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Learning to Detect an Odd Markov Arm

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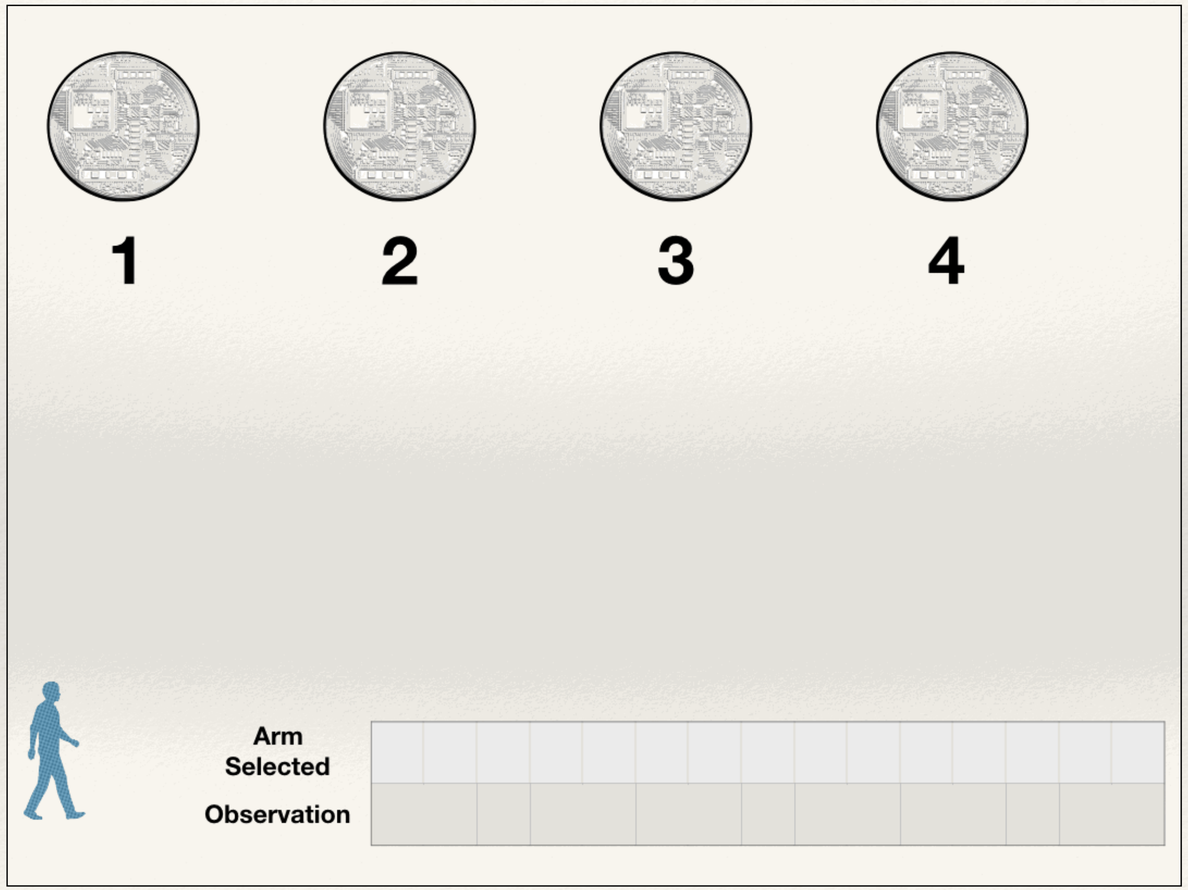
Problem Setup

- A multi-armed bandit with K independent arms
- Arm: an ergodic DTMC on a finite state space
- Common state space for all the arms
- One of the arms (**odd arm**) is governed by tpm P_1
- Rest of the arms governed by tpm P_2 ($P_2 \neq P_1$)
- Only one arm may be observed at any time
- The observed arm undergoes transition. The unobserved arms remain frozen (rested)

GOAL:
Given an error tolerance level $\epsilon > 0$, identify the odd arm as quickly as possible without the knowledge of P_1 or P_2, subject to $P_e \leq \epsilon$



Comparison With Existing Works

		Problem Setting		Nature of observations		Arm distributions	
		Regret Minimisation	Optimal Stopping	IID	Markov (rested)	Known	Unknown
Odd Arm Identification	Gittins[1]	✓	✗	✗	✓	✓	✗
	Agarwal et al.[2] Anantharam et al.[3]	✓	✗	✗	✓	✗	✓
	Vaidhiyan et al.[4] Prabhu et al.[5]	✗	✓	✓	✗	✗	✓
	Current work	✗	✓	✗	✓	✗	✓

Finite parameter space

Uncountably Infinite parameter space

[1] J. C. Gittins, "Bandit processes and dynamic allocation indices," Journal of the Royal Statistical Society. Series B (Methodological), pp. 148–177, 1979.

[2] R. Agrawal, D. Teneketzis, and V. Anantharam, "Asymptotically efficient adaptive allocation schemes for controlled Markov chains: Finite parameter space," IEEE Trans. on Automatic Control, vol. 34, no. 12, pp. 1249–1259, 1989.

[3] Anantharam V, Varaiya P, Walrand J. Asymptotically efficient allocation rules for the multiarmed bandit problem with multiple plays-Part II: Markovian rewards. IEEE Trans. on Automatic Control. 1987 Nov, 32(11), pp. 977-982.

[4] N. K. Vaidhiyan and R. Sundaresan, "Learning to detect an oddball target," IEEE Trans. on Information Theory, vol. 64, no. 2, pp. 831–852, 2018.

[5] G. R. Prabhu, S. Bhashyam, A. Gopalan, and R. Sundaresan, "Learning to detect an oddball target with observations from an exponential family," 2017.

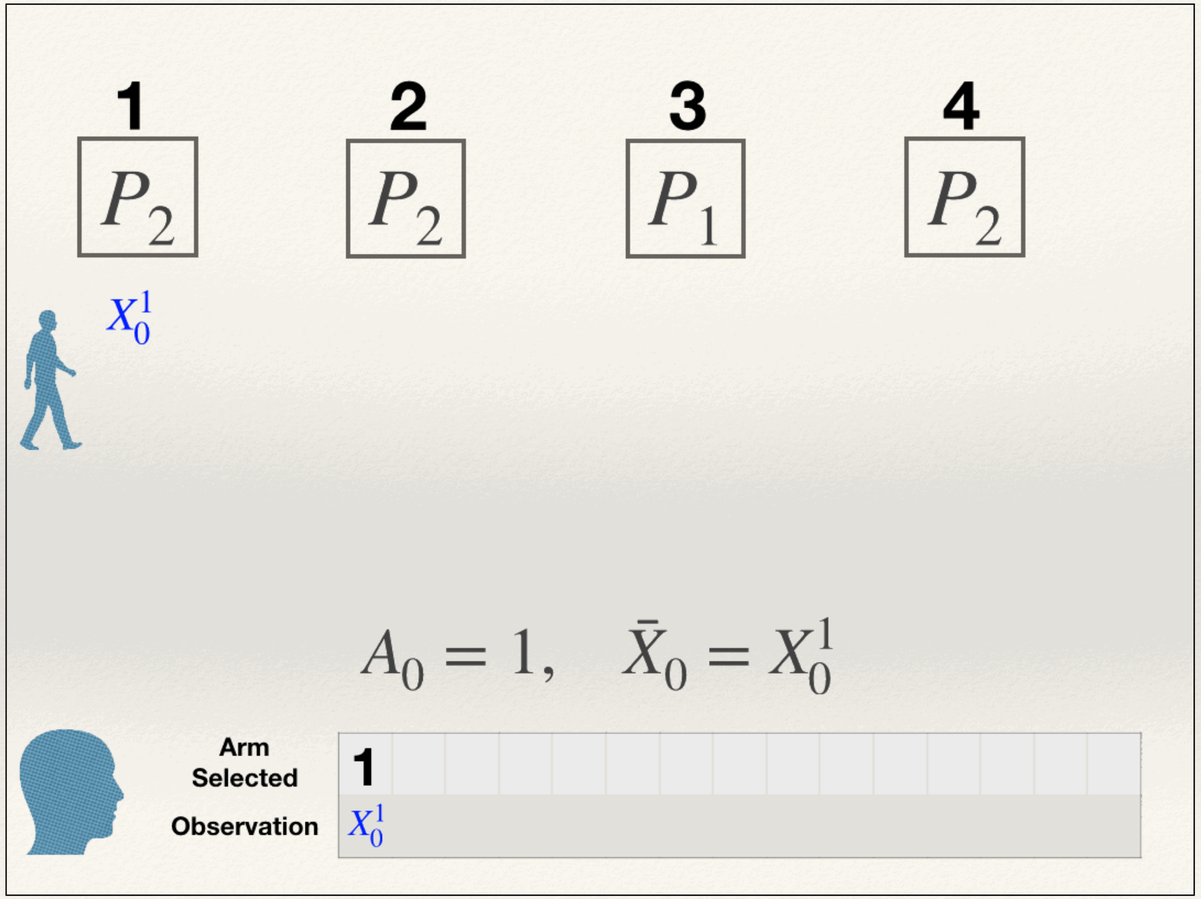
Main Contributions

- An asymptotic[1] lower bound on the expected number of samples required to identify the odd arm as a function of error tolerance
- An asymptotically optimal scheme which respects the desired error tolerance
- The scheme is a modification of the classical GLRT with forced exploration
- Key challenges in the Markov setting identified. A first step towards solving the more difficult case of “restless” Markov arms

[1]The asymptotics is as the error tolerance vanishes.

Notations

- $(X_k^a)_{k\geq 0}$: Markov process of arm a
- TPM of odd arm: P_1; TPM of other arms: P_2
- $(A_k)_{k\geq 0}$: arm selection process
- $(\bar{X}_k)_{k\geq 0}$: observation process
- $A_k \in \sigma(\bar{X}_0^{k-1}, A_0^{k-1})$ for all k



The Lower Bound

Suppose $C = (h, P_1, P_2)$ denotes the underlying configuration of the arms. Let μ_i denote the stationary distribution of P_i for $i = 1,2$.

For any $\epsilon > 0$, let $\Pi(\epsilon)$ denote the set of policies

$$\Pi(\epsilon) = \left\{ \pi : P^{\pi}(\text{error} \,|\, C) \leq \epsilon \;\forall\; C = (h, P_1, P_2) \right\}$$

Lower bound:

$$\lim_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} \frac{E[\tau(\pi) \,|\, C]}{\log(1/\epsilon)} \geq \frac{1}{D^*(h, P_1, P_2)}$$

$$D^*(h, P_1, P_2) = \max_{0 \leq \lambda \leq 1} \left\{ \lambda \, D(P_1 \,||\, P_\lambda \,|\, \mu_1) + (1 - \lambda) \frac{(K-2)}{(K-1)} D(P_2 \,||\, P_\lambda \,|\, \mu_2) \right\}$$

$$P_\lambda(j \,|\, i) = \frac{\lambda_1 \mu_1(i) P_1(j \,|\, i) + (1 - \lambda_1) \frac{(K-2)}{(K-1)} \mu_2(i) P_2(j \,|\, i)}{\lambda_1 \mu_1(i) + (1 - \lambda_1) \frac{(K-2)}{(K-1)} \mu_2(i)}$$

For any two transition matrices P and Q, and a probability distribution μ on the states, define the quantity $D(P \,||\, Q \,|\, \mu)$ as

$$D(P \,||\, Q \,|\, \mu) = \sum_i \mu(i) \sum_j P(j \,|\, i) \log \frac{P(j \,|\, i)}{Q(j \,|\, i)}$$

Remarks

- $D^*(h, P_1, P_2)$ **is a measure of effort required to guard against the nearest alternative**
- **For small values of error tolerance** ϵ**,**

$$E^{\pi}[\tau(\pi) \,|\, C] \gtrapprox \frac{\log(1/\epsilon)}{D^*(h, P_1, P_2)}, \quad \pi \in \Pi(\epsilon)$$

Lower Bound: Key Steps in the Proof

- If $C = (h, P_1, P_2)$ and $C' = (h', P_1', P_2')$, where $h' \neq h$, then using a result of Kaufmann et al.[1], we have

$$\min_{C'=(h',P_1',P_2')} E^{\pi}\left[\sum_{t=0}^{\tau(\pi)} \log \frac{f(\bar{X}_t \,|\, \bar{X}_0^{t-1}, A_0^{t-1}, C)}{f(\bar{X}_t \,|\, \bar{X}_0^{t-1}, A_0^{t-1}, C')}\right] \quad \geq \quad \underbrace{\epsilon \log \frac{\epsilon}{1-\epsilon} + (1-\epsilon)\log \frac{1-\epsilon}{\epsilon}}_{d(\epsilon, 1-\epsilon)}$$

- For any given arm, the long-term fraction of exits of a state i is equal to the long-term fraction of entries into i. This common fraction is the stationary probability of observing i. **This is a consequence of the rested nature of the arms**

- Wald's identity not applicable. A generalisation of a change of measure argument in Kaufmann et al.[1] to Markov processes used

- $\dfrac{d(\epsilon, 1-\epsilon)}{\log(1/\epsilon)} \to 1 \text{ as } \epsilon \downarrow 0$

Lower bound:

$$\lim_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} \frac{E^{\pi}[\tau(\pi) \,|\, C]}{\log(1/\epsilon)} \geq \frac{1}{D^*(h, P_1, P_2)}$$

$$D^*(h, P_1, P_2) = \max_{0 \leq \lambda \leq 1} \left\{ \lambda \, D(P_1 \,||\, P_\lambda \,|\, \mu_1) + (1-\lambda)\frac{(K-2)}{(K-1)} D(P_2 \,||\, P_\lambda \,|\, \mu_2) \right\}$$

$$P_\lambda(j \,|\, i) = \frac{\lambda_1 \mu_1(i) P_1(j \,|\, i) + (1-\lambda_1)\frac{(K-2)}{(K-1)}\mu_2(i) P_2(j \,|\, i)}{\lambda_1 \mu_1(i) + (1-\lambda_1)\frac{(K-2)}{(K-1)}\mu_2(i)}$$

$$E^{\pi}\left[\sum_{t=0}^{\tau(\pi)} \log \frac{f(\bar{X}_t \,|\, \bar{X}_0^{t-1}, A_0^{t-1}, C)}{f(\bar{X}_t \,|\, \bar{X}_0^{t-1}, A_0^{t-1}, C')}\right] = E^{\pi}[\tau(\pi)] \cdot E^{\pi}\left[\log \frac{f(\bar{X}_t \,|\, A_t, C)}{f(\bar{X}_t \,|\, A_t, C')}\right]$$

Wald's identity for IID processes

[1] E. Kaufmann, O. Cappe, and A. Garivier, "On the complexity of best-arm identification in multi-armed bandit models," The Journal of Machine Learning Research, vol. 17, no. 1, pp. 1–42, 2016.

Upper Bound: Preliminaries

- The learner does not know the underlying configuration of the arms
- Suppose the configuration is $C = (h, P_1, P_2)$
- Suppose observations $\bar{X}_0, \ldots, \bar{X}_n$ have been obtained under arm selections $A_0, \ldots, A_n$
- The joint likelihood of all observations, given arm selections, under config. C is

$$f(\bar{X}_0^n \,|\, A_0^n, C) = \prod_{a=1}^{K} \nu(X_0^a) \prod_{i,j} (P_1(j\,|\,i))^{N_h(n,i,j)} \; (P_2(j\,|\,i))^{\sum_{a \neq h} N_a(n,i,j)}$$

- Introduce a joint prior on (P_1, P_2) as $\mathscr{D}(P_1, P_2) = \Gamma(P_1) \cdot \Gamma(P_2)$
- **Average likelihood** of all observations, given actions, when h is the odd arm is

$$f_h(\bar{X}_0^n \,|\, A_0^n) = \prod_{a=1}^{K} \nu(X_0^a) \int_{(P_1,P_2)} \prod_{i,j} (P_1(j\,|\,i))^{N_h(n,i,j)} \; (P_2(j\,|\,i))^{\sum_{a \neq h} N_a(n,i,j)} \, \mathscr{D}(P_1, P_2) \, dP_1(j\,|\,i) \, dP_2(j\,|\,i)$$

$N_h(n, i, j)$: no. of $i \to j$ transitions observed on arm h

Prior $\Gamma(\,\cdot\,)$ on the space of all TPMs:

Pick each row of a TPM independently according to uniform distribution on the probability simplex.

Cannot be computed since the actual configuration is not known

Can be computed for each odd arm index h

Upper Bound: Preliminaries

- **Average likelihood** of all observations, given actions, when h is the odd arm is

$$f_h(\bar{X}_0^n \mid A_0^n) = \prod_{a=1}^{K} \nu(X_0^a) \int_{(P_1,P_2)} \prod_{i,j} (P_1(j \mid i))^{N_h(n,i,j)} \, (P_2(j \mid i))^{\sum_{a \neq h} N_a(n,i,j)} \, \mathscr{D}(P_1, P_2) \, dP_1(j \mid i) \, dP_2(j \mid i)$$

- When h is the odd arm, the **maximum likelihood** of all observations, given actions, is

$$\hat{f}_h(\bar{X}_0^n \mid A_0^n) = \prod_{a=1}^{K} \nu(X_0^a) \prod_{i,j} \left(\frac{N_h(n,i,j)}{N_h(n,i)} \right)^{N_h(n,i,j)} \left(\frac{\sum_{a \neq h} N_a(n,i,j)}{\sum_{a \neq h} N_a(n,i)} \right)^{\sum_{a \neq h} N_a(n,i,j)}$$

When h is the odd arm:

$\frac{N_h(n,i,j)}{\sum_j N_h(n,i,j)}$ = ML estimate of (i,j) entry of the TPM of h

$\frac{\sum_{a \neq h} N_a(n,i,j)}{\sum_{a \neq h} \sum_j N_h(n,i,j)}$ = ML estimate of (i,j) entry of the TPM of $a \neq h$

Modified GLR Statistic

$$M_{ha}(n) = \log \frac{f_h(\bar{X}_0^n \mid A_0^n)}{\hat{f}_a(\bar{X}_0^n \mid A_0^n)}, \quad a \neq h$$

Usual GLR Statistic

$$M_{ha}(n) = \log \frac{\hat{f}_h(\bar{X}_0^n \mid A_0^n)}{\hat{f}_a(\bar{X}_0^n \mid A_0^n)}, \quad a \neq h$$

Policy: Modified GLRT + Forced Exploration

Policy $\pi^\star(L, \delta)$

- Fix parameters $L \geq 1$ and $\delta \in (0,1)$
- Select each arm once in the first K time instants
- Repeat the following steps until stoppage for all $n \geq K-1$:
  - Compute

$$\hat{h}(n) = \arg\max_{h} \min_{a \neq h} M_{ha}(n)$$

  - if $\min_{a \neq \hat{h}(n)} M_{\hat{h}(n)a}(n) \geq \log((K-1)L)$, stop and declare $\hat{h}(n)$ as the odd arm
  - if $\min_{a \neq \hat{h}(n)} M_{\hat{h}(n)a}(n) < \log((K-1)L)$, select next arm as follows:
    - Toss a coin with bias δ
    - If coin falls heads, select the next arm uniformly at random
    - If coin falls tails, select the next arm according to distribution $\lambda_{opt}(\hat{h}(n), \hat{P}_1^n, \hat{P}_2^n)$

L : error tolerance control parameter

δ : forced exploration parameter

Modified GLR Statistic

$$M_{ha}(n) = \log \frac{f_h(\bar{X}_0^n \,|\, A_0^n)}{\hat{f}_a(\bar{X}_0^n \,|\, A_0^n)}, \quad a \neq h$$

$$D^*(h, P_1, P_2) = \max_{0 \leq \lambda \leq 1} \left\{ \lambda\, D(P_1 \,||\, P_\lambda \,|\, \mu_1) + (1-\lambda) \frac{(K-2)}{(K-1)} D(P_2 \,||\, P_\lambda \,|\, \mu_2) \right\}$$

$$P_\lambda(j \,|\, i) = \frac{\lambda_1 \mu_1(i) P_1(j \,|\, i) + (1-\lambda_1)\frac{(K-2)}{(K-1)} \mu_2(i) P_2(j \,|\, i)}{\lambda_1 \mu_1(i) + (1-\lambda_1)\frac{(K-2)}{(K-1)} \mu_2(i)}$$

Suppose $\hat{\lambda} = \arg\max \lambda$ in the expression for $D^*(\hat{h}(n), \hat{P}_1^n, \hat{P}_2^n)$

Then, let

$$\lambda_{opt}(\hat{h}(n), \hat{P}_1^n, \hat{P}_2^n)(a) = \begin{cases} \hat{\lambda}, & a = \hat{h}(n), \\ \frac{1-\hat{\lambda}}{K-1}, & a \neq \hat{h}(n)\,. \end{cases}$$

Salient Features of Policy $\pi^\star(L, \delta)$

- The policy stops in finite time a.s.
- If $C = (h, P_1, P_2)$ is the actual configuration, then a.s.:

$$\hat{h}(n) \to h$$
$$\hat{P}_1^n(j \mid i) \to P_1(j \mid i) \quad \text{for all } i, j$$
$$\hat{P}_2^n(j \mid i) \to P_2(j \mid i) \quad \text{for all } i, j$$

- Given error tolerance $\epsilon > 0$, for any $\delta \in (0,1)$,

$$L = (1/\epsilon) \implies P_e(\pi^\star(L, \delta)) \leq \epsilon$$

- **Asymptotic optimality**:

$$\lim_{\delta \downarrow 0} \lim_{L \to \infty} \frac{E[\tau(\pi^\star(L, \delta)) \mid C]}{\log L} = \frac{1}{D^*(h, P_1, P_2)}$$

Policy $\pi^\star(L, \delta)$

- Fix parameters $L \geq 1$ and $\delta \in (0,1)$
- Select each arm once in the first K time instants
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  - Compute

$$\hat{h}(n) = \arg\max_h \min_{a \neq h} M_{ha}(n)$$

  - if $M_{\hat{h}(n)}(n) \geq \log((K-1)L)$, stop and declare $\hat{h}(n)$ as the odd arm
  - if $M_{\hat{h}(n)}(n) < \log((K-1)L)$, select next arm as follows:
    - Toss a coin with bias δ
    - If coin falls heads, select the next arm uniformly at random
    - If coin falls tails, select the next arm according to distribution $\lambda_{opt}(\hat{h}(n), \hat{P}_1^n, \hat{P}_2^n)$

Conclusions and Future Work

- Analysed the problem of odd arm identification for rested Markov arms
- Provided an asymptotic lower bound on the expected number of arm selections needed to identify the odd arm
- Proposed a scheme based on a modification of GLRT with forced exploration, and proved that it is asymptotically optimal
- Future work: to analyse the more difficult problem of odd arm identification in the setting of "restless" Markov arms. The analysis in the current paper serves as a key first step towards this

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