Eg 1 find F.T of

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

W.K.T

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is} \right]$$

$$\because \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\,\frac{e^{isa} - e^{-isa}}{2\,is} \right]$$

$$= \frac{2}{\sqrt{2\pi}\, s} \sin sa$$

$$= \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\cdot\sqrt{\pi}} \cdot \frac{\sin sa}{s} = \frac{\sqrt{2}}{\sqrt{\pi}}\, \frac{\sin sa}{s}$$

Eg: 7 Find F.S.T and F.C.T of

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a 1 \cdot \sin sx \, dx + \cancelto{0}{\int_a^\infty 0 \cdot \sin sx \, dx} \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sa}{s} + \frac{\cos 0}{s} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos sa}{s} \right]_{//}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \, dx \Rightarrow \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - \cancelto{0}{\frac{\sin 0}{s}} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} \right]_{//}$$

Eg 2: Find the F.T of

$$f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

W.K.T

$$F(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

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$$= \sqrt{\frac{1}{2\pi}} \int_{-1}^{1} \underbrace{(1-x^2)}_{u} \underbrace{e^{isx}\, dx}_{dv}$$

$$= \sqrt{\frac{1}{2\pi}} \left[(1-x^2)\frac{e^{isx}}{is}\Bigg|_{-1}^{1} - \int_{-1}^{1} \overbrace{\frac{e^{isx}}{is}\,dx}^{dv}\, \underbrace{(-2x)}_{u} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[(1-x^2)\frac{e^{isx}}{is}\Bigg|_{-1}^{1} - \left[(-2x)\frac{e^{isx}}{(is)^2}\Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{e^{isx}}{(is)^2}(-2)\,dx \right] \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\cancel{(1-x^2)\frac{e^{isx}}{is}\Bigg|_{-1}^{1}} \to 0 + 2x \cdot \frac{e^{isx}}{(is)^2}\Bigg|_{-1}^{1} - 2\, \frac{e^{isx}}{(is)^3}\Bigg|_{-1}^{1} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{2e^{is}}{(is)^2} - \frac{2(-1)\,e^{-is}}{(is)^2} - 2\frac{e^{is}}{(is)^3} + 2\frac{e^{-is}}{(is)^3} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2e^{is}}{s^2} - \frac{2e^{-is}}{s^2} + \frac{2e^{is}}{is^3} - \frac{2e^{-is}}{is^3} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2}{s^2}\left[2\underbrace{\frac{e^{is}+e^{-is}}{2}}_{\to \cos s}\right] + \frac{2}{s^3}\left[2\underbrace{\frac{e^{is}-e^{-is}}{2i}}_{\to \sin s}\right] \right]$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{-2}{s^2} \cdot 2\cdot\cos s + \frac{2}{s^3} \cdot 2\sin s \right]$$

$$= \sqrt{\frac{1}{2\pi}}\, \frac{4}{s^3}\left[-s\cos s + \sin s\right] //$$

Eg:3 find F.S.T and F.C.T of $f(x) = e^{-ax}$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} \left[-a\sin sx - s\cos sx\right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cancelto{1}{e^0}}{a^2+s^2} \left[-a \cancelto{0}{\sin 0} - s \cancelto{1}{\cos 0}\right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{a^2+s^2} \left[-s\right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2} \right] //$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} \left(-a\cos sx + s\sin sx\right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{\cancelto{1}{e^0}}{a^2+s^2} \left(-a \cancelto{1}{\cos 0} + s \cancelto{0}{\sin 0}\right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{a^2+s^2} (-a) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2} \right] //$$

Eg: 5 Find the F.C.T of $f(x) = e^{-2x} + 4e^{-3x}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-2x} + 4e^{-3x}) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty e^{-2x} \cos sx + 4 \int_0^\infty e^{-3x} \cos sx \, dx \right]$$

W.K.T $\displaystyle \int_0^\infty e^{-ax} \cos sx = \frac{a}{a^2+s^2}$ //

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{2^2+s^2} + 4 \cdot \frac{3}{3^2+s^2} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{1}{4+s^2} + \frac{6}{9+s^2} \right] //$$

Eg. 4 Find F.S.T of $f(x) = \frac{1}{x}$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} \, dx$$

Putting $sx = \phi$ $\qquad dx = d\phi/s$

$x = \frac{\phi}{s}$ $\qquad \begin{array}{l|l} x \to 0 & x \to \infty \\ \phi \to 0 & \phi \to \infty \end{array}$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\phi \cdot \cancel{s}}{\phi} \cdot \frac{d\phi}{\cancel{s}}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\phi}{\phi}\, d\phi.$$

$$= \sqrt{\frac{2}{\pi} \cdot \frac{\pi}{2}}$$

$$= \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}} \cdot \sqrt{\pi}}{\cancel{\sqrt{2}} \cdot \sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2}} \text{ or } \sqrt{\frac{\pi}{2}}$$

Eg 5: find F.S.T of $f(x) = \frac{e^{-ax}}{x}$

W.K.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx\, dx \quad \text{— (1)}$$

Diff (1) w.r.t ds on both sides we have

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} dx \cdot \frac{d}{ds}(\sin sx)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} dx \cdot \cos sx \cdot \cancel{x}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

Integrating both sides w.r.t s

$$\int \cancel{d} F_s(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2} \cdot ds$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

$\therefore$ @ $s=0$ $\quad F_s(0) = \sqrt{\frac{2}{\pi}} \cancel{\tan^{-1}(0)}^{\,0} + c$

$$\therefore c = 0$$

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right)$$ //.

Eg 8: Find the F.T of Dirac Delta fn =

$\delta(x-a)$

$$\delta(x-a) = \lim_{h\to 0} f(x)$$

$$f(x) = \begin{cases} \frac{1}{h} & \text{for } a < x < a+h \\ 0 & \text{for } x < a, \; x > a+h \end{cases}$$

$$F(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \sqrt{\frac{1}{2\pi}} \; \underset{h \to 0}{Lt} \int_{a}^{a+h} \frac{1}{h} e^{isx} \, dx$$

$$= \sqrt{\frac{1}{2\pi}} \; \underset{h \to 0}{Lt} \; \frac{1}{h} \left[\frac{e^{isx}}{is} \right]_{a}^{a+h}$$

$$= \sqrt{\frac{1}{2\pi}} \; \underset{h \to 0}{Lt} \; \frac{1}{h} \left[\frac{e^{is(a+h)}}{is} - \frac{e^{isa}}{is} \right]$$

$$= \sqrt{\frac{1}{2\pi}} \; \underset{h \to 0}{Lt} \; e^{isa} \underbrace{\left[\frac{e^{ish} - 1}{ish} \right]}$$

$$\underset{\theta \to 0}{Lt} \; \frac{e^{\theta} - 1}{\theta} = 1$$

$$= \sqrt{\frac{1}{2\pi}} \; e^{isa} \cdot \underset{h \to 0}{Lt} \cancelto{1}{\left[\frac{e^{ish} - 1}{ish} \right]}$$

$$= \sqrt{\frac{1}{2\pi}} \cdot e^{isa} \;//$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$e^{\theta} - 1 = \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$\frac{e^{\theta} - 1}{\theta} = 1 + \cancelto{0}{\frac{\theta}{2!} + \frac{\theta^2}{3!} + \frac{\theta^3}{4!} + \cdots}$$

$$\underset{\theta \to 0}{Lt} \; \frac{e^{\theta} - 1}{\theta} = 1 \;//$$

Eg 9: S.T $F_s\{xf(x)\} = -\frac{d}{ds}F_c(s)$

$$F_c\{xf(x)\} = \frac{d}{ds}F_s(s)$$ ✓

hence find the F.C.T and F.S.T of

$f(x) = xe^{-ax}$

W.K.T

$$F_c\{f(x)\} = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$\frac{d}{ds}F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot dx\cdot\frac{d}{ds}\cos sx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\cdot(-\sin sx)\cdot x$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin sx\,dx$$

$$-\frac{d}{ds}F_c(s) = F_s\{xf(x)\}$$

next proof

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$\frac{d}{ds}F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\,\frac{d}{ds}(\sin sx)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx\cdot\cos sx\cdot x$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$\frac{d}{ds} \underbrace{F_s(s)}_{F_s\{f(x)\}} = F_c\{x f(x)\}$$

W.K.T

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_c(x \cdot \underbrace{e^{-ax}}_{f(x)}) = \frac{d}{ds} F_s\{e^{-ax}\}$$

$$= \frac{d}{ds}\left[\frac{s}{a^2+s^2}\right]$$

$$\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$= \frac{(a^2+s^2)\cdot 1 - s(2s)}{(a^2+s^2)^2}$$

$$= \frac{a^2+s^2-2s^2}{(a^2+s^2)^2}$$

$$\boxed{F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(a^2+s^2)^2}}$$

$$F_s\{x e^{-ax}\} = -\frac{d}{ds} F_c\{e^{-ax}\}$$

$$= -\frac{d}{ds}\left[\frac{a}{a^2+s^2}\right]$$

$$= -\frac{\cancel{(a^2+s^2)0}^{0} - a(2s)}{(a^2+s^2)^2}$$

$$F_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}$$

Eg 10: Find F.C.T of $e^{-a^2x^2}$ and hence find F.S.T of $x \cdot e^{-a^2x^2}$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot\cos sx\,dx$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot(\cos sx + i\sin sx)\,dx$$

$$= R \cdot P\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cdot e^{isx}\,dx$$

Standard Integral formula

$$\int_0^\infty e^{-ax^2}\cdot e^{bx}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{\frac{b^2}{4a}}$$

$$= R \cdot P\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a^2}}\,e^{(is)^2/4a^2}$$

$$= R \cdot P\sqrt{2/\pi}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a^2}}\cdot e^{-\frac{s^2}{4a^2}}$$

$$= R.P \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{\cancel{\sqrt{2}}\sqrt{2}} \cdot \frac{\sqrt{\pi}}{\sqrt{a^2}} \cdot e^{-\frac{s^2}{4a^2}}$$

$$= R.P \frac{1}{\sqrt{2}\cdot a} e^{-\frac{s^2}{4a^2}}$$

$$F_c\{e^{-a^2x^2}\} = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2}$$

Find the $F_s\{x \cdot \underbrace{e^{-a^2x^2}}\}$

W.K.T

$$F_s\{x \cdot f(x)\} = -\frac{d}{ds} F_c\{f(x)\}$$

$$F_s\{x \cdot e^{-a^2x^2}\} = -\frac{d}{ds} F_c\{e^{-a^2x^2}\}$$

$$= -\frac{d}{ds}\left[\frac{1}{\sqrt{2}\,a} \cdot e^{-\frac{s^2}{4a^2}}\right]$$

$$= -\frac{1}{\sqrt{2}\,a} e^{-\frac{s^2}{4a^2}} \cdot \left(-\frac{2s}{4a^2}\right)$$

$$= \frac{\cancel{\sqrt{2}} \cdot \sqrt{2} \cdot s}{\cancel{\sqrt{2}} \cdot a \cdot 4a^2} \cdot e^{-s^2/4a^2}$$

$$= \frac{\sqrt{2}s}{4a^3} \cdot e^{-s^2/4a^2}$$

$$F_s\{x \cdot e^{-a^2x^2}\} = \frac{\cancel{\sqrt{2}}s}{\cancel{\sqrt{2}} \cdot \sqrt{2} \cdot 2a^3} e^{-s^2/4a^2} = \boxed{\frac{s}{2\sqrt{2}a^3} e^{-s^2/4a^2}}$$