Fourier Half range Series: $2T$

Fourier full range series. $\begin{cases} -\pi < x < \pi \\ 0 < x < 2\pi \end{cases}$

$f(x)$ defined in the $0 < x < \pi$

$-\pi/2 < x < \pi/2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \checkmark$$

T or π

Fourier cosine series $b_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \checkmark \quad ; \quad a_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

Eg: Find the Fourier sine series for the fn: $f(x) = e^{ax}$ for $0 < x < \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - n\cos nx\right]_0^{\pi}\right.$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}\left(a\sin n\pi - n\cos n\pi\right) - \left[\frac{1}{a^2+n^2}\cdot - n(\cos(0))\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}\left(-n(-1)^n\right) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\left\{\frac{n}{a^2+n^2}\left[e^{a\pi} - (-1)^n + 1\right]\right]$$

$$b_n = \frac{2n}{\pi(a^2+n^2)}\left[1 - e^{a\pi}(-1)^n\right]$$

$$b_1 = \frac{2}{\pi(a^2+1)}\left[1 + e^{a\pi}\right] \; ; \; b_2 = \frac{2\cdot 2}{\pi(a^2+4)}\left[1 - e^{a\pi}\right]$$

$$f(x) = \sum_{n=1}^{p} b_n \sin nx$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2}{a^2+2^2}\sin 2x + \cdots\right]$$

Change of Intervals:

$$-c \le x \le c \qquad f(x) = x \; ;$$

Full range	Half range	
$-\pi < x < \pi$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$\underset{-c}{-2} < x < \underset{c}{2}$
$0 < x < 2\pi$	$0 < x < \pi$	

Full range series

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Half Range Series

$$a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx$$

$-\frac{1}{2} < x < \frac{1}{2}$

$0 < x < 1$

$0 < x < \cancel{l}$

$\uparrow$

c

$$a_n = \frac{2}{c}\int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{2}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Full Range Series

Eg: find the Fourier Series $f(x) = |x|$ $\quad -\underset{c}{2} < x < \underset{c}{2}$

$$a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx \; ; \; a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$f(x) = \begin{cases} -x & -\underset{c}{2} < x < 0 \\ x & 0 < x < \underset{c}{2} \end{cases}$$

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx\right]$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} -x \cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x \cos\frac{n\pi x}{2}\,dx\right]$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} -x \frac{\sin n\pi x}{2}\,dx + \int_{0}^{2} x \frac{\sin n\pi x}{2}\,dx\right]$$

$|x|$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

Example for Half Range Series.

Find the F.S $f(x) = \left(\frac{-x}{l} + 1\right) \quad 0 \leq x \leq l$

$$a_0 = \frac{2}{c}\int_{0}^{c} f(x)\,dx \; ; \quad a_n = \frac{2}{c}\int_{0}^{c} f(x) \cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{2}{c}\int_{0}^{c} f(x) \sin\frac{n\pi x}{c}\,dx$$

$$a_0 = \frac{2}{l}\int_{0}^{l}\left(\frac{-x}{l} + 1\right)dx \; ; \quad b_n = \frac{2}{l}\int_{0}^{l}\left(\frac{-x}{l} + 1\right) \sin\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{2}{l}\int_{0}^{l}\left(\frac{-x}{l} + 1\right) \cos\frac{n\pi x}{l}\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{p} a_n \cos nx + \sum_{n=1}^{p} b_n \sin nx$$

$$\int e^{ax} \sin nx \, dx = \frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx) + c$$

$$\int e^{ax} \cos nx \, dx = \frac{e^{ax}}{a^2+n^2}(a \cos nx + n \sin nx) + c$$