TA Session 15

Agenda : Problems on Poisson process

A quick recap of important definitions and results

1. A simple point process is a countable collection of distinct random variables $\phi = \{S_n \in \mathbb{R}_+ : n \in \mathbb{N}\}$ s.t. $|S_n| \to \infty$ as $n \to \infty$.

2. A Counting process wrt $\phi = \{S_n \in \mathbb{R}_+ : n \in \mathbb{N}\}$ is a random process $\{N(t) : t \in \mathbb{R}_+\}$ s.t. for any $t \in \mathbb{R}_+$,

$$N(t) = \sum_{k=1}^{n} \mathbb{1}_{\{S_n \leq t\}}$$

$=$ # random variables S_n that are lesser than or equal to t.

Interpretation : S_n's can be thought of as the time instants of arrivals of customers at a bank. That is, S_1 is the time instant of arrival of the 1^{st} customer, S_2 is the time instant of arrival of the 2^{nd} customer, and so on. Clearly,

$$S_1 \leq S_2 \leq S_3 \leq \cdots$$

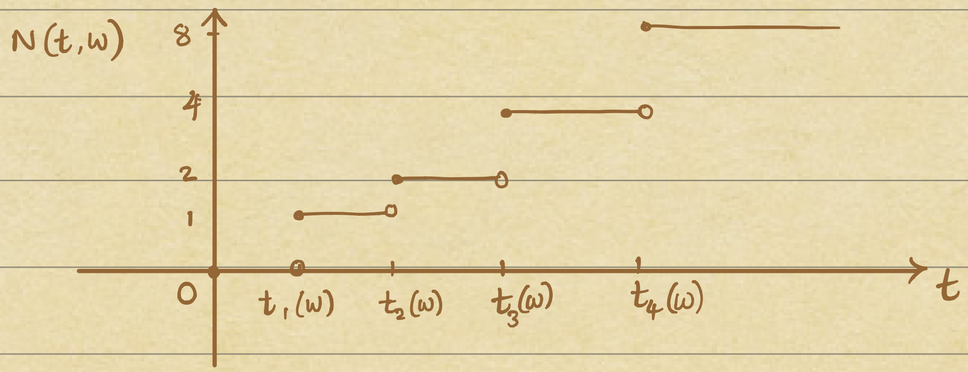
$N(t)$ then is the number of customers who arrived in the time interval $I = (0, t]$, with the interpretation that $N(0) = 0$.

3. The above is the def^n of a Counting process with respect to an underlying point process. If $\phi = \{S_n \in \mathbb{R}_+ : n \in \mathbb{N}\}$ is a simple point process, then $\{N(t) : t \in \mathbb{R}_+\}$ is said to be a simple Counting process.

4. Alternatively, a counting process may be defined (without the aid of any underlying point process) as a stochastic process $\{N(t) : t \in \mathbb{R}_+\}$ s.t.

a) $N(0) = 0$, and $N(t) \in \{0, 1, 2, \cdots\}$ $\forall$ $t \in \mathbb{R}_+$,

b) For each $\omega \in \Omega$, the sample path $\{N(t,\omega) : t \in \mathbb{R}_+\}$ is a non-decreasing, right continuous function of t.



In this def^n of counting process, a counting process is called simple if

$$P(\{\omega : \text{all jumps of } N(t,\omega) \text{ are of unit size}\}) = 1.$$

5. A counting process $\{N(t) : t \in \mathbb{R}_+\}$ is said to have independent increments if:

$\forall$ $0 = t_0 < t_1 < t_2 < \cdots < t_m$, where $m \geq 1$ is any integer, we have that the collection

$$\{N(t_i) - N(t_{i-1}) : 1 \leq i \leq m\}$$

of random variables is mutually independent.

6. A counting process $\{N(t) : t \in \mathbb{R}_+\}$ is said to have stationary increments if for all $s, t \in \mathbb{R}_+$,

the distribution of the random variable

$$N(t+s) - N(t)$$

does not depend on t but is a function only of s.

7. A Counting process $\{N(t) : t \in \mathbb{R}_+\}$ that

a) Is simple,

b) Has independent increments, and

c) Has stationary increments

is called a homogeneous Poisson process.

Further, if $\{N(t) : t \in \mathbb{R}_+\}$ satisfies

$$P\left(N(t+s) - N(t) = n\right) = e^{-\lambda s} \cdot \frac{(\lambda s)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$\left(N(t+s) - N(t) \sim Poi(\lambda)\right)$$

for some $\lambda > 0$ and for all $s, t \in \mathbb{R}_+$, then it is called a homogeneous Poisson process of rate λ.

8. Let $\{S_n \in \mathbb{R}_+ : n \in \mathbb{N}\}$ be a simple point process, and let $\{N(t) : t \in \mathbb{R}_+\}$ be its associated Counting process. If $\{N(t) : t \in \mathbb{R}_+\}$ satisfies the Conditions in point 7. above, then $\{N(t) : t \in \mathbb{R}_+\}$ is called a Poisson point process w.r.t the point process $\{S_n \in \mathbb{R}_+ : n \in \mathbb{N}\}$

Further, in this case, it can be shown that the collection

$$\left\{X_n = S_n - S_{n-1}, \quad n \geq 1\right\} \quad (S_0 := 0)$$

is iid and distributed as $\text{Exp}(\lambda)$ for all $n \geq 1$. This follows from a very important property known as the <u>strong independent increments</u> <u>property</u> of the Poisson process $\{N(t) : t \in \mathbb{R}_+\}$.

<u>Strong independent increments</u>: Let T be any stopping time with respect to the process $\{N(t) : t \in \mathbb{R}_+\}$ such that $P(T < \infty) = 1$. Then, $\forall s \in \mathbb{R}_+$, the random variable $N(T+s) - N(T)$ is independent of the history $\{N(u) : u \leq T\}$.

Furthermore, if $\{N(t) : t \in \mathbb{R}_+\}$ is a homogeneous Poisson process of rate λ, then

$$N(T+s) - N(T) \sim \text{Poi}(\lambda s) \qquad \forall s > 0 \ .$$

Using this, we can prove that $\{X_n = S_n - S_{n-1} : n \geq 1\}$ is iid $\text{Exp}(\lambda)$ as follows:

$$\begin{aligned} P(X_n > t) &= P\left(N(S_{n-1} + t) - N(S_{n-1}) = 0\right) \\ &= P\left(N(t) = 0\right) \\ &= e^{-\lambda t}. \end{aligned}$$

this step follows by noting that S_{n-1} is a stopping time wrt the process $\{N(t) : t \in \mathbb{R}_+\}$, and that $P(S_{n-1} < \infty) = 1$ (see homework 7, Question 4)

Similarly, $\forall s, t \in \mathbb{R}_+$,

$$\begin{aligned} &P(X_n > t, X_{n-1} \leq s) \\ &= E\left[\mathbb{1}_{\{X_n > t,\ X_{n-1} \leq s\}}\right] \\ &= E\left[E\left[\mathbb{1}_{\{X_n > t,\ X_{n-1} \leq s\}} \,\middle|\, \sigma(N(u) : u \leq S_{n-1})\right]\right] \\ &= E\left[\mathbb{1}_{\{X_{n-1} \leq s\}} \cdot E\left[\mathbb{1}_{\{X_n > t\}} \,\middle|\, \sigma(N(u) : u \leq S_{n-1})\right]\right] \end{aligned}$$

$$= E\left[1_{\{X_{n-1} \le s\}} \cdot E\left[1_{\{N(S_{n-1}+t) - N(S_{n-1}) = 0\}} \,\middle|\, \sigma(N(u) : u \le S_{n-1}) \right] \right]$$

$$= E\left[1_{\{X_{n-1} \le s\}} \cdot P\left(N(S_{n-1}+t) - N(S_{n-1}) = 0\right) \right]$$

follows from strong independent increments property

$$= E\left[1_{\{X_{n-1} \le s\}} \cdot P(N(t) = 0) \right]$$

$$= e^{-\lambda t} \cdot P(X_{n-1} \le s)$$

$$= P(X_n > t)\, P(X_{n-1} \le s).$$

Conditional distributions of arrival times

Let $\{N(t) : t \in \mathbb{R}_+\}$ be a homogeneous Poisson process of rate $\lambda > 0$.

Let

$$S_n := \inf\{t \in \mathbb{R}_+ : N(t) \ge n\}$$

denote the n^{th} arrival time.

We now want to compute the conditional distribution of S_n, given that we know $N(t) = n$, i.e., we would like to compute $\underbrace{f_{S_n | \{N(t)=n\}}(s)}$, $s \ge 0$.

This quantity is "like" asking for the probability of S_n to be close to s, given that n arrivals have happened in $(0, t]$.

This can be derived using the theory of <u>order statistics</u>.

Order Statistics

Let $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} F_Y(y)$ be n iid random samples drawn according to a distribution $F_Y(y)$. Let's assume that the cdf F_Y admits a pdf f_Y or pmf p_Y as is the case.

Now, let us arrange the samples $Y_1, \ldots, Y_n$ in increasing order. Let the ordered samples be denoted as $Y_{(1)}, \ldots, Y_{(n)}$, where $Y_{(1)} \leq Y_{(2)} \leq Y_{(3)} \leq \cdots \leq Y_{(n)}$, with

$$Y_{(1)} = \min\{X_1, \ldots, X_n\}$$ → called 1^{st} order statistic

$$Y_{(2)} = \text{second min}\{X_1, \ldots, X_n\}$$ → 2^{nd} order statistic

$$\vdots$$

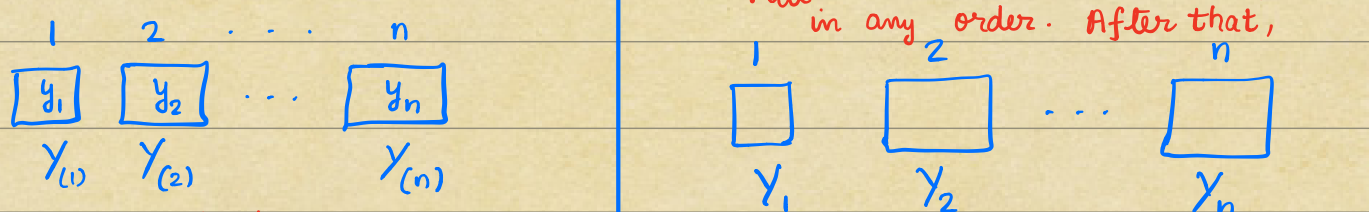
$$Y_{(n)} = \max\{X_1, \ldots, X_n\}.$$ → nth order statistic

wlog, we will consider only pdf case below

① Now, what is the joint density of $Y_{(1)}, \ldots, Y_{(n)}$?

Ans: We want to compute $f_{Y_{(1)}, \ldots, Y_{(n)}}(y_1, \ldots, y_n)$, where we would like to impose the constraint that $y_1 \leq y_2 \leq \cdots \leq y_n$ (otherwise the joint density is 0). This means that if we draw $y_1, \ldots, y_n$ as samples and if we rearrange these samples in increasing order, we will see $Y_{(1)} = y_1, \ldots, Y_{(n)} = y_n$.

Thus, we have the following picture:



We want this

Fill these n boxes with $y_1, \ldots, y_n$ in any order. After that, arrange in increasing order to get the picture on the left.

From the above figure, we notice that the probability of getting $y_1, \cdots, y_n$ in any order is

$$\left(f_{Y_1,\dots,Y_n}(y_1, \cdots, y_n)\right) \times n!$$

$$= n! \prod_{i=1}^{n} f_Y(y_i) \quad \left(\because Y_1, \cdots, Y_n \overset{iid}{\sim} f_Y(\cdot)\right)$$

Thus,

$$f_{Y_{(1)},\dots,Y_{(n)}}(y_1, \cdots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f_Y(y_i), & \text{if } y_1 \le y_2 \le \cdots \le y_n, \\ 0, & \text{otherwise.} \end{cases}$$

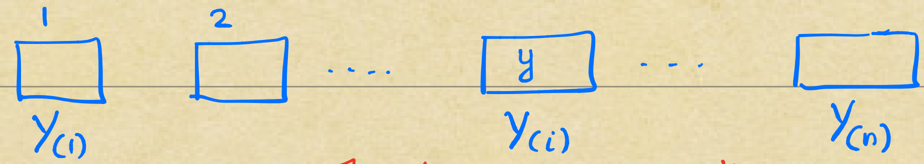
Result: The joint distribution of $S_1, \dots, S_n$ (the n arrival times of a homogeneous PP $\{N(t): t \in \mathbb{R}_+\}$ with rate $\lambda > 0$), given that $N(t) = n$, is given by

$$f_{S_1,\dots,S_n \mid \{N(t)=n\}}(s_1, \dots, s_n) = \begin{cases} \frac{n!}{t^n}, & 0 < s_1 \le s_2 \le \cdots \le s_n \le t \\ 0, & \text{otherwise.} \end{cases}$$

That is, given $\{N(t)=n\}$, the conditional joint distribution of the n arrival times $S_1, \dots, S_n$ is identical to the joint distribution of the n order statistics coming from unif $(0,t]$ distribution.

② What is the distribution of $Y_{(i)}$?

Ans: We want to compute $f_{Y_{(i)}}(y)$. To get this, we have to draw n iid samples from $f_Y(\cdot)$ such that one of them is 'y', $(i-1)$ of remaining $(n-1)$ samples are $\le y$, and rest of the samples are $\ge y$. If we



This is what we want

do so, then, after rearranging the samples in ascending order, we will get the above picture. Thus,

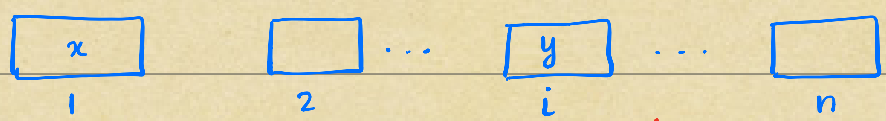
$$f_{Y_{(i)}}(y) = \underbrace{{}^{n}C_{1}\, f_Y(y)}_{\text{① probability of getting `y' as one of the samples}} \;\; \underbrace{{}^{n-1}C_{i-1}\, \big(P(Y\le y)\big)^{i-1}}_{\text{② prob. of getting (i-1) of (n-1) samples to be } \le y} \;\; \underbrace{\big(P(Y\ge y)\big)^{n-i}}_{\text{③ Prob. of getting remaining n-i samples to be } \ge y.}$$

<u>Result</u>: Applying the above result with $f_Y(\cdot) = \text{unif}\,(0,t]$, we get

$$f_{S_i \mid \{N(t)=n\}}(s) = \begin{cases} {}^{n}C_{1} \cdot \dfrac{1}{t} \cdot {}^{n-1}C_{i-1} \cdot \left(s/t\right)^{i-1} \left(1 - s/t\right)^{n-i}, & \text{if } 0<s\le t \\ 0 & \text{Otherwise.} \end{cases}$$

③ What is the joint distribution of $Y_{(1)}$ and $Y_{(i)}$, $i \neq 1$?

<u>Ans</u>: We want to compute $f_{Y_{(1)},Y_{(i)}}(x,y)$.



This is what we want

If we draw n iid samples from $f_Y(\cdot)$ such that one of them is `x', one of the remaining $n-1$ is 'y', $i-2$ of the remaining $n-2$ are $\ge x$ and $\le y$, and rest are $\ge y$, we get the above picture after rearranging the samples in ascending order.

Thus,

$$f_{Y_{(1)},Y_{(i)}}(x,y) = \begin{cases} {}^{n}C_1\, f_Y(x) \cdot {}^{n-1}C_1\, f_Y(y) \cdot {}^{n-2}C_{i-2} \big(P(x \le Y \le y)\big)^{i-2} \cdot \big(P(Y \ge y)\big)^{n-i}, & \text{if } x \le y \\ 0 & \text{Otherwise.} \end{cases}$$

<u>Result</u>: Applying the above result to $f_Y(\cdot) = \text{unif}\,[0,t]$, we get that the conditional joint distribution of S_1 and S_i (1^{st} & i^{th} arrival times of a homogeneous PP $\{N(t) : t \in \mathbb{R}_+\}$), given that $N(t) = n$, is given by

$$f_{S_1,S_i|\{N(t)=n\}}(u,v) = \begin{cases} {}^{n}C_1 \cdot \left(\frac{1}{t}\right) \cdot {}^{n-1}C_1 \cdot \left(\frac{1}{t}\right) \cdot {}^{n-2}C_{i-2} \left(\frac{v-u}{t}\right)^{i-2} \cdot \left(1 - \frac{v}{t}\right)^{n-2}, & \text{if } 0 < u \le v \le t, \\ 0, & \text{Otherwise.} \end{cases}$$

<u>Exercise</u>: Work out the joint distribution of $Y_{(i)}, Y_{(j)}$, $i \neq j$, and subsequently write $f_{S_i,S_j|\{N(t)=n\}}(u,v)$.

<u>Splitting and Merging of Poisson Processes</u>

Let $\{N_1(t) : t \in \mathbb{R}_+\}$ and $\{N_2(t) : t \in \mathbb{R}_+\}$ be two independent homogeneous Poisson processes of rates λ_1 and λ_2 respectively.

A merging of the two processes is a new process $\{N(t) : t \in \mathbb{R}_+\}$ such that

$$N(t) = N_1(t) + N_2(t) \quad \text{for each } t \in \mathbb{R}_+.$$

Remarks: ① It can be shown that $N(t)$ is also a homogeneous PP, but of rate $\lambda_1 + \lambda_2$.

② Let the arrival times of N_1 process be denoted $\{S_n^{(1)}\}_{n\geq 1}$ and those of N_2 be $\{S_n^{(2)}\}_{n\geq 1}$. Then, it follows that

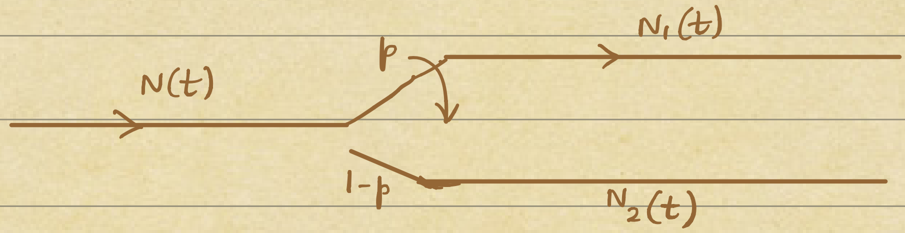
$$S_n^{(1)} \perp\!\!\!\perp \{N_1(S_n^{(1)}+u) - N_1(S_n^{(1)}) : u \geq 0\}$$
$$\perp\!\!\!\perp \{N_2(S_n^{(1)}+u) - N_2(S_n^{(1)}) : u \geq 0\}$$

$$S_n^{(2)} \perp\!\!\!\perp \{N_2(S_n^{(2)}+u) - N_2(S_n^{(2)}) : u \geq 0\}$$
$$\perp\!\!\!\perp \{N_1(S_n^{(2)}+u) - N_1(S_n^{(2)}) : u \geq 0\}.$$

Further, for all m, n, $S_m^{(1)}$ and $S_n^{(2)}$ are independent, and
$$P(S_m^{(1)} = S_n^{(2)}) = 0.$$

Conversely, consider a homogeneous Poisson process $\{N(t) : t \in \mathbb{R}_+\}$ of rate $\lambda > 0$. A splitting of N into two independent processes $\{N_1(t) : t \in \mathbb{R}_+\}$ and $N_2(t) : t \in \mathbb{R}_+\}$ is carried out as follows:

each arrival in $N(t)$ is independently labelled $N_1(t)$ with prob. p and labeled $N_2(t)$ with prob. $1-p$.



Remarks: It can be shown that N_1 and N_2 are homogeneous PPs that are independent of each other, with rates $p\lambda$ & $(1-p)\lambda$ resp.

Compound Poisson Process

A real-valued Stochastic process $\{Z(t) : t \in \mathbb{R}_+\}$ is called a Compound Poisson process if for each $t \in \mathbb{R}_+$, $Z(t)$ Can be expressed in the form

$$Z(t) = \sum_{k=1}^{N(t)} X_i$$

where $N(t)$ is a homogeneous Poisson process, and X_i are iid random variables independent of the Poisson process $\{N(t) : t \in \mathbb{R}_+\}$

Two general remarks on order statistics

① Recall that

$$f_{Y_{(1)}, \cdots, Y_{(n)}}(y_1, \cdots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f_Y(y_i), & \text{if } y_1 \le y_2 \le \cdots \le y_n, \\ 0, & \text{otherwise.} \end{cases}$$

This is equivalent to saying that

$$f_{Y_{(1)}, \cdots, Y_{(n)}}(y_1, \cdots, y_n) \overset{(a)}{=} \sum_{\sigma} f_{Y_{\sigma(1)}, \cdots, Y_{\sigma(n)}}(y_1, \cdots, y_n)$$

$$\overset{(b)}{=} \sum_{\sigma} \prod_{i=1}^{n} f_{Y_{\sigma(i)}}(y_i)$$

$$\overset{(c)}{=} \sum_{\sigma} \prod_{i=1}^{n} f_Y(y_i)$$

$$= n! \prod_{i=1}^{n} f_Y(y_i),$$

Where in (a)-(c), $\sigma = (\sigma(1), \dots, \sigma(n))$ denotes a permutation of

$(1, \dots, n)$, (b)-(c) follows by noting that for each permutation $\sigma = (\sigma(1), \dots, \sigma(n))$, $y_{\sigma(1)}, \dots, y_{\sigma(n)} \overset{iid}{\sim} f_y(\cdot)$, and the last line follows from the fact that there are $n!$ permutations in all. This is just another, perhaps more detailed, way of understanding the joint distribution of all n order statistics.

② What is $E[y_{(1)} + \dots + y_{(n)}]$?

This is same as $E[y_1 + \dots + y_n]$. This is because adding all order statistic samples is just the same as adding all the n iid samples $y_1, \dots, y_n$ (note that in trying to find $f_{y_{(1)}, \dots, y_{(n)}}(y_1, \dots, y_n)$, we wanted $y_{(1)} = y_1, \dots, y_{(n)} = y_n$ in particular. However, once we take the samples $y_1, \dots, y_n$, whether we add $y_{(1)} = y_1, \dots, y_{(n)} = y_n$ or just $y_1, \dots, y_n$, we will end up with $y_1 + \dots + y_n$ in the end anyway).

This is useful in the following case:

<u>Example</u>: Consider homework 7, question 3, part (a).

Here, we may compute $E[W]$ by using the law of iterated expectations as follows: $E[W] = E[E[W | N(t)]]$.

Now,

$$E[W | N(t) = n] = E[t - S_1 + \dots + t - S_n \,|\, N(t) = n]$$

$$= nt - E[S_1 + \dots + S_n \,|\, N(t) = n]$$

$$= nt - E[y_{(1)}, \dots, y_{(n)}]$$, where $y_{(1)}, \dots, y_{(n)}$ are the ordered statistics of $y_1, \dots, y_n \overset{iid}{\sim} \text{unif}[0, t]$

$$= nt - E[y_1 + \dots + y_n]$$

$$= nt - nt/2$$

$$= nt/2.$$

Thus,

$$E[W \mid N(t)] = N(t) \cdot t/2.$$

Therefore, we have

$$E[W] = E[E[W \mid N(t)]]$$

$$= E[N(t) \cdot t/2]$$

$$= E[N(t)] \cdot t/2$$

$$= \lambda t \cdot t/2 \quad (\because N(t) \sim Poi(\lambda t))$$

$$= \frac{\lambda t^2}{2}.$$

Problems:

1. Let $\{N(t) : t \in \mathbb{R}_+\}$ be a Poisson process of rate $\lambda = 2$. Let X_i denote the i^{th} inter-arrival time.

(i) Find $P(X_1 > 0.5)$

Ans: $P(X_1 > 0.5) = e^{-\lambda \cdot 0.5} = e^{-1}$.

(ii) Given that we had no arrivals before $t=1$, find $P(X_1 > 3)$.

Ans: $P(X_1 > 3 \mid N(1) = 0) = P(N(3) - N(1) = 0 \mid N(1) = 0)$

$$= P(N(3) - N(1) = 0)$$

$$= P(N(2) = 0)$$

$$= e^{-2 \cdot 2} = e^{-4}.$$

(iii) Given that 3^{rd} arrival occurred at $t=2$, find prob. that fourth arrival occurs after $t=4$.

Ans:

$$P(X_4 > 4 \mid N(2) = 3) = P(N(4) - N(2) = 0 \mid N(2) = 3)$$

$$= P(N(4) - N(2) = 0)$$

$$= P(N(2) = 0)$$

$$= e^{-4}.$$

(iv) I start watching the process at time $t = 10$. Let T be the time of the first arrival I see. Find $E[T]$ and $Var(T)$.

Ans:

$T = X_1 + 10$, where X_1 is the first inter-arrival time.

We know $X_1 \sim Exp(\lambda)$.

Thus, $E[T] = E[X_1] + 10 = 10.5$

$Var(T) = Var(X_1) = 1/4$.

2. During working hours (9 am – 5pm), customer arrivals at a certain post office follows a Poisson process with arrival rate of 20 customers per hour.

a) What is the expected time of arrival of the fifth customer?

Ans: $S_5 = \sum_{i=1}^{5} X_i$, where $X_i \overset{iid}{\sim} Exp(20)$.

Thus,

$$E[S_5] = 5 . E[X_1] = 5 . \frac{1}{20} = \frac{1}{4} \text{ hours.}$$

b) Given that only 10 customers arrived between 12pm and 1pm, what is the expected no of customers arriving between 1pm & 2pm?

Ans:

$$N((12pm, 1pm]) = 10.$$

We have

$$P\left(N((1pm, 2pm]) = n \,\middle|\, N((12pm, 1pm]) = 10\right)$$

$$= P\left(N((1pm, 2pm]) = n\right) \quad (\text{indep. incr.})$$

$$= \text{Poisson}\left(20 \text{ cust/hr} \times 1 \text{ hour}\right)$$

$$= Poi(20).$$

Hence, $E\left[N((1pm, 2pm]) \,\middle|\, N((12pm, 1pm]) = 10\right] = 20.$

3. You enter a bank, and find that there are 3 tellers, all of which are busy serving one customer each. As soon as

any one of the customer leaves, you get served. Assume that the service times of the tellers are iid Exp(1) random variables. What is the probability that you will be the last to leave?

<u>Ans</u>: Let A, B, C be the customers already being served. Let T_A, T_B and T_C denote their respective service times. Let T denote your service time. We want to compute

$$\left(P\left(T+T_A > \max\{T_B, T_C\} \,\middle|\, T_A < \min\{T_B, T_C\}\right) \cdot P\left(T_A < \min\{T_B, T_C\}\right)\right) \times 3$$

The details are left as exercise.

4. A person arrives at a pedestrian crossing line on a road. Assume that c seconds are required to cross the road, and that the crossing is a zero-width line. The person waits until the time for the next vehicle crossing the line is > c seconds.

a) Consider a one-way, single lane road. The instants at which vehicles cross the line is as per a Poisson process of rate λ.

(i) Write an expression for dist^n of the time until the first vehicle crosses the road after arrival of pedestrian.

<u>Ans</u>: Exp(λ). This is because:

Let T denote the time at which person arrives. This is independent of Poisson process arrivals. Now, let

$Y(t)$ = time until the first vehicle arrives, when the person arrived at time t.

We know

$$P(Y(t) > s) = e^{-\lambda s}.$$

Thus, we have

$$P(Y(T)>s) = E\left[1_{\{Y(T)>s\}}\right]$$

$$= E\left[\underbrace{E\left[1_{\{Y(T)>s\}} \mid T\right]}_{\exp(\lambda)}\right]$$

$= e^{-\lambda s}$, which follows the fact that

$E\left[1_{\{Y(T)>s\}} \mid T\right] = e^{-\lambda s}$ since now, upon Conditioning on T, T can be viewed as a deterministic Constant.

(ii) What is the probability that the person waits for k cars before he crosses the road.

Ans: k cars have to arrive within c seconds of each other, and the $(k+1)^{st}$ car has to take $> c$ seconds to arrive.

Therefore the required probability is

$$\left(1-e^{-\lambda c}\right)^k e^{-\lambda c}.$$

5. Let $\{N_1(t) : t \in \mathbb{R}_+\}$ and $\{N_2(t) : t \in \mathbb{R}_+\}$ be two indep. PPs with rates λ_1 and λ_2 respectively. Consider their merging

$\{N_1(t) + N_2(t) : t \in \mathbb{R}_+\}$. Suppose that an observer arrives at $S_n^{(1)}$, the time of n^{th} arrival of N_1 process.

a) What is the dist^n of the time until the next arrival in N_2 process?

Ans: Let $Y_2(t) :=$ time until next arrival in N_2 process when the observer starts observing from time t onwards.

We want the distribution of $Y_2(S_n^{(1)})$.

Notice that $S_n^{(1)}$ is independent of N_2 process.

Thus, we can use the same arguments as in Q4) above.

Furthermore, from homework 7, q4, we have

$$P(S_n^{(1)} < \infty) = 1.$$

Thus,

$$P(Y_2(S_n^{(1)}) > s) = E\left[\mathbb{1}_{\{Y_2(S_n^{(1)}) > s\}}\right]$$

$$= E\left[E\left[\mathbb{1}_{\{Y_2(S_n^{(1)}) > s\}} \,\middle|\, S_n^{(1)}\right]\right]$$

$$= E\left[e^{-\lambda s}\right],$$

where the last line follows by noting that

$P(Y_2(t) > s) = e^{-\lambda s}$, and by conditioning on $S_n^{(1)}$,

$P(Y_2(S_n^{(1)}) > s \mid S_n^{(1)}) = e^{-\lambda s}$ since now, $S_n^{(1)}$ can be treated as a deterministic Constant.

b) What is the probability that the next arrival in the merged process is an arrival in the N_1 process?

Ans: We want the prob. that the first arrival in N_1 happens before the first arrival in N_2 process.

Thus, the required prob. is

$$P\left(Y_1(S_n^{(1)}) < Y_2(S_n^{(1)})\right) = \int_0^\infty \underbrace{P(Y > t)}_{X \perp\!\!\!\perp Y} \cdot \lambda_1 e^{-\lambda_1 t}\, dt$$

$$\| $$

$$P(X < Y)$$

where $X \overset{d}{=} Y_1(S_n^{(1)})$, $Y \overset{d}{=} Y_2(S_n^{(1)})$, $Y \perp\!\!\!\perp X$.

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)t}\, dt$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

c) What is the distribution of time until the next arrival in the merged process.

Ans: The next arrival time for the merged process is the min. of arrival times of N_1 & N_2 processes. Thus, we have

$$\text{dist}^n\left(\min\{\underbrace{Y_1(S_n^{(1)})}_{\exp(\lambda_1)}, \underbrace{Y_2(S_n^{(1)})}_{\exp(\lambda_2)}\}\right)$$

$$= \exp(\lambda_1+\lambda_2).$$

d) Obtain the mean # arrivals in N_1 process that the observer will see before the next arrival in N_2 process.

Ans:

Let $M_1 :=$ # arrivals in N_1 process before next arrival in N_2 process.

Then,

$$E[M_1] = E\left[E\left[M_1 \mid Y_2(S_n^{(1)})\right]\right].$$

Now,

$$E\left[M_1 \mid Y_2\left(S_n^{(1)}\right) = t\right] = E\left[\operatorname{Poi}(\lambda_1 t)\right] = \lambda_1 t.$$

Thus,

$$E\left[M_1 \mid Y_2\left(S_n^{(1)}\right)\right] = \lambda_1 \, Y_2\left(S_n^{(1)}\right).$$

Hence,

$$E[M_1] = \frac{\lambda_1}{\lambda_2} \quad \left(\because \underbrace{Y_2(S_n^{(1)}) \sim \exp(\lambda_2)}_{\text{from part (a)}}\right).$$

6. Let $\{N_1(t) : t \in \mathbb{R}_+\}$ and $\{N_2(t) : t \in \mathbb{R}_+\}$ be two indep. homogeneous PPs with rates 1 and 2 respectively. Let

$$N(t) = N_1(t) + N_2(t).$$

a) Find $P\left(N(1) = 2,\ N(2) = 5\right)$.

Ans: N process has rate $\lambda = 1 + 2 = 3$.

$$P\left(N(1) = 2,\ N(2) = 5\right) = P\left(N(1) = 2,\ N(2) - N(1) = 3\right)$$

$$= e^{-\lambda} \cdot \frac{\lambda^2}{2!} \cdot e^{-\lambda} \cdot \frac{\lambda^3}{3!}$$

b) Given that $N(1) = 2$, find the prob. that $N_1(1) = 1$.

Ans:

$$P\left(N_1(1) = 1 \mid N(1) = 2\right) = P\left(N_1(1) = 1,\ N(1) = 2\right) \Big/ P\left(N(1) = 2\right)$$

$$= \frac{P(N_1(1)=1,\ N_2(1)=1)}{P(N(1)=2)}$$

$$= \frac{e^{-1} \cdot e^{-2} \cdot 2}{e^{-3} \cdot \frac{3^2}{2!}} = \frac{4}{9}.$$

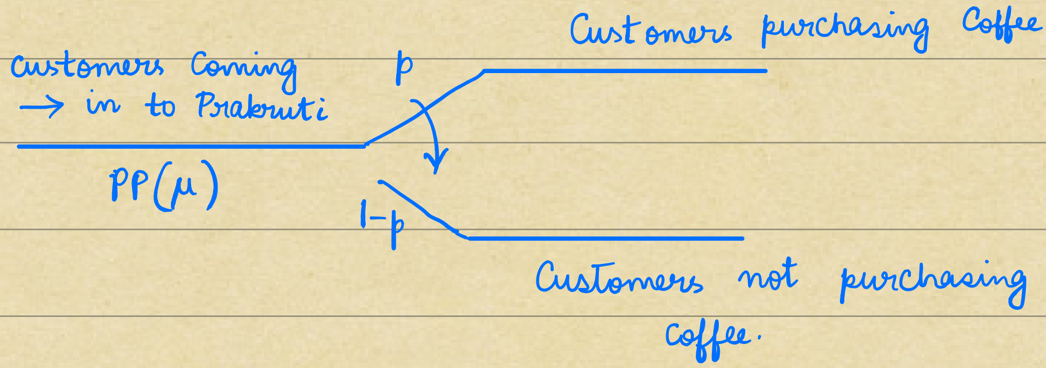
7. Suppose that the # customers visiting Prakruti canteen between 10 am to 4 pm is a $Poi(\mu)$ random variable. Assume that each customer purchases coffee with probability p, independent of all other customers & independently of N. Let X be the total # customers who purchased coffee from 10am to 4pm.

a) Show that $X \sim Poi(p\mu)$.

b) Let $Y = N - X$ = # customers who did not purchase coffee. Show that $Y \sim Poi((1-p)\mu)$ and prove that $X \perp\!\!\!\perp Y$.

Ans:

This can be viewed as a splitting.



$$P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n) \cdot P(N=n)$$

$$= \sum_{n=k}^{\infty} {}^nC_k\, p^k (1-p)^{n-k} \cdot e^{-\mu} \cdot \frac{\mu^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!\,(n-k)!} p^k (1-p)^{n-k} e^{-\mu} \cdot \frac{\mu^n}{\cancel{n!}}$$

$$= \frac{p^k \mu^k e^{-\mu p}}{(1-p)^k \cdot k!} \sum_{n=k}^{\infty} \frac{(\mu(1-p))^{n-k} e^{-(\mu-p)}}{(n-k)!}$$

$$= e^{-\mu p} \cdot \frac{(\mu p)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Thus, $X \sim Poi(\mu p)$.

b) Exercise. For independence, show that

$$P(X=k, Y=m) = P(X=k)\,P(Y=m) \quad \forall\, m, k.$$

8. Let $N_1(t)$ and $N_2(t)$ be two independent homogeneous PPs with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Find the probability that the 3^{rd} arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$.

Ans: Let $N(t)$ be a homogeneous PP with rate $\lambda = 3$. Consider a splitting of this process into N_1 & N_2, with split probability of $p = 1/3$ towards N_1. In other words, we label each arrival in the N process as N_1 with prob. p and N_2 with prob. $1-p$.

$\{3^{rd}$ arrival in N_1 before 3^{rd} arrival in $N_2\}$

= seeing one of the following patterns of heads (1) and tails (0)

$$\Big\{111,\ 0111,\ 1011,\ 1101,\ \underbrace{00111,\ 01011,\ \ldots}_{{}^5C_2 - [01110,\ 10110,\ 11010,\ 11100]}\Big\}$$

Required probability $= p^3 + 3p^3(1-p) + 6p^3(1-p)^2$

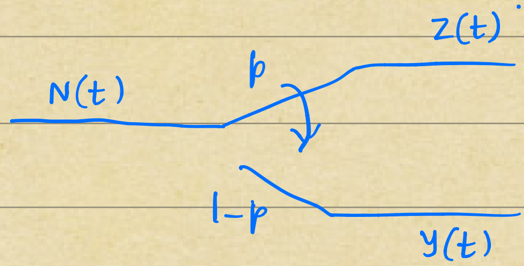
9. Let $\{Z(t) : t \geq 0\}$ be a Compound Poisson process with

$$Z(t) = \sum_{k=1}^{N(t)} X_i ,$$

where $X_i \overset{iid}{\sim} \text{Ber}(p)$ and $\{N(t) : t \in \mathbb{R}_+\}$ is a homogeneous PP of rate $\lambda > 0$. Find the distribution of $Z(t)$. What is the interpretation of $Z(t)$?

Ans: $P(Z(t) = k \mid N(t) = n) = {}^nC_k\, p^k (1-p)^{n-k}$

we want prob. of seeing k heads out of n independent tosses of a coin of bias p, where toss outcomes are independent of $\{N(t) : t \in \mathbb{R}_+\}$.



$Z(t)$ has the interpretation that it is a split of $N(t)$ with prob. p.

Thus, $Z(t) \sim \text{Poi}(\lambda p t)$.

10. A batch of customers arrive at a shop at time instants $(S_n \in \mathbb{R}_+ : n \in \mathbb{N})$, where $0 = S_0 < S_1 < S_2 < \cdots$ Furthermore, the inter-arrival times

$$X_n = S_n - S_{n-1}, \quad n \geq 1,$$

are iid $\exp(\lambda)$ rvs.

At the n^{th} arrival time, a batch of B_n Customers arrive, where B_n is independent of $B_1, \ldots, B_{n-1}$ and of the arrival times $(S_n : n \in \mathbb{N})$. Further, all B_n's have a Common distribution given by

$$P(B_n = j) = p_j \quad , \quad j = 1, \ldots, M.$$

a) Fix $k \in \{1, \ldots, M\}$. Let $Z_k(t)$ denote the number of batches of size k that have arrived in the interval $(0, t]$. What is $E[Z_k(t)]$?

Ans: We can write

$$Z_k(t) = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}} 1_{\{B_n = k\}}.$$

Thus,

$$E[Z_k(t)] = E\left[\sum_{n=1}^{\infty} 1_{\{S_n \leq t\}} 1_{\{B_n = k\}}\right]$$

$$= \sum_{n=1}^{\infty} E\left[1_{\{S_n \leq t\}} 1_{\{B_n = k\}}\right] \quad \left(\text{by } \underset{\text{monotone Convergence thm}}{\text{MCT}}\right)$$

$$= \sum_{n=1}^{\infty} E\left[1_{\{S_n \leq t\}}\right] E\left[1_{\{B_n = k\}}\right] \quad \left(\because B_n\text{'s are independent of } S_n\text{'s}\right)$$

$$= \sum_{n=1}^{\infty} E\left[1_{\{S_n \leq t\}}\right] \cdot p_k$$

$$= p_k \sum_{n=1}^{\infty} E\left[1_{\{S_n \leq t\}}\right]$$

$$= p_k \, E\left[\sum_{n=1}^{\infty} 1_{\{S_n \leq t\}}\right]$$

$$= p_k \, E\left[\sum_{n=1}^{\infty} 1_{\{\ldots\}}\right]$$

$$p_k \; E\left[\sum_{n=1} \mathbb{1}_{\{N(t)\geq n\}}\right]$$

$$= p_k \, E\left[\sum_{n=1}^{N(t)} 1\right] = p_k E[N(t)] = p_k \lambda t.$$

b) What is $E[Z_k(t) Z_l(t)]$ for $k \neq l$?

Ans:

$$E[Z_k(t) Z_l(t)] = E\left[\left(\sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}} \mathbb{1}_{\{B_n \leq k\}}\right)\left(\sum_{m=1}^{\infty} \mathbb{1}_{\{S_m \leq t\}} \mathbb{1}_{\{B_m \leq l\}}\right)\right]$$

If $k \neq l$, then $n \neq m$ and vice versa

$$= E\left[\sum_{n \neq m} \mathbb{1}_{\{S_n \leq t\}} \mathbb{1}_{\{S_m \leq t\}} \mathbb{1}_{\{B_n \leq k\}} \mathbb{1}_{\{B_m \leq l\}}\right]$$

$$\overset{MCT}{=} \sum_{n \neq m} p_k \, p_l \; E\left[\mathbb{1}_{\{S_n \leq t\}} \mathbb{1}_{\{S_m \leq t\}}\right] \quad (\because B_n \perp\!\!\!\perp B_m, \; B_n, B_m \perp\!\!\!\perp S_n, S_m)$$

$$= p_k \, p_l \; E\left[\sum_{n \neq m} \mathbb{1}_{\{S_n \leq t, \, S_m \leq t\}}\right]$$

$$= p_k \, p_l \; E\left[\left(\sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}\right)^2 - \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}\right]$$

$$= p_k \, p_l \; E\left[(N(t))^2 - N(t)\right]$$

$$= p_k \, p_l \left[(\lambda t)^2 + \lambda t - \lambda t\right]$$

$$= (p_k \lambda t)(p_l \lambda t)$$

$$= E[Z_k(t)] \; E[Z_l(t)].$$

Alternatively, think of a splitting of $N(t)$ into k independent Poisson processes $(N_i(t))_{i=1}^{k}$, where the splitting happens with a k-ary switch whose prob. are $p_1, \ldots, p_k$. Then, it follows that $Z_k(t)$ and $Z_l(t)$ are independent $\forall \; k \neq l$.