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1 Glide Reflections

Now we will introduce the concept of a glide reflection, and prove which combinations of isometries are equivalent to a single glide reflection.

Definition 1. Imagine any shape in the plane as having a front and a back face; if you start looking front of the shape, then changing the orientation will cause you to see the back of the shape. This happens as a result of a reflection.

As we saw in a previous section, create any isometry of the plane where the orientation is reversed can be made by three reflections. In the case where all three lines are parallel, they can be reduced to a single reflection, and if there is only one intersection point, they are one reflection line, but in the case where there are at least two points of intersection among the lines of reflection, they represent an operation that cannot be reduced into a single one of any of the isometries that we have introduced; therefore we will reduce them to a representation where each isometry has a unique identity and refer to this class of isometries as glide reflections.

Definition 2. A glide reflection is a form of isometry represented by a composition of a reflection and a translation parallel to the line of reflection; it is a fundamental isometry which is distinct from the other three.

The fundamental nature of a glide reflection will be easy to see, but first we will look at how a glide reflection can be formed from three reflections, as this will demonstrate the reasons for our choice of representation.

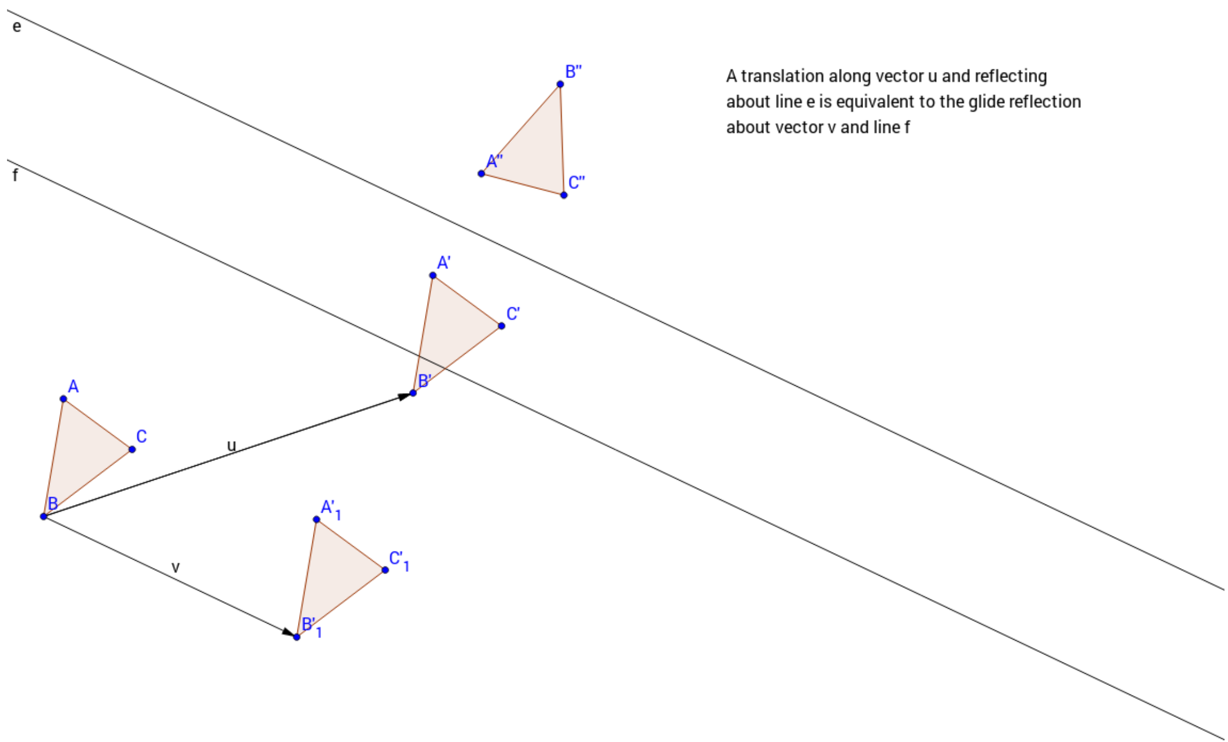
Theorem 1.1. *A glide reflection is formed by the composition of three reflections that have at least two points of intersection among them.*

First we will consider the case where two of the reflection lines are parallel, but first we need a supporting theorem which will be required to completely reduce the reflections:

Theorem 1.2. *A glide reflection can be formed from one reflection and one translation, as long as the angle between the reflection line and translation vector is not 90 degrees.*

Proof. A reflection and a translation perpendicular to each other can be combined by moving the reflection by half of the translation's magnitude in the opposite direction.

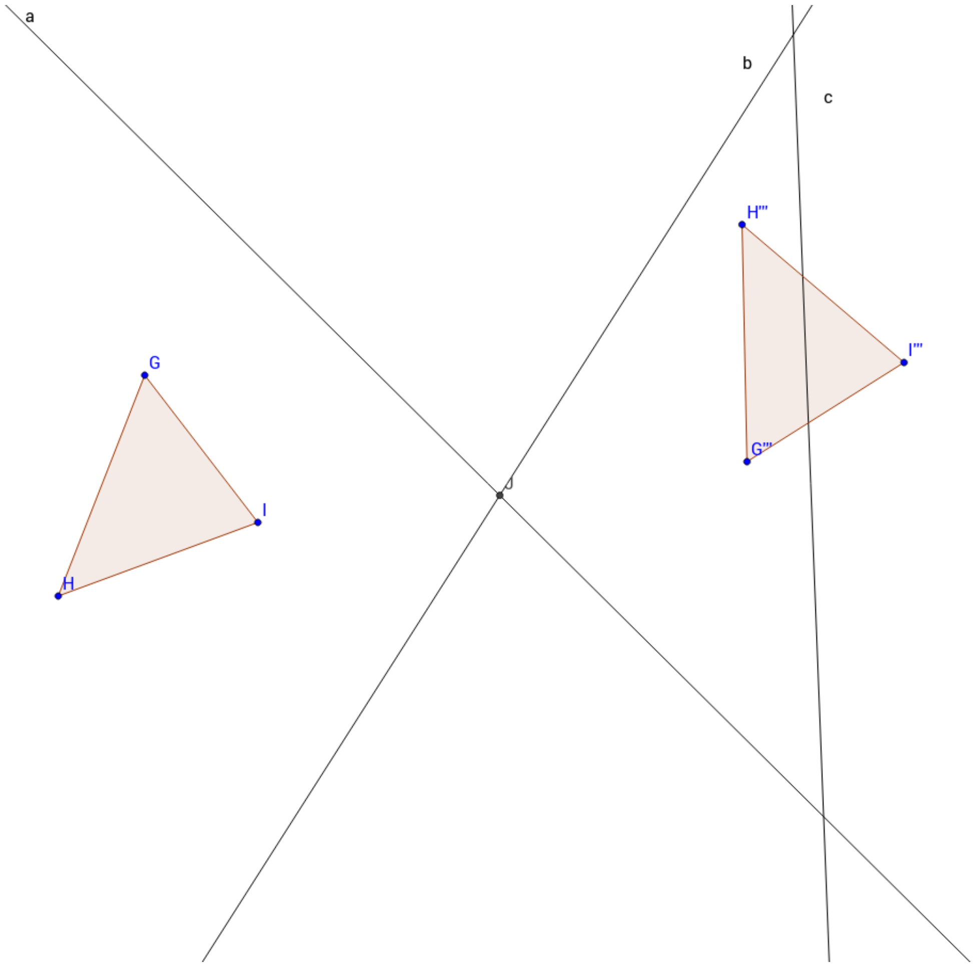
Therefore, a reflection line and a translation vector with an arbitrary angle between them can be reduced to a glide reflection by taking the components of the translation vector both parallel and perpendicular to the reflection line, which compose to form the first translation; the perpendicular component can be absorbed by moving the reflection line, and since the parallel component must be non-zero due to the restriction on the angle of the translation vector, the remaining two isometries make up our definition of a glide reflection. □



Now we can prove Theorem 5.1 for the following case:

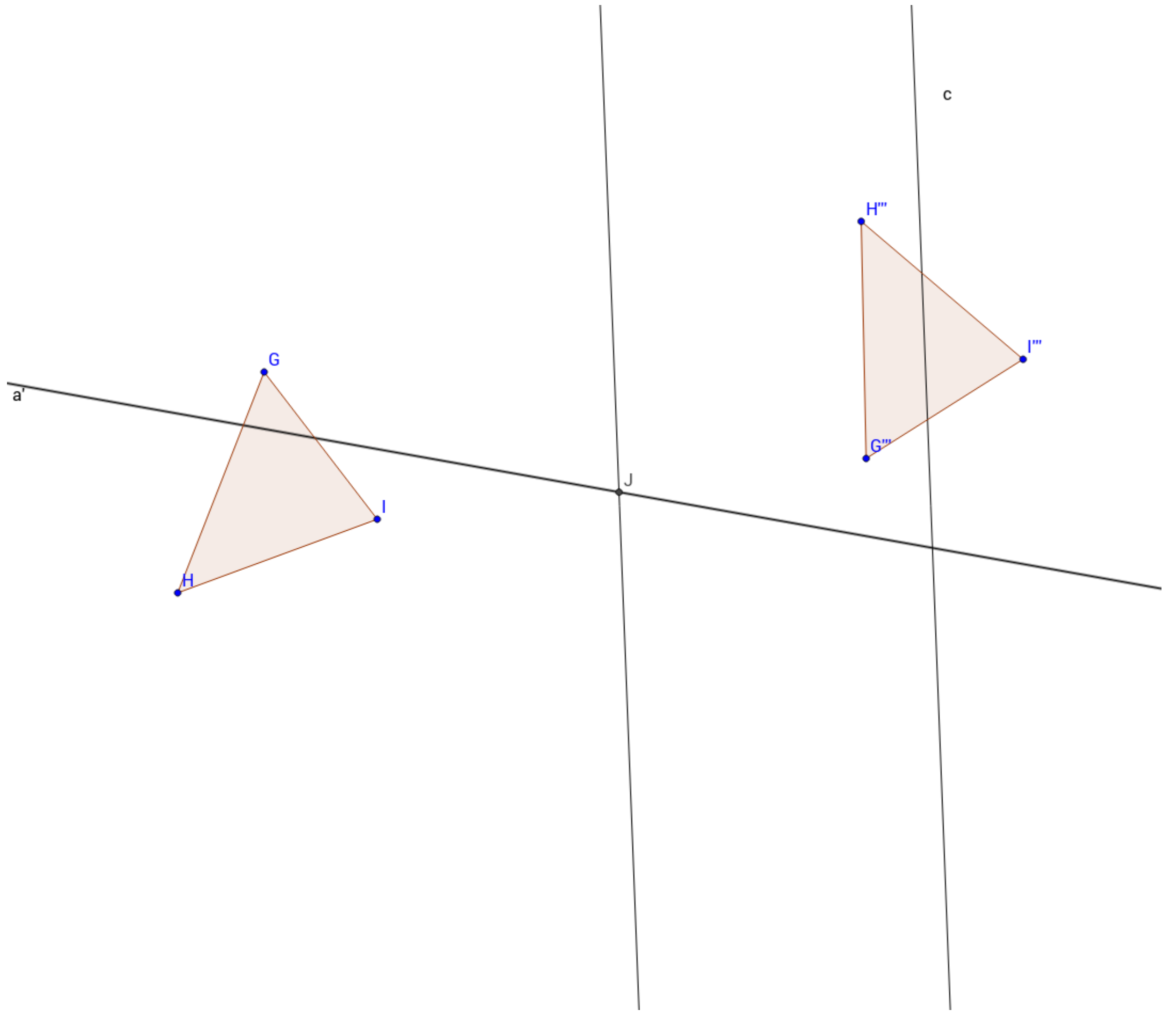
Proof. For a set of reflection lines L_1, L_2, and L_3, where the first two are parallel and the third intersects both, the composition is a glide reflection since the parallel reflection lines, L_1 and L_2, are a single translation. The resulting translation vector will never be at a 90 degree angle with the remaining reflection line, L_3, because that reflection line started off at some angle with respect to L_1 and L_2, and as shown above, this forms a glide reflection. □

In order to show that any three reflections can be made into a glide reflection even if none are parallel, we leverage the fact that two intersecting reflections form a rotation.



Proof. For a set of reflections across lines L_1, L_2, and L_3, where there are exactly three intersection points, the isometries can be reduced to a glide reflection by rotating two of the reflection lines such that one of them is parallel to the third. This can be done because two intersecting reflection lines compose to form a rotation around the intersection point: first transform the two reflections into the rotation, then choose a new reflection line through the center of rotation that is parallel to the third reflection line. It is guaranteed that a reflection line exists such that when composed with the reflection line that you chose it will reproduce the rotation around that center point. Now that the lines that formed the rotation have been transformed, you have reached the case where two reflection lines are parallel and the third intersects both, which we have already shown to be a glide reflection.

$\square$



1.1 Exercises

Now that we've introduced the concept of glide reflections, there are a few things which can increase your understanding of them and other isometries:

1. Add a column for "glide reflections" to your composition table. Are there compositions you couldn't do before that are simpler with glide reflections? What sorts of compositions can you have with glide reflections?

2. prove that the reflection and translation used to make a glide reflection commute with each other, so there is no need for ordering them.

3. Why do we require the translation to be parallel to the reflection line? What properties do we gain by doing so? What would it mean for glide reflections if we didn't require this?

4. Think about how to find the isometries you get when there are less than two points of intersection.