TA Session 16 24/11/2018

Agenda : Problems on Convergence of sequences of random variables

1. $\{X_n : n \geq 1\}$ is an independent sequence of random variables, with marginal pmfs given by

$$P\left(X_n = \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right)\right) = P\left(X_n = \tfrac{1}{2}\left(1 + \tfrac{1}{n}\right)\right) = \tfrac{1}{2}.$$

a) Show whether the sequence converges in mean-squared sense.

b) Show whether the sequence converges almost surely.

Ans : a) Intuitively, as $n \to \infty$, X_n's tend to take the value $\frac{1}{2}$. We now demonstrate that this is indeed the case.

We have

$$E\left[\left(X_n - \tfrac{1}{2}\right)^2\right] = \frac{1}{2} \cdot \frac{1}{4n^2} + \frac{1}{2} \cdot \frac{1}{4n^2} = \frac{1}{4n^2} \xrightarrow[n \to \infty]{} 0.$$

Thus, $X_n \xrightarrow[n \to \infty]{m.s.} \frac{1}{2}$.

b) By Chebyshev's inequality, for every $\varepsilon > 0$

$$P\left(\left|X_n - \tfrac{1}{2}\right| > \varepsilon\right) \leq \frac{E\left[\left(X_n - \frac{1}{2}\right)^2\right]}{\varepsilon^2} = \frac{1}{4n^2 \varepsilon^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} P\left(\left|X_n - \tfrac{1}{2}\right| > \varepsilon\right) \leq \frac{1}{4\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus, by the first part of Borel-Cantelli lemma, we conclude that $P\left(\left\{\left|X_n - \tfrac{1}{2}\right| > \varepsilon\right\} \text{ i.o.}\right) = 0 \ \forall\, \varepsilon > 0$. This in turn implies that $X_n \xrightarrow[n\to\infty]{a.s.} 1/2$.

Alternative solution for B:

Let $\Omega_1^{(n)} = \{\omega\in\Omega : X_n(\omega) = 1/2\left(1-\frac{1}{n}\right)\}$

$\Omega_2^{(n)} = \{\omega\in\Omega : X_n(\omega) = 1/2\left(1+\frac{1}{n}\right)\}$

Clearly, $\Omega_1^{(n)} \cap \Omega_2^{(n)} = \phi$, and $P(\Omega_1^{(n)} \uplus \Omega_2^{(n)}) = 1$.

Further, let

$$A_n = \Omega_1^{(n)} \cup \Omega_2^{(n)}.$$

Then, $P(A_n) = 1 \quad \forall\ n\geq 1$

$$\Rightarrow \quad P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1.$$

Let $A = \bigcap_{n=1}^{\infty} A_n$. Then,

$\omega\in A \Rightarrow \omega\in A_n \quad \forall\ n\geq 1$

$\Rightarrow \omega\in \Omega_1^{(n)} \uplus \Omega_2^{(n)} \quad \forall\ n\geq 1$

$\Rightarrow \omega\in\Omega_1^{(n)}$ $\boxed{\text{OR}}$ $\omega\in\Omega_2^{(n)}$, $\forall\ n\geq 1$

$\Rightarrow X_n(\omega) \in \frac{1}{2}\left(1-\frac{1}{n}\right)$ $\boxed{\text{OR}}$ $X_n(\omega) = 1/2\left(1+\frac{1}{n}\right)$, $\forall\ n\geq 1$

$\Rightarrow \lim_{n\to\infty} X_n(\omega) = 1/2$.

Thus, we have shown that

$$A \subseteq \{\omega\in\Omega : \lim_{n\to\infty} X_n(\omega) = 1/2\}.$$

$\Rightarrow$

$P\left(\lim_{n\to\infty} X_n = 1/2\right) = 1$ (since $P(A)=1$).

$\Rightarrow X_n \to 1/2$ almost surely.

2. $\{X_n : n \geq 2\}$ is an independent sequence of random variables, with marginal pmfs given by

$$P\left(X_n = \tfrac{1}{2}(1 - 1/n)\right) = P\left(X_n = \tfrac{1}{2}(1 + 1/n)\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right),$$

$$P(X_n = 1) = 1/n.$$

a) Sketch the CDF of X_n, and show whether the sequence converges in distribution.

Ans:

$$F_{X_n}(x) = \begin{cases} 0, & \text{if } x < \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right), \\ \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right), & \text{if } \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right) \leq x < \tfrac{1}{2}(1 + 1/n) \\ 1 - \tfrac{1}{n}, & \text{if } \tfrac{1}{2}(1 + 1/n) \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

We see that $X_n \xrightarrow{d} 1/2$.

b) Show whether the sequence converges almost surely.

Ans: We see that

$$\sum_{n=1}^{\infty} P(X_n = 1) = \sum_{n=1}^{\infty} 1/n = \infty,$$

and since X_n's are independent, by second part of Borel-Cantelli lemma, we have $P(\{X_n = 1\} \text{ i.o.}) = 1$. This clearly implies

$$X_n \overset{a.s.}{\not\to} 1/2$$

3. Let $U \sim \text{unif}[0,1]$, and $\forall n \geq 1$, let $X_n = (-1)^n \frac{U}{n}$.

a) Show whether X_n converges almost surely.

Ans: We note that

$|X_n| = \frac{U}{n} \leq \frac{1}{n}$ $\forall n \geq 1$, and hence we get

$$\lim_{n\to\infty} |X_n| = 0 \quad \text{a.s.}$$

because $f(x) = |x|$ is a continuous function of x $\Rightarrow f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$.

$$\Rightarrow \left|\lim_{n\to\infty} X_n\right| = 0 \quad \text{a.s} \quad \Rightarrow \lim_{n\to\infty} X_n = 0 \quad \text{a.s.}$$

b) Show whether the sequence converges in mean squared sense.

Ans: Note that

$$|X_n| = \frac{U}{n} \leq \frac{1}{n} \leq 1 \quad \text{a.s. for all } n \geq 1$$

$$\Rightarrow P(|X_n| \leq 1) = 1.$$

Also,

$$P(|X_n - 0| > \varepsilon) = P(U > n\varepsilon)$$

$$= 0 \quad \forall n \text{ s.t. } \frac{1}{n} < \varepsilon.$$

$$\Rightarrow P(|X_n - 0| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0.$$

Thus, $X_n \xrightarrow{p} 0$, $P(|X_n| \leq 1)$ $\forall n \geq 1$ $\Rightarrow X_n \xrightarrow{m.s.} 0$.

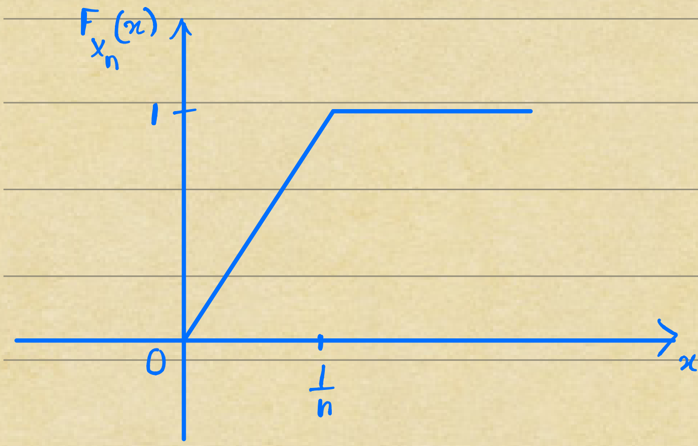
Alternatively,

$$E[X_n^2] = E\left[\frac{U^2}{n^2}\right] = \frac{1/12 + 1/4}{n^2} = \frac{1}{3n^2} \xrightarrow[n\to\infty]{} 0.$$

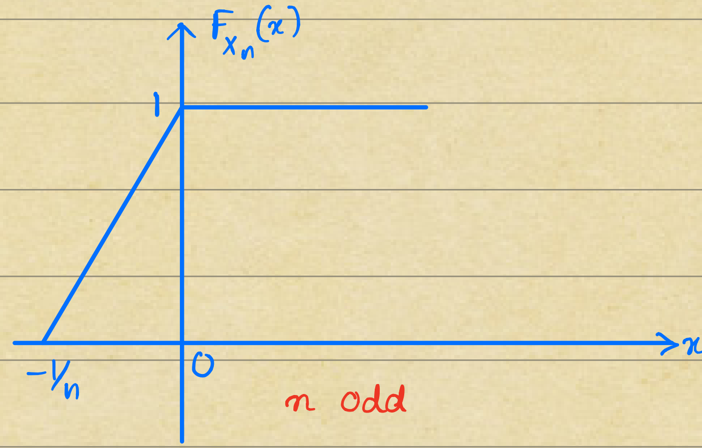
Thus, $X_n \xrightarrow[m.s.]{} 0$.

c) Demonstrate formally that $X_n \xrightarrow{d} 0$.

Ans: The CDF of X_n is graphed below:



n even



n odd

We see that $F_{X_n}(x) \longrightarrow 0 \quad \forall\ x<0,$

$F_{X_n}(x) \longrightarrow 1 \quad \forall\ x \geq 0.$

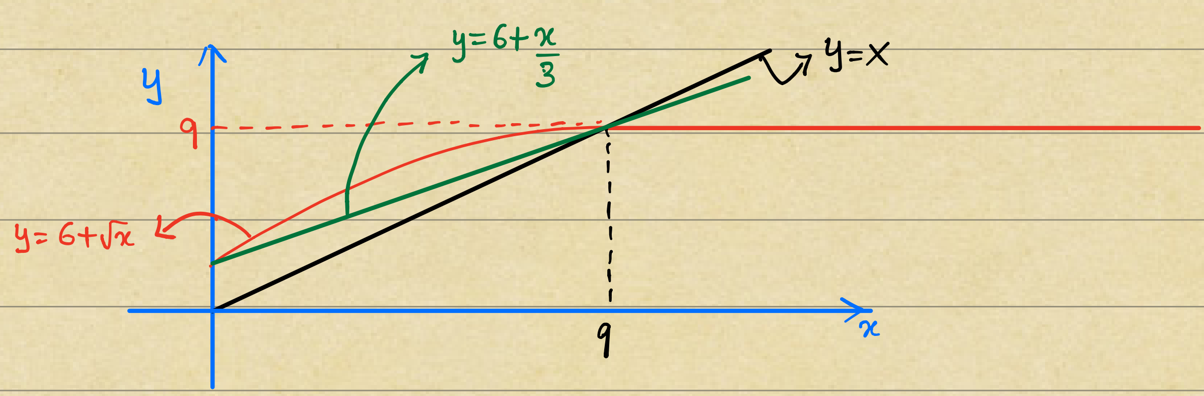
Thus, $X_n \xrightarrow[d]{} 0$, since 0 is the only discontinuity point of the CDF of $X \equiv 0$.

4. Let X_0 be a non-negative random variable s.t. $P(X_0 \geq 0) = 1$.

$$X_n = 6 + \sqrt{X_{n-1}}, \quad n \geq 1.$$

Show that $X_n \xrightarrow[a.s.]{} 9$.

Ans:



We first notice a couple of points:

a) If $X_0 = 9$, then $X_n = 9 \ \forall\ n \geq 1$.

b) If $X_0 < 9$, then $X_n \leq X_{n+1} \ \forall\ n \geq 1$, and X_n seems to increase to 9

c) If $X_0 > 9$, then $X_n \geq X_{n+1}$ $\forall n \geq 1$, and X_n seems to decrease towards 9.

Formally, we note that

$$6+\sqrt{x}-9 \geq 6+\frac{x}{3}-9 \quad \forall \quad x \leq 9,$$

(for small x, $\sqrt{x}$ will be larger than an appropriately scaled version of x)

$$6+\sqrt{x}-9 \leq 6+\frac{x}{3}-9 \quad \forall \quad x \geq 9,$$

($\sqrt{x}$ grows slower than x for large x. That is the idea)

and thus, for all $x \geq 0$,

$$\left|6+\sqrt{x}-9\right| \leq \left|6+\frac{x}{3}-9\right|.$$

Hence,

$$\left|X_n - 9\right| \leq \left|6+\frac{X_{n-1}}{3}-9\right|$$

$$= \frac{1}{3}\left|X_{n-1}-9\right|$$

By induction, we get

$$\left|X_n - 9\right| \leq \frac{1}{3^n}\left|X_0 - 9\right|.$$

Hence,

$$\lim_{n\to\infty}\left|X_n(\omega)-9\right| = 0 \quad \forall \ \omega \text{ s.t. } X_0(\omega) \geq 0.$$

Hence, $X_n \xrightarrow[a.s.]{} 9$.

5. Let $\{U_n : n \geq 1\} \overset{iid}{\sim} \text{unif}[0,1]$. Define

$$X_n = \min\{U_1, \dots, U_n\}.$$

a) Show whether X_n converges almost surely.

<u>Ans</u>: Intuitively, if the one of the U_n's is 0, then $X_n = 0$ for all n after some stage. Also, we have $X_n \geq X_{n+1}$ $\forall n \geq 1$. Thus, as a first guess, we can take 0 to be a possible limit for convergence.

Let us first check for convergence in prob. to 0. We have

$$P(X_n > \varepsilon) = P(\min\{U_1, \dots, U_n\} > \varepsilon)$$

$$= \begin{cases} (1-\varepsilon)^n, & \text{if } 0 < \varepsilon < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\xrightarrow[n\to\infty]{} 0 \quad \forall\, \varepsilon > 0.$$

Thus, indeed, $X_n \xrightarrow[P]{} 0$. Furthermore,

$$\sum_{n=1}^{\infty} P(X_n > \varepsilon) < \infty \quad \forall\, \varepsilon > 0.$$

Thus, by the first part of Borel-Cantelli lemma, we get

$$P(\{X_n > \varepsilon\} \text{ i.o.}) = 0 \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \quad X_n \xrightarrow[a.s.]{} 0.$$

b) Let $Y_n = n \times \min\{U_1, \dots, U_n\}$, $n \geq 1$.

Show that Y_n converges in distribution. What is the limit distribution?

<u>Ans</u>: Clearly, Y_n's are non-negative random variables. Furthermore, the support of Y_n is $[0, n]$, which grows to $[0, \infty)$ as $n \to \infty$. Thus, the limit distribution must be a distribution on $[0, \infty)$.

Fix any arbitrary $y \geq 0$. Then, there exists n_0 large enough s.t. $\frac{y}{n_0} \le 1 \quad \forall\ n \ge n_0$ (this n_0 may depend on the choice of y).

$$\Rightarrow \quad P(Y_n > y) = \left(1 - \frac{y}{n}\right)^n \quad \forall\ n \ge n_0$$

$$\Rightarrow \quad \lim_{n \to \infty} P(Y_n > y) = e^{-y}, \quad \forall\ y \ge 0.$$

Thus, $Y_n \to Y$, where $Y \sim Exp(1)$.

6. Let $W_1, W_2, \ldots$ be a sequence of iid $N(0, \sigma^2)$ random variables. Let $X_0 = 0$, and define

$$X_{n+1} = \frac{X_n + W_{n+1}}{2}, \quad n \ge 0.$$

Argue that X_n converges in distribution. What is the limit distribution?

Ans: First, we note that for each $n \ge 1$, we have

$$X_n = \frac{W_1}{4^n} + \frac{W_2}{4^{n-1}} + \cdots + \frac{W_n}{4}.$$

Thus, it follows that $X_n \sim N\left(0, \sigma^2\left(\frac{1}{4^n} + \cdots + \frac{1}{4}\right)\right)$.

Therefore,

$$P(X_n \le x) = P\left(\frac{X_n}{\sqrt{Var X_n}} \le \frac{x}{\sqrt{Var X_n}}\right)$$

$$= \phi\left(\frac{x}{\sqrt{Var X_n}}\right).$$

Thus, $\lim_{n \to \infty} P(X_n \le x) = \lim_{n \to \infty} \phi\left(\frac{x}{\sqrt{Var X_n}}\right)$

$$= \phi\left(\lim_{n\to\infty} \frac{x}{\sqrt{\operatorname{Var} X_n}}\right) \quad (\because \phi \text{ is a continuous function})$$

Now,

$$\lim_{n\to\infty} \operatorname{Var}(X_n) = \lim_{n\to\infty} \sigma^2\left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n}\right)$$

$$= \sigma^2 \cdot \sum_{n=1}^{\infty} \frac{1}{4^n}$$

$$= \sigma^2/3.$$

Thus, $\displaystyle\lim_{n\to\infty} P(X_n \leq x) = \phi\left(\frac{x}{\sigma^2/3}\right).$

Thus, $X_n \xrightarrow{d} X$, where $X \sim N\left(0, \frac{\sigma^2}{3}\right).$