Discrete Mathematics

Model Question Paper

Section — A

① (a) Show that $P \to Q$ and $\neg Q \to \neg P$ are logically Equivalent.

Sol Let us construct a truth table for $P \to Q$ and $\neg Q \to \neg P$.

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since bothe $P \to Q$ & $\neg Q \to \neg P$ columns are having identical truth values for all the combinations of the truth values of P and Q, $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent.

(b) Show that the relation $\leq$ (less than or equal) defined on the set of positive integers Z^+ is a partial order relation.

Sol Let $a \in Z^+$

Since $a \leq a \quad \forall a \in Z^+$

$\therefore$ "$\leq$" is a reflexive relation on Z^+

Let $a, b \in Z^+$

if $a \leq b$ and $b \leq a$ then $a = b$ for every integer a, b.

$\therefore$ "$\leq$" is an anti-symmetric relation on Z^+

Let $a, b, c \in Z^+$

if $a \leq b$ and $b \leq c$ then

$\Rightarrow a \leq c$ for all $a, b, c \in Z^+$

$\therefore$ "$\leq$" is a transitive relation on Z^+

Since "$\leq$" is reflexive, anti-symmetric and transitive

$\therefore$ "$\leq$" is a Partial order relation on Z^+.

(C) Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Sol

We can derive $R \wedge (P \vee Q)$ from the given premises in the following steps.

{1}	(1)	$P \to M$	Rule P	
{2}	(2)	$\neg M$	Rule P	
{1,2}	(3)	$\neg P$	Rule T (1),(2)	$P \to M, \neg M \Rightarrow \neg P$
{4}	(4)	$P \vee Q$	Rule P.	
{1,2,4}	(5)	Q	Rule T (3),(4)	$\neg P, P \vee Q \Rightarrow Q$
{6}	(6)	$Q \to R$	Rule P	
{1,2,4,6}	(7)	R	Rule T (5),(6)	$Q, Q \to R \Rightarrow R$
{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T (7),(4)	$R, P \vee Q \Rightarrow R \wedge (P \vee Q)$

$\therefore$ $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

(d) If R be a relation on the set of integers Z, defined by $R = \{(x,y) / x \in Z, y \in Z, (x-y) \text{ is divisible by } 6\}$ is an equivalence relation.

Sol

$R = \{(x-y) / x \in Z, y \in Z, (x-y) \text{ is divisible by } 6\}$

Let $x \in Z$ then

$x - x = 0$, which is divisible by 6

$\therefore$ $(x,x) \in R \; \forall x,y \in Z$.

$\therefore$ R is reflexive

Let $x, y \in Z$

if $(x-y)$ is divisible by 6 then $(x,y) \in R$

Since $x-y$ is divisible by 6

$\Rightarrow x-y = 6a$ ie multiple of 6

$\Rightarrow y-x = -6a$ which is also divisible by 6

$\therefore (y,x) \in R$ whenever $(x,y) \in R \ \forall x,y \in Z$

$\therefore$ R is symmetric.

Let $x, y, z \in Z$

and $(x,y) \in R$ and $(y,z) \in R$

~~$x-y = 6a$~~

$x-y$ is of the form $x-y = 6a$ — ①

$y-z$ is of the form $y-z = 6b$ — ②

add ①, ②

$x-\cancel{y}+\cancel{y}-z = 6a+6b$

$\Rightarrow x-z = 6(a+b)$

$\therefore (x-z)$ is divisible by 6.

$\therefore (x,z) \in R$ whenever $(x,y) \in R$ & $(y,z) \in R \ \forall x,y,z \in Z$

$\therefore$ R is transitive.

Since R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation

(2) (a) Solve the recurrence relation $a_n = a_{n-1} + 2$, $n \geq 2$ subject to initial condition $a_1 = 3$

Sol

$$a_n = a_{n-1} + 2$$

Since the above recurrence relation is non-homogeneous the required solution is

$$a_n = a_n^{[h]} + a_n^{[p]} \quad \text{—} (1)$$

Here $a_n^{[h]}$ is the solution for the homogeneous part i.e the associated homogeneous recurrence relation.

i.e. $a_n = a_{n-1}$

$$\Rightarrow a_n - a_{n-1} = 0 \quad \text{—} (2)$$

The characteristic equation for eq (2) is

$$r - 1 = 0$$
$$\Rightarrow r = 1$$

Now, the solution for eq (2) is $a_n^{[h]} = \alpha_1 \cdot 1^n$

$$= \alpha_1 \cdot 1 = \alpha_1 \quad \text{—} (3)$$

Next, we have to find a particular polynomial solution i.e. $a_n^{[p]}$.

Since $F(n) = 2$, i.e a polynomial in n of degree zero.

the reasonable solution is

$$a_n^{[p]} = n^m \left(A_0 + A_1 n + \cdots A_2 n^2 + \cdots A_q n^q \right)$$

Here m is the multiplicity of the root which is '1'

$$\therefore a_n^{[p]} = n^1 (A_0) = A_0 n \quad \text{—} (4)$$

~~There~~ Since $a_n = a_{n-1} + 2$

Substitute $a_n = A_0 n$

$\Rightarrow A_0 n = A_0(n-1) + 2$

$\Rightarrow A_0 \cancel{n} = A_0 \cancel{n} - A_0 + 2$

$\Rightarrow A_0 = 2$

$\therefore a_n^{[p]} = 2n$ — (5)

Since from eq (1) $a_n = a_n^{[h]} + a_n^{[p]}$

$\Rightarrow a_n = \alpha_1 + 2n$ — (6)

Since the initial condition $a_1 = 3$

$\Rightarrow$ put $n=1$ in eq (6)

$\Rightarrow a_1 = \alpha_1 + 2(1)$

$\Rightarrow a_1 = \alpha_1 + 2$

Substitute $a_1 = 3$

$\Rightarrow 3 = \alpha_1 + 2$

$\Rightarrow \alpha_1 = 3 - 2 = 1$

$\therefore$ The required solution is $a_n = 1 + 2n$

Iteration Method:

$a_n = a_{n-1} + 2, \quad n \geq 2, \; a_1 = 3$

$a_n = a_{n-1} + 2$

$= (a_{n-2} + 2) + 2$

$= ((a_{n-3} + 2) + 2) + 2$

$\ldots\ldots\ldots$

$= a_{n-(n-1)} + (n-1)2$

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$= a_1 + (n-1)2$

$= 3 + 2n - 2$

$= 2n+1$

(b) How many ways are there to assign five different jobs to four different employees if every employee assigned atleast one job?

Sol

We know that if m and n be positive integers with $m \geq n$. Then the expansion

$$n^m - nC_1(n-1)^m + nC_2(n-2)^m + \dots + (-1)^{n-1} C_{n-1}$$

gives the onto functions from a set with m elements to a set with n elements.

Now assume the assignment of jobs as a function from the set of five jobs to the set of four employees. An assignment in which every employee gets atleast one job is the same as onto function from the set of jobs to the set of employees.

$n=4,\ m=5$

$$\therefore\ 4^5 - 4C_1(3)^5 + 4C_2(2)^5 - 4C_3(1)^5$$

$= 1024 - 4 \cdot 243 + 6 \cdot 32 - 4 \cdot 1$

$= 1024 - 972 + 192 - 4 = 240$

$\therefore$ 240 ways to assign jobs so that each employee is assigned at least one job.

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(c) Applying pegion hole principle show that of any 14 integers are selected from the set $S = \{1, 2, 3, \ldots 25\}$ there are atleast two whose sum is 26. Also write a statement that generalizes this result.

Sol Let the pegions be the numbers selected. Define 13 pegion holes corresponding to the 13 sets: $\{1,25\}, \{2,24\}, \{3,23\}, \{4,22\}, \{5,21\}, \{6,20\}, \{7,19\}, \{8,18\}, \{9,17\}, \{10,16\}, \{11,15\}, \{12,14\}, \{13\}$.

Notice that the numbers in each of the first 12 sets sum is 26 and there is no pair of distinct numbers containing 13 that sum to 26.

When a number is selected, it gets placed into the pegionhole corresponding to the set that contains it. Atmost one number can go into the pegion hole corresponding to $\{13\}$

Since 14 numbers are selected and placed in 13 pegionholes, some pegion hole contains two numbers and those two numbers sum to 26.

$\therefore$ If $n+1$ numbers are selected from $\{1, 2, \ldots 2n-1\}$ then some two of these numbers sum to $2n$

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(2) (d) In a class of 25 students, 12 have taken mathematics, 8 have taken mathematics but not biology. Find the number of students who have taken mathematics and biology and those who have taken biology but not mathematics.

Sol

Let $n(M)$ = no. of students taken Mathematics = 12.

$n(B)$ = no. of students taken Biology

Since there are 25 students in the class

then $n(M \cup B) = 25$.

No. of students taken Mathematics but not Biology

$$n(M \cap \bar{B}) = 8.$$

No. of students taken both Mathematics and Biology

$$n(M \cap B) = n(M) - n(M \cap \bar{B})$$

$$= 25 - 8$$

$$= 4.$$

Since w.k.T. $n(M \cup B) = n(M) + n(B) - n(M \cap B)$

$$\Rightarrow 25 = 12 + n(B) - 4$$

$$\Rightarrow 25 = 8 + n(B)$$

$$\Rightarrow n(B) = 17$$

Now, No. of students taken Biology but not Mathematics

$$n(\bar{M} \cap B) = n(B) - n(M \cap B)$$

$$= 17 - 4$$

$$= 13$$

$\therefore$ 4 students have taken both Mathematics & Biology.

13 students have taken Biology but not Mathematics.

③ (a) If $G = (V, E)$ be a directed graph with 'e' edges then $\sum_{v \in V} deg^+_G(v) = \sum_{v \in V} deg^-_G(v) = e$

Sol

Let G be a directed graph and it has 'e' edges. Since each edge incident on two vertices and each edge is a directed edge, it contributes '1' to the indegree of its terminal vertex and '1' to the outdegree of its initial vertex.

While counting the degree sum of the vertices of the graph, each edge is counted twice and each edge contributes '1' to the indegree sum and '1' to the outdegree sum of the vertices.

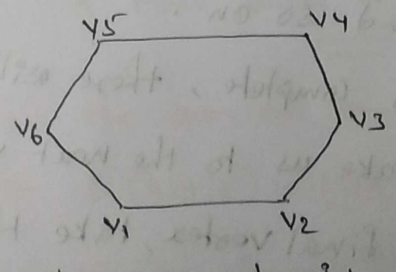
$\therefore$ The sum of indegrees of the vertices of the graph $G = \sum_{v \in V} deg^-_G(v) = e$

and

The sum of outdegrees of the vertices of the graph $G = \sum_{v \in V} deg^+_G(v) = e$

(b) Show that C_6 is a bipartite graph.

Sol



The above graph is a cycle on 6 vertices i.e. C_6.

We can partition its vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

into two subsets $V_1 = \{v_1, v_3, v_5\}$ and

$V_2 = \{v_2, v_4, v_6\}$

Observe here, each and every edge of C_6 connects a vertex in V_1 and a vertex in V_2. There are no edges in between the vertices of V_1 as well as in V_2.

$\therefore$ C_6 is a bi-partite graph.

(3) (c) Show that the complete graph K_n has a Hamiltonian cycle.

Sol Complete graph on n vertices K_n is a graph that has n vertices and each vertex is connected to every other vertex by an edge.

A hamiltonion cycle is a cycle which contains all the vertices in the given graph.

For K_n, $n \geq 3$, there are n choices for where to begin the cycle, then $(n-1)$ choices for which vertex to visit next, then $(n-2)$ choices for which vertex to visit, and so on.

Since the graph is complete, there will always be an edge that will take us to the next vertex in the list. After the final vertex, take the edge that connects back to the starting vertex.

Hence, we ~~can~~ find a hamiltonion cycle.

③ (d) Prove that a tree with 'n' vertices has n-1 edges.

Sol: We will prove this theorem using Mathematical Induction

Basis step:

Let $n=1$

Then a tree with $n=1$ vertex has no edges

∴ It is true for $n=1$.

Induction Hypothesis:

Assume that every tree with K vertices has K-1 edges, where K is a positive integer.

Induction Step:

Suppose T is a tree with K+1 vertices

Let v is a leaf of T and w be the parent of v.

Now, Remove vertex v and the edge connecting w and v from T. Now, the resultant tree T' is a tree with K vertices.

By the induction hypothesis T' has K-1 edges.

It follows that T has K edges, since it has one more than +1, the edge connecting v and w.

∴ A tree with 'n' vertics has n-1 edges.

④ (a) Find the sum of products expansion for the function $F(x,y,z) = (x+y)\bar{z}$

Sol: $F(x,y,z) = (x+y)\bar{z}$

$= x\bar{z} + y\bar{z}$ [∵ Distributive Law

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(5) (1) Define Hamilton circuit, Hamilton graph give examples

$= x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z}$ [∵ Identity Laws]

$= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z}$ [∵ Distributive Laws]

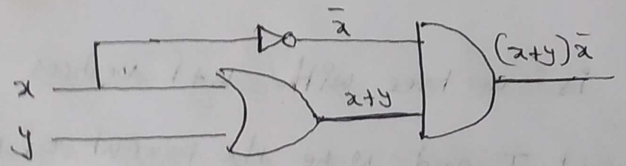
$= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$ [∵ Idempotent Laws]

(4) (b) Construct circuits that produce the following outputs.

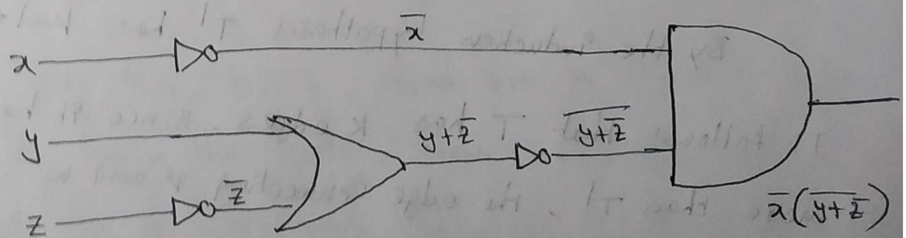
(i) $(x+y)\bar{x}$ (ii) $\bar{x}(\overline{y+\bar{z}})$

Sol

(i) $(x+y)\bar{x}$



(ii) $\bar{x}(\overline{y+\bar{z}})$



4 (c) Show that distributive law $x(y+z) = xy + xz$ is valid.

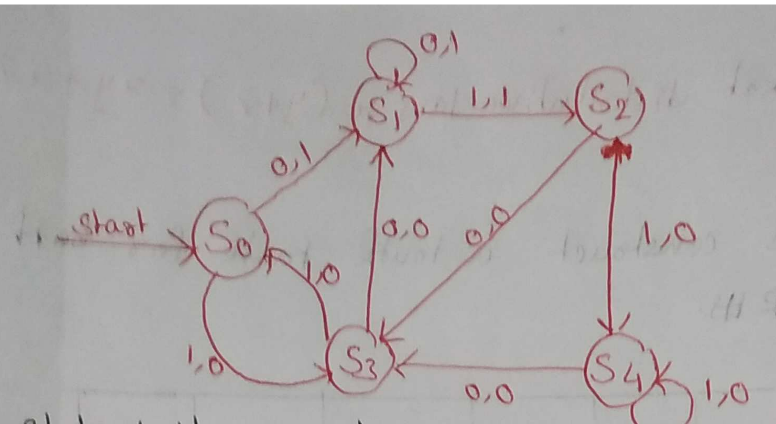
Sol
Let us construct a truth table for both LHS & RHS.

x	y	z	$(y+z)$	$x(y+z)$	xy	xz	$(xy+xz)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Since both $x(y+z)$ and $(xy+xz)$ columns have identical truth values for all combinations of truth values of x, y and z

$x(y+z) = xy + xz$ is valid.

4 (d) Construct the state table for the finite state machine with the state diagram shown in the following figure.



State table for the finite state machine is

State	State Transition function f		output function g	
	Input=0	Input=1	Input=0	Input=1
S_0	S_1	S_3	1	0
S_1	S_1	S_2	1	1
S_2	S_3	S_4	0	0
S_3	S_1	S_0	0	0
S_4	S_3	S_4	0	0

Section – B

5 (a) construct the truth table for $P \wedge (\sim q \vee q)$

Sol

p	q	$\neg q$	$q \vee q$	$P \wedge (\neg q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

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5 (b) write the following in symbolic form.

Every person is precious

Sol

Let our univers of discourse is, the people in the world.

Let $P(x)$ be a predicate "x is precious"

$\therefore$ we can write the given statement in the following symbolic form

$$\forall x\, P(x)$$

5 (c) compute $\frac{20!}{18!}$

Sol

$$\frac{20!}{18!} = \frac{20 \times 19 \times \cancel{18!}}{\cancel{18!}}$$

$$= 20 \times 19$$

$$= \underline{380}$$

5 (d) Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Sol

$$A \cup (B \cap C) = \{x \mid x \in A \cup (B \cap C)\}$$

$$= \{x \mid x \in A \vee x \in B \cap C\}$$

$$= \{x \mid x \in A \vee (x \in B \wedge x \in C)\}$$

$$= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\}$$

$= \{x / x \in (A \cup B) \wedge x \in (A \cup C)\}$

$= \{x / x \in (A \cup B) \cap (A \cup C)\}$

$= (A \cup B) \cap (A \cup C)$

(5) (e) State and prove Handshaking Theorem.

Sol Statement:

Let $G(V, E)$ be an undirected graph with 'e' number of edges then $\sum_{v \in V} deg_G(v) = 2e$

i.e. The degree sum of the vertices of the given undirected graph is equal to twice the number of edges.

Proof: This is true because while counting the degree sum of the vertices, each edge is exactly counted twice. These fore the degree sum of vertices is equal to $2 \times$ (the no. of edges).

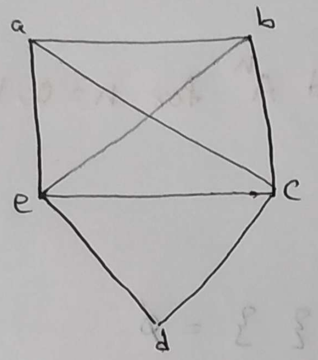
(5) (f) Define Hamilton circuit, Hamilton graph give examples to each.

Sol

Hamilton Path: — A simple path that covers all the vertices in the given graph is called Hamilton path.

Hamilton Circuit: — A simple circuit that covers all the vertices in the given graph exactly once, except starting vertex is called Hamilton circuit.

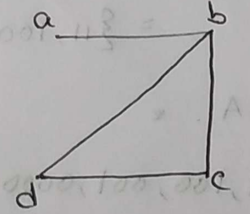
Ex: 1



In the above graph there exist a Hamilton circuit

a — b — c — d — e — a

Ex: 2



In the above graph no hamilton circuit, but a Hamilton path a — b — c — d

A graph is a Hamilton graph if there exist a Hamilton circuit in the graph. Hence the graph in Ex-1 is a Hamilton graph.

(5) (g) Find the duals of $x(y+0)$ and $\bar{x}\cdot 1 + (\bar{y}+z)$

Sol The dual of a Boolean expression is the expression that is obtained by interchanging '+' and '.', and interchangin 0's and 1's.

Now the dual of $x(y+0) = x + y\cdot 1$

The dual of $\bar{x}\cdot 1 + (\overline{y+z}) = (\bar{x}+0)\cdot(\overline{y\cdot z})$

(5) (f) Let $A = \{1, 00\}$, find A^n for $n = 0, 1, 2$ and 3

Sol

Since $A = \{1, 00\}$

If $n = 0$ then $A^0 = \{\ \} = \phi$

$n = 1$ then $A^1 = A = \{1, 00\}$

$n = 2$ then $A^2 = A\cdot A = \{1,00\}\cdot\{1,00\}$

$= \{11, 100, 001, 0000\}$

$n = 3$ then $A^3 = A^2\cdot A$

$= \{11, 100, 001, 0000\}\cdot\{1, 00\}$

$= \{111, 1100, 1001, 10000, 0011, 00100, 00001, 000000\}$