Lecture 7: Transformation Properties of Vectors and Scalars (*RHB 7.14; 19.10*)

7. 1. Transformation of vector components

Let $\underline{A}$ be any vector, having components A_i in the $\underline{e}_i$ basis and components A'_i in the $\underline{e}_i{}'$ basis. *i.e.*

$$\boxed{\underline{A} \; = A_i\, \underline{e}_i \; = A'_i\, \underline{e}_i{}'} \;.$$

Since both bases are orthonormal, $A_j = \underline{A}\cdot\underline{e}_j$ and $A'_j = \underline{A}\cdot\underline{e}_j{}'$.
The components are related as follows:–

$$A'_i \;=\; \underline{A}\cdot\underline{e}_i{}' \;=\; (A_j\underline{e}_j)\cdot\underline{e}_i{}' = (\underline{e}_i{}'\cdot\underline{e}_j)A_j \;= \lambda_{ij}A_j .$$

$$\therefore \quad \boxed{A'_i = \lambda_{ij}A_j}$$

$$A_i \;=\; \underline{A}\cdot\underline{e}_i \;=\; (A'_k\underline{e}_k{}')\cdot\underline{e}_i = \lambda_{ki}A'_k \;= \lambda^T_{ik}A'_k .$$

Note carefully that we do *not* put a prime on the vector itself – there is only one vector, $\underline{A}$, in the above discussion.

However, the *components* of this vector are different in different bases, and so are denoted by A_i in the $\underline{e}_i$ basis, A'_i in the $\underline{e}_i{}'$ basis, etc.

In matrix form we can write these relations as

$$\boxed{\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}}$$

Example: Consider a rotation of the axes about $\underline{e}_3$

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \cos\theta A_1 + \sin\theta A_2 \\ \cos\theta A_2 - \sin\theta A_1 \\ A_3 \end{pmatrix}$$

A direct check of this using trigonometric considerations is significantly harder (tutorial 4)!

7. 2. The Transformation of the Scalar Product

Let $\underline{A}$ and $\underline{B}$ be vectors with components A_i and B_i in the $\underline{e}_i$ basis and components A'_i and B'_i in the $\underline{e}_i{}'$ basis.
In the $\underline{e}_i$ basis, the scalar product, denoted by $(\underline{A}\cdot\underline{B})$, is:–

$$(\underline{A}\cdot\underline{B}) \;= A_iB_i \,.$$

In the $\underline{e}_i{}'$ basis, we denote the scalar product by $(\underline{A} \cdot \underline{B})'$, and we have

$$\begin{aligned} (\underline{A} \cdot \underline{B})' &= A'_i B'_i = \lambda_{ij} A_j \lambda_{ik} B_k = \delta_{jk} A_j B_k \\ &= A_j B_j = (\underline{A} \cdot \underline{B}) . \end{aligned}$$

Thus the scalar product is the same evaluated in any basis. This is of course expected from the geometrical interpretation of scalar product which is independent of basis.

Summary We have now obtained an algebraic definition of scalar and vector quantities: Under the orthogonal transformation from the $\underline{e}_i$ basis to the $\underline{e}_i{}'$ basis, defined by the transformation matrix $\lambda : \underline{e}_i{}' = \lambda_{ij}\, \underline{e}_j$, we have that:–

- A **scalar** is defined a number ϕ which transforms to ϕ' where:

$$\boxed{\phi' = \phi} \, .$$

- A **vector** is defined as an 'ordered triple' of numbers A_i which transforms to A'_i :

$$\boxed{A'_i = \lambda_{ij} A_j} \, .$$

7. 3. Transformation of the Vector Product

Care is needed with vector product under transformation!

Consider the inversion $\underline{e}_i{}' = -\underline{e}_i$ (i.e. $\lambda_{ij} = -\delta_{ij}$). Now, $A'_i = -A_i$ and $B'_i = -B_i$ and, for example,

$$A'_i \underline{e}_i{}' = (-A_i)(-\underline{e}_i) = A_i \underline{e}_i = \underline{A} \, .$$

so the vectors $\underline{A}$ and $\underline{B}$ are unchanged by the transformation, as they should be.

However if we calculate the vector product in the new basis using the formula

$\underline{C} = \begin{vmatrix} \underline{e}_1{}' & \underline{e}_2{}' & \underline{e}_3{}' \\ A'_1 & A'_2 & A'_3 \\ B'_1 & B'_2 & B'_3 \end{vmatrix}$ (or equivalently $C'_i = \epsilon_{ijk} A'_j B'_k$)

we obtain $\begin{vmatrix} \underline{e}_1{}' & \underline{e}_2{}' & \underline{e}_3{}' \\ A'_1 & A'_2 & A'_3 \\ B'_1 & B'_2 & B'_3 \end{vmatrix} = (-)^3 \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$

Which is $-\underline{C}$ as calculated in the original basis!

The explanantion is that if $\underline{e}_i$ was a **R.H. basis**, $\underline{e}_i{}'$ is now a **L.H. basis** since λ is an *improper* transformation. The formula we used for the vector product holds in a right-handed basis. If we use this formula in a left handed basis the direction of the vector product is reversed (it is equivalent to using left-hand rule rather than right-hand rule to calculate the vector product). Since the calculation of the vector product depends on the handedness of the basis it is called a 'pseudovector' or an 'axial vector' —for more details see 'Tensors and Fields' course in third year.

7. 4. Summary of story so far

We take the opportunity to summarise some key-points of what we have done so far. N.B. this is NOT a list of everything you need to know.

Key points from geometrical approach
You should recognise on sight that

$$\begin{aligned} \underline{x} \times \underline{b} = \underline{c} &\quad \text{is a line } (\underline{x} \text{ lies on a line}) \\ \underline{x} \cdot \underline{a} = d &\quad \text{is a plane } (\underline{x} \text{ lies in a plane}) \end{aligned}$$

Useful properties of vector products to remember

$$\begin{aligned} \underline{a} \cdot \underline{b} = 0 &\quad \Leftrightarrow \quad \text{vectors orthogonal} \\ \underline{a} \times \underline{b} = 0 &\quad \Leftrightarrow \quad \text{vectors collinear} \\ \underline{a} \cdot (\underline{b} \times \underline{c}) = 0 &\quad \Leftrightarrow \quad \text{vectors co-planar or linearly dependent} \\ \underline{a} \times (\underline{b} \times \underline{c}) &\quad = \quad \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \end{aligned}$$

Key points of suffix notation
We label basis vectors $1, 2, 3$ thus

$$\underline{A} = \sum_{i=1}^{3} A_i \underline{e}_i$$

The kronecker delta δ_{ij} can be used to define an orthonormal basis

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

δ_{ij} has a very useful sifting property

$$\sum_j [\cdots]_j \delta_{jk} = [\cdots]_k$$

and is represented in matrix form by the identity matrix.

$$(\underline{e}_1, \underline{e}_2, \underline{e}_3) = \pm 1 \quad \text{determines whether the basis is right or left-handed}$$

Key points of summation convention
Using the summation convention we have for example

$$\underline{A} = A_i \underline{e}_i$$

and the sifting property of δ_{ij} becomes

$$[\cdots]_j \delta_{jk} = [\cdots]_k$$

We introduce ϵ_{ijk} to enable us to write the vector products of basis vectors in a uniform way

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k$$

.

The vector products and scalar triple products are

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad \text{or equivalently} \quad (\underline{A} \times \underline{B})_i \; = \; \epsilon_{ijk} A_j B_k$$

$$\underline{A} \cdot (\underline{B} \times \underline{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad \text{or equivalently} \quad \underline{A} \cdot (\underline{B} \times \underline{C}) \; = \; \epsilon_{ijk} A_i B_j C_k$$

Key points of change of basis

The new basis is written in terms of the old through

$$\underline{e}_i{}' = \lambda_{ij} \underline{e}_j \quad \text{where } \lambda \text{ is the transformation matrix}$$

λ is an orthogonal matrix, the defining property of which is

$$\lambda^{-1} = \lambda^T$$

and this can be written as

$$\lambda \lambda^T = \mathbb{1} \quad \text{or} \quad \lambda_{ik} \lambda_{jk} = \delta_{ij}$$

$|\lambda| = \pm 1$ decides whether the transformation is proper or improper i.e. whether the handedness of the basis is changed

Key points of algebraic approach

A **scalar** is defined as a number that is invariant under an orthogonal transformation

A **vector** is defined as an object $\underline{A}$ represented in a basis by numbers A_i which to A_i' through

$$A_i' \; = \lambda_{ij} A_j .$$

or in matrix form

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$