Constructing Pairing-Friendly Elliptic Curves

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Abstract

The goal of this report is to....

Contents

1 Introduction

1.1 Elliptic Curves

For our project, we shall define an elliptic curve to be a curve of the form:

$$E: \; y^2 = x^3 + Ax + B \tag{1}$$

where A and B are elements of some field $\mathbb{F}$, with $\text{char}(\mathbb{F}) \neq 2, 3$. The curve E is nonsingular if $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are not simultaneously 0 for all points on E. It follows that E is nonsingular $\iff$ $x^3 + Ax + B$ has distinct roots. Through Vieta's formulas, E has distinct roots $\iff$ $((r_1 - r_2)(r_1 - r_3)(r_2 - r_3))^2 = -(4A^3 + 27B^2)$ is nonzero. Therefore, we shall also require that the discriminant of E,

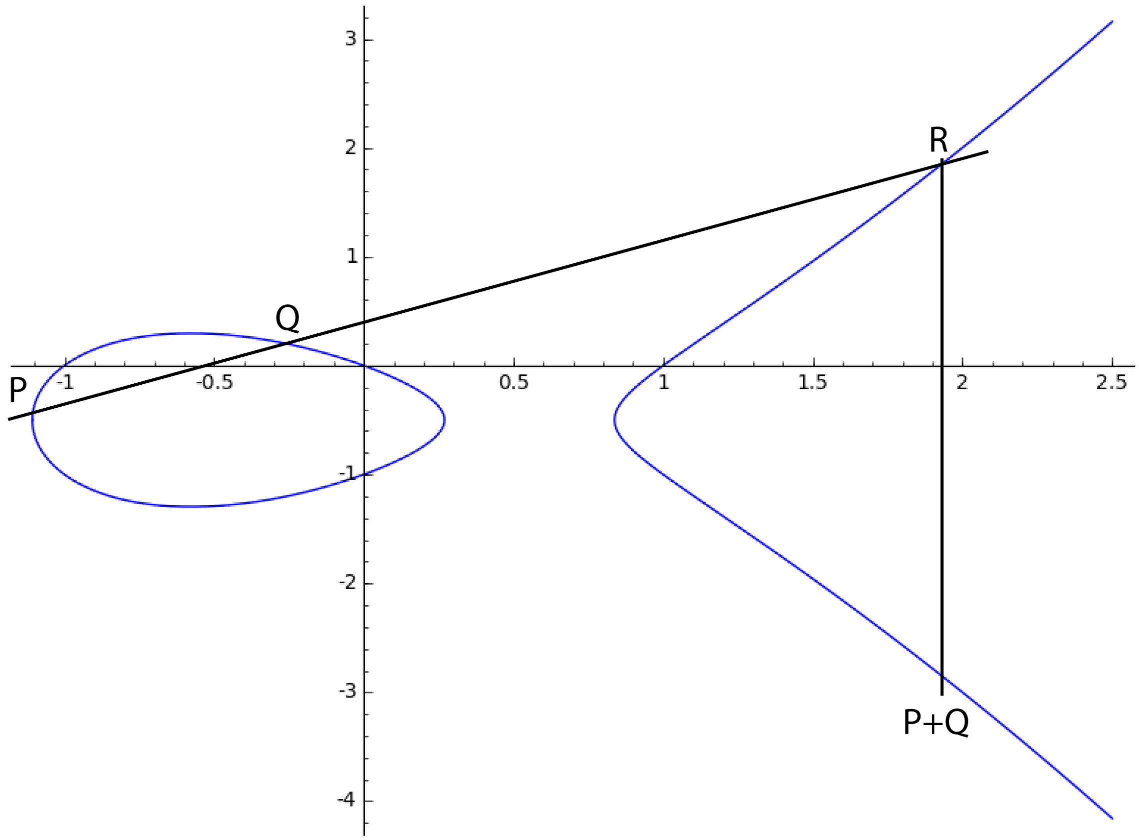
$$\Delta = -16(4A^3 + 27B^2) \tag{2}$$

is nonzero. The j-invariant of E is defined by:

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2} \tag{3}$$

1.2 Group Law

The points on an elliptic curve form an additive abelian group. We shall define the group law geometrically.
Let $P = (x_p, y_p), Q = (x_q, y_q)$. A line through P and Q intersects E at a third point, $R = (x_r, y_r)$. We define $P + Q := (x_r, -y_r)$. Pictorally, this looks like



The group law can also be defined in terms of algebraic formulas, which can be found in [SIL08].

1.3 Notation

- $\mathbb{F}$ is a field
- $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$
- $\mathbb{F}_p$ is a field with p elements, where p is prime
- $E(\mathbb{F}) = \{\mathbb{F} \times \mathbb{F} \mid E(x, y) = 0\}$
- ϕ is an isogeny (or endomorphism)
- ϕ_p is the Frobenius endomorphism
- $[n]$ is the multiplication by n map

2 Background

In this section, we shall define key concepts needed for our report. We will prove some of the more important results, and cite a source otherwise.

2.1 Isogenies

An isogeny of two elliptic curves E_1 and E_2 defined over a field $\mathbb{F}$ is a nonconstant morphism $\phi : E_1 \to E_2$, where ϕ is a group homomorphism from $E_1(\overline{\mathbb{F}}) \to E_2(\overline{\mathbb{F}})$. E_1 and E_2 are isomorphic if $\exists \phi_1 : E_1 \to E_2$ and $\phi_2 : E_2 \to E_1$, isogenies, such that $\phi_2 \circ \phi_1 = \text{Identity}$.

2.1.1 Separable and Inseparable Isogenies

Any isogeny ϕ can be expressed as $\phi(x, y) = (\frac{u(x)}{v(x)}, \frac{s(x)}{t(x)}y)$, where $u, v, s, t \in \mathbb{F}[x]$, and $\gcd(u, v) = \gcd(s, t) = 1$. An isogeny is separable if $(\frac{u}{v})' = 0$, and is inseparable otherwise. The degree of an isogeny is defined as $\deg(\phi) := \max(\deg(u), \deg(v))$. For any separable isogeny ϕ, $\deg(\phi) = |\ker \phi|$. [SUTH15]

2.1.2 Dual isogenies

Theorem 1 *Let $\phi : E_1 \to E_2$ be an isogeny. Then $\exists$ a unique $\hat{\phi} : E_2 \to E_1$ such that $\hat{\phi} \circ \phi = [n]$, where $n = \deg(\phi)$.*

The proof of this can be found in either [SIL08] or [SUTH15]. Furthermore, for any two isogenies ϕ_1 and ϕ_2, $\widehat{\phi_1 + \phi_2} = \hat{\phi}_1 + \hat{\phi}_2$.

2.2 Endomorphisms

An endomorphism is an isogeny from E to itself. The endomorphisms of E form a ring, where addition is addition of functions and multiplication is function composition.

2.2.1 Examples

The map $[n] : E \to E$, where $[n]P = P + P + \cdots + P$ (n times) is an endomorphism.

If E is defined over $\mathbb{F}_p$, the Frobenius map $\phi_p : E \to E$ defined by $\phi_p(x, y) := (x^p, y^p)$ is an (inseparable) endomorphism.

2.2.2 Trace of an Endomorphism

Theorem 2 *For any endomorphism ϕ, $\phi + \hat{\phi} = 1 + \deg(\phi) - \deg(1 - \phi)$, where we can regard the RHS as an endomorphism by the map $n \mapsto [n]$.*

Proof: As endomorphisms,

$$\begin{aligned}
&[\deg(1-\phi)] = (\widehat{1-\phi})(1-\phi) = (\hat{1} - \hat{\phi})(1-\phi) = (1-\hat{\phi})(1-\phi) \\
&= 1 - \hat{\phi} - \phi + \hat{\phi} \circ \phi = 1 - \hat{\phi} - \phi + [\deg(\phi)] \\
&\implies \phi + \hat{\phi} = 1 + [\deg(\phi)] - [\deg(1-\phi)]
\end{aligned}$$

$\square$

By the above theorem, we can now define $\text{trace}(\phi) := \phi + \hat{\phi}$.

Theorem 3 *$\#E(\mathbb{F}_p) = p + 1 - t$, where $t = \text{trace}(\phi_p)$*

Proof: The fixed field of ϕ_p is $\mathbb{F}_p$, and $1 - \phi_p$ is separable (see [SUTH15]). Therefore, $\ker(1 - \phi_p) = \#E(\mathbb{F}_p)$. It is clear that $\deg(\phi_p) = p$ by definition ($u(x) = x^p$ and $v(x) = 1$). We have that

$$\begin{aligned}
&\ker(1 - \phi_p) = \deg(1 - \phi_p) = 1 + \deg(\phi_p) - \text{trace}(\phi_p) = p + 1 - t \\
&\implies \#E(\mathbb{F}_p) = p + 1 - t
\end{aligned}$$

$\square$

2.3 j-invariant

Theorem 4 *Two elliptic curves $E_1(\mathbb{F})$ and $E_2(\mathbb{F})$ are isomorphic over $\overline{\mathbb{F}}$ $\iff$ $j(E_1) = j(E_2)$. Furthermore, $\forall j_0 \in \overline{\mathbb{F}}$, $\exists$ an elliptic curve $E(\mathbb{F})$ such that $j(E) = j_0$.*

The proof requires some lengthy algebraic manipulation, which can be found in [SIL08]. As a consequence of the proof, for any $j \in \overline{\mathbb{F}}$, we can define an canonical elliptic curve E associated with this j-invariant. We see that

$$\begin{aligned}
&E : y^2 = x^3 + \frac{3j}{1728 - j}x + \frac{2j}{1728 - j} \text{ if } \ j \neq 0, 1728 \qquad (4) \\
&E : y^2 = x^3 + 1 \text{ if } \ j = 0 \\
&E : y^2 = x^3 + x \text{ if } \ j = 1728
\end{aligned}$$

2.4 Twists

Two curves $E_1(\mathbb{F})$ and $E_2(\mathbb{F})$ are *twists* if they are isomorphic over $\overline{\mathbb{F}}$ but not over $\mathbb{F}$.

2.4.1 Quadratic Twists

In particular, we are interested in quadratic twists. If $E : y^2 = x^3 + Ax + B$ is an elliptic curve defined over $\mathbb{F}$, and $d \in \mathbb{F}$ is a nonsquare, then we define the *quadratic twist of E* as $\tilde{E} : y^2 = x^3 + d^2Ax + d^3B$.

Theorem 5 *If $E : y^2 = x^3 + Ax + B$ is an elliptic curve over $\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = p + 1 - t$, then $\#\tilde{E}(\mathbb{F}_p) = p + 1 + t$*

Proof: Let $\left(\frac{\cdot}{p}\right)$ be the legendre symbol mod p. For any $x \in \mathbb{F}_p$, we see that $1 + \left(\frac{x^3+Ax+B}{p}\right) = \#$ of points on E with x-coordinate x. Therefore,

$$\#E(\mathbb{F}_p) = 1 + \sum_{x \in \mathbb{F}_p} (1 + \left(\frac{x^3 + Ax + B}{p}\right)) = p + 1 + \sum_{x \in \mathbb{F}_p} \left(\frac{x^3 + Ax + B}{p}\right)$$

Since $\mathbb{F}_p$ is a field, $\forall x \in \mathbb{F}_p$, $\exists x' \in \mathbb{F}_p$ such that $dx' = x$. Therefore, for $\tilde{E}(\mathbb{F}_p$, we have that

$$\#\tilde{E}(\mathbb{F}_p) = 1 + \sum_{x\in\mathbb{F}_p} (1 + \left(\frac{(dx)^3 + A(dx) + B}{p}\right)) = p + 1 + \sum_{x\in\mathbb{F}_p} \left(\frac{d^3(x^3 + Ax + B)}{p}\right)$$
$$= p + 1 + \sum_{x\in\mathbb{F}_p} \left(\frac{d}{p}\right)\left(\frac{d^2}{p}\right)\left(\frac{x^3 + Ax + B}{p}\right) = p + 1 - \sum_{x\in\mathbb{F}_p} \left(\frac{x^3 + Ax + B}{p}\right)$$

□

2.5 Pairings and Pairing-Friendly Curves

Let G be an additive abelian group of order p, a prime, and let G' be a multiplicative group of order p. A pairing is a bilinear map $e : G \times G \to G'$ satisfying:

1. (bilinearity) $e(P_1 + P_2, P_3) = e(P_1, P_3)e(P_2, P_3)$ $\forall P_1, P_2, P_3 \in G$
2. (non-degeneracy) $e(P, P) \neq 1$, where $G = <P>$
3. (computability) e is efficiently computable

[PAIR91]

2.5.1 Weil and Tate Pairings

Let $E[r] = \{P \in E(\mathbb{F}_p) \mid rP = O\}$ be the r-torsion group of $E(\mathbb{F}_p)$.

The Weil Pairing is a map $e_r : E[r] \times E[r] \to \mu_r$, where μ_r is the set of rth roots of unity in $\overline{\mathbb{F}_p}$.

The Tate Pairing is a map: $\tau_r : E(F_p)[r] \times E(F_p)[r]/rE(F_p) \to \mu_r$, where $\tau_r(P, Q) = e_r(P, R - \phi_p(R))$, where R satisfies $rR = Q$. [WASH08]

2.5.2 Embedding Degree

We do not need the full algebraic closure $\overline{\mathbb{F}_p}$ to determine μ_r. Instead, we can find a positive integer k such that $\mu_r \subset \mathbb{F}_{p^k}$, that is, we require only a finite degree algebraic extension. We define the embedding degree k of $E(\mathbb{F}_p)$ with respect to r as $k := [\mathbb{F}_p(\mu_r) : \mathbb{F}_p]$, the degree of the extension field. Therefore, we can regard k as the smallest positive integer such that $\mu_r \subset \mathbb{F}_{p^k}$.

Alternatively, suppose that k is the smallest positive integer such that $\mu_r \subset \mathbb{F}_{p^k}$. $\mu_r \subset \mathbb{F}_{p^k} \iff r \Big| \#\mathbb{F}^*_{p^k}$ (since multiplicative groups of finite fields are cyclic) $\iff$ $\gcd(r, p^k - 1) = r \iff p^k - 1 \equiv 0 \mod r \iff p$ is a primitive kth root of unity mod r (since we picked k to be minimal).

Therefore, the embedding degree k is the smallest positive integer satisfying $p^k \equiv 1$ mod r.

2.5.3 Pairing-Friendly Curves

The Weil/Tate Pairing is efficiently computable when k is small, as computing the pairings requires computation in $\mathbb{F}_{p^k}$. A curve with efficiently computable pairings is *pairing-friendly*. Pairing-friendly curves are rare. In general, if $r \approx p$, then Pr[pairing-friendly curve] $= O(\frac{\log^3 M}{M})$. [IM98]

3 Constructing Pairing-Friendly Curves

3.1 Complex Multiplication Method

Input: p, a prime, and N a positive integer
Output: An elliptic curve $E(\mathbb{F}_p)$ where $\#E(\mathbb{F}_p) = N$
$t := p + 1 - N$;
$D =$ square free part of $t^2 - 4p$;
$H_D(x) =$ Hilbert class polynomial
Pick $j \in \mathbb{F}_p$ such that $H_D(j) = 0$ Compute E according to (4).
if $\#E(\mathbb{F}_p) = N$ **then**
  return E
else
  return quadratic twist of E
end

As a result from the theory of complex multiplication, the above algorithm will always succeed provided that such a curve E exists and will be efficient if D is not too large [SUTH15]. A slight subtlety arises if $j = 0$ or $j = 1728$. If $j = 0$, then there are 6 classes of curves, corresponding to sextic twists of E by $1, \zeta, \zeta^2, \ldots, \zeta^5$, for ζ, a primitive root of unity in $\mathbb{F}_p$. If $j = 1728$, then there are 4 classes of curves, corresponding to quartic twists of E by $1, \zeta, \ldots \zeta^3$, for ζ, a primitive root of unity in $\mathbb{F}_p$. If a curve E exists, then one of these curves will satisfy the conditions of the algorithm. TODO NEED SOURCE
Rationale for this algorithm:

3.2 Cocks Pinch Method

We wish to construct a curve with a subgroup of size r and embedding degree k. Suppose we also have chosen a CM discriminant D such that $\left(\frac{D}{r}\right) = 1$. The Cocks Pinch method finds a prime p and the trace of Frobenius t such that $\exists$ an elliptic curve E over $\mathbb{F}_p$ with trace t, and a subgroup of size r with embedding degree k. There are 3 conditions on p, t:

1. $t^2 - 4p = f^2 D$, for some f. This means that E has CM discriminant D

2. $p + 1 - t \equiv 0 \mod r$. This means that $r \Big| \#E(\mathbb{F}_p) \implies$ there is a subgroup of size r by the characterization of finitely generated abelian groups

3. p is a primitive kth root of unity mod r. As mentioned earlier, this condition is equivalent to E having embedding degree k with respect to the subgroup of size r.

We see that we only need to satisfy the above 3 conditions for p, t. To do this, we first choose g, a primitive kth root of unity mod r. We know that $t-1 \equiv p \equiv g \mod r$. Using (1), we see that $\frac{(t^2-f^2D)}{4} = p$. We can set $a = 2^{-1}(g+1) \mod r$ (as integers). Then, $2a \equiv g+1 \equiv t \mod r \implies a^2 \equiv \frac{t^2}{4} \mod r$. We can also set $f_0 = \frac{2(a-1)}{\sqrt{D}} \mod r$. Suppose $p = \frac{(t^2-f_0^2D)}{4}$ is prime, and $t = 2a$. Then $p+1-t \equiv a^2-(a-1)^2+1-t \equiv 2a-1+1-t \equiv 0 \mod r$, so that (2) is satisfied, and (2) $\implies$ (3) since $t-1 \equiv g$. (1) is satisfied by construction of p, and so we have the desired output. If p is not prime, then we can compute $p = \frac{(t^2-f^2D)}{4}$ for $f = f_0 + ir$, which gives new values for p while preserving the above congruences mod r. If the algorithm succeeds in finding a prime p then it outputs p, t, and will output $\perp$ if it fails. The algorithm is given below in pseudocode:

Input: k, embedding degree, r, size of subgroup, D, CM discriminant with $\left(\frac{D}{r}\right) = 1$
Output: p, a prime, and t a trace of Frobenius
$a := 2^{-1}(g+1) \mod r$ (as integers)
$f = \frac{2(a-1)}{d}$ (as integers where $d \equiv \sqrt{D} \mod r$)
$t := 2a$
$p = \frac{(t^2-f^2D)}{4}$
while *p is not prime* **do**
  $f = f + r$
  $p = \frac{(t^2-f^2D)}{4}$
  if *running for too long* **then**
    return $\perp$
  end
end
return p, t

If the algorithm succeeds, we can use the CM method to construct the desired elliptic curve E, provided that E exists. We see that $\frac{t^2}{4} \leq \frac{t^2-f^2D}{4} = p^2 \implies t \leq 2\sqrt{p}$ since $D < 0$. As part of the proof of the CM method [SUTH15], a curve E with trace t exists if (1) is satisfied, $t \leq 2\sqrt{p}$, and $t \not\equiv 0 \mod p$. The last equation is satisfied since $t > 0$ and $p > t$. Therefore, E exists.

3.3 Dupont Enge Morain Method

4 Applications

4.1 Elliptic Curve Discrete Logarithm Problem

The Elliptic Curve Discrete Logarithm Problem (ECDLP) is formalized as follows: Given two points P, Q, find an integer k such that $kP = Q$.

Currently, the fastest known method for solving ECDLP is Pollard's ρ method, which runs in $O(\sqrt{p})$ time.

4.1.1 Pohlig-Hellman Method

Suppose $N = \#E(\mathbb{F}_p)$, and write $N = \prod_i q_i^{e_i}$ as a product of primes. To determine k, all we need to do is find $k \mod q_i^{e_i}$ and then construct k using the Chinese Remainder theorem. This is the main idea behind the Pohlig-Hellman Method, which is efficient provided that the prime factors of N are small. [WASH08]

4.1.2 MOV attack

The Menezes-Okamoto-Vanstone (MOV) attack relies on using pairings to solve the ECDLP. The idea behind the attack is to map the DLP on E to the DLP in $\mathbb{F}_{p^k}$, where k is the embedding degree, and then use the index calculus method to solve the DLP in subexponential time. This attack is efficient provided that k is small. However, elliptic curves generally have large embedding degree with respect to any large subgroup, so this attack is only useful against pairing-friendly elliptic curves.

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