Lecture 8: Tensors of Second Rank

8. 1. Nature of Physical Laws

Let us consider what is a 'physical law'. Firstly we should realise that it is not an absolute truth but a simple accurate description valid in some regime. For example special relativity shows that Newton's law, valid at low speeds, must be modified at high speeds. Simple examples of physical laws are

$$\begin{aligned} \underline{F} &= m\underline{a} && \text{Newton's Law} \\ \underline{J} &= g\underline{E} && \text{Ohm's Law, } g \text{ is conductivity} \\ \underline{F} &\simeq k\underline{v} && \text{Highly damped motion (m}\underline{a}\text{ term neglected)} \end{aligned}$$

The last 'law' may be unfamiliar but is relevant to the lecture demonstration

Notes

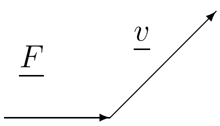
(i) All these laws take the form *vector* = *scalar* × *vector*

(ii) They are all linear *e.g.* $\underline{F}^a + \underline{F}^b = k\underline{v}^a + k\underline{v}^b = k(\underline{v}^a + \underline{v}^b)$

(iii) They relate two vectors in the *same* direction

8. 2. Examples of more complicated laws

Consider a hypothetical example . Imagine some strange and highly damped medium wherein a force produces a velocity in a direction different from the force. *e.g.* pushing in the 1 direction with force F_1 produces velocity components $v_1 = K_{11}F_1$ in the 1 direction and $v_2 = K_{21}F_1$ in the 2 direction, say. Here we discuss the two dimensional case that corresponds to the lecture demonstration, normally we will work in 3d.



$$\begin{aligned} v_1 &= K_{11}F_1 + K_{12}F_2 \\ v_2 &= K_{21}F_1 + K_{22}F_2 \end{aligned}$$

We may write this relation in a matrix form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \qquad \text{or} \qquad v_i = K_{ij}F_j$$

We say that K_{ij} is a matrix representing a tensor. However if we change basis the components of the vectors will change as will the matrix representation of the tensor.

Note that

(i) $\hat{\underline{v}} \neq \hat{\underline{F}}$
(ii) we have linearity $K_{ij}(F^a_j + F^b_j) = K_{ij}F^a_j + K_{ij}F^b_j = v^a_j + v^b_j$

Another example: anisotropic Ohm's law

Anisotropy means that 'the physics is different in different directions'

Suppose we have a material whose molecular bonding is such that electrons and therefore current can only flow in direction $\underline{n}$. A reasonable physical law relating current and electric field might be that

$$\underline{J} \propto \text{ projection of } \underline{E} \text{ onto } \underline{n}$$

so that

$$\begin{aligned} \underline{J} &= g(\underline{E} \cdot \underline{n})\underline{n} \\ J_i &= g E_j n_j n_i \end{aligned}$$

In general we can write the relation between $\underline{J}$ and $\underline{E}$ in the form

$$J_i = G_{ij} E_j$$

where G_{ij} is the conductivity tensor. For the particular system above we see

$$G_{ij} = g\, n_i n_j \ .$$

where g is a conductivity constant.

8. 3. Summary of why we need tensors

(i) Physical laws often relate two vectors

(ii) A Tensor provides a linear relation between two vectors which may be in different directions

(iii) Tensors allow the generalisation of isotropic laws ('physics the same in all directions') to anisotropic laws ('physics different in different directions')

8. 4. Tensors as mathematical objects

We may define a tensor as a linear map T between vector spaces

$$V \xrightarrow{T} W$$

Where $\underline{y} = T(\underline{x})$ for $\underline{x} \in V$ and $\underline{y} \in W$

e.g. $\underline{J} = G(\underline{E})$ — the space of electric field vectors is mapped onto the space of current vectors.

The definition of linearity is

$$\begin{aligned} T(\underline{x} + \underline{y}) &= T(\underline{x}) + T(\underline{y}) \\ T(\alpha \underline{x}) &= \alpha T(\underline{x}) \\ T(\underline{0}) &= \underline{0} \end{aligned}$$

8. 5. Tensor components and transformation law

Due to linearity we deduce

$$\begin{aligned} \underline{y} &= T(\underline{x}) = T(x_j \underline{e}_j) = x_j T(\underline{e}_j) \\ y_i &= \underline{e}_i \cdot \underline{y} = \underline{e}_i \cdot T(\underline{e}_j) x_j \end{aligned}$$

We write this as

$$\boxed{\begin{aligned} y_i &= T_{ij} x_j \\ T_{ij} &= \underline{e}_i \cdot T(\underline{e}_j) \end{aligned}}$$

Examples:

Projection tensor $P(\underline{x}) = \underline{n}(\underline{n} \cdot \underline{x})$ projects $\underline{x}$ onto $\underline{n}$.

$$P_{ij} = \underline{e}_i \cdot P(\underline{e}_j) = (\underline{e}_i \cdot \Big(\underline{n}(\underline{n} \cdot \underline{e}_j)\Big) = n_i n_j$$

Identity tensor $T(\underline{x}) = \underline{x}$

$$T_{ij} = \underline{e}_i \cdot T(\underline{e}_j) = \underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

Consider the projection tensor in $\underline{e}_i$ basis $\quad y_i = P_{ij} x_j \quad$ and $\quad P_{ij} = n_i n_j$

in $\underline{e}_i{}'$ basis $\quad y'_i = P'_{ij} x'_j \quad$ and $\quad P'_{ij} = n'_i n'_j$

Therefore

$$P'_{ij} = n'_i n'_j = \lambda_{ik} \lambda_{jl} n_k n_l = \lambda_{ik} \lambda_{jl} P_{kl}$$

To see the general case use the vector transformation law

$$\begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \lambda T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda T \lambda^T \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

where we have used the inverse vector transformation law. Thus in matrix form the tensor transformation law for an arbitrary tensor (of rank two) is

$$\boxed{T' = \lambda T \lambda^T}$$

or if we write it out in components

$$\boxed{T'_{ij} = \lambda_{ik} \lambda_{jl} T_{kl}}$$

Notes

(i) It is inaccurate to say a tensor is a matrix; rather the tensor is the fundamental object and is *represented* in a given basis by a matrix which transforms according to the above law.

(ii) It is wrong to say a matrix is a tensor *e.g.* the transformation matrix λ is not a tensor but nine numbers defining the transformation

8. 6. Invariants

Trace of a tensor The trace of a matrix is defined as the sum of the diagonal elements T_{ii}. Consider the trace of the matrix representing the tensor in the transformed basis

$$\begin{aligned} T'_{ii} &= \lambda_{ir}\lambda_{is}T_{rs} \\ &= \delta_{rs}T_{rs} = T_{rr} \end{aligned}$$

Thus the trace is the same, evaluated in any basis and is a scalar invariant.

Determinant It can be shown that the determinant is also an invariant

Symmetry of tensor If a matrix T_{ij} representing the tensor is symmetric then

$$T_{ij} = T_{ji}$$

Under a change of basis

$$\begin{aligned} T'_{ij} &= \lambda_{ir}\lambda_{js}T_{rs} \\ &= \lambda_{ir}\lambda_{js}T_{sr} \quad \text{using symmetry} \\ &= \lambda_{is}\lambda_{jr}T_{rs} \quad \text{relabelling} \\ &= T'_{ji} \end{aligned}$$

Therefore a symmetric tensor remains symmetric under a c.o.b. Similarly (exercise) an antisymmetric tensor $T_{ij} = -T_{ji}$ remains antisymmetric.

In fact one can decompose an arbitrary tensor T_{ij} into a symmetric part S_{ij} and an anti-symmetric part A_{ij} through

$$\boxed{S_{ij} = \frac{1}{2}\left[T_{ij} + T_{ji}\right] \quad A_{ij} = \frac{1}{2}\left[T_{ij} - T_{ji}\right]}$$

8. 7. Eigenvectors

In general a tensor maps a given vector onto a vector in a different direction: if a vector $\underline{n}$ has components n_i then

$$T_{ij}n_j = m_i ,$$

where m_i are components of $\underline{m}$, the vector that $\underline{n}$ is mapped onto.

However some special vectors called **eigenvectors** may exist such that $m_i = t\,n_i$ *i.e.* the new vector is in the *same* direction as the original vector.
Eigenvectors usually have special physical significance (see later).