On Detecting an Anomalous Arm in a Multi-armed Bandit with Markov Observations

Joint work with Rajesh Sundaresan

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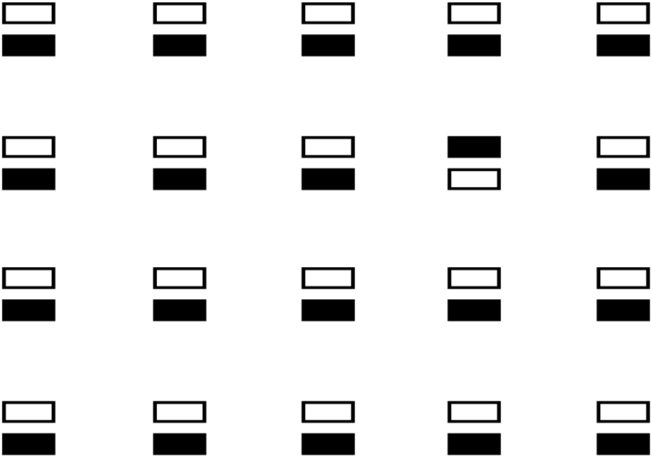
Motivation

A Visual Search Experiment



Can you identify the location of the “odd” image as quickly as possible?

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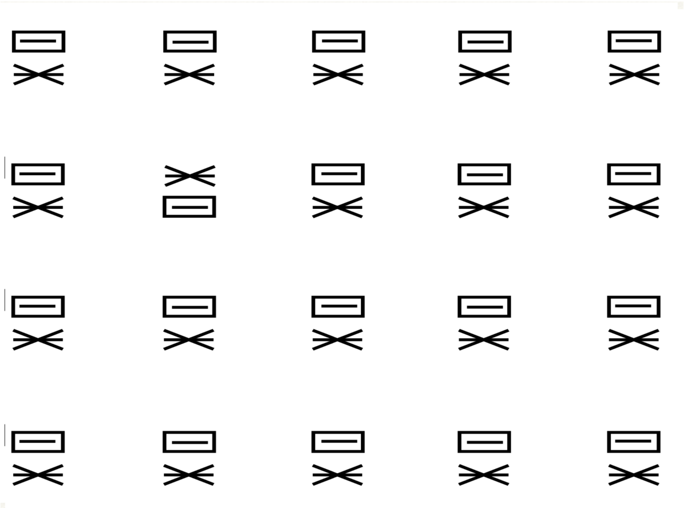

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Can we make precise this notion of closeness?

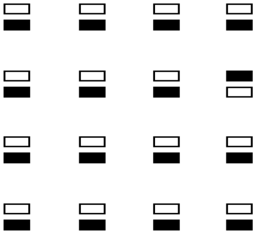
- If the static images in the previous experiments were replaced with dynamic images (movies)[†], what is the notion of closeness of the odd and the non-odd movies?

†Krueger, Paul M., et al. "Evidence accumulation detected in BOLD signal using slow perceptual decision making." Journal of neuroscience methods 281 (2017): 21-32.

Problem Setup: Odd Arm Identification

Problem Formulation

- A multi-armed bandit with $K \geq 3$ arms
- Each arm is either an iid process or a Markov process on a finite state space
- The law of one of the arms (the **odd** arm) is P_1, which is different from P_2, the common law of each of the other arms
- P_1 and P_2 may or may not be known
- Sequential arm selections



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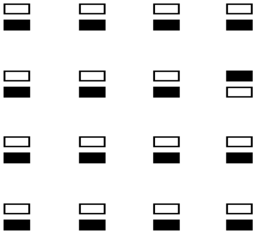

Two quantities of interest:

(a) (Average) time taken to identify the odd arm

(b) Probability of error

Problem Formulation

This talk:

- **Fixed confidence setting**
- **Asymptotics as (b) $\rightarrow 0$.**

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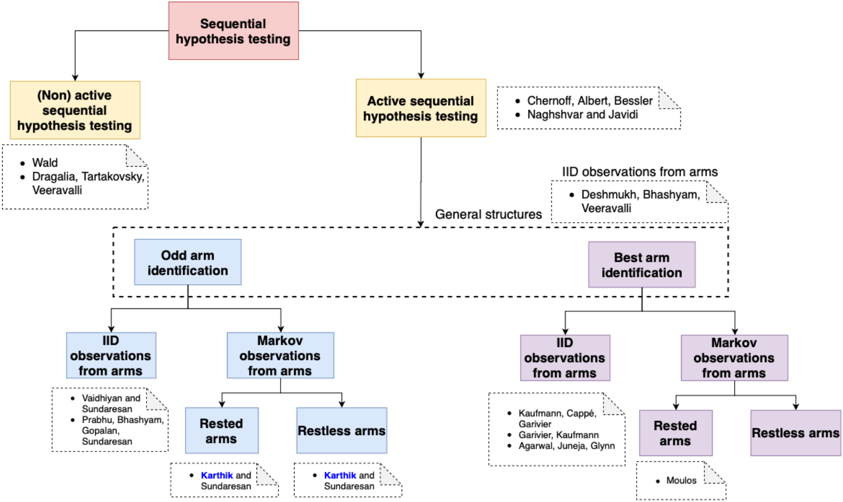
The Big Picture



OAI as a Composite Hypothesis Testing Problem

When P_1 and P_2 are known: K simple hypotheses

$$\begin{aligned}
\mathcal{H}_1 &: \text{arm 1 is the odd arm} \\
\mathcal{H}_2 &: \text{arm 2 is the odd arm} \\
&\vdots \\
\mathcal{H}_K &: \text{arm } K \text{ is the odd arm}
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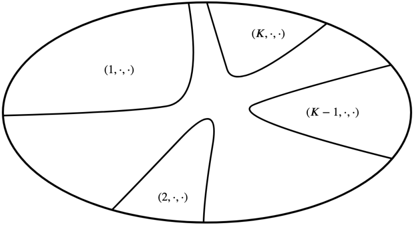
When neither P_1 nor P_2 is known: K composite hypotheses



$$\mathcal{H}_1 : (1, \cdot, \cdot)$$
$$\mathcal{H}_2 : (2, \cdot, \cdot)$$
$$\vdots$$
$$\mathcal{H}_K : (K, \cdot, \cdot)$$

Notation : (odd arm, P_1, P_2)

Overview of Classical Results

Humble Beginnings

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- SPRT
- Only one arm to choose
- Optimal

- Procedure A
- Multiple arms
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All's well that starts well!

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$$R_1 = P_{FA} + c\,\mathbb{E}[N|\mathcal{H}_1], \quad R_2 = P_{MD} + c\,\mathbb{E}[N|\mathcal{H}_2]$$

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- Goal: minimise $w\,R_1 + (1-w)\,R_2$

Wald's SPRT: Main Idea

At time n, construct the statistic

$$S_n = \log \frac{P_{\theta_2}(X_1, \ldots, X_n)}{P_{\theta_1}(X_1, \ldots, X_n)}$$

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$$\mathbb{E}[S_n|\mathcal{H}_1] = -n\,D(P_{\theta_1}\|P_{\theta_2}), \quad \mathbb{E}[S_n|\mathcal{H}_2] = n\,D(P_{\theta_2}\|P_{\theta_1})$$

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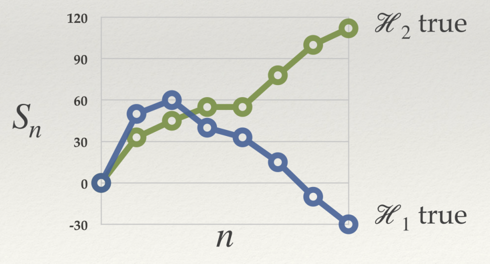
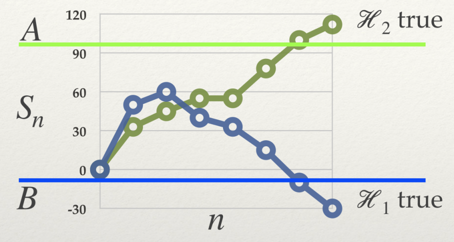
By the law of large numbers, we expect the following behaviour:



Wald's SPRT: Algorithm



At time n:

If $S_n \geq A$, stop and declare $\mathcal{H}_2$ true

If $S_n \leq B$, stop and declare $\mathcal{H}_1$ true

If $B < S_n < A$, take one more observation

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By Wald's identity, if SPRT stops at a random time N, then

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Combining, we get

$$\boxed{\frac{\mathbb{E}[N|\mathcal{H}_1]}{\log \frac{1}{c}} \approx \frac{1}{D(P_{\theta_1} \| P_{\theta_2})}, \quad \frac{\mathbb{E}[N|\mathcal{H}_2]}{\log \frac{1}{c}} \approx \frac{1}{D(P_{\theta_2} \| P_{\theta_1})}}$$

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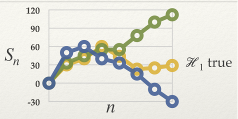
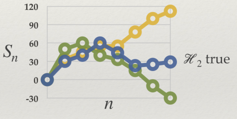
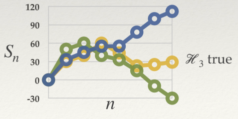
A behaviour we would like S_n to have (shown for $M = 3$):







M-SPRT†

Construct the statistic

$$S_n^{(i)} = \log \frac{P_{\theta_i}(X_1, \ldots, X_n)}{\max\limits_{j \neq i} P_{\theta_j}(X_1, \ldots, X_n)}, \quad i = 1, \ldots, M$$

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At time n, do the following:

Let $i^*(n) \in \arg\max\limits_i S_n^{(i)}$
If $S_n^{(i^*(n))} \geq -logc$, stop and declare $\mathcal{H}_{i^*(n)}$ true
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It can be shown that if N is the random stopping time of M-SPRT, then

$$\frac{\mathbb{E}[N|\mathcal{H}_i]}{\log \frac{1}{c}} \approx \frac{1}{\min_{j \neq i} D(P_{\theta_i} \| P_{\theta_j})}, \quad i = 1, \ldots, M$$

† Draglia, V. P., Alexander G. Tartakovsky, and Venugopal V. Veeravalli, "Multihypothesis sequential probability ratio tests - Part I: Asymptotic optimality", IEEE Transactions on Information Theory 45.7 (1999): 2448-2461.

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Chernoff's Procedure A and its Asymptotic Optimality

Algorithm described by Chernoff:

$$S_n^{(i)} = \log \frac{P_{\theta_i}(X_1, E^{(1)}, \ldots, X_n, E^{(n)})}{\max\limits_{j \neq i} \; P_{\theta_j}(X_1, E^{(1)}, \ldots, X_n, E^{(n)})}$$

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Chernoff showed that

$$\frac{\mathbb{E}[N|\mathcal{H}_i]}{\log \frac{1}{c}} \approx \frac{1}{\max\limits_{\lambda} \min\limits_{j \neq i} \sum\limits_{l=1}^{L} \lambda(E_l)\, D(P_{\theta_i, E_l} \| P_{\theta_j, E_l})}, \quad i = 1, \ldots, M$$

Back to Odd Arm Identification

Visual Search with Static Images

- Solved by Vaidhiyan and Sundaresan
- Experiments $\equiv$ image locations
- $c \to 0 \equiv$ prob. of error $\to 0$
- Observation: number of neuron firings in the brain corresponding to an image observed
- Model: no. of firings is Poisson[†]
- Odd arm: Poisson(R_1), non-odd arms: Poisson(R_2)

† A. P. Sripati and C. R. Olson, "Global image dissimilarity in macaque inferotemporal cortex predicts human visual search efficiency," J. Neurosci., vol. 30, no. 4, pp. 1258–1269, Jan. 2010.

Visual Search with Static Images

When R_1 and R_2 are known[1] (prob. of error: ϵ)

$$\frac{\mathbb{E}[N|\mathcal{H}_i]}{\log \frac{1}{\epsilon}} \approx \frac{1}{\max\limits_{\lambda} \min\limits_{j \neq i} \sum_{a=1}^{K} \lambda(a)\, D(\mathrm{Poi}(R_{i,a}) \| \mathrm{Poi}(R_{j,a}))}, \quad i = 1, \ldots, M$$

[1] Vaidhiyan, Nidhin Koshy, Sripati P. Arun, and Rajesh Sundaresan. "Neural dissimilarity indices that predict oddball detection in behaviour." IEEE Transactions on Information Theory 63.8 (2017): 4778-4796.

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When neither R_1 nor R_2 is known[2]:

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A policy similar to Chernoff's with forced exploration works

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- For two images $\mathcal{I}_1$ (odd) and $\mathcal{I}_2$ (non-odd) whose Poisson firing rates R_1 and R_2 are **not known**,

$$D^*(i, R_1, R_2) = \max_{\lambda} \min_{j \neq i, R_1', R_2'} \sum_{a=1}^{K} \lambda(a)\, D(\mathrm{Poi}(R_{i,a}) \| \mathrm{Poi}(R_{j,a}))$$

From Static Images to Movies

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Rested Markov Arms: Setup

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- Each arm is a homogeneous and ergodic DTMC on a finite state space
- State space is common to all the arms
- Odd arm has TPM P_1, non-odd arms have TPM P_2
- The Markov chain of each arm evolves only when the arm is selected; otherwise, state of the arm is frozen

Rested Markov Arms: Results[3]

When P_1 and P_2 are known:

$$D^*(i, P_1, P_2) = \max_{\lambda} \min_{j \neq i} \sum_{a=1}^{K} \lambda(a)\, D(P_{i,a} \| P_{j,a} | \mu_{i,a}), \quad i = 1, \ldots, M$$

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When neither P_1 nor P_2 is known:

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A policy similar to Chernoff's with forced exploration works

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- Wald's identity not applicable since the observations are Markov. A generalisation needed
- The optimum distribution does not depend on where the movie of each arm was paused. This is because of ergodicity of each arm

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We have

$$(\underline{d}(t), \underline{s}(t)) \longrightarrow A_t \longrightarrow (\underline{d}(t+1), \underline{s}(t+1)) \longrightarrow A_{t+1}$$

Restless Markov Setting: Key Findings[4]

- $\{(\underline{d}(t), \underline{s}(t))\}$ is a controlled Markov chain, with $\{A_t\}$ as the sequence of controls

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- When A_t's are stationary control strategies, we have an MDP on a countable state space (HARD!)
- Our objective is to characterise $D^*(i, P_1, P_2)$, which is quite non-standard in MDP literature

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Restless Markov Setting: Key Findings

- When A_t is chosen according to a distribution of the form

$$P(A_t = a \mid \underline{d}(t) = \underline{d}, \underline{s}(t) = \underline{s}) = \frac{\eta}{K} + (1 - \eta)\, \lambda(a|\underline{d}, \underline{s})$$

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- When P_1 and P_2 are known,

$$D^*(i, P_1, P_2) = \sup_{\lambda(\cdot|\cdot)} \min_{j \neq i} \sum_{(\underline{d}, \underline{s})} \sum_{a=1}^{K} \nu^{\lambda}(\underline{d}, \underline{i}, a)\, D(P_{i,a}^{d_a}(\cdot|s_a) \| P_{j,a}^{d_a}(\cdot|s_a))$$

- Sample complexity-type results
- Generalises all the previously known results

Open Threads

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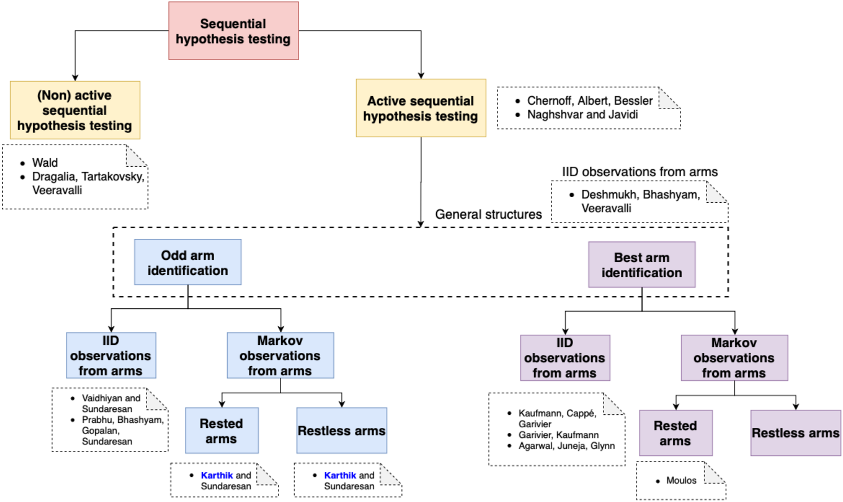
- Analysis for the case when each arm is a Markov process on a general state space
- General structures for Markov observations

A Final Glimpse of Where We Stand



Thank You!