Biased Random Walk in a box

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Abstract

1 Simple Random Walk and Diffusive Behavior

A walker takes on discrete positions on a line $X(t) \in \mathbb{Z}$. He starts at the origin: $X(0) = 0$ and at each time step $\Delta t = 1$, moves either to the left or to the right with probabilities $p_{L,R}$, respectively, so that $p_L + p_R = 1$. At a time $t = T$, the walker is at some position $X(T)$ which is not predetermined, i.e. the dynamics is intrinsically stochastic. It is then more suitable to ask the following question: how often does the walker reach X after T steps, given that it started the walk at $X = 0$?

At each time step there are two possibilities: to go left or to go right. Thus, the trajectory of the walk is a succession of moves $\underbrace{RLLLRRLRLRLL...RL}_{T \text{ elements}}$. Note that the end position is independent of the particular sequence of moves, as long as the number of moves of each type is the fixed:

$$X = N_R - N_L = 2N_R - T, \tag{1}$$

where N_R is the number of moves to the right and N_L is the number of moves to the left ($N_R + N_L = T$). A given sequence occurs with probability $p_R^{N_R} p_L^{N_L}$. Any permutation of the sequence of T steps (obtained by exchanging a R move with a L move) leads the walker to the same position after T steps. The probability for a walker to be at X is related to the probability of going to the right N_R times:

$$P(N_R) = \frac{T!}{(T - N_R!)N_R!)} p_R^{N_R} (1 - p_R)^{T - N_R} \tag{2}$$

The distribution is normalized which is easy to show using the binomial formula.

$$\sum_{N_R=0}^{T} \frac{T!}{(T - N_R!)N_R!} p_R^{N_R} (1 - p_R)^{T - N_R} = (p_R + 1 - p_R)^T = 1 \tag{3}$$

Similarly, the average number of steps to the right after a time T is given by

$$\langle N_R \rangle = \sum_{n=0}^{T} \frac{T!}{(T - n!)n!} p_R^n p_L^{T-n} n = p_R \frac{d(p_R + p_L)^T}{dp_R} = p_R T \tag{4}$$

Moreover, the variance $\sigma_{N_R}^2 = \langle N_R^2 \rangle - \langle N_R \rangle^2$ is obtained by iterating equation (4).

$$\langle N_R^2 \rangle = p_R d_{p_R} p_R d_{p_R} (p_R + p_L)^T = T p_R d_{p_R} \left[p_R (p_R + p_L)^{T-1} \right] = p_R T (1 + p_R (T - 1)) \tag{5}$$

The second term cancels so that $\sigma_{N_R}^2 = p_R T - p_R^2 T = p_R (1 - p_R) T = p_R p_L T$. From equation (1), it is easy to see that a factor of 4 appears in the variance of X, so that

$$\sigma_X^2 = 4 p_R p_L T \tag{6}$$

Changing the variable $N_R \to X$, we obtain the distribution

$$P(X,T) = \frac{T!}{(\frac{T+X}{2}!)(\frac{T-X}{2}!)} p_R^{(T+X)/2} p_L^{(T-X)/2}, \tag{7}$$

with average $\langle X \rangle = (p_R - p_L)T$. The average drifts towards $p_R \neq p_L$, while the width increases like $\sqrt{T}$, as we found in equation (6).

The long time behavior of $P(X,T)$ is obtained by taking a continuum limit. The distribution is always in $[-T,T]$, but it is only significantly different from zero in a region of size σ_X around the average. Thus, we introduce a variable $x = X/T$, which takes us to the continuum limit. Stirling's formula

$$\ln N! = NlnN - N + \frac{1}{2}\ln(2\pi N) \tag{8}$$

allows us to find the desired limit.

$$\begin{aligned}\ln P(xT,T) &= \ln\left(\frac{T!}{(\frac{T+xT}{2})!(\frac{T-xT}{2})!}p_R^{(T+xT)/2}(1-p_R)^{(T-xT)/2}\right)\\ &= -\frac{1}{2}\ln\left(2\pi T\frac{1-x^2}{4}\right) + Tf(x),\end{aligned} \tag{9}$$

where

$$f(x) = \frac{1+x}{2}\ln\left(\frac{2p_R}{1+x}\right) + \frac{1-x}{2}\ln\left(\frac{2p_L}{1-x}\right) \tag{10}$$

In the large T limit, it is equivalent to maximize the $P(X,T)$ or $f(x)$.

$$d_x f|_{x_m} = 0 \to x_m = p_R - p_L \tag{11}$$

which coincides with the mean $\langle x\rangle$, as it should. Developing $f(x)$ around its maximum, we obtain

$$f(\langle x\rangle + \delta) = -\frac{1}{2}\frac{\delta^2}{1-(p_R-p_L)^2} = -\frac{\delta^2}{8p_Rp_L} \tag{12}$$

leading to the Gaussian distribution

$$P(X,T) = \frac{\exp\left(-\frac{(X-T(p_R-p_L))^2}{8p_Rp_LT}\right)}{\sqrt{2\pi Tp_Rp_L}} \tag{13}$$

We usually describe the behavior of this distribution in terms of a coexistence of two regimes:

- The maximum displacement characterized by a constant velocity $v = p_R - p_L$.
- The diffusion of the probability (which corresponds to the widening of the distribution with $\sqrt{T}$) described by a diffusion constant $D = 2p_Rp_L$.

To reach the continuum limit, we multiply by $T/2$ since in the discrete case X only takes on values separated by 2.

$$P(x) = \frac{\exp\left(-\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}, \tag{14}$$

where $\langle x\rangle = p_R - p_L$ and $\sigma^2 = 4p_Rp_LT$. In 'intensive units', the distribution function is centered in the same position, but it becomes narrower as time progresses.

2 Master equation

It is possible to reformulate our problem as the time evolution of an initial probability distribution $P(X,0) = \delta_{X,0}$. The time evolution is defined by the conditioned probabilities $p_{L,R}$. To formalize the probabilistic concepts we used in the previous section, we introduce some notation. Let $P(X_1,t_1;X_2,t_2)$ be the probability of being at X_1 at time t_1 and X_2 at time t_2. By marginalizing with respect to either of the variables, we obtain

$$\sum_{X_{1(2)}} P(X_1,t_1;X_2,t_2) = P(X_{2(1)},t_{2(1)}) \tag{15}$$