Large Scale Geometry of Randomly Growing Interfaces

Riddhipratim Basu

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

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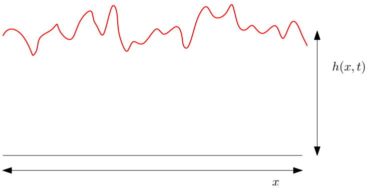
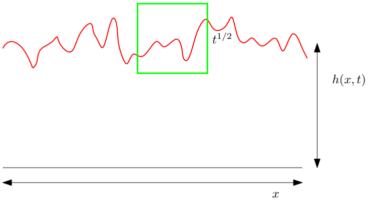
Interface growth in $(1+1)$ dimension

- Flat substrate at time 0.
- For $x \in \mathbb{Z}$, the height of the column at x increases independently at rate 1 by i.i.d. random amounts.
- Height function $h(x,t)$ is the height of the column at x at time t.



The Gaussian universality class

- $h(x,t)$ has diffusive (i.e., of order $\sqrt{t}$) fluctuation.
- By Central Limit Theorem, has Gaussian scaling limit.
- Universal behaviour: does not depend on the increment distribution.



Gaussian universality class does not cover all random growth processes of interest.

A different universal behaviour

Many naturally occurring growth models exhibit the following additional features:

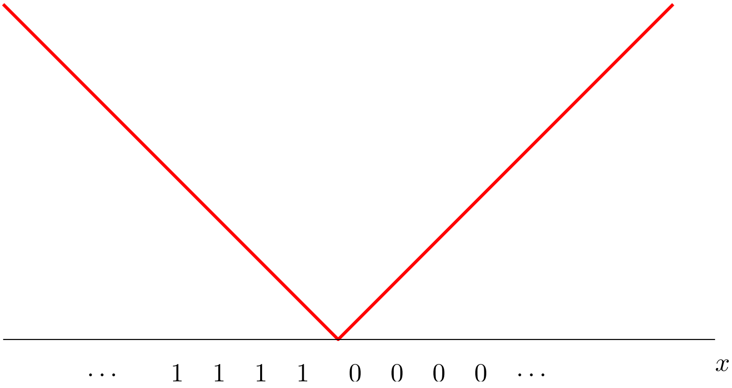
1. Locality of growth (no long range interaction).
2. Independent space-time noise.
3. Lateral Growth with slope dependent speed.
4. Relaxation mechanism (valleys are filled quickly).

These large scale behaviour of these models are different.

- Height fluctuations are sub-diffusive.
- Scaling limit is non-Gaussian.

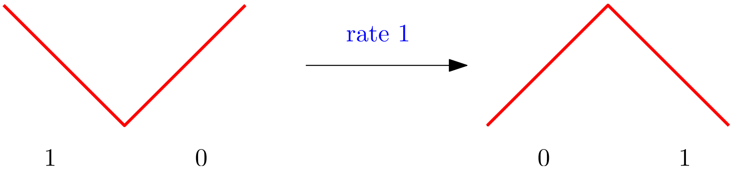
Example: Ballistic deposition model, corner growth model.

Corner Growth Model



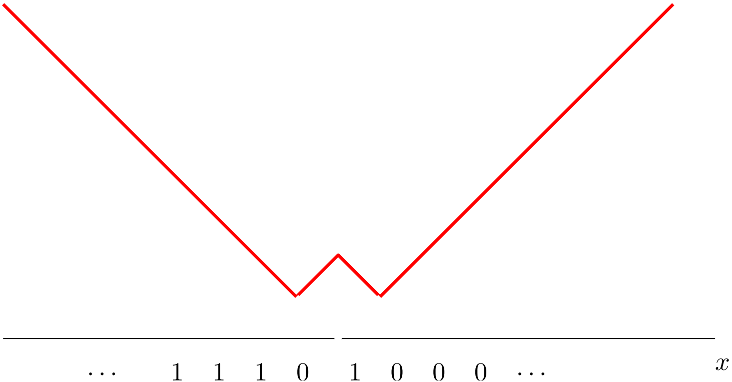
Step initial condition

Corner Growth Model



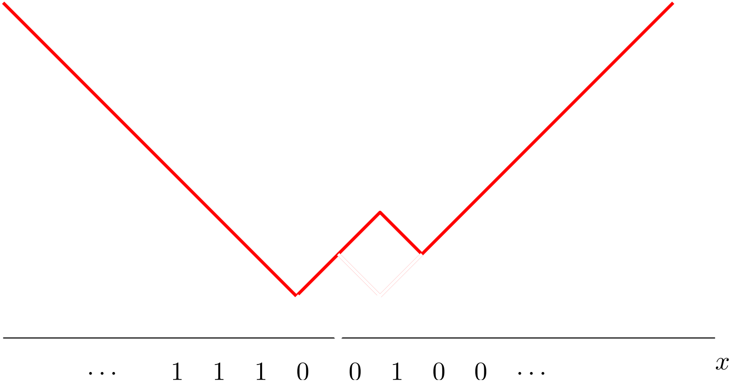
Corners are filled at rate 1

TASEP as a Corner Growth Model



Evolution of Height Function

TASEP as a Corner Growth Model



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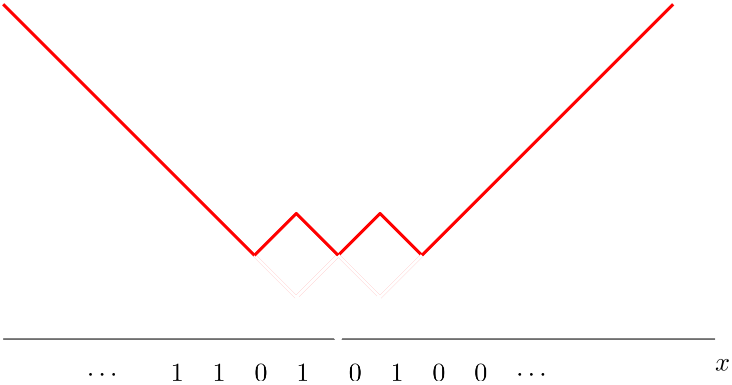

Evolution of Height Function

The KPZ equation and the universality class

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Dynamic Scaling of Growing Interfaces

Mehran Kardar
Physics Department, Harvard University, Cambridge, Massachusetts 02138

Giorgio Parisi
Physics Department, University of Rome, I-00173 Rome, Italy

and

Yi-Cheng Zhang
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 61.15.Jj

$$\frac{\partial}{\partial t}h(x,t) = \nu\frac{\partial^2}{\partial x^2}h(x,t) + \lambda(\frac{\partial}{\partial x}h(x,t))^2 + \xi(x,t).$$

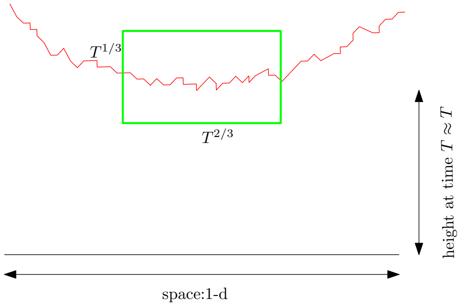
Kardar, Parisi, Zhang (1986)

$\xi :=$ independent space-time white noise.

KPZ universality: Exponents in (1+1) dimension

A non-rigorous renormalization group analysis suggests

- Scaling exponent of $1/3$ for fluctuation.
- Scaling exponent of $2/3$ for correlation length.



KPZ scaling in real world phenomenon

1. Mutant Bacterial Colony Formation. Wakita et al. (1994)
2. Slow combustion of paper. Maunuksela et al. (1997), Myllys et al. (1997)
3. Interface between Dynamic Scattering Modes Takeuchi et al. (2011)
4. Coffee ring effect: Yunker et al. (2013)

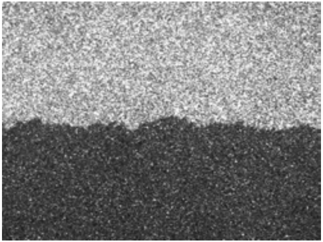


Image: Takeuchi et al.

Image: Yodh Lab press release

Back to the KPZ equation

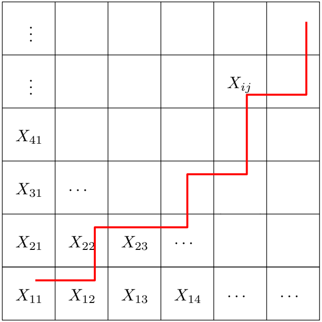
$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + (\frac{\partial}{\partial x}h(x,t))^2 + \xi(x,t).$$

- Ill-posed.
- Non-linear term creates the problem.
- Existence, uniqueness, regularity theory developed in Hairer's fields medal winning works.

Directed Last Passage Percolation (LPP) on $\mathbb{Z}^2$

- Put i.i.d. exponential weights on each vertex of in $\mathbb{Z}^2$.
- π: directed path from $(1,1)$ to (n,n).
- The last passage time from $(1,1)$ to (n,n).

$$T_{n,n} = \max_{\pi} \sum_{i} X_{i,\pi(i)}.$$



$$X_{ij} \sim \text{i.i.d. Exp}(1)$$

$$\lim_{n\to\infty} \frac{T_{nx,ny}}{n} = (\sqrt{x} + \sqrt{y})^2.$$

Rost (1981)

KPZ scaling

One point convergence

$$2^{-4/3}n^{-1/3}\left(T_{n,n}-4n\right)\xrightarrow{d}F_{TW}$$

where F_{TW} is the GUE Tracy-Widom distribution. Johansson (2000)

Process convergence

$$n^{-1/3}\left(T_{n+xn^{2/3},n-xn^{2/3}}-4n\right)\xrightarrow{d}\mathcal{A}(x)-x^2$$

for some stationary process $\mathcal{A}$ (Airy$_2$ process).

Prahofer, Spohn (2002), Borodin, Ferrari (2008)

Exactly solvable models

- The same scaling and limit is believed to hold under very mild assumptions on the distribution of the vertex weights.
- Rigorously known only for Geometric, Exponential and Bernoulli weights and also for continuum Poissonian LPP.
- These are so-called *exactly solvable models* for which exact formulae are available via some remarkable bijections.
- Much of KPZ literature is studying these exactly solvable models and taking appropriate limits.

Integrable probability: A KPZ revolution(1999-)

- Determinantal integrable models: Tracy Widom limit for Directed last passage percolation
  - Poissonian last passage percolation. Baik, Deift, Johansson (1999)
  - Exponential and Geometric last passage percolation. Johansson (2000)
  - Longest increasing subsequence of a random involution. Baik, Rains (2000)
- Two-point Correlation and Airy process limit
  - Prahofer, Spohn (2002),
  - Ben Arous, Corwin (2010)
- Non-determinantal integrable models:
  - Stochastic Heat Equation Amir, Corwin, Questel (2010)
  - q-TASEP Borodin, Corwin (2011)
  - Log-gamma polymer Corwin, O-Connell, Seppäläinen, Zygouras (2011)
- And many others...

Transversal fluctuations and polymer coalescence

Polymer geometry and the exponent 2/3

- Transversal fluctuations TF_n measure the maximum distance of the polymer between $(0,0)$ and (n,n) from the diagonal.
- $TF_n \approx n^{2/3+o(1)}$. Johansson (2000)

Quantitative estimates: Upper tail

$$\mathbb{P}(TF_n \geq kn^{2/3}) \leq e^{-ck^2}.$$

B., Sidoravicius, Sly (2014)

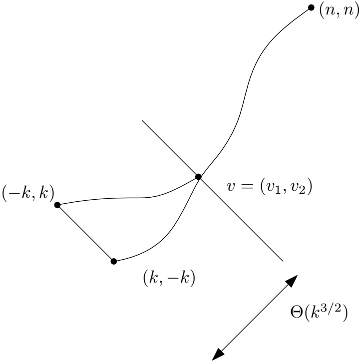
Quantitative estimates: Lower tail

$$\mathbb{P}(TF_n \leq n^{2/3-\varepsilon}) \leq e^{-cn^{\varepsilon/10}}.$$

B., Ganguly, Hammond (2017)

Coalescence of Polymers

- Consider polymers to (n, n) from $(k, -k)$ and $(-k, k)$ where $n \gg k$.
- $v = (v_1, v_2) :=$ the point of coalescence.
- Natural guess is that $|v|$ scales as $k^{3/2}$.



Coalescence of Polymers

Distance to Coalescence: Upper Bounds

Uniformly in all large k

$$\limsup_{n\to\infty} \mathbb{P}(v_1 + v_2 > rk^{3/2}) \leq Cr^{-\alpha}$$

for some $\alpha > 0$. B., Sarkar, Sly (2017)

- There is a corresponding lower bound. Pimentel (2016)

Local Fluctuations time correlations

Brownian fluctuations for point-to-line profile

- It is known that the Airy process looks locally Brownian.

Brownian fluctuation upper bounds

Uniformly in all large n and $|r| \ll n^{2/3}$ with high probability we have

$$T_{n+r,n-r} - T_{n,n} = O(r^{1/2}).$$

B., Ganguly (2018)

- Similar estimates were previously known only for Brownian LPP.

Hammond (2017)

Aging properties of the profile

- Let $r_n(\theta) := \mathrm{Corr}(T_n, T_{n(1+\theta)})$.

Time correlation exponents

Uniformly in all large n

$$1 - r_n(\theta) = \Theta(\theta^{2/3}) \text{ as } \theta \to 0;$$

$$r_n(\theta) = \Theta(\theta^{-1/3}) \text{ as } \theta \to \infty.$$

B., Ganguly (2018)

Applications of the geometric approach

- TASEP with a slow bond. B., Sidoravicius, Sly (2014) B., Sarkar, Sly (2017)
- TASEP on a relaxation time scale. Baik, Liu (2016)
- Wulff shape fluctuation for area-constrained polymers. B., Ganguly, Hammond (2018)
- Transition to shock in TASEP. Najjer (2017)
- Non-existence of bigeodesics B., Hoffman, Sly (2018+)

Summary

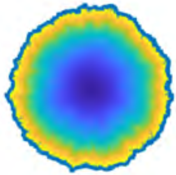
- KPZ universality is an important phenomenon in statistical mechanics that is mathematically challenging to understand.
- One studies the PDE aspects of the KPZ equation as well as the discrete pre-limiting models, these complement each other and often feed into one another.
- The study of discrete models are mostly based on exact algebraic formulae and remarkable bijections of integrable probability.
- Importance of geometric understanding and its usefulness beyond the integrable setting has recently started to be explored.

Major challenges: non-integrable models

- First passage percolation on $\mathbb{Z}^2$.
- Put i.i.d. weights on the edges and let

$$T_{u,v} := \text{weight of the minimum weight path from } u \text{ to } v.$$

- Believed to be in the same universality class but almost no rigorous evidence.



Thank You

Questions?