Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

Date : **05-Nov-2020**

Attendance : **9**

Example 69. *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.d\vec{r}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*

$x = t,\ y = t^2,\ z = t^3.$ *(Uttarakhand, I Semester, Dec. 2006)*

$$\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$

parametric equations.

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\int \vec{A}\cdot d\vec{r} = \int 3x^2\,dx + \int 6y\,dx - \int 14yz\,dy + \int 20xz^2\,dz$$

$x = t \qquad y = t^2 \qquad z = t^3$

$t = 0\ ;\ x = 0 \qquad t = 0,\ y = 0 \qquad t = 0,\ z = 0$

$t = 1\ ;\ x = 1 \qquad t = 1,\ y = 1 \qquad t = 1,\ z = 1$

$y = t^2$

$dy = 2t\cdot dt$

$z = t^3$

$dz = 3t^2\,dt$

$$\int \vec{A}\cdot d\vec{r} = \int_0^1 3t^2\,dt + \int_0^1 6t^2\,dt - \int_0^1 14\cdot(t^2)(t^3)(2t\,dt) + \int_0^1 20(t)(t^3)^2\,3t^2\,dt$$

$$= 3 \cdot \frac{t^3}{3} \Big|_0^1 + 6 \cdot \frac{t^3}{3} \Big|_0^1 - 28 \cdot \frac{t^7}{7} \Big|_0^1 + 60 \frac{t^{10}}{10} \Big|_0^1$$

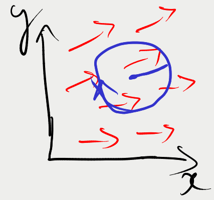
$$= 1^3 + 2 \cdot 1^3 - 4 \cdot 1^7 + 6 \cdot 1^{10}$$

$$= 3 - 4 + 6$$

$$= 3 + 2$$

$$= 5 \,//.$$

Example 71. *Compute* $\int_c \vec{F} \cdot \vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*



$$\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2 + y^2} \qquad x^2 + y^2 = 1$$

Parametric form for circle:

$$x^2 + y^2 = r^2$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

if $\Rightarrow$ is a unit circle the $r = 1$

$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\} \text{unit circle } //$$

$$\vec{F} \cdot d\vec{r} = \frac{y}{\underbrace{x^2+y^2}_{=1}} dx - \frac{x\,dy}{\underbrace{x^2+y^2}_{=1}}$$

$$= y\,dx - x\,dy$$

Parametric form

$$x = \cos\theta \quad ; \quad dx = -\sin\theta\, d\theta$$
$$y = \sin\theta \quad ; \quad dy = \cos\theta\, d\theta$$

$$\theta \to 0 \sim 2\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta)\, d\theta - \cos\theta(\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(-\sin^2\theta - \cos^2\theta\right) d\theta$$

$$= -\int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta\right) d\theta$$

$$= -\int_0^{2\pi} d\theta$$

$$= -\theta\Big|_0^{2\pi}$$

$$= -2\pi + 0$$

$$\oint \vec{F} \cdot d\vec{r} = -2\pi \quad A.$$

Example 72. *Show that the vector field* $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ *is conservative. Find its scalar potential and the work done in moving a particle from* (–1, 2, 1) to (2, 3, 4).

(*A.M.I.E.T.E. June 2010, 2009*)

$$\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$$

$$\vec{F} = \vec{\nabla}\phi$$

pontential

conservative force

$\vec{F}$ depends only on the r or $\frac{1}{r}$

$\vec{F}$ depends on $\vec{v}$ $\frac{dr}{dt}$

$$\vec{\nabla}\times\vec{F} = 0$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(2x^2y)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3x^2z^2) - \frac{\partial}{\partial z}(2x(y^2+z^3))\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2x(y^2+z^3))\right]$$

$$= \hat{i}\,[\cancel{0-0}]^{0} - \hat{j}\,[\cancel{6xz^2 - 6xz^2}]^{0} + \hat{k}\,[\cancel{4xy - 4xy}]^{0}$$

$$\vec{\nabla}\times\vec{F} = 0$$

so $\vec{F}$ is conservative.

scalar field

$$\vec{F} = \vec{\nabla}\phi$$

$$d\phi = \left[\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz\right]$$

$$= \underbrace{\left[\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right]}\cdot \left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} \qquad \vec{F} = \vec{\nabla}\phi$$

$$d\phi = \vec{F}\cdot d\vec{r}$$

$$\phi = \int d\phi = \int \vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = 2x(y^2+z^3)\,dx + 2x^2y\,dy + 3x^2z^2\,dz$$

$$\phi = \int d\phi = \int \vec{F}\cdot d\vec{r} = \int 2x(y^2+z^3)\,dx + \int 2x^2y\,dy + \int 3x^2z^2\,dz$$

$$= \int 2xy^2\,dx + \int 2xz^3\,dx + \int 2x^2y\,dy + \int 3x^2z^2\,dz$$

$$= 2y^2\int x\,dx + 2z^3\int x\,dx + 2x^2\int y\,dy + 3x^2\int z^2\,dz$$

$$= \cancel{2}xy^2 \cdot \frac{x^2}{\cancel{2}} + \cancel{2}z^3 \cdot \frac{x^2}{\cancel{2}} + \cancel{2}x^2 \cdot \frac{y^2}{\cancel{2}} + \cancel{3}x^2 \cdot \frac{z^3}{\cancel{3}}$$

$$= y^2x^2 + x^2z^3 + x^2y^2 + x^2z^3 + C$$

$$\phi = 2\left(y^2x^2 + x^2z^3\right) + C$$

$$\begin{array}{ccc} x_u & y_u & z_u \\ 2, & 3, & 4 \end{array}$$

$$\phi\Big|_{-1,2,1}^{2,3,4} = 2\left(y^2x^2 + x^2z^3\right)\Big|_{-1,2,1}^{2,3,4}$$

$$\begin{array}{ccc} -1, & 2, & 1 \\ x_l & y_l & z_l \end{array}$$

$$= 2\cdot\left[3^2\cdot(2)^2 + 2^2\cdot 4^3 - \left(2^2\cdot(-1)^2 + (-1)^2\cdot(1)^3\right)\right]$$

$$= 2\cdot\left[9\cdot 4 + 4\cdot 64 - (4 + 1)\right]$$

2

$$= 2\cdot\left[36 + 256 - 5\right]$$

$$\phi\Big|_{-1,2,1}^{2,3,4} = 574 \checkmark$$

291 ← text book answer (wrong)

Ex 5.10

5. Find the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$, under the field of force given by $\vec{F} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$. Is the field of force conservative?
(*A.M.I.E.T.E., Winter 2000*) **Ans.** 40 π

$$\oint \vec{F}\cdot d\vec{r}$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F}\cdot d\vec{r} = (2x - y + z)\,dx + (x + y - z^2)\,dy + \cancel{(3x - 2y + 4z)\,dz}^{\,0}$$

$$z = 0 \qquad \frac{x^2}{25} + \frac{y^2}{16} = 1$$

given in the problem.

$$\vec{F}\cdot d\vec{r} = (2x - y)\,dx + (x + y)\,dy$$

Equation of circle:

$$x^2 + y^2 = r^2$$

Equation of ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

General form

Parametric form

our curve 'C'

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$x = a\cos\theta$$

$$y = b\sin\theta$$

For our case $a = 5$; $b = 4$

$$\therefore\ x = 5\cos\theta \ ; \quad dx = -5\sin\theta\, d\theta$$

$$y = 4\sin\theta \ ; \quad dy = 4\cos\theta\, d\theta$$

We have

$$\vec{F}\cdot d\vec{r} = (2x - y)\,dx + (x + y)\,dy$$

So, $\vec{F}\cdot d\vec{r} = 2\cdot(5\cos\theta\cdot(-5\sin\theta\, d\theta))$

$-4\sin\theta(-5\sin\theta\, d\theta)$

$+5\cos\theta\cdot 4\cos\theta\, d\theta$

$+4\sin\theta\cdot 4\cos\theta\, d\theta$

$= -50\cos\theta\sin\theta\, d\theta$

$+20\sin^2\theta\, d\theta + 20\cos^2\theta\, d\theta$

$+16\cos\theta\sin\theta\, d\theta$

$= (16-50)\cos\theta\sin\theta\, d\theta$

$+20(\underbrace{\sin^2\theta+\cos^2\theta}_{1})\, d\theta$

$\vec{F}\cdot d\vec{r} = -34\cos\theta\sin\theta\, d\theta + 20\, d\theta$

$W.d = \int_0^{2\pi}\vec{F}\cdot d\vec{r} = -34\int_0^{2\pi}\cos\theta\sin\theta\, d\theta$

$+20\int_0^{2\pi} d\theta$

$= -34\left.\frac{\sin^2\theta}{2}\right|_0^{2\pi}$

$+20\cdot\theta\Big|_0^{2\pi}$

$= 0 + 20\cdot 2\pi = 40\pi$ //