ME 639: INTRO TO ROBOTICS

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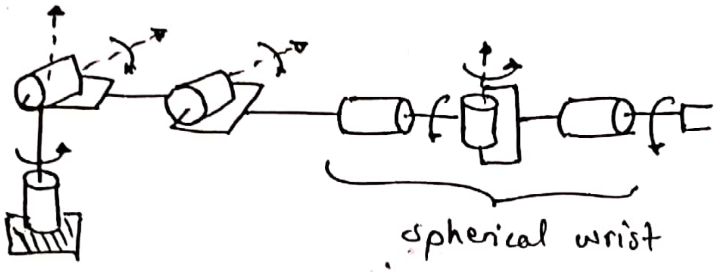
Problem 3-7: Three-link articulated manipulator + spherical wrist



$T_3^6 \longrightarrow$ for spherical wrist

$T_0^3 \longrightarrow$ for 3-link articulated manipulator

$$\Rightarrow T_0^6 = T_0^3 \cdot T_3^6$$

From eq (3.15) of the text,

we have
$$T_3^6 = A_4 A_5 A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

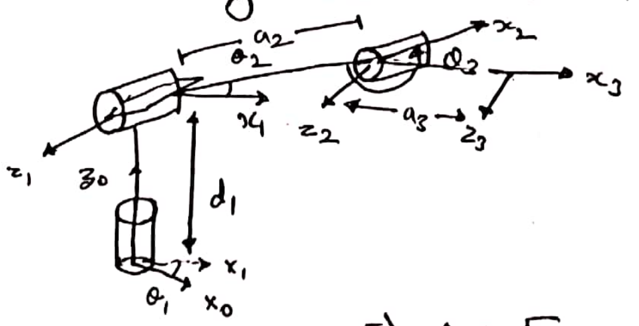
Or equivalently, the D-H parameters are:

Link	a_i	α_i	d_i	θ_i
4	0	-90°	0	θ_4
5	0	+90	0	θ_5
6	0	0	d_6	θ_6

Or
$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we only need to find A_1, A_2 & A_3 for 3-link articulated manipulator



link	a_i	α_i	d_i	θ_i
1	0	90	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$\Rightarrow A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_2 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_0^6 = A_1 A_2 A_3 A_4 A_5 A_6 = T_0^3 \, T_3^6$$

where
$$T_0^3 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & a_2 c_1 c_2 + a_3 c_1 c_{23} \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & a_2 c_2 s_1 + a_3 s_1 c_{23} \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_0^6 = T_0^3 \, T_3^6$$

$$= \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & a_2 c_1 c_2 + a_3 c_1 c_{23} \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & a_2 c_2 s_1 + a_3 s_1 c_{23} \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= T_0^6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $r_{11} = c_1 c_{23} [c_4 c_5 c_6 - s_4 s_6] - c_1 s_{23} [s_4 c_5 c_6 + c_4 s_6] + s_1 (-s_5 c_6)$

$= c_1 s_6 [-c_{23} s_4 - s_{23} c_4] + c_1 c_5 c_6 [c_{23} c_4 - s_{23} s_4] - s_1 s_5 c_6$

$$\boxed{r_{11} = c_1 [c_5 c_6 c_{234} - s_6 s_{234}] - s_1 s_5 c_6}$$

$r_{12} = c_1 c_{23} [-c_4 c_5 s_6 - s_4 c_6] - c_1 s_{23} [-s_4 c_5 s_6 + c_4 c_6] + s_1 s_5 s_6$

$$\boxed{r_{12} = -c_1 [c_5 s_6 c_{234} + c_6 s_{234}] + s_1 s_5 s_6}$$

$r_{13} = c_1 c_{23} c_4 s_5 - c_1 s_{23} s_4 s_5 + s_1 c_5$

$$\boxed{r_{13} = c_1 s_5 c_{234} + s_1 c_5}$$

$d_x = a_2 c_1 c_2 + a_3 c_1 c_{23} + d_6 [c_1 s_5 c_{234} + s_1 c_5]$

$r_{21} = s_1 c_{23} (c_4 c_5 c_6 - s_4 s_6) - s_1 s_{23} (s_4 c_5 c_6 + c_4 s_6) + c_1 s_5 c_6$

$$\boxed{r_{21} = s_1 (c_5 c_6 c_{234}) - s_1 s_6 (s_{234}) + c_1 s_5 s_6}$$

Scanned with CamScanner

$$\boxed{r_{22} = -c_1 s_5 s_6 - s_1 c_5 c_6 c_{234} - s_1 c_6 s_{234}}$$

$$\boxed{r_{23} = s_1 s_5 c_{234} - c_1 c_5}$$

$$\boxed{d_y = -d_6 [c_1 c_5 - s_1 s_5 c_{234}] + a_2 c_2 s_1 + a_3 s_1 c_{23}}$$

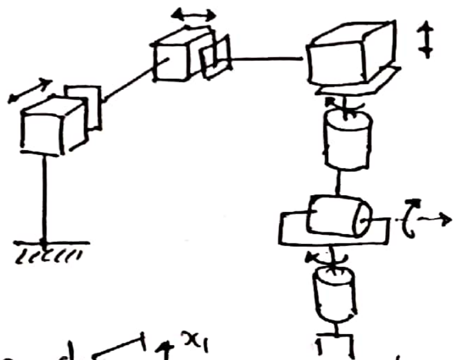
$$\boxed{r_{31} = c_5 c_6 s_{234} + s_6 c_{234}} \quad \cancel{- s_5 c_6 s_{234}}$$

$$\boxed{r_{32} = -c_5 s_6 s_{234} + c_6 c_{234}}$$

$$\boxed{r_{33} = s_5 s_{234}}$$

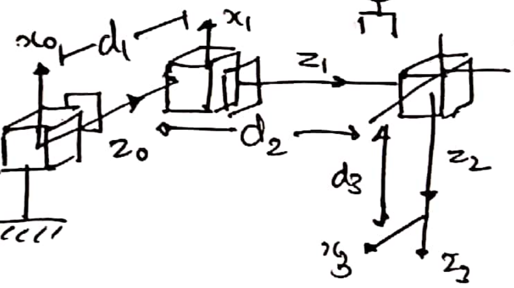
and $$\boxed{d_z = a_2 s_2 + a_3 s_{23} + s_5 s_{234} d_6}$$

Problem 3-8: 3-link Cartesian Manipulator + spherical wrist.



spherical wrist $\rightarrow T_3^6 \rightarrow$ already defined

Again $T_0^6 = T_0^3 \, T_3^6$



$\rightarrow$ 3-link cartesian

link	a_i	α_i	d_i	θ_i
1	0	$-90°$	d_1	0
2	0	$90°$	d_2	$90°$
3	0	0	d_3	$-90°$

$$\Rightarrow A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1 A_2 A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & d_3 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & d_3 \\ -1 & 0 & 0 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_0^6 = T_0^3 \cdot T_3^6$$

$$= \begin{bmatrix} 0 & 0 & 1 & d_3 \\ -1 & 0 & 0 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4S_5S_6 - S_4S_6 & C_4S_5 & C_4S_5d_6 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & S_4S_5d_6 \\ -S_5C_6 & S_5S_6 & C_5 & C_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_5C_6 & S_5S_6 & C_5 & C_5d_6 + d_3 \\ -C_4C_5C_6 + S_4S_6 & C_4S_5S_6 + S_4C_6 & -C_4S_5 & d_2 - C_4S_5d_6 \\ -S_4C_5C_6 - C_4S_6 & S_4C_5S_6 - C_4C_6 & -S_4S_5 & d_1 - S_4S_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\hookrightarrow T_0^6$

$\hookrightarrow$ for 3-link Cartesian + spherical wrist

10) we are given $D(q)$ & $V(q)$

$D(q)$: $n \times n$ inertia matrix; $V(q) \hookrightarrow$ potential

$$\Rightarrow \quad K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

$$\mathcal{L} = K - P = K - V = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j - V(q)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt}d_{kj}\,\dot{q}_j$$

$$= \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j + \sum_j d_{kj}\ddot{q}_j$$

and
$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k}$$

$\Rightarrow$ For each, k, the eq^n (Euler-Lagrange) can be written as:

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial V}{\partial q_k} = \tau_k$$

Now, using symmetry,

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right]\dot{q}_i\dot{q}_j\,.$$

$$\Rightarrow \quad c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\} \longrightarrow \underline{\text{Christoffel symbols}}$$

$\Rightarrow$ <u>Euler-Lagrange</u> <u>eqn</u>:

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + \frac{\partial V}{\partial q_k} = \tau_k$$

$\frac{\partial V}{\partial q_k} \hookrightarrow g_k(q)$ (defined)

8) Derivation of 2R manipulator's dynamic equations using D-H notation.

$i = 1,2 \rightarrow q_i \rightarrow$ joint angles

$l_i \rightarrow$ denotes the link length

$l_{ci} \rightarrow$ distance from the previous joint to the center of mass of link i

$I_i \rightarrow$ moment of inertia of link i about an axis coming out of the page, passing through the center of mass of link i

$$v_{ci} = \underbrace{J_{v_{ci}}}_{3\times 2 \text{ matrix}} \dot{q} \rightarrow \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

($\because$ planar & 2 joints)

where $J_{v_{c1}} = \begin{bmatrix} -l_c \sin q_1 & 0 \\ l_{c_1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$

Similarly, $v_{c_2} = J_{v_{c_2}} \dot{q}$ where $J_{v_{c2}} = \begin{bmatrix} -l_1 \sin q_1 - l_{c_2} \sin(q_1+q_2) & -l_{c_2}\sin(q_1+q_2) \\ l_1 \cos q_1 + l_{c_2}\cos(q_1+q_2) & l_{c_2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}$

Translational Kinetic Energy $= \frac{1}{2} m_1 v_{c_1}^T v_{c_1} + \frac{1}{2} m_2 v_{c_2}^T v_{c_2}$

$$= \frac{1}{2} \dot{q}^T \{ m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} \} \dot{q}$$

Now, $\omega_1 = \dot{q}_1 \hat{k}$ & $\omega_2 = (\dot{q}_1 + \dot{q}_2)\hat{k}$

Rotational Kinetic Energy $= \frac{1}{2} I_i \omega_i^2$ for $i = 1,2$

$$= \frac{1}{2} \dot{q}^T \left\{ I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \dot{q}$$

$$\Rightarrow D(q) = m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

$$\Rightarrow p(q) = m_1 \begin{bmatrix} -l_{c_1}\sin q_1 & l_{c_1}\cos q_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -l_{c_1}\sin q_1 & 0 \\ l_{c_1}\cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ m_2 \begin{bmatrix} -l_1\sin q_1 - l_{c_2}\sin(q_1+q_2) & l_1\cos q_1 + l_{c_2}\cos(q_1+q_2) & 0 \\ -l_{c_2}\sin(q_1+q_2) & l_{c_2}\cos(q_1+q_2) & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -l_1\sin q_1 - l_{c_2}\sin(q_1+q_2) & -l_{c_2}\sin(q_1+q_2) \\ l_1\cos q_1 + l_{c_2}\cos(q_1+q_2) & l_{c_2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

$$\Rightarrow \boxed{d_{11} = m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2}\cos q_2) + I_1 + I_2}$$

$$\boxed{d_{12} = d_{21} = m_2(l_1 l_{c_2}\cos q_2 + l_{c_2}^2) + I_2}$$

$$\boxed{d_{22} = m_2 l_{c_2}^2 + I_2}$$

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \quad ; \quad c_{121} = c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 l_1 l_{c_2}\sin q_2$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\underbrace{\frac{\partial d_{22}}{\partial q_1}}_{\to 0} = -m_2 l_1 l_{c_2}\sin q_2$$

$$c_{112} = \underbrace{\frac{\partial d_{21}}{\partial q_1}}_{\to 0} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = +m_2 l_1 l_{c_2}\sin q_2$$

$$c_{122} = c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{222} = 0 = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2}$$

$$V_1 = m_1 g l_{c_1}\sin q_1$$

$$V_2 = m_2 g(l_1\sin q_1 + l_{c_2}\sin(q_1+q_2)) \qquad V = V_1 + V_2$$

$$V = m_1 g l_{c_1}\sin q_1 + m_2 g(l_1\sin q_1 + l_{c_2}\sin(q_1+q_2))$$

$$\frac{\partial P}{\partial q_1} = m_1 g l_{c_1} \cos q_1 + m_2 g l_1 \cos q_1 + m_2 g l_{c_2} \cos(q_1+q_2) = g_1$$

$$\frac{\partial P}{\partial q_2} = m_2 g l_{c_2} \cos(q_1+q_2) = g_2$$

$$\Rightarrow \quad d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + \underbrace{c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1}_{\text{same}} + c_{221}\dot{q}_2^2 + g_1 = \tau_1$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 = \tau_2$$

$$\Rightarrow \left[m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2}\cos q_2) + I_1 + I_2\right]\ddot{q}_1 + \left[m_2(l_1 l_{c_2}\cos q_2 + l_{c_2}^2) + I_2\right]\ddot{q}_2$$

$$- 2 m_2 l_1 l_{c_2} \sin q_2 \, \dot{q}_1 \dot{q}_2 - m_2 l_1 l_{c_2} \sin q_2 \, \dot{q}_2^2 + (m_1 l_{c_1} + m_2 l_1) g \cos q_1$$

$$+ m_2 g l_{c_2} \cos(q_1+q_2) = \tau_1 \quad \text{— } (1)$$

$$\left[m_2(l_1 l_{c_2}\cos q_2 + l_{c_2}^2) + I_2\right]\ddot{q}_1 + (m_2 l_{c_2}^2 + I_2)\ddot{q}_2$$

$$+ m_2 l_1 l_{c_2} \sin q_2 \, \dot{q}_1^2 + m_2 g l_{c_2} \cos(q_1+q_2) = \tau_2 \quad \text{— } (2)$$

There are more number of terms in the equations to the equations of motion derived in the mini project. These additional terms may arise due to the different approach that we have taken for calculating the velocities, i.e. the D-H convention where the frames are chosen with a particular convention.

Question 1: Singularities and Jacobian Matrix

There are internal singularities, which are caused by the axes of the robot aligning in space. At that time, the rotation of one joint is countered by another, and thus the location of the effector indeterminate. Boundary singularities occur when the there is full extension of a joint, and the manipulator is commanded to move to a point outside the workspace. Inverse mapping from Cartesian space to joint space becomes a problem in case of singularities. Infinite inverse kinematic solutions occur at a singularity, and the joint velocity becomes infinite in the joint space. This occurs whenever the Jacobian matrix has a zero determinant. Therefore, if the determinant is close to zero, or equivalently the joint velocities come out to be very high, then the point is near the singularity point.

Question 7: Comparing the Three Different Configurations of 2R Manipulator

1. Planar Elbow Manipulator: In the dynamics of the planar elbow manipulator with both the actuators at the respective joint locations, there are Coriolis forces showing up in the equations of motion. Additionally, there are centrifugal forces that are also present in the equations. Centrifugal forces arise due to the measurements in the rotating frame, while the Coriolis forces arise because the second link has motion relative to an already rotating frame (frame 1). This makes the equations of motion very complex and coupled, i.e. the torques are not dependent on both the angles and vice versa.

2. Planar Elbow Manipulator with Remotely Driven Link: When one of the link is remotely driven (using a timing belt or a gearing mechanism) by situating both the actuators at the base of the manipulator, then the generalised coordinates are changed and the corresponding Jacobian matrix also changes. As a result, there is a simplification in the equations of motion. The Coriolis terms disappear while the centrifugal force still remains, which is actually coupling the joints.

3. Five-Bar Linkage: Here there are four links (not counting the ground link) that make a parallelogram with an extended arm in one of the links. Note that the degrees of freedom still remain two in this case. The goal here is to decouple the two joint variables, so that they can be independently controlled. Since the mass distribution need not be same for all the links, the centre of masses need not be same for all the links, even though the lengths are equal for opposite links. Here the partial differentiation of the potential energy with respective to i^{th} joint angle is dependent only on the

i^{th} angle and with this simplification, the equations of motion become decoupled. This is important as now one of the angles can be controlled without thinking about how it would affect the other angle. This is a very advantageous situation and hence the parallelogram configuration is a popular one in industrial robots.