(f) Here are pictures of some easy coverings.

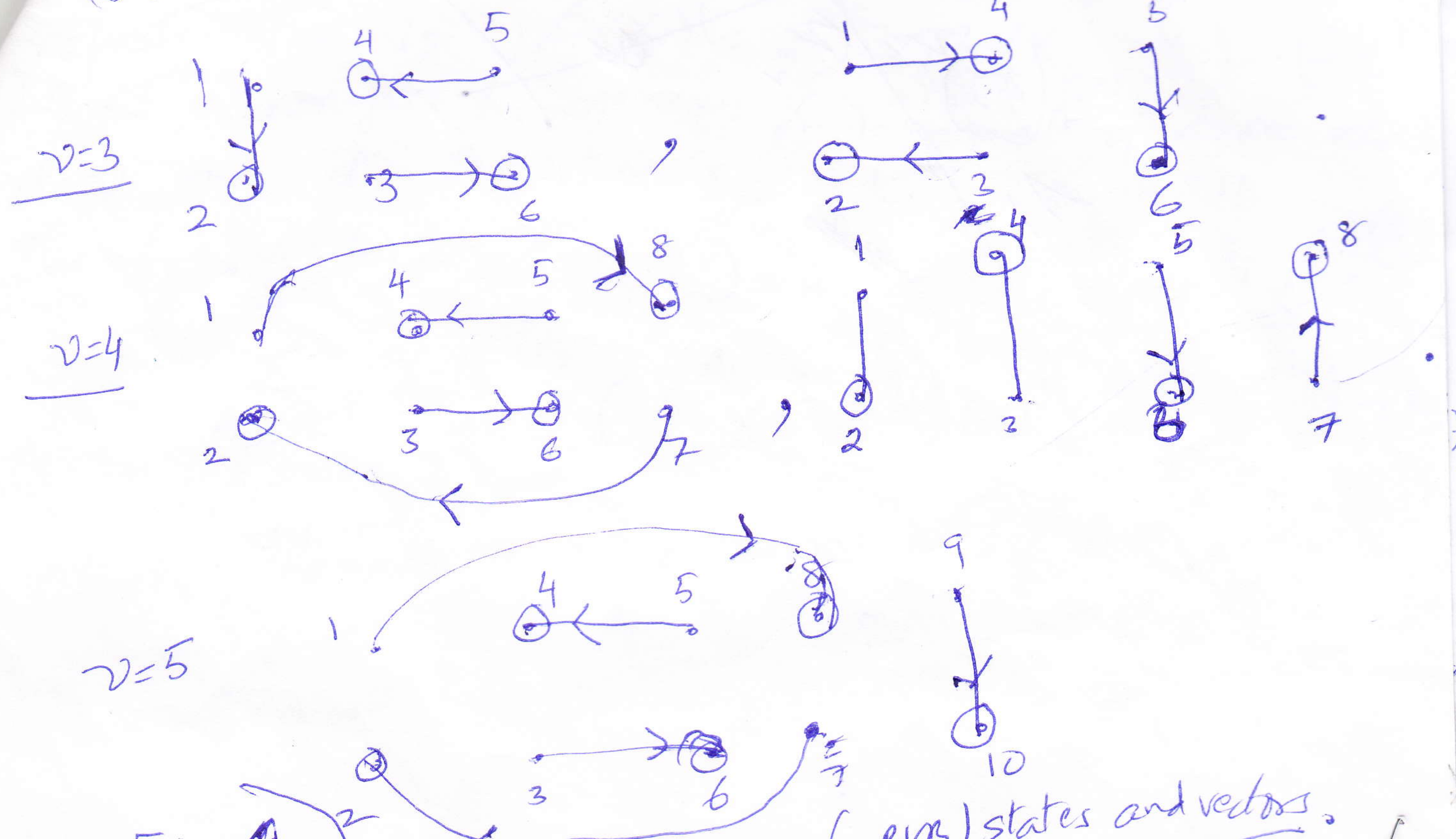

Figure 3

(3.1.2) Resonance valence bond (RVB) states and vectors.

For $a \in A$, $b \in B$, we write $H_{a,b}$ for $\mathcal{H}_a \otimes \mathcal{H}_b$. We follow the polynomial representation as set out in §2 above.

(a) A dimer or a singlet at a, b is the unit vector

$$[a,b] = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_a \otimes |\downarrow\rangle_b - |\downarrow\rangle_a \otimes |\uparrow\rangle_b\right) = \frac{1}{\sqrt{2}}\left(|0\rangle_a \otimes |1\rangle_b - |1\rangle_a \otimes |0\rangle_b\right) \text{ in } \mathcal{H}_{a,b}.$$

Its polynomial representation is given by

$$F_{a,b}(X_a, X_b) = \frac{1}{\sqrt{2}}(X_b - X_a). \qquad (3.3)$$

(b) Let C be a covering of Γ_ν i.e. a bijective map ψ of A to B. We let $|C\rangle = |\psi\rangle$ be the vector in $\mathcal{H}$ identified with $\bigotimes_{a \in A} \mathcal{H}_{a,\psi(a)}$ given by $\bigotimes_{a \in A} [a, \psi(a)]$, i.e.,

$$2^{-\nu/2} \bigotimes_{a \in A} \left(|\uparrow\rangle_a \otimes |\downarrow\rangle_{\psi(a)} - |\downarrow\rangle_a \otimes |\uparrow\rangle_{\psi(a)}\right)$$

$$= 2^{-\nu/2} \bigotimes_{a \in A} \left(|0\rangle_a \otimes |1\rangle_{\psi(a)} - |1\rangle_a \otimes |0\rangle_{\psi(a)}\right).$$

ψ

It is a unit vector and its polynomial representation is $F_{\Psi}(\underline{X}) = 2^{-\nu/2} \prod_{a \in A} (X_{\psi(a)} - X_a)$ --(3.4)

It is a non-zero homogeneous polynomial of degree ~~~~ $d = \nu$ and its support is $S_{F_\psi} = \Gamma_n$, i.e. $m = n$.

Indeed, $\cancel{F_{\Psi}(\underline{X}) = 2^{-\nu}}$ $F_\psi(\underline{X}) = 2^{-\nu/2} \sum_{A_1 \subset A} (-1)^{\# A_1} \underline{X}^{A_1} \underline{X}^{B \setminus \psi(A_1)}$

$$= 2^{-\nu/2} \sum_{\substack{A_1 \subset A \\ \# A_1 \le \nu/2, \\ 1 \in A_1 \text{ if } \# A_1 = \nu/2}} (-1)^{\# A_1} \left(\underline{X}^{A_1} \underline{X}^{B \setminus \psi(A_1)} + (-1)^{\nu} \underline{X}^{A \setminus A_1} \underline{X}^{\psi(A_1)} \right) \text{ --(3.5)}$$

Clearly, the number t of terms in F_ψ is 2^{ν} and $\underline{X}^K$ is not a factor of $F_\psi(\underline{X})$ for any $\phi \ne K \subsetneq \Gamma_n$.

(c) Let $\nu \geq 2$ and Ψ a set of coverings of Γ_n with $2 \le \# \Psi$. Then we write $|\Psi\rangle$ for the RVB vector $\frac{1}{\# \Psi} \sum_{\psi \in \Psi} |\psi\rangle$ in $\mathcal{H}$. We note that $\| \Psi \| \le 1$ and we will come ~~back~~ to its normalization later in (3.1.4).

(d) ~~Its~~ The polynomial representation of F_Ψ is given by

$$F_{\Psi}(\underline{X}) = \frac{1}{\# \Psi} \sum_{\psi \in \Psi} F_\psi(\underline{X}) = \frac{1}{\# \Psi} \sum_{\substack{A_1 \subset A, \\ B_1 \subset B, \# B_1 = \# A_1}} (-1)^{\# A_1} \# \{\psi \in \Psi : \psi(A_1) = B_1\} \underline{X}^{A_1} \underline{X}^{B \setminus B_1} \text{ --(3.6)}$$

Then $|\Psi\rangle \neq 0$ simply because of $2^{-\nu/2} (\underline{X}^B + (-1)^{\nu} \underline{X}^A)$ is a part of $F_\Psi(\underline{X})$.

Indeed, this also gives that $S_{F_\Psi} = \Gamma_n$, i.e., $m = n$ and $\underline{X}^K$ is not a factor of $F_\Psi(\underline{X})$ for any $\phi \ne K \subsetneq \Gamma_n$.

(e) Let $\nu \geq 2$, Ψ_{NN} = the set of nearest neighbour (NN) coverings of Γ_n. ~~Then $|\Psi_{NN}\rangle$~~ and Ψ_{PNN} = the set of periodic nearest neighbour ~~(NN)~~ (PNN) coverings of Γ_n. Then $|\Psi_{NN}\rangle$ and $|\Psi_{PNN}\rangle$ will be called the NN-RVB and the PNN-RVB vectors (in the context of n-qubit system).

We record some immediate applications of Theorem 2.1, labelled as Theorem 3.1 and Example 3.1.

See ... 2.2 above in §2

Theorem 3.1. We consider the vectors given in (3.1.2) above in their respective spaces.

(i) A dimer $[a,b]$, $|\psi\rangle, |\Psi\rangle$ are all entangled.

(ii) Let $\nu \geq 2$ and (E,E') a bipartite cut of Γ_n.

(a) $|\psi\rangle$ is ~~a~~ a product vector in the cut (E,E') if and only if $\psi(E\cap A) = E\cap B$ if and only if
$$\psi(E'\cap A) = E'\cap B) \quad \text{--- (3.7)}$$
In this case we have $\# E$ and $\# E'$ are both even.

(b) If $|\Psi\rangle$ is a product vector in the cut (E,E') then $\psi(E\cap A) = E\cap B$ (equivalently, $\psi(E'\cap A) = E'\cap B$) for each ψ in Ψ --- (3.8).
In this case, $\# E$ and $\# E'$ are both even.

(iii) For $\nu = 2$, $|\Psi\rangle$ is genuinely entangled.

(iv) For Ψ NN or PNN, Ψ is genuinely entangled.

(v) (a) The converse of ii(b) holds for $\nu = 3$ but not in general.

(b) The converse of (ii)(b) holds for all $\nu \geq 3$ if we consider special cuts with $\min\{\# E, \# E'\} = 2$.

Proof. (i) It follows from Theorem 2.1(i)(a) because $d = \nu < 2\nu = n$ for the corresponding polynomials.

(ii) (a) The second equivalence is trivial. For the first one, the 'if' part is immediate. For the 'only if' part, let $p(\underline{X}_E), q(\underline{X}_{E'})$ be polynomials that satisfy $F(\underline{X}) = p(\underline{X}_E)q(\underline{X}_{E'})$. Then by Theorem 2.1 (iv) (b), $p(\underline{X}_E)$ and $q(\underline{X}_{E'})$ are both homogeneous, say, of degree ν_1 and ν_2 respectively with $\nu_1 \geq 1$, $\nu_2 \geq 1$, $\nu_1 + \nu_2 = \nu$, $S_p = E$ and $S_q = E'$. Now by (3.1.2)(b), $\underline{X}^A$ and $\underline{X}^B$ both occur in $F_\psi(\underline{X})$. The one and only one way it can happen is that $\underline{X}^{E\cap A}$, $\underline{X}^{E\cap B}$ occur in $p(\underline{X}_E)$, and, $\underline{X}^{E'\cap A}$ and $\underline{X}^{E'\cap B}$ occur in $q(\underline{X}_{E'})$. This gives that $\# E\cap A = \# E\cap B = \nu_1$ and $\# E'\cap A = \# E'\cap B = \nu_2$. Now $\underline{X}^{E\cap A}\underline{X}^{E'\cap B}$ occurs in (c). But this can happen for $F_\psi(\underline{X})$ if and only if $B \setminus \psi(E \cap A) = E' \cap B$, i.e., $\psi(E \cap A) = E \cap B$. As a consequence $\psi(E' \cap A) = E' \cap B$ as well. This confirms the assertion already made about the relationships amongs cardinalities of the sets, ν_1, ν_2. Clearly $\# E = \#(E \cap A) + \#(E \cap B) = 2\nu_1$.

(b) As noted in (3.1.2)(c), $\underline{X}^A$ and $\underline{X}^B$ both occur in $F_\Psi(\underline{X})$. So arguments in the proof of part (ii)(a) above by replacing F_ψ by F_Ψ ~~give~~ with the addition of "for some $\psi \in \Psi$ in the sentence beginning with 'But'. Now consider any $\varphi \in \Psi$. Then $\underline{X}^{E \cap A} \underline{X}^{B \setminus \varphi(E \cap A)}$ occurs in $F_\Psi(\underline{X})$ and, therefore, in $p(\underline{X}_E)\, q(\underline{X}_{E'})$. The only way this can happen is that $B \setminus \varphi(E \cap A) = E' \cap B$, which gives, in turn, $\varphi(E \cap A) = E \cap B$. This completes the proof of the first assertion. The second statement follows immediately.

(iii) For $\nu = 2$, the conditions in (ii)(b) is not satisfied for any $\phi \neq E \subset \Gamma_4$ as is clear from 3.1.1(e) above.

(iii) It follows from (ii)(b) and (c).

(iv) In view of (iii), it is enough to consider the case $\nu \geq 3$. Let, if possible, the condition in (ii)(b) be satisfied for some E with $\phi \neq E \subsetneq \Gamma_n$ and $\Psi = \Psi_{NN}$.

Consider any $2s-1 \in E \cap A$ with $1 \leq s \leq \nu$. If $s < \nu$, then there exist $\psi_1, \psi_2 \in \Psi$ with $\psi_1(2s-1) = 2(s+1)$ and $\psi_2(2s+1) = 2(s+1)$. So $2(s+1) \in E \cap B$, which in turn, gives that $2s+1 \in E \cap A$. On the other hand, if $s > 1$, then there exist $\psi_3, \psi_4 \in \Psi$ satisfying $\psi_3(2s-1) = 2(s-1)$ and $\psi_3(2s-3) = 2(s-1)$. So $2(s-1) \in E \cap B$ which leads to $2s-3 \in E \cap A$. Repeating the arguments iteratively we arrive at the conclusion that $\{2t-1 : 1 \leq t \leq \nu\} \subset E \cap A$. But that is not so. Hence Ψ_{NN} is not a product vector in any bipartite cut and is, therefore genuinely entangled.

Now we come to Ψ_{PNN}. Because $\Psi_{PNN} \supset \Psi_{NN}$, it cannot satisfy (3.6) in (ii)(b) for any E with $\phi \neq E \subsetneq \Gamma_n$. Hence Ψ_{PNN} is genuinely entangled.

(v)(a) For any bipartite cut (E, E') of Γ_6 with $\# E$ and $\# E'$ both even, $\min\{\# E, \# E'\} = 2$. Suppose $\# E' = 2$. Then for any $\psi \in \Psi$, $\psi(E' \cap A) = E' \cap B$, say, $\psi(a) = b$ where $E' \cap A = \{a\}$, $E' \cap B = \{b\}$. So $X_b - X_a$ is a factor of $F(\underline{X})$. Similarly for the case of $\# E = 2$. So $|\Psi\rangle$ is a product vector in the cut (E, E') by Theorem 2.1 (ii)(a).

~~(b)~~ We ~~recall~~ shall give an examples in Example 3.1 below to show that the converse of (ii)(b) in Theorem here does not hold in general.

(b) We can formulate the proof on the lines of that of (a) above. $\square$

<u>Definition 3.1</u> (i) We call Ψ <u>decomposable via E</u> if it satisfies the condition (3.6) in Theorem 3.1(ii)(b) above, i.e., $\psi(E \cap A) = E \cap B$ for $\psi \in \Psi$.
(ii) Ψ will be called <u>decomposable</u> if it is decomposable via E for some bipartite cut (E, E').

<u>Example 3.1</u>. We consider (3.1.1) (f) above and use Theorem 3.1 above freely.

(i) Let $\nu=3$ and Ψ consist of the two coverings given in ~~the picture~~ figure 2. Then Ψ is not decomposable. So $|\Psi\rangle$ is genuinely entangled.

(ii) Let $\nu=4$ and Ψ consist of the two coverings in ~~the picture~~ Figure 3. Then Ψ is decomposable with $E=\{3,4,5,6\}$ serving the desired purpose. However, $|\Psi\rangle$ is not a product vector in the bipartite cut (E,E') as can be seen by computing F_Ψ. In fact,

$$F_\Psi(\underline{X}) = (X_2X_8+X_1X_7-X_1X_2-X_7X_8)(X_4X_6+X_3X_5-X_3X_4-X_5X_6)+(X_2X_8+X_1X_7-X_1X_8-X_2X_7)(X_4X_6+X_3X_5-X_3X_6-X_4X_5).$$

This can not be factored as $p(\underline{X}_E)q(\underline{X}_{E'})$ for any polynomials p and q. Moreover, apart from E there is no other subset G with $\phi\neq G\subsetneq\Gamma_8$ that satisfies the condition (3.6) of decomposability. This shows that the converse of Theorem 3.1(ii)(b) does not hold in general.

(3.1.3). Concepts and properties of Ψ related to decomposability.

Let $\nu\geq 3$ and $\#\Psi\geq 2$.

(a) Let $\mathcal{A}_1=\{A_1\subset A : \#\{\psi(A_1):\psi\in\Psi\}=1\}$. For $A_1\in\mathcal{A}_1$, we write B_1 for $\psi(A_1)$ for any (and thus for all) $\psi\in\Psi$. $\mathcal{A}_1$ is an algebra of subsets of A. Then $\phi, A\in\mathcal{A}_1$ and $\mathcal{A}_1$ is ~~also~~ an ... As a consequence, $\mathcal{B}_1=\{B_1=\psi(A_1):A_1\in\mathcal{A}_1\}$ is also an algebra of subsets of B. In particular, $\#\mathcal{A}_1\geq 4$ ~~if~~ and only if $\#\mathcal{A}_1\geq 3$ if and only if $\#\mathcal{B}_1\geq 3$. In this case, let $\mathcal{A}_1^{min}$ be the subset of $\mathcal{A}_1$ consisting of its minimal subsets. $\#\mathcal{B}_1\geq 4$ if and only if ...

(b) Now suppose $\#\mathcal{A}_1\geq 3$ and consider any $A_1\in\mathcal{A}_1$ with $\phi\neq A_1\neq A$. We set $E=A_1\cup B_1$. Then $\phi\neq E\neq\Gamma_n$. Also $\psi(E\cap A)=E\cap B$ for $\psi\in\Psi$. In other words, Ψ is decomposable via E. On the other hand, if Ψ is decomposable via E then $\phi\neq E\cap A\neq A$ and $E\cap A\in\mathcal{A}_1$. This, in turn, gives that $\#\mathcal{A}_1\geq 3$.

c) Because $\#\Psi \geq 2$, there exist $\psi_1, \psi_2 \in \Psi$, $a_1 \in A$, $b_1 \neq b_2$ in B with $\psi_1(a_1) = b_1$, $\psi_2(a_1) = b_2$. So $\{a_1\} \notin \mathcal{Q}_1$. Hence $\mathcal{Q}_1 \neq \mathcal{P}_A$.

Can deduce from/or $\mathcal{Q}_1$ is an algebra of subsets of A too!

(d) Consider a_1 as in (c) above and $a_2 \neq a_1$ in A. If $\{a_1, a_2\} \in \mathcal{Q}_1$ then $\psi_1\{a_1, a_2\} = \{b_1, \psi_1(a_2)\}$ has to be $\psi_2\{a_1, a_2\} = \{b_2, \psi_2(a_2)\}$ so that $\psi_1(a_2) = b_2$ and $\psi_2(a_2) = b_1$ and, thus, $\{a_2\} \notin \mathcal{Q}_1$. On the other hand, if $\{a_2\} \in \mathcal{Q}_1$, then $\psi_1\{a_1, a_2\} = \{b_1, \psi_1(a_2)\} = \{b_1, \psi_2(a_2)\} \neq \{b_2, \psi_2(a_2)\} = \psi_2\{a_1, a_2\}$ and, thus $\{a_1, a_2\} \notin \mathcal{Q}_1$. Hence, at most one of $\{a_2\}, \{a_1, a_2\}$ is in $\mathcal{Q}_1$.

(e) Let $\mathcal{Q}_2 = \{A_2 \subset A : \#\{\psi(A_2) : \psi \in \Psi\} \geq 2\}$. Then $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \phi$, ~~and~~ $\mathcal{Q}_1 \cup \mathcal{Q}_2 = \mathcal{P}_A$ and for $A_2 \in \mathcal{Q}_2$, $A \setminus A_2 \in \mathcal{Q}_2$. By (c) above $\{a_1\} \in \mathcal{Q}_2$ and therefore, $A \setminus \{a_1\} \in \mathcal{Q}_2$. Now by (d), for $a_2 \in A \setminus \{a_1\}$, at least one of $\{a_2\}$ and $\{a_1, a_2\}$ is in $\mathcal{Q}_2$ and neither of them or their complements in A equal $\{a_1\}$ or $A \setminus \{a_1\}$ because $\nu \geq 3$. Hence $\#\mathcal{Q}_2 \geq 4$.

(f) ϕ and A are in $\mathcal{Q}_1$ and, therefore, not in $\mathcal{Q}_2$. So for $A_2 \in \mathcal{Q}_2$ $\nu - 1 \geq \# A_2 \geq 1$ ~~for $A_2 \in \mathcal{Q}_2$~~ and, therefore, ${}^{\nu}C_{\#A_2} \geq \nu$.

(g)
$$F_{\Psi}(\underline{X}) = 2^{-\nu/2} \sum_{A_1 \in \mathcal{Q}_1} (-1)^{\#A_1} \underline{X}^{A_1} \underline{X}^{B \setminus B_1} +$$
$$2^{-\nu/2} \frac{1}{\#\Psi} \sum_{A_2 \in \mathcal{Q}_2} (-1)^{\#A_2} \underline{X}^{A_2} \left(\sum_{B_2 \subset B} \#\{\psi \in \Psi : \psi(A_2) = B_2\} \underline{X}^{B \setminus B_2} \right) \quad \ldots (3.9)$$

(h) Let $A_2 \in \mathcal{Q}_2$. We write, for $B_2 \subset B$, c_{A_2, B_2} for $\frac{1}{\#\Psi} \#\{\psi \in \Psi : \psi(A_2) = B_2\}$, which is ≥ 0.

We note that $c_{A_2, B_2} \neq 0$ if and only if $B_2 \in \{\psi(A_2) : \psi \in \Psi\}$.

In particular, $c_{A_2, B_2} = 0$ for $\# A_2 \neq \# B_2$. Moreover,

$\sum_{B_2 \subset B} c_{A_2,B_2} = 1$. So $^{\nu}C_{\#A_2} \geq \#\{B_2 \subset B : c_{A_2,B_2} \neq 0\} \geq 2$.

This, in turn, gives that $c_{A_2,B_2} > 0$ for at least two distinct subsets B_2 of B. As a consequence, $c_{A_2,B_2} < 1$ for all $B_2 \subset B$.

(i) Let $F^1_{\Psi}(\underline{X}) = 2^{-\nu/2} \sum_{A_1 \in \mathcal{A}_1} (-1)^{\#A_1} \underline{X}^{A_1} \underline{Y}^{B \setminus B_1}$,

$$F^2_{\Psi}(\underline{X}) = 2^{-\nu/2} \sum_{A_2 \in \mathcal{A}_2} \sum_{\substack{B_2 \subset B \\ \#B_2 = \#A_2}} c_{A_2,B_2} (-1)^{\#A_2} \underline{X}^{A_2} \underline{X}^{B \setminus B_2}. \quad (3.10)$$

So $F_{\Psi}(\underline{X}) = F^1_{\Psi}(\underline{X}) + F^2_{\Psi}(\underline{X})$.

Let $t, t^{(1)}, t^{(2)}$ be the number of terms in $F_{\Psi}(\underline{X})$, $F^{(1)}_{\Psi}(\underline{X})$ and $F^2_{\Psi}(\underline{X})$ respectively. Then $t^{(1)} = \#\mathcal{A}_1$. $\sum_{A_2 \in \mathcal{A}_2} {}^{\nu}C_{\#A_2} \geq t^{(2)} \geq 2\#\mathcal{A}_2$. But by Using (h), we have Hence $2^{\nu} + \sum_{A_2 \in \mathcal{A}_2} \left(^{\nu}C_{\#A_2} - 1\right)$

(e), $\#\mathcal{A}_2 \geq 4$ and, therefore, $t^{(2)} \geq 8$.

$= \#\mathcal{A}_1 + \sum_{A_2 \in \mathcal{A}_2} {}^{\nu}C_{\#A_2} \geq t^{(1)} + t^{(2)} = t \geq \#\mathcal{A}_1 + 2\#\mathcal{A}_2 = \#\mathcal{P}_A + \#\mathcal{A}_2$

$= 2^{\nu} + \#\mathcal{A}_2 \geq 2^{\nu} + 4$ — (3.11)

(j) If $\{a\} \in \mathcal{A}_1$ for some $a \in A$, then by (b) above and Theorem 3.1(v)(b), $|\Psi\rangle$ is a product ~~vector~~ in the bipartite cut $(E, \bar{E})$ with $E = \{a\}$. $\{a, b\}$ ~~with~~ by (a) for some and thus all ?

~~So we may~~

(k) We confine our attention to the case that $\{a\} \notin \mathcal{A}_1$ for any $a \in A_1$. In that case, $\{a\} \in \mathcal{A}_2$ for $a \in A$. So, $\#\mathcal{A}_2 \geq 2\nu$ because $\{A \setminus \{a\}, \{a\} : a \in A\}$ are all different sets ~~simply~~ in view of ~~because~~ $\nu \geq 3$. This refines the second part of (3.11) ~~to~~ ~~2^{ν}~~ viz.,

$$t \geq 2^{\nu} + \#\mathcal{A}_2 \geq 2^{\nu} + 2\nu = 2(2^{\nu-1} + \nu) \quad \text{---} \quad (3.12)$$

We record a few tests of decomposability which are implicit in 3.1.3(a)&(b), (i) and (k) above.

Theorem 3.2. (Tests of decomposability). Let $\nu \geq 3$ and $\# \Psi \geq 2$.

(i) The following are equivalent.

(a) Ψ is decomposable.

(b) $\# \mathcal{Q}_1 \geq 3$

(c) $\# \mathcal{Q}_1 \geq 4$.

(d) The number s of terms in $F_\Psi(\underline{X})$ with $|\text{coefficient}| = 2^{-\nu/2}$ is ≥ 3.

(e) The number s as in (d) above is ≥ 4.

(ii) If $t < 2^\nu + 2\nu$, then Ψ is decomposable via ~~via (E,E') with~~ $E = \{a,b\}$ for some $a \in A$, $b \in B$. Further, in this case $|\Psi\rangle$ is a product vector in the bipartite cut (E,E'). $\square$

(3.1.4). Normalization of $|\Psi\rangle$. Let $\nu \geq 2$ and $\# \Psi \geq 2$.

(a) By 3.1.1(e) for $\nu = 2$, there is only one Ψ. The polynomial representation of $|\Psi\rangle$ is given by

$$F_\Psi(\underline{X}) = \frac{1}{4}\left((X_2 - X_1)(X_4 - X_3) + (X_4 - X_1)(X_2 - X_3)\right)$$

$$= \frac{1}{4}\left(2(X_1X_3 + X_2X_4) - X_1X_4 - X_2X_3 - X_1X_2 - X_3X_4\right)$$

So $\| |\Psi\rangle \| = \frac{1}{4}\sqrt{(4+4+1+1+1+1)} = \frac{\sqrt{3}}{2}$ — (3.13)

Hence the corresponding RVB state is $|\tilde{\Psi}\rangle = \frac{2\sqrt{3}}{3}|\Psi\rangle$ and its polynomial representation is given by

$$\tilde{F}_\Psi(\underline{X}) = \frac{\sqrt{3}}{6}\left(2(X_1X_3 + X_2X_4) - X_1X_2 - X_2X_3 - X_1X_4 - X_3X_4\right) \quad (3.14)$$

) Let $\nu \geq 3$. We use (3.1.3)(i). The terms of the polynomial representation of $F_\Psi(\underline{X})$ coming from non-zero values of c_{A_2,B_2}'s are mutually orthogonal in $L^2(\mathbb{T}^n)$. So

$$\| |\Psi\rangle \| = \| F_\Psi(\underline{X}) \|_2 = 2^{-\nu/2} \sqrt{\left(\# \mathcal{Q}_1 + \sum_{A_2 \in \mathcal{Q}_2} \left(\sum_{B_2 \subset B} c_{A_2,B_2}^2 \right)\right)} \quad \text{--- (3.15)}$$

But for each $A_2 \in \mathcal{Q}_2$, $\#\{B_2 \subset B : c_{A_2,B_2} \neq 0\} \geq 2$ and $\sum_{B_2 \subset B} c_{A_2,B_2} = 1$; by (3.1.3)(h). So $\sum_{B_2 \in B} c_{A_2,B_2}^2 < 1$ for $A_2 \in \mathcal{Q}_2$.

Hence $\| |\Psi\rangle \| < 2^{-\nu/2} (\# \mathcal{Q}_1 + \# \mathcal{Q}_2) = 1$ --- (3.16).

The corresponding RVB state, say $|\tilde{\Psi}\rangle = \frac{1}{\| |\Psi\rangle \|} |\Psi\rangle$ and its polynomial representation is ~~$\frac{1}{\|\Psi\|} F_\Psi(\underline{X})$~~:

$$\tilde{F}_\Psi(\underline{X}) = \frac{1}{\| |\Psi\rangle \|} F_\Psi(\underline{X}) \quad \text{--- (3.17)}$$

(3.1.5) Motivation for factorability of sets of coverings.

Theorem 3.1 and (3.1.3) permit us to confine our attention to $\nu \geq 3$ and decomposable coverings Ψ with $\#\Psi \geq 2$ and $\{a\} \notin \mathcal{Q}_1$ for any $a \in A$ on the one hand and motivate the new concept of factorability of Ψ on the other hand.

(A) Suppose Ψ is decomposable via E. Let $\nu_1 = \# E \cap A$, $\nu_2 = \# E' \cap A$.

~~Let $\Psi_E = \{\psi/E : \psi \in \Psi\} = \{\psi_j^E : 1 \leq j \leq j'\}$, say~~, Labelling

~~$\Psi_{E'} = \{\psi/E' : \psi \in \Psi\} = \{\psi_k^{E'} : 1 \leq k \leq k'\}$, say~~, with a