An invitation to ergodic theory

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- Closely connected to the subject of *Dynamical Systems*.
- Connections to Geometry, Number Theory and many other subjects.

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- $n \log 6 \mod 1 \in [\log 9, \log 10)$.
- $T(x) = x + \log 6 \pmod 1$.

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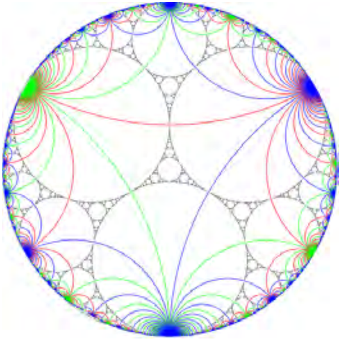
- We get that 9 appears as the first digit in powers of 6 with frequency $\log(10/9) \asymp 4.58\%$.

Geodesics and horocycles



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- Give rise to a rich class of dynamical systems.
- The ergodic theory of these actions
- Has been extensively studied over the last few decades.

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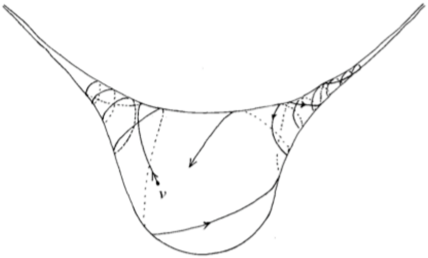
- Connected to KAM theory.

Excursions in hyperbolic 3 manifolds



Bounded geodesics

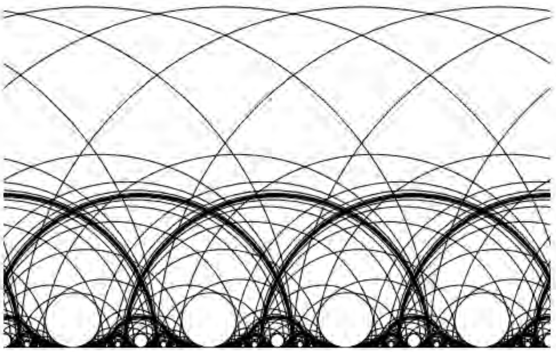
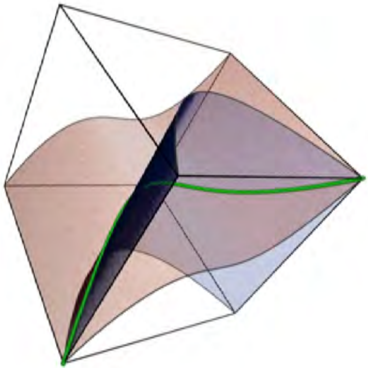


Figure 1. A long, bounded geodesic on $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ defined over $\mathbb{Q}(\sqrt{5})$.

Curves and varieties



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- Especially the work of Harish Chandra.
- Connections with the theory of automorphic forms and Ramanujan's conjectures.