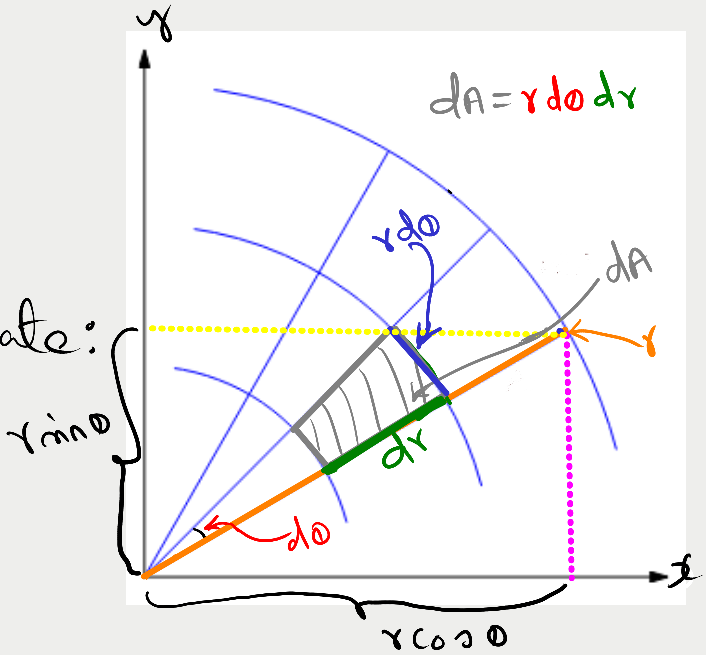
In Cartesian co-ordinates:

$$dA = dx\,dy$$

Elemental Area

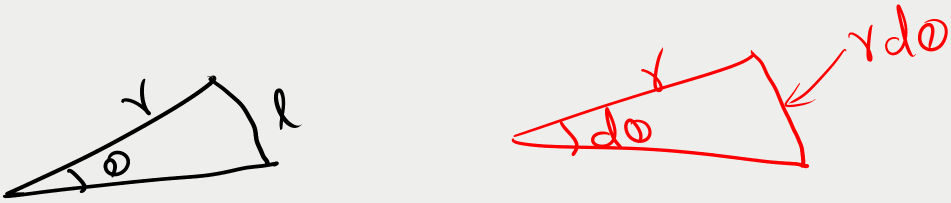


In cylindrical coordinate:

$$dA = r\,d\theta\,dr$$

Elemental area

Sector



$$l = r\theta$$

Transformation from Cartesian to Cylindrical coordinate

$$x \longrightarrow r\cos\theta$$

$$y \longrightarrow r\sin\theta$$

Elemental area

$$dA = dx\,dy \longrightarrow r\,d\theta\,dr$$

Department of Medical Physics
Bharathidasan University

Mathematical Physics (MP101)

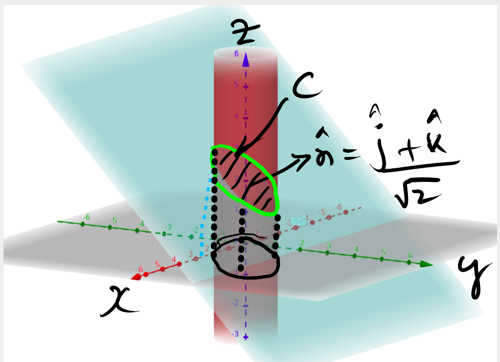
Date : **02-Dec-20**

Attendance : **9**

Example 87. *Evaluate* $\int_C \vec{F} \cdot d\vec{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$. (Gujarat, I sem. Jan. 2009)

Stoke's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds$$



$$\vec{F} = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = (1+2y)\hat{k} \quad ; \quad \phi = y + z - 2$$

$$\vec{\nabla}\phi = \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{j} + \hat{k}}{\sqrt{1^2 + 1^2}} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}$$

$$\hat{n} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}$$

$$\hat{n} \cdot \hat{k} = \frac{1}{\sqrt{2}}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = (\hat{x} + 2y\hat{k})\cdot\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$$

$$= \frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k} + 2y\cdot\frac{1}{\sqrt{2}}\hat{k}\cdot\hat{k}$$

$$\boxed{(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{2}}(1+2y)}$$

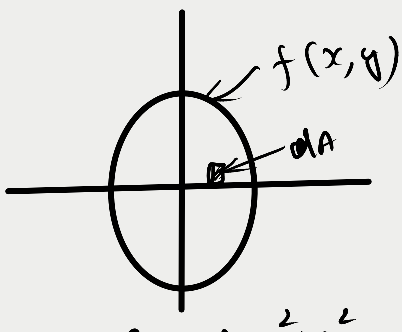
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,dx\,dy$$

$$\iint_S (x^2+y^2)\underbrace{dx\,dy}_{dA}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1+2y}{\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\,dx\,dy$$

$$= \iint_S (1+2y)\,\underbrace{dx\,dy}_{dA}$$



$f(x,y) = x^2 + y^2$

In cylinderical coordinate

$$x \to r\cos\theta \;;\quad y \to r\sin\theta$$

$$dx\,dy \to r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{r=1} \left(1+2(r\sin\varphi)\right)\cdot r\, d\varphi\, dr$$

$$= \int_0^{2\pi} d\varphi \int_0^1 \left[r\,dr + 2\cdot r^2 \sin\varphi\, dr\right]$$

$$= \int_0^{2\pi} d\varphi \left[\frac{r^2}{2}\Big|_0^1 + 2\cdot\frac{r^3}{3}\sin\varphi\Big|_0^1\right]$$

$$= \int_0^{2\pi} d\varphi \left[\frac{1}{2} + \frac{2}{3}\sin\varphi\right]$$

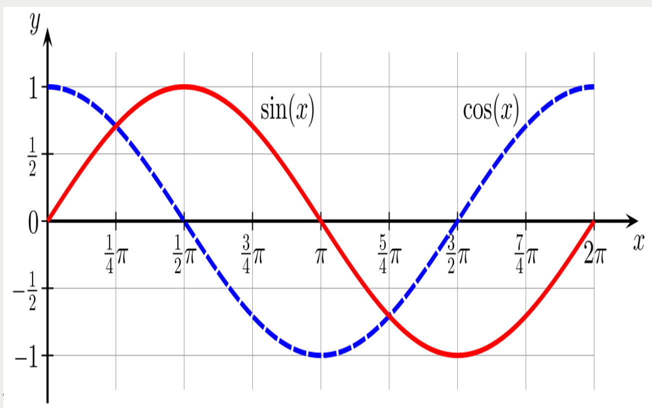
$$= \frac{1}{2}\varphi\Big|_0^{2\pi} + \frac{2}{3}(-\cos\varphi)\Big|_0^{2\pi}$$

$$= \left[\pi - 0 - \left(\frac{2}{3}\cancelto{1}{\cos 2\pi} - \frac{2}{3}\cancelto{1}{\cos 0}\right)\right]$$

$$= \pi - \cancel{\frac{2}{3}(1)} + \cancel{\frac{2}{3}(1)}$$

$$= \pi$$

$$\oint_C \left(-y^2\hat{i} + x\hat{j} + z^2\hat{k}\right)\cdot d\vec{r} = \pi$$

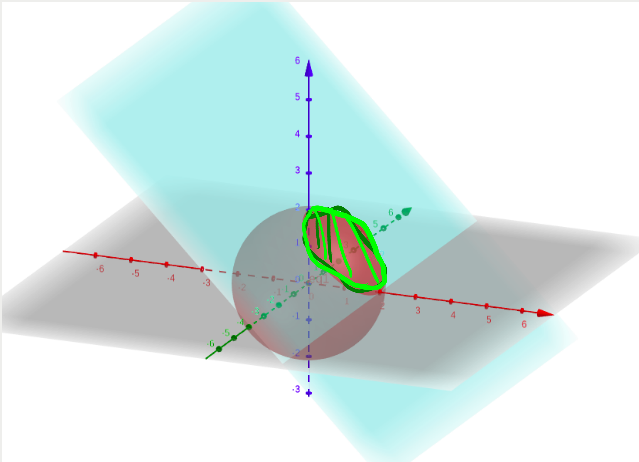


Example 88. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ *and* $x + z = a$. (*Nagpur, Summer 2001*)

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (y\,dx + z\,dy + x\,dz)$$



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$$

$$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\vec{\nabla}\times\vec{F} = -(\hat{i}+\hat{j}+\hat{k}) \;; \qquad \phi = x + z - a$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \qquad \hat{n} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{2}}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds = \iint_S \frac{-2}{\sqrt{2}}\cdot ds$$

$$= -\frac{2}{\sqrt{2}} \underbrace{\iint_S ds}_{\text{area of the disc formed by the intersection the plane } x+z=a, \text{ and } x^2+y^2+z^2=a^2}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds = \frac{-2}{\sqrt{2}} A_d$$

W.K.T area of a disc $= \pi r^2$

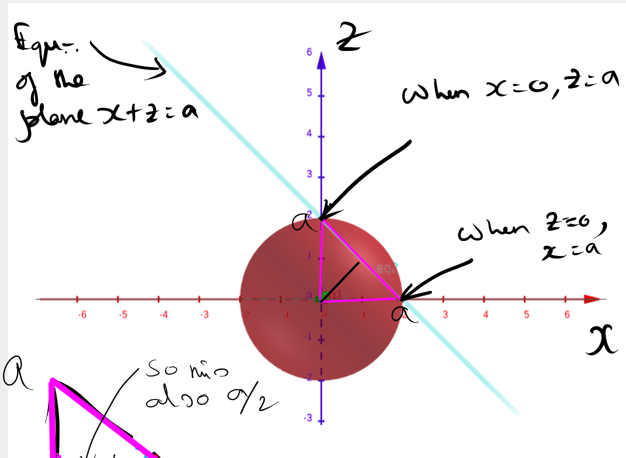
If "r_d" is the radius of the disc

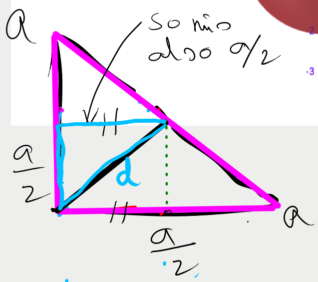
radius of the sphere

distance bt the center of the disc and the center of the sphere

$$\boxed{a^2 = r_d^2 + d^2}$$

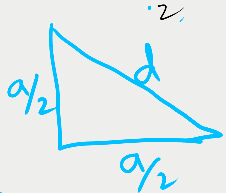
radius of the disc



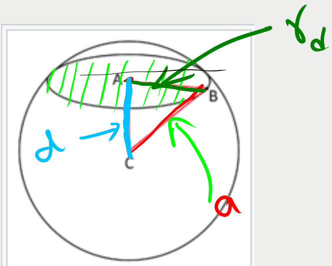


$$d^2 = \frac{a^2}{4} + \frac{a^2}{4}$$

$$d^2 = \frac{2a^2}{4}$$



$$\boxed{\therefore d^2 = \frac{a^2}{2}}$$



$BC^2 = AB^2 + AC^2$, where C is the center of the sphere, A is the center of the small circle, and B is a point in the boundary of the small circle. Therefore, knowing the radius of the sphere, and the distance from the plane of the small circle to C, the radius of the small circle can be determined using the Pythagorean theorem.

$$\therefore r_d^2 = a^2 - d^2$$

$$= a^2 - \frac{a^2}{2}$$

$$\boxed{\therefore r_d^2 = \frac{a^2}{2}}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, ds = -\frac{2}{\sqrt{2}} \cdot A_d$$

$$A_d = \pi r_d^2$$

$$r_d^2 = \frac{a^2}{2}$$

$$\oint_c \vec{F} \cdot dr = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, ds = -\frac{2}{\sqrt{2}} \pi \cdot \frac{a^2}{2}$$

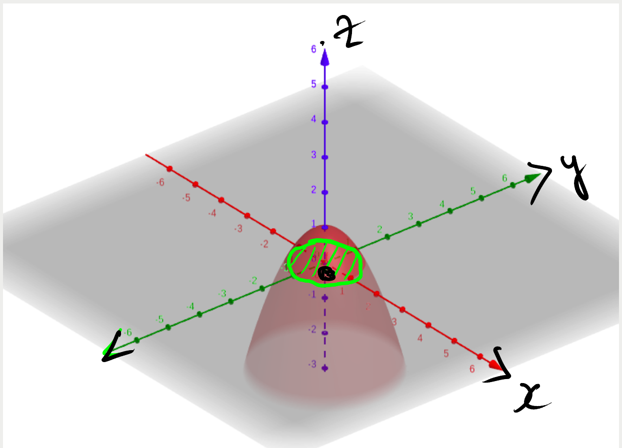
$$= -\frac{\cancel{2}}{\sqrt{2}} \pi \cdot \frac{a^2}{\cancel{2}}$$

$$= -\frac{\pi a^2}{\sqrt{2}} \quad //$$

Example 89. *Directly or by Stoke's Theorem, evaluate* $\iint_s curl\ \vec{v} \cdot \hat{n}\, ds$, $\vec{v} = \hat{i}y + \hat{j}z + \hat{k}x$, *s is the surface of the paraboloid* $z = 1 - x^2 - y^2$, $z^3 \geq 0$ *and* $\hat{n}$ *is the unit vector normal to s.*

$0 = 1 - x^2 - y^2$

$x^2 + y^2 = 1$



$$\vec{v} = y\hat{i} + z\hat{j} + x\hat{k}$$

Since this surface is on the xy plane $\hat{n} = \hat{k}$

$$\vec{\nabla} \times \vec{v} = -(\hat{i} + \hat{j} + \hat{k})$$

$$(\vec{\nabla} \times \vec{v}) \cdot \hat{n} = -1$$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dx\,dy}{1}$$

$$\iint_S \underbrace{(\vec{\nabla}\times\vec{v})\cdot\hat{n}}\, ds = \iint_S (-1)\cdot \frac{dx\,dy}{1}$$

$$= -1\iint_S dx\,dy$$

$$= -1\cdot\pi(1)^2$$

$$= -\pi$$

Example 90. *Use Stoke's Theorem to evaluate* $\int_c \vec{v}\cdot\vec{dr}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c* is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

$$\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$$

From Stoke's theorem

$$\oint_C \vec{v}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{v})\cdot\hat{n}\, ds$$

On the xy plane

$z = 0$

$x^2 + y^2 + \cancel{z^2} = 9$ ($\to 0$)

$\underbrace{x^2 + y^2 = 3^2}$

radius of the circle

$$\vec{\nabla}\times\vec{v} = -z\hat{j} - y\hat{k} \quad ;$$

$$\phi = x^2 + y^2 + z^2 - 9$$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4x^2 + 4y^2 + 4z^2}$$

$$= 2\cdot\sqrt{x^2+y^2+z^2}$$

$$= 2\cdot\sqrt{9}$$

$$|\vec{\nabla}\phi| = 6$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{6}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

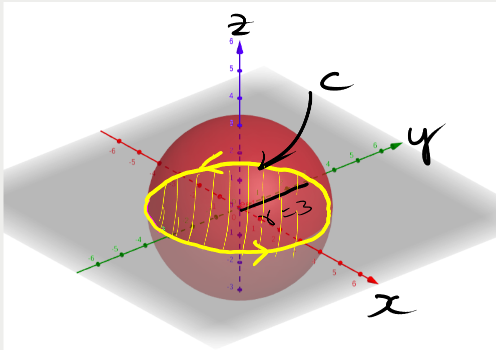
$$(\vec{\nabla}\times\vec{v})\cdot\hat{n} = -\frac{2}{3}zy$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\frac{z}{3}} = \frac{3}{z}\,dx\,dy$$

$$\iint_S \underbrace{(\vec{\nabla}\times\vec{v})\cdot\hat{n}}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint_S -\frac{2}{\cancel{3}}\cancel{z}y\cdot\frac{\cancel{3}}{\cancel{z}}\,dx\,dy$$

$$= -2\iint_S y\cdot dx\,dy$$

In cylindrical coordinate



$$y \to r\sin\phi$$

$$dx\,dy \to r\,d\phi\,dr$$

$$= -2\int_0^{2\pi}\int_0^{3} r\sin\phi\, r\,d\phi\,dr$$

$$= -2\int_0^{2\pi} d\phi \int_0^{3} r^2\sin\phi\,dr$$

$$= -2\int_0^{2\pi}\sin\phi\,d\phi\cdot\left[\frac{r^3}{3}\right]_0^3$$

$$= -2\int_0^{2\pi}\sin\phi\,d\phi\cdot\frac{3^{\cancel{3}\,2}}{\cancel{3}}$$

$$= -18\left(-\cos\phi\right)_0^{2\pi}$$

$$= 18\left[\underset{1}{\cos 2\pi} - \cos 0\right]$$

$$= 18\left[1-1\right]$$

$$= 0$$