September 15 **Semester 2018-II**

Tutorial 8: Random Processes

Lecturer: Utpal Mukherji/ Parimal Parag *Prepared by: Karthik*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

8.1 Agenda

- Problems

1. Let X be Gaussian distributed with mean 0 and variance 1, i.e., $X \sim \mathcal{N}(0,1)$. Define a new random variable Y as $Y = X^2$. Write down the CDF and pdf of Y.

Solution:
Let us first evaluate the CDF of Y, and subsequently obtain the pdf by differentiating the CDF. Towards this, note that Y is a nonnegative random variable. Therefore, for any $y < 0$, we have

$$P(Y \leq y) = 0.$$

Now, fix an arbitrary $y \geq 0$. Then, we have

$$\begin{aligned}
P(Y \leq y) &= P(X^2 \leq y) \\
&= P(|X| \leq \sqrt{y}) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx \\
&= \int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx - \int_{-\infty}^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx \\
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),
\end{aligned}$$

where we define

$$\Phi(c) := \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx$$

to be the CDF of a Gaussian distribution with mean 0 and variance 1 evaluated at $c \in \mathbb{R}$. Therefore, we have

$$F_Y(y) = P(Y \leq y) = \begin{cases} \Phi(\sqrt{y}) - \Phi(-\sqrt{y}), & y \geq 0, \\ 0, & y < 0. \end{cases}$$

We now proceed to compute the pdf of Y. Note that in order to do so, we need to differentiate $F_Y(y)$. Clearly, since $F_Y(y) = 0$ for $y \in (-\infty, 0)$, it follows that $f_Y(y) = 0$ for all $y < 0$. Therefore, we focus on the case when $y \geq 0$. Notice that $F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$ for any $y \geq 0$. Defining

$$g(y) := \Phi(\sqrt{y}) - \Phi(-\sqrt{y}), \quad y \geq 0,$$

we see that in order to compute the pdf of Y for $y \geq 0$, it suffices to differentiate $g(\cdot)$. Notice that $g(\cdot)$ is a function defined on the interval $[0, \infty)$. In order to talk about differentiation of $g(\cdot)$ at any point $y \geq 0$, we need to talk about limits from the left and right of the point y. However, at the point $y = 0$, we can only talk about limit from the right.

Therefore, the pdf of Y may be computed by differentiating $g(y)$ only for $y \in (0, \infty)$. Let us evaluate the pdf in this interval. Towards this, for any $y \in (0, \infty)$, we have

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy} g(y) = \frac{d}{dy}\left(\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right) \\
&= \frac{d}{dy} g(y) \\
&= \frac{d}{dy}\left(\int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}}\, dx - \int_{-\infty}^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}}\, dx\right) \\
&= \frac{1}{\sqrt{2y\pi}}\, e^{\frac{-y}{2}},
\end{aligned}$$

and for other values of y, we take $f_Y(y) = 0$. Therefore, we have

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2y\pi}}\, e^{\frac{-y}{2}}, & y \in (0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.

Exercise: For $X \sim \mathcal{N}(\mu, \sigma^2)$, check that the CDF of $Y = X^2$ is

$$P(Y \leq y) = \begin{cases} \Phi\left(\frac{\sqrt{y}-\mu}{\sigma}\right) - \Phi\left(\frac{-\sqrt{y}-\mu}{\sigma}\right), & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.

In the above problem, the distribution of Y is called the "Chi-squared" distribution. Alternatively, Y is referred to as a Chi-squared random variable.

2. Let Θ be uniformly distributed over the interval $[0, \pi]$. Let $a > 0$ be any scalar. Define a new random variable B as

$$B = a \cos \Theta.$$

Find the CDF and pdf of B.

Solution:
Let us first compute the CDF of B, and then obtain the pdf by differentiating the CDF. Towards this, notice the following points:

(a) For $\theta \in [0, \pi]$, $\cos\theta$ is a strictly decreasing function of θ.

(b) Since $-1 \le \cos\theta \le 1$, B takes values between $-a$ and a (both inclusive).

From the above, we immediately get

$$P(B \le b) = \begin{cases} 0, & b < -a, \\ 1, & b \ge a. \end{cases}$$

We now focus on the case when $B \in [-a, a)$. Fix an arbitrary $b \in [-a, a)$. Then,

$$\begin{aligned} P(B \le b) &= P(a\cos\Theta \le b) \\ &= P\left(\cos\Theta \le \frac{b}{a}\right) \text{ (if } a \text{ was negative, then we would get } P\left(\cos\Theta \ge \frac{b}{a}\right) \text{ in this step)} \\ &\overset{(a)}{=} P\left(\Theta \ge \cos^{-1}\left(\frac{b}{a}\right)\right) \\ &= \frac{\pi - \cos^{-1}\left(\frac{b}{a}\right)}{\pi} \\ &= 1 - \frac{\cos^{-1}\left(\frac{b}{a}\right)}{\pi}, \end{aligned}$$

where (a) above follows from the strictly decreasing property of $\cos(\cdot)$. Therefore, we get

$$F_B(b) = P(B \le b) = \begin{cases} 0, & b < -a, \\ 1 - \frac{\cos^{-1}\left(\frac{b}{a}\right)}{\pi}, & -a \le b < a, \\ 1, & b \ge a. \end{cases}$$

We now proceed to evaluate the pdf of B. Again, as before, notice that for the function

$$h(b) := 1 - \frac{\cos^{-1}\left(\frac{b}{a}\right)}{\pi}, \quad -a \le b < a,$$

it is not possible to speak of its derivative at $b = -a$ since only limit from the right is well-defined. Hence, $f_B(b)$ may be computed by differentiating $h(b)$ only for $b \in (-a, a)$, in which case we get

$$f_B(b) = \frac{d}{db}h(b) = \frac{1}{\pi\sqrt{a^2 - b^2}},$$

and for other values of b, we take $f_B(b) = 0$. Therefore, we have

$$f_B(b) = \begin{cases} \frac{1}{\pi\sqrt{a^2-b^2}}, & -a < b < a, \\ 0, & \text{otherwise.} \end{cases}$$

3. Let Θ be uniformly distributed over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Define a new random variable X as

$$X = \tan\Theta.$$

Find the CDF and pdf of X. What is $E[X]$?

Solution:
Let us first evaluate the CDF of X, and subsequently obtain the pdf by differentiating the CDF. Notice that $\tan(\cdot)$ is a strictly increasing function on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and therefore forms a bijective map

from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ onto $\mathbb{R}(-\infty, \infty)$. Thus, X may take any value in $\mathbb{R} = (-\infty, \infty)$. Fix an arbitrary $x \in \mathbb{R}$. Then, we have

$$\begin{aligned} P(X \le x) &= P(\tan\Theta \le x) \\ &\overset{(a)}{=} P\left(\Theta \le \tan^{-1}(x)\right) \\ &= \frac{\tan^{-1}(x) - \left(-\frac{\pi}{2}\right)}{\pi} \\ &= \frac{\tan^{-1}(x) + \frac{\pi}{2}}{\pi}, \end{aligned}$$

where (a) above follows from the strictly increasing property of $\tan(\cdot)$. From this, we get

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

This pdf is known as the "Cauchy" pdf, or alternatively, X is referred to as a Cauchy random variable.

We now proceed to evaluate $E[X]$. We have

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx \\ &= \frac{1}{2\pi} \log(1+x^2)\Big|_{-\infty}^{\infty} \\ &= \frac{1}{2\pi} \left(\infty - \infty\right), \end{aligned}$$

which is undefined. To see this more clearly, let us evaluate $E[X_+]$ and $E[X_-]$ separately, where

$$X_+ = \max\{X, 0\}, \quad X_- = -\min\{X, 0\}.$$

We have

$$\begin{aligned} E[X_+] = E[\max\{X,0\}] &= \int_{-\infty}^{\infty} \max\{x, 0\}\, f_X(x)\, dx \\ &= \int_{0}^{\infty} x\, f_X(x)\, dx \\ &= \frac{1}{2\pi} \int_{0}^{\infty} \frac{2x}{1+x^2}\, dx \\ &= \infty. \end{aligned}$$

On similar lines, we have

$$\begin{aligned}
E[X_-] = E[-\min\{X,0\}] &= \int_{-\infty}^{\infty} -\min\{x,0\}\, f_X(x)\, dx \\
&= \int_{-\infty}^{0} -x\, f_X(x)\, dx \\
&= \frac{1}{2\pi}\int_{0}^{\infty} \frac{2x}{1+x^2}\, dx \\
&= \infty.
\end{aligned}$$

Since $E[X_+] = \infty = E[X_-]$, we conclude that $E[X] = E[X_+] - E[X_-]$ is undefined.

Remark 3. $E[X]$ is always defined when X is a nonnegative random variable. That is, the problem of $E[X_+] = \infty = E[X_-]$ does not occur when X is nonnegative since for such X, $E[X_-] = 0$. However, $E[X]$ may be equal to ∞.

Remark 4. For a random variable X (not necessarily nonnegative), if $E[X] = \infty$ (or $E[X] = -\infty$), it means that $E[X]$ is defined and its value is equal to ∞ (or $-\infty$).

4. Give examples of distributions for which

   (a) Mean and variance are both finite.

   (b) Mean is finite, variance is infinite

   (c) Mean is infinite

   (d) Mean is undefined.

   Solution:

   (a) Gaussian distribution with mean 0 and variance 1.

   (b) Let X be a random variable whose pdf is given by

   $$f_X(x) = \begin{cases} \frac{2}{x^3}, & x \in [1,\infty), \\ 0, & \text{otherwise.} \end{cases}$$

   Then, we have

   $$E[X] = \int_{1}^{\infty} x\, f_X(x)\, dx = 2\int_{1}^{\infty} \frac{1}{x^2}\, dx = 2,$$

   whereas

   $$E[X^2] = \int_{1}^{\infty} x^2\, f_X(x)\, dx = 2\int_{1}^{\infty} \frac{1}{x}\, dx = \infty.$$

   Therefore, we get

   $$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \infty.$$

(c) We may modify the preceding example to have the following pdf for X:

$$f_X(x) = \begin{cases} \frac{1}{x^2}, & x \in [1, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

It then follows that

$$E[X] = \int_1^\infty x\, f_X(x)\, dx = \int_1^\infty \frac{1}{x}\, dx = \infty.$$

Remark 5. For a random variable X, if $E[X] = \infty$, then by Jensen's inequality, since

$$E[X^2] \geq (E[X])^2,$$

we get $E[X^2] = \infty$. Now, this seems to suggest that

$$\mathrm{var}(X) = E[X^2] - (E[X])^2$$

is not defined since the right hand side is of the form $\infty - \infty$. This is however not true. When $E[X] = \infty$ (or even when $E[X] = -\infty$), then the correct formula to use for variance is

$$\mathrm{var}(X) = E[(X - E[X])^2].$$

This formula can be reduced to $\mathrm{var}(X) = E[X^2] - (E[X])^2$ if and only if $-\infty < E[X] < \infty$.

Remark 6. For a real-valued random variable X (i.e., X satisfying $P(X < \infty) = 1$), if $E[X] = \infty$, then $\mathrm{var}(X) = \infty$.

(d) We already saw an example (the Cauchy distribution) when mean is undefined. The following is an example for the discrete case. Let N be an integer-valued random variable with the pmf

$$P(N = n) = \frac{3}{\pi^2} \frac{1}{n^2} = P(N = -n), \quad n \in \{1, 2, \ldots\}.$$

Then, we have

$$E[N] = \sum_{n=-\infty}^{-1} n\, P(N = n) + \sum_{n=1}^{\infty} n\, P(N = n).$$

The first term on the right hand side of the above equation is given by

$$\sum_{n=-\infty}^{-1} n\, P(N = n) = \frac{3}{\pi^2} \sum_{n=-\infty}^{-1} \frac{1}{n} = -\frac{3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = -\infty,$$

while the second term is given by

$$\sum_{n=1}^{\infty} n\, P(N = n) = \frac{3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Therefore, we have

$$E[N] = -\infty + \infty,$$

which implies that $E[N]$ is undefined.

5. Let random variables X and Y have the following joint pdf:

$$f_{X,Y}(x, y) = \begin{cases} cy, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c, and write down the marginal pdfs of X and Y.

(b) Evaluate $P(X \geq Y + 0.5)$.

(c) Evaluate $P(X > 0.75 | Y > 0.5)$.

Solution:
We note that the region over which the joint pdf is specified is as depicted by the shaded region in Figure 8.1.

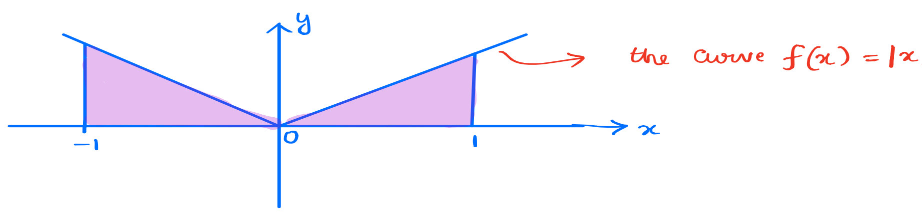

Figure 8.1: The region over which the joint pdf of X and Y is specified.

Let

$$R := \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1,\ 0 \leq y \leq |x|\}$$

denote the shaded region in the figure.

(a) To evaluate c, we shall integrate the joint pdf over R and set this integral to be equal to 1. We then have

$$1 = \int_{-1}^{1} \int_{0}^{|x|} f_{X,Y}(x, y)\, dy\, dx = c \int_{-1}^{1} \int_{0}^{|x|} y\, dy\, dx = c \int_{-1}^{1} \frac{x^2}{2}\, dx = \frac{c}{3},$$

from which it follows that $c = 3$.

We now proceed to evaluate the marginal pdf of Y. Fix an arbitrary $y \in [0, 1]$. Then, noting that the set of all feasible values of x is given by

$$\mathcal{X} := [-1, -y] \cup [1, y],$$

we have

$$f_Y(y) = \int_{\mathcal{X}} f_{X,Y}\,dx = \int_{-1}^{-y} 3y\,dx + \int_{y}^{1} 3y\,dx = 6y(1-y),$$

and for other values of x, we take $f_Y(y) = 0$. Therefore, we have

$$f_Y(y) = \begin{cases} 6y(1-y), & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

On similar lines, to evaluate the marginal pdf of X, we first fix an arbitrary $x \in [-1,1]$, and note that the set of all feasible of y is given by

$$\mathcal{Y} := [0,\ |x|].$$

Thus, we have

$$f_X(x) = \int_{\mathcal{Y}} f_{X,Y}\,dy = \int_{0}^{|x|} 3y\,dy = \frac{3x^2}{2},$$

and we take $f_X(x) = 0$ for other values of x. Therefore, we have

$$f_X(x) = \begin{cases} \frac{3x^2}{2}, & x \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

(b) In order to evaluate the required probability, we first compute the conditional pdf of X given Y, whose definition is as below:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

for all x, y such that

i. the denominator is not zero, and

ii. the numerator and the denominator are not simultaneously zero.

Note that the numerator and the denominator are both simultaneously zero for $y = 0$, and the denominator is zero for $y = 1$. Thus, the conditional pdf $f_{X|Y}(x|y)$ is defined for all $y \in (0,1)$ and for all $x \in [-1,-y] \cup [y,1]$.

In order to evaluate the conditional pdf, let us fix an arbitrary $y \in (0,1)$. Then, noting that the feasible values of x are $x \in\in [-1,-y] \cup [y,1]$, we get

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3y}{6y(1-y)} = \frac{1}{2(1-y)},$$

and we take the conditional pdf to be 0 otherwise. Therefore, we have

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{2(1-y)}, & y \in (0,1),\ x \in [-1,-y] \cup [y,1] \\ 0, & \text{otherwise.} \end{cases}$$

We now proceed to evaluate the required probability. By the law of total probability, we have

$$\begin{aligned} P(X \ge Y + 0.5) &= \int_0^1 P(X \ge y + 0.5 | Y = y)\, f_Y(y)\,dy \\ &= \int_0^{0.5} P(X \ge y + 0.5 | Y = y)\, f_Y(y)\,dy \end{aligned}$$

since for $y \in [0.5, 1)$, we note that $P(X \geq y + 0.5 | Y = y) = 0$ as X can take values only up to 1. In the above equation,

$$P(X \geq y + 0.5 | Y = y) = \int_{y+0.5}^{1} f_{X|Y}(x|y)\, dx = \int_{y+0.5}^{1} \frac{1}{2(1-y)}\, dx = \frac{1-2y}{4(1-y)}.$$

Substituting this back, we get

$$P(X \geq Y + 0.5) = \frac{3}{2} \int_{0}^{0.5} y(1-2y)\, dy = \frac{1}{16}.$$

(c) Exercise.

6. Let X be Poisson distributed with parameter $\lambda > 0$. Show that for any $r \in \{1, 2, 3, \ldots\}$,

$$E[X(X-1)\cdots(X-r+1)] = \lambda^r.$$

Solution:

Noting that the pmf of X is given by

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \{0, 1, 2, \ldots\},$$

we have

$$\begin{aligned}
E[X(X-1)\cdots(X-r+1)] &= \sum_{k=0}^{\infty} k(k-1)\cdot(k-r+1) e^{-\lambda} \frac{\lambda^k}{k!} \\
&\overset{(a)}{=} \sum_{k=r}^{\infty} k(k-1)\cdot(k-r+1) e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \sum_{k=r}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-r)!} \\
&= \lambda^r \sum_{k=r}^{\infty} e^{-\lambda} \frac{\lambda^{k-r}}{(k-r)!} \\
&\overset{(b)}{=} \lambda^r \underbrace{\sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!}}_{=1} \\
&= \lambda^r,
\end{aligned}$$

where (a) above follows by noting that

$$k(k-1)\cdots(k-r+1) = 0 \quad \text{for } k = 1, 2, \ldots, r-1,$$

and (b) follows by making the substitution $m = k - r$.

7. Let X be a random variable whose CDF is as depicted in Figure 8.2.

(a) What is $P(X \leq 0.8)$.

(b) Compute $E[X]$.

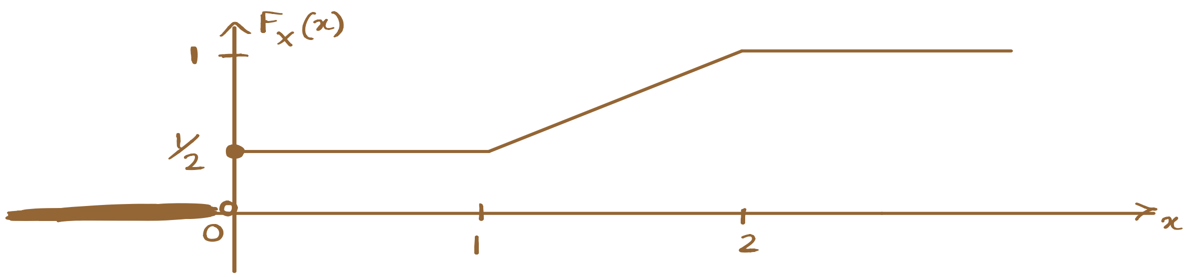

Figure 8.2: CDF of the random variable X.

(c) Compute var(X).

Solution:

(a) We first note a few points.

i. Since the CDF of X has a jump at $x = 0$, with the size of jump equal to 0.5, we may conclude that $P(X = 0) = 0.5$.

ii. The CDF of X may mathematically be expressed as follows:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5, & 0 \le x < 1, \\ \frac{x}{2}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

From the above, it follows that $F_X(x)$ is not differentiable at $x = 0, 1, 2$, and is differentiable elsewhere. This in particular implies that we may derive the pdf of X by differentiating $F_X(x)$ as

$$f_X(x) = \begin{cases} \frac{1}{2}, & 1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have a special type of a distribution in this example. It has jumps and also admits pdf in a certain region. Such distributions are neither purely continuous nor purely discrete,

and are referred to as "mixed" distributions. Notice that the CDF puts a mass of 0.5 at $x = 0$ and distributes the remaining mass of 0.5 over the interval $(1, 2)$.
From the CDF, we have

$$P(X \leq 0.8) = F(0.8) = 0.5.$$

(b) We have

$$E[X] = 0 \cdot P(X = 0) + \int_1^2 x\, f_X(x)\, dx = \int_1^2 \frac{x}{2}\, dx = \frac{3}{4}.$$

(c) In order to compute the variance of X, we first evaluate $E[X^2]$ as follows:

$$E[X^2] = 0^2 \cdot P(X = 0) + \int_1^2 x^2\, f_X(x)\, dx = \int_1^2 \frac{x^2}{2}\, dx = \frac{7}{6}.$$

Therefore, we have

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \frac{29}{48}.$$

Exercises:

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $A, B \in \mathcal{F}$ be any two events. Define $X = 1_A$ and $Y = 1_B$. Write down the joint CDF of X and Y and the marginal CDFs of X and Y. What are $E[X]$ and $E[Y]$?

2. Let X and Y have the following joint pdf:

$$f_{X,Y}(x, y) = \begin{cases} cx(y - x)e^{-y}, & 0 \leq x \leq y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the constant c, and evaluate the marginals $f_X(\cdot)$ and $f_Y(\cdot)$.

   (b) Show that

$$f_{X|Y}(x|y) = 6x(y - x)y^{-3},\ 0 \leq x \leq y < \infty,$$
$$f_{Y|X}(y|x) = (y - x)e^{x-y},\ 0 \leq x \leq y < \infty.$$