delta_{ij} - x_i^\alpha x_j^\alpha \right\} \\
&= \sum_\alpha m_\alpha \left\{ (\underline{R} + \underline{s}^\alpha)^2\, \delta_{ij} - (\underline{R} + \underline{s}^\alpha)_i (\underline{R} + \underline{s}^\alpha)_j \right\} \\
&= M \left\{ R^2 \delta_{ij} - R_i R_j \right\} + \sum_\alpha m_\alpha \left\{ (\underline{s}^\alpha \cdot \underline{s}^\alpha)\, \delta_{ij} - s_i^\alpha s_j^\alpha \right\} \\
&\qquad + 2\, \delta_{ij}\, \underline{R} \cdot \sum_\alpha m_\alpha \underline{s}^\alpha - R_i \sum_\alpha m_\alpha s_j^\alpha - R_j \sum_\alpha m_\alpha s_i^\alpha \\
&= M \left\{ R^2 \delta_{ij} - R_i R_j \right\} + I_{ij}(G)
\end{aligned}
$$

the crossed terms vanishing since

$$
\sum_\alpha m_\alpha s_i^\alpha \;=\; \sum_\alpha m_\alpha \left( r_i^\alpha - R_i \right) \;=\; 0 \,.
$$

**Example of use of Parallel-Axis Theorem.**
Consider the same arrangement of masses as before but with $O$ at one corner of the square. *i.e.* a (massless) lamina of side $2a$, with masses $m$ at each corner and the origin

$O$ at the bottom, left so that the masses are at $(0,0,0)$, $(2a,0,0)$, $(0,2a,0)$ and $(2a,2a,0)$




We have $M = 4m$ and

$$
\begin{aligned}
\underline{OG} \;=\; \underline{R} &= \frac{1}{4m} \left\{ m(0,0,0) + m(2a,0,0) + m(0,2a,0) + m(2a,2a,0) \right\} \\
&= (a, a, 0)
\end{aligned}
$$

and so $G$ is at the centre of the square and $R^2 = 2a^2$ . We can now use the parallel axis theorem to relate the inertia tensor of the previous example to that of the present

$$
I_{ij}(O) - I_{ij}(G) \;=\; 4m \left\{ 2a^2\, \delta_{ij} - a^2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \;=\; \underline{4ma^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \,.
$$

From the previous example,

$$
I_{ij}(G) \;=\; 4ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \,.
$$

therefore

$$
I_{ij}(O) \;=\; 4ma^2 \begin{pmatrix} 1+1 & 0-1 & 0 \\ 0-1 & 1+1 & 0 \\ 0 & 0 & 2+2 \end{pmatrix} \;=\; \underline{4ma^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}
$$