# What is finiteness?

(Abhishek Banerjee)

(Indian Institute of Science)

# Infinities

## You can get as much of them as you want!

## Literally!

- From an infinite set, you can get part which is equal to whole.
- $\mathbb{N}$ = collection of natural numbers

$$\{1, 2, 3, ...\}$$

- Consider : $1 \mapsto 2,\ 2 \mapsto 4,\ 3 \mapsto 6,\ ...$
- So:

$$\{1, 2, 3, ...\} = \{2, 4, 6, ...\}$$

## “Bigger” infinities?

• $\mathbb{R}$ = collection of all real numbers.

• $\mathbb{N} < \mathbb{R}$

• You cannot set up a correspondence between all naturals and all reals.

• Fascinating Cantor’s diagonal argument ( Please please look it up!)

## Do finites get enough respect?

• Finiteness is easy, no?

• Just count whether 1, 2, 3,...

• But then we miss out on the true richness of the concept of finitness.

• There's more finiteness around. In fact, finiteness is what helps us really understand things.

• Take two sets with 3 elements each and a function $f$ between them:

$$\{a, b, c\} \quad \xrightarrow{f} \quad \{x, y, z\}$$

• Suppose $f$ is "one-one", i.e., no two elements of $\{a, b, c\}$ go to the same place.

• Notice that this forces $f$ to be "onto", i.e, the images of the elements $\{a, b, c\}$ cover all elements in $\{x, y, z\}$.

• Obviously, this happened because both sets are finite and of the same size.

• Now we will see that $\mathbb{R}$, the collection of real numbers, is also "finite".

• For this we will not take ALL functions, just linear functions ... functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which do this:

$$f(x+y) = f(x) + f(y) \qquad f(rx) = rf(x)$$

• So that's really easy because it means every

$$f(x) = xf(1)$$

• Once we choose the image of 1, the whole function $f$ is fixed.

• To make $f$ one-one, we must choose $f(1) \neq 0$.

- Say $f(1) = 5$. Then, $f$ is simply multiplication by 5.
- Magically : making $f$ one-one **forced** $f$ to become onto:

$$\frac{x}{5} \mapsto x$$

- $\mathbb{R}$ is finite!

- Take the 2-dimension plane

$$\mathbb{R}^2 = \{(x, y) \mid x \text{ and } y \text{ are real}\}$$

- Elements are pairs $\bar{x} = (x, y)$
- Consider linear functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y}) \qquad f(r \cdot \bar{x}) = r f(\bar{x})$$

- Wonderful exercise : If $f$ is one-one, it must be onto!
- Also wonderful exercise If $f$ is onto, it must be one-one!

- Where is this coming from?
- The fact that every pair $(x, y)$ can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Taking linear functions forces $\mathbb{R}^2$ to behave exactly like a set with two elements.
- We need a richer notion of "finiteness"
- We need **Category Theory**

## ... sa structure la plus evidente ...

- When working with sets, take functions between them.
- When working with linear spaces, say $\mathbb{R}^n$, take linear functions between them.
- When working with rings, take ring maps.
- When working with topological spaces, take continuous maps.

- Find out what finiteness really looks like :
- Take a finite set $S$
- Take a "chain" of sets:

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

- Call the union of this chain $T$.
- Suppose there is a map $S \longrightarrow T$.
- This map **must** pass through some link in the chain.
- Why? Take one element in $S$, say $s_1$. It's image must lie in $T$ and therefore in some link ... say $T_5$.

- Take the next element $s_2$ in $S$. Its image lies in some link, say $T_9$ and keep going.
- But we have only finitely many elements.
- So there's a max, say $T_{100}$ which contains images of all elements of $S$.
- This is an **if and only if** condition for finiteness.

- If we replace the function with linear maps, this same notion works with linear spaces.
- Say $\mathbb{R}^2$
- The linear map is determined purely by what it does to $(1, 0)$ and $(0, 1)$.

## Same notion of finitness

- Sets & linear maps : Finite sets.
- Linear spaces & linear maps : Finite dimensional linear spaces.
- Topological spaces & continuous functions : Quasi-compact spaces (*)
- And much more : In every category, we know who are the finite objects.

# Finiteness helps us hold things

- Algebraic Geometry is built out of Quasi-compact pieces.
- Manifolds are built out of locally compact pieces.

## We knew this all along!

• If you want to understand something, measure it!

• If you want to understand something, find its finite pieces!

• Bonus fact : You can measure geometric spaces with linear things like $\mathbb{R}^n$. That's called cohomology!

## Orders of finitness

- $f : X \longrightarrow Y$ onto : if $X$ is finite, so is $Y$ (almost always true)
- $f : X \longrightarrow Y$ one-one : if $Y$ is finite, $X$ isn't always finite.
- Finite things can contain infinite things.
- **Noetherian** = Finite things contain only finite things.
- By the way, what's "onto"?

- $\{$Set of integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ..\}\} \hookrightarrow \{$Set of all fractions $\mathbb{Q}\}$ is onto when viewed as a function of rings!

**?????????????**

- Because take two maps of rings:

$$\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} R$$

- Suppose $f \circ i = g \circ i$. Then, $f = g$.
- Reverse the direction of arrows here and you get the real definition of "one-one".
- So inclusion of integers into fractions is a one-one onto map of rings without actually making the two rings identical.

Like Grothendieck said, we are only looking for

*“sa structure la plus evidente...”*

# The Rising Sea

*L'image qui m'était venue il y a quelques semaines était différente encore ... La mer s'avance insensiblement et sans bruit, rien ne semble se casser rien ne bouge l'eau est si loin on l'entend à peine. . . Pourtant elle finit par entourer la substance rétive, celle-ci peu à peu devient une presqu'île, puis une île, puis un îlot, qui finit par être submergé à son tour, comme s'il s'était finalement dissous dans l'océan...*

— Alexandre Grothendieck (Récoltes et Semailles)

(The image that came to me a few weeks ago was different still ... The sea rises almost imperceptibly, without noise, nothing breaks and nothing moves; the water seems so far away that you can barely hear it ... Yet, it ends up by surrounding the resistant substance, first makes it nearly an island, then an island, then an islet and finally it becomes as if it was always dissolved in the ocean)