Ax=b: A Familiar Setup, Axioms and An Open Question

Karthik P. N.

Joint work with Prof. Rajesh Sundaresan

An example

An example

- Consider a problem of <u>image reconstruction</u>

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- Image is represented as a function

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- Image is represented as a function
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$$f = \sum_{j=1}^{n} x_j f_j, \quad f_j = \text{ indicator of the } j\text{th pixel}$$

- Measurements

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✓ A 🔍 x b ✓

- Goal: Recover $x = (x_1, \ldots, x_n)^T$ from the measurements

Criterion for / goodness of recovery

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- Typically, one first defines a cost function

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$$L = \{w \in \mathbb{R}^n : Aw = b\}$$

$$x = \arg\min_{w \in L} F(w)$$

Function-minimisation approach: examples

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- Maximum entropy:

$$x = \arg \min_{w \in L \cap \mathbb{R}_+^n} \sum_{i=1}^{n} w_i \log w_i$$

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$$x = \arg \min_{w \in L \cap \mathbb{R}_+^n} \sum_{i=1}^{n} w_i \log \frac{w_i}{x_i^\star} - w_i + x_i^\star$$

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Thus, function-minimisation defines a projection rule

Projection rules: formal definition

- Let

$$\mathscr{L} = \left\{ L = \{w \in \mathbb{R}^n : Aw = b\} : A \text{ is a } k \times n \text{ matrix having rank } k,\ b \in \mathbb{R}^k \right\}$$

where $k = 0,1,2,\ldots,n$

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$$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n$$

$$(L, x^\star) \mapsto \Pi(L \,|\, x^\star) \in L$$

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$$\text{If } x^\star \in L, \text{ then } \Pi(L \,|\, x^\star) = x^\star$$

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$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n$ projection rule

<u>Definition:</u> <u>Π is generated by F</u>, if

$$\Pi(L \,|\, x^\star) = \arg\min_{w \in L} F(w \,|\, x^\star)$$

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WHY LEAST SQUARES AND MAXIMUM ENTROPY? AN AXIOMATIC APPROACH TO INFERENCE FOR LINEAR INVERSE PROBLEMS[1]

BY IMRE CSISZÁR

Mathematical Institute of the Hungarian Academy of Sciences

An attempt is made to determine the logically consistent rules for selecting a vector from any feasible set defined by linear constraints, when either all n-vectors or those with positive components or the probability vectors are permissible. Some basic postulates are satisfied if and only if the selection rule is to minimize a certain function which, if a "prior guess" is available, is a measure of distance from the prior guess. Two further natural postulates restrict the permissible distances to the author's f-divergences and Bregman's divergences, respectively. As corollaries, axiomatic characterizations of the methods of least squares and minimum discrimination information are arrived at. Alternatively, the latter are also characterized by a postulate of composition consistency. As a special case, a derivation of the method of maximum entropy from a small set of natural axioms is obtained.

Axiomatic approach

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Axioms satisfied by a projection rule	Nature of the function generating the projection rule
Regularity + Locality	$F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star)$, f_i continuously differentiable and strictly convex
Regularity+ Locality + Subspace Transitivity	$F(w \mid x^\star) =$ Bregman's divergence
Regularity+ Locality + Subspace Transitivity + Statistical	$F(w \mid x^\star) =$ I-divergence
Regularity + Locality + Subspace Transitivity + Location Invariance + Scale Invariance	$F(w \mid x^\star) =$ Euclidean distance

The axiom of regularity

$$\mathscr{M} = \left\{ \{w \in \mathbb{R}^n : a^T w = b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R} \right\}$$

The axiom of regularity

$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n$ projection rule

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Π satisfies regularity if, for all $x^\star \in \mathbb{R}^n$,

Regularity	(Consistency) $L' \subset L,\ \Pi(L \mid x^\star) \in L' \implies \Pi(L' \mid x^\star) = \Pi(L \mid x^\star)$
	(Distinctness) $L, \tilde{L} \in \mathscr{M},\ L \neq \tilde{L},\ x^\star \notin L \cap \tilde{L} \implies \Pi(L \mid x^\star) \neq \Pi(\tilde{L} \mid x^\star)$
	(Continuity) $\Pi(\,\cdot \mid x^\star)$ restricted to any fixed dimension is continuous

The axiom of consistency

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Additional observations do not give a reason to change the original selection

The axiom of distinctness

$$L, \tilde{L} \in \mathscr{M},\ L \neq \tilde{L},\ x^\star \notin L \cap \tilde{L} \implies \Pi(L \,|\, x^\star) \neq \Pi(\tilde{L} \,|\, x^\star)$$

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$$Y = \left\{ [(A, b)] \in \mathbb{R}^{k \times n} \times \mathbb{R}^k : A \text{ has rank } k \right\}$$

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- Equivalence classes

$$Y = \left\{ [(A, b)] \in \mathbb{R}^{k \times n} \times \mathbb{R}^k : A \text{ has rank } k \right\}$$

- Topology on Y is the quotient topology derived out of the Euclidean topology on X

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collection of all $(n-2)$-dimensional sets

$\mathcal{M}$ = collection of all $(n-1)$-dimensional sets

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$\mathscr{M} =$ collection of all $(n-1)$-dimensional sets

$\Pi(\,\cdot\,|x^\star)$ restricted to any fixed dimension is continuous

Axiom of subspace transitivity

$$L' \subset L \implies \Pi(L' \,|\, x^\star) = \Pi(\Pi(L \,|\, x^\star) \,|\, x^\star)$$

Why are these axioms of interest to us?

Kumar, M. Ashok, and Rajesh Sundaresan. "Minimization Problems Based on Relative $\alpha $-Entropy I: Forward Projection." *IEEE Transactions on Information Theory* 61.9 (2015): 5063-5080.

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Locality	✗

- These projection rules are generated by a divergence known as Sundaresan's divergence (relative α-entropy)

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$$\mathscr{I}_\alpha(w, x) = \frac{\alpha}{1-\alpha} \log\left(\sum_{i=1}^{n} \frac{w_i}{||w||} \left(\frac{x_i}{||x||} \right)^{\alpha-1} \right),$$

$$\text{where } ||x|| = \left(\sum_{i=1}^{n} x_i^\alpha \right)^{1/\alpha}, \quad \alpha > 0,\ \alpha \neq 1$$

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- This is **<u>not a Bregman's divergence</u>**. Hence, the corresponding projection rule is not local

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Remainder of the talk is about regularity, and more….

What we want to know about regularity

- Csiszár's results provide a necessary and sufficient axiomatic characterisation of many projection rules

Regularity + Locality $\Longleftrightarrow$ Projection rule generated by $F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star)$

Regularity + Locality + Subspace Transitivity $\Longleftrightarrow$ Projection rule generated by Bregman's divergence

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Regularity $\Longleftrightarrow$ Projection rule generated by ???

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$$\Pi(\,\cdot\,|\,x^\star)\colon \mathscr{M} \backslash \mathscr{L}^0(x^\star) \to \mathbb{R}^n \backslash \{x^\star\}$$

is a homeomorphism

$\Pi(\,\cdot\,|x^\star)\colon \mathscr{M}\backslash\mathscr{L}^0(x^\star) \to \mathbb{R}^n\backslash\{x^\star\}$ is a homeomorphism

Implications of regularity

This means that for every $w \neq x^\star$,
there exists a unique set L of dimension $(n-1)$
such that $\Pi(L \,|\, x^\star) = w$. Denote this L as $L(w \,|\, x^\star)$

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$$
\begin{aligned}
L(w|x^\star) &= \{ y \in \mathbb{R}^n : a^T y = b \} \\
&= \{ y \in \mathbb{R}^n : a^T y = a^T w \} \\
&= \{ y \in \mathbb{R}^n : \frac{a^T}{\|a\|}(y - w) = 0 \} \\
&= \{ y \in \mathbb{R}^n : \tilde{a}(w|x^\star)^T (y - w) = 0 \}
\end{aligned}
$$

Regularity and vector fields

- Mere regularity helps us extract a **<u>vector field</u>**

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$$\tilde{a}(\,\cdot\,|\,x^\star) : \mathbb{R}^n \to \mathbb{R}^n$$

$$\tilde{a}(\,\cdot\,|\,x^\star) \text{ continuous}$$

$$\tilde{a}(x^\star\,|\,x^\star) = 0,$$

$$\tilde{a}(w\,|\,x^\star) \neq 0 \text{ for all } w \neq x^\star$$

Csiszár's results and vector fields

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$\Rightarrow$

There exists a smooth function

$$F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star) \text{ such that}$$

$$\nabla F(w \mid x^\star) = \lambda \tilde{a}(w \mid x^\star) \text{ for all } w \neq x^\star$$

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$\lambda \neq 0$, could depend on $w, x^\star$

Thus, $\nabla F(w\,|\,x^\star) = \lambda(w\,|\,x^\star)\tilde{a}(w\,|\,x^\star)$

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$$F(w\,|\,x^\star) = \sum_{i=1}^{n} f_i(w_i\,|\,x_i^\star) \text{ such that}$$

$$\nabla F(w\,|\,x^\star) = \lambda \tilde{a}(w\,|\,x^\star) \text{ for all } w \neq x^\star$$

$\lambda \neq 0$, could depend on $w, x^\star$

Thus, $\nabla F(w\,|\,x^\star) = \lambda(w\,|\,x^\star)\tilde{a}(w\,|\,x^\star)$

for all $w \neq x^\star, \lambda(w\,|\,x^\star) \neq 0$ and continuous at w

Csiszár's results and vector fields

- The proof of Theorem 1 in Csiszár's paper reveals the following:

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This particular form is due to <u>locality</u> axiom

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Open question

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$$a : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous}$$

$$\lambda : \mathbb{R}^n \to \mathbb{R} \text{ continuous}$$

$$\lambda \cdot a = \nabla F?$$

Under more assumption

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$$\begin{aligned}
a_1(w)\frac{\partial \lambda(w)}{\partial w_2} - a_2(w)\frac{\partial \lambda(w)}{\partial w_1} &= \lambda(w)\left(\frac{\partial a_2(w)}{\partial w_1} - \frac{\partial a_1(w)}{\partial w_2}\right), \\
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a_3(w)\frac{\partial \lambda(w)}{\partial w_1} - a_1(w)\frac{\partial \lambda(w)}{\partial w_3} &= \lambda(w)\left(\frac{\partial a_1(w)}{\partial w_3} - \frac{\partial a_3(w)}{\partial w_1}\right)
\end{aligned}$$

Summary

- Axiomatic characterisation of projection rules
- Regularity: a fundamental axiom
- Regularity has connections with conservative vector fields
- Given a continuous vector field that is not necessarily conservative, is there a continuous scaling that can make the product vector field conservative? Open!

Thank you

The axiom of locality

- Consider two sets L and $\tilde{L}$ of the form

$$L = \{w \in \mathbb{R}^n : w_J \in L_0, w_{J^c} \in L'\}$$

$$\tilde{L} = \{w \in \mathbb{R}^n : w_J \in L_0, w_{J^c} \in L''\}$$

where $J \subset \{1,2,\ldots,n\}$ is arbitrary

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Then, $(\Pi(L|x^\star))_J = (\Pi(\tilde{L}|x^\star))_J$