

MODEL QUESTION PAPER - I
SUBCODE: BMATS201/C201/M201
Integral calculus.

MODULE-1

Q. 01

a. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

$$\Rightarrow \int_0^a \int_0^x \int_0^{x+y} e^x \cdot e^y \cdot e^z dz dy dx = I$$

$$\Rightarrow \int_0^a \int_0^x e^x e^y [e^z]_0^{x+y} dy dx$$

$$\Rightarrow \int_0^a \int_0^x e^{2x} \left[\frac{e^{2y}}{2} \right]_0^x - e^x [e^y]_0^x dy dx$$

$$\Rightarrow \int_0^a \frac{e^{2x}}{2} (e^{2x} - 1) - e^x (e^x - 1) dx$$

$$\Rightarrow \int_0^a \left(\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x \right) dx$$

$$\Rightarrow \left[\frac{e^{4x}}{8} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^x \right]_0^a$$

$$\Rightarrow \left[\frac{e^{4a}}{8} - \frac{3}{4} \cdot e^{2a} + e^a - \left(\frac{1}{8} - \frac{3}{4} + 1 \right) \right]$$

$$\Rightarrow \left(\frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^a - \frac{3}{8} \right)$$

b. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ by changing into polar coordinates

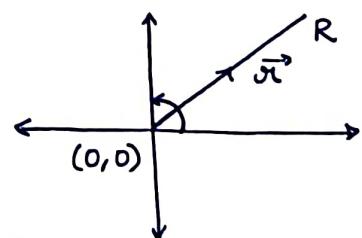
$$\Rightarrow I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

$$\Rightarrow \text{Here, } x: 0 \longrightarrow \infty \\ y: 0 \longrightarrow \infty$$

Now, limits:

$$\theta: 0 \longrightarrow \pi/2$$

$$\rho: 0 \longrightarrow \infty$$



$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

$$\text{Let } t = r^2$$

$$dt = 2r dr$$

$$r dr = dt/2$$

Limits, when $r=0, t=0$

$r=\infty, t=\infty$

$$\int_0^{\pi/2} \int_0^\infty \frac{e^{-t}}{2} dt d\theta$$

$$\Rightarrow \frac{1}{2} \int_0^{\pi/2} [e^{-t}]_0^\infty d\theta$$

$$\Rightarrow -\frac{1}{2} \int_0^{\pi/2} (-1) d\theta$$

$$\Rightarrow \frac{1}{2} [\theta]_0^{\pi/2}$$

$$\underline{\underline{\frac{\pi}{4}}}$$

c. Show that $\beta(m, n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

$$\Gamma(m) = 2 \int_0^\infty e^{-x^2} x^{2m-1} dx \rightarrow ①$$

$$\Gamma(n) = 2 \int_0^\infty e^{-y^2} y^{2n-1} dy \rightarrow ②$$

$$\Gamma(m+n) = 2 \int_0^\infty e^{-r^2} r^{2(m+n)-1} dr \rightarrow ③$$

consider,

$$\begin{aligned} \Gamma(m)\Gamma(n) &= 2 \int_0^\infty e^{-x^2} x^{2m-1} dx \times 2 \int_0^\infty e^{-y^2} y^{2n-1} dy \\ &= 4 \iint_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dy dx \rightarrow ④ \end{aligned}$$

The above integral is solved by changing to polar coordinates,

$$x = r \cos \theta, y = r \sin \theta$$

$$r : 0 \longrightarrow \infty$$

$$\theta : 0 \longrightarrow \pi/2$$

$$\begin{aligned} \textcircled{4} \Rightarrow \Gamma(m)\Gamma(n) &= 4 \int_0^{\infty} \int_0^{\pi/2} e^{-r^2} (r \cos \theta)^{am-1} (r \sin \theta)^{an-1} r dr d\theta \\ &= 2 \int_0^{\pi/2} \cos^{am-1} \theta \sin^{an-1} \theta d\theta \times 2 \int_0^{\infty} e^{-r^2} r^{am+an-1} r dr \\ &= \beta(m, n) \times 2 \int_0^{\infty} e^{-r^2} r^{am+an-1} dr \end{aligned}$$

$$[\Gamma(m)\Gamma(n) = \beta(m, n) \times \Gamma(m+n)]$$

OR

Q. 02

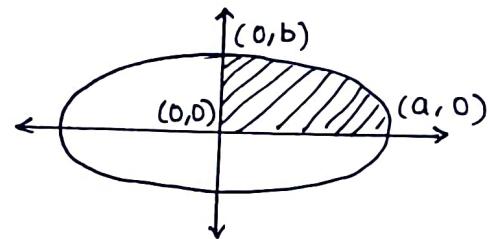
b. Using double integration find the area of a plane in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{Area} = 4 \iint_R dy dx$$

Limits,

$$x : 0 \longrightarrow a$$

$$y : 0 \longrightarrow \frac{b}{a} \sqrt{a^2 - x^2}$$



$$A = 4 \int_0^a \int_0^{b/a \sqrt{a^2 - x^2}} dy dx$$

$$= 4 \int_0^a [y]_0^{b/a \sqrt{a^2 - x^2}} dx$$

$$= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} - 0 dx$$

$$= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a} \left[(0-0) + \frac{a^2}{2} (\sin^{-1} 1 - \sin^{-1} 0) \right]$$

$$= \frac{4b}{a} \times \frac{a^2}{2} \left(\frac{\pi}{2} - 0 \right)$$

$$= \underline{\underline{\pi ab \text{ sq units}}}$$

c. Write the codes to find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$ and $z=0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ using Mathematical tools.

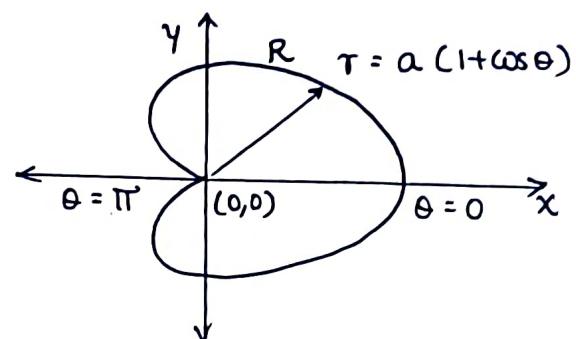
Ans: Area = $\iint r dr d\theta$

Limits,

$$\theta : 0 \rightarrow \pi$$

$$r : 0 \rightarrow a(1 + \cos\theta)$$

$$\begin{aligned} A &= \int_0^\pi \int_0^{a(1+\cos\theta)} r dr d\theta \\ &= \int_0^\pi \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta \\ &= \frac{1}{2} \int_0^\pi (a(1 + \cos\theta))^2 d\theta \\ &= \frac{a^2}{2} \int_0^\pi (2\cos^2 \frac{\theta}{2})^2 d\theta \\ &= \frac{a^2}{2} \times 4 \int_0^\pi \cos^4 \frac{\theta}{2} d\theta \end{aligned}$$



$$\text{Let } t = \theta/2$$

$$dt = d\theta/2 \Rightarrow d\theta = 2dt$$

$$\text{When, } \theta = 0, t = 0$$

$$\theta = \pi, t = \pi/2$$

$$\Rightarrow 2a^2 \int_0^{\pi/2} 2 \cos^4 t dt$$

$$\Rightarrow 4a^2 \int_0^{\pi/2} (\cos t)^4 dt$$

$$\Rightarrow \text{Reduction Form: } \int_0^{\pi} \cos^m x dx = \frac{(m-1)(m-3)\dots}{m(m-2)(m-4)\dots} \times k$$

$$\Rightarrow 4a^2 \left[\frac{3}{4} \frac{1}{1} \times \frac{\pi}{2} \right]$$

$$\Rightarrow \underline{\underline{\frac{3\pi a^2}{4}}}$$

MODEL QUESTION PAPER -II

MODULE-1

Q. 01

a. Evaluate $\iiint_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dx dy dz$

$$\underline{\text{Ans: }} I = \int_{-c}^c \int_{-b}^b \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_{-a}^a dy dx$$

$$= \int_{-c}^c \int_{-b}^b \left[x^2(2a) + y^2(2a) + \frac{2a^3}{3} \right] dy dx$$

$$= 2a \int_{-c}^c \int_{-b}^b x^2 + y^2 + \frac{a^2}{3} dy dx$$

$$= 2a \int_{-c}^c \left[x^2 y + \frac{y^3}{3} + \frac{a^2}{3} y \right]_{-b}^b dx$$

$$= 4ab \int_{-c}^c \left(x^2 + \frac{b^2}{3} + \frac{a^2}{3} \right) dx$$

$$= 4ab \left[\frac{x^3}{3} + \frac{b^2 x}{3} + \frac{a^2 x}{3} \right]_{-c}^c$$

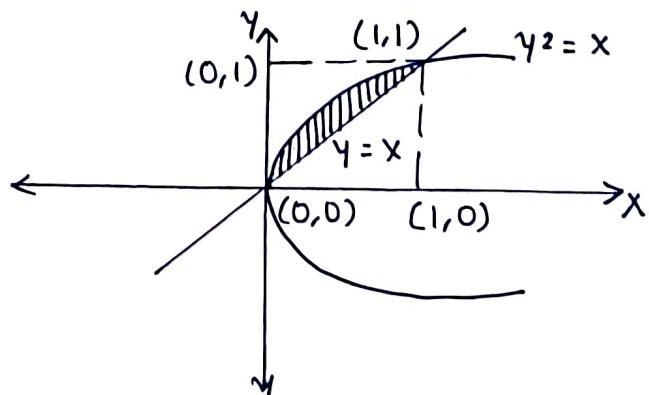
$$= \underline{\underline{\frac{8abc}{3} (a^2 + b^2 + c^2)}}$$

b. Evaluate $\iint_x^{\sqrt{x}} xy dy dx$ by changing the order of integration.

$$\underline{\text{Ans: }} I = \int_0^1 \int_x^{\sqrt{x}} xy dy dx$$

$$\begin{aligned} x &: 0 \longrightarrow 1 \\ y &: x \longrightarrow \sqrt{x}. \end{aligned}$$

$$\begin{aligned} \text{Now, } y &: 0 \longrightarrow 1 \\ x &: y^2 \longrightarrow y \end{aligned}$$



1c. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\text{We know that, } \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\text{Put } m=n=\frac{1}{2}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \longrightarrow ①$$

Let us consider,

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

$$m=n=\frac{1}{2}$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2 [\theta]_0^{\pi/2} = \frac{2\pi}{2} = \underline{\underline{\pi}}$$

$$① \Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \quad \rightarrow$$

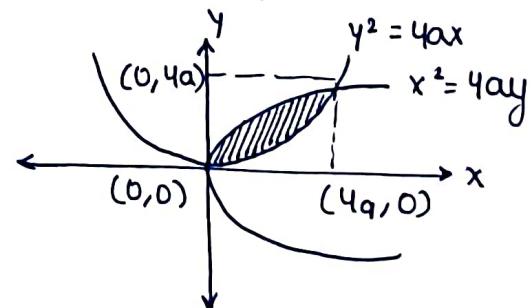
$$\Rightarrow \underline{\underline{\Gamma\left(\frac{1}{2}\right)}} = \sqrt{\pi}$$

hence proved

Q0.2

b. Find by double integration, the area between the parabolas $y^2=4ax$ and $x^2=4ay$.

$$\begin{aligned} A &= \iint_R dy dx \\ &= \int_{x^2/4a}^{4a} \int_{y^2=4ax}^{2\sqrt{ax}} dy dx \\ &= \int_{x^2/4a}^{4a} [y]_{y^2=4ax}^{2\sqrt{ax}} dx \\ &= \int_{x^2/4a}^{4a} 2\sqrt{ax} - \frac{x^2}{4a} dx \end{aligned}$$



$$\begin{aligned}
 &\Rightarrow 2 \int_0^{4a} \frac{2}{3} \sqrt{a} x^{3/2} - \frac{1}{4a} \frac{x^3}{3} \\
 &\Rightarrow \frac{4}{3} \left[\sqrt{a} x^{3/2} \right]_0^{4a} - \frac{1}{12a} \left[x^3 \right]_0^{4a} \\
 &\Rightarrow \frac{4}{3} \left[a^{1/2} \cdot (4a)^{3/2} \right] - \frac{1}{12a} 4 \times 4 \times 4 a^3 \\
 &\Rightarrow \frac{4}{3} \left[2^{2 \times \frac{3}{2}} a^{\frac{1}{2} + \frac{3}{2}} \right] \Rightarrow \frac{4}{3} \times 8 \times 16 a^2 \\
 &\Rightarrow \frac{32}{3} a^2 - \frac{16a^2}{3} = \underline{\underline{\frac{16a^2}{3}}} \text{ sq units}
 \end{aligned}$$

c. Using Mathematical tools, write the code to find the area of an ellipse by double integration, $A = 4 \int_0^a \int_0^{\sqrt{a^2-x^2}} dy dx$.

$$I = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dy dx$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

$$dy dx = r d\theta dr$$

limits here, $x : -a \rightarrow a$
 $y : 0 \rightarrow \sqrt{a^2-x^2}$

Now,

$$\theta : 0 \rightarrow \pi$$

$$r : 0 \rightarrow a$$

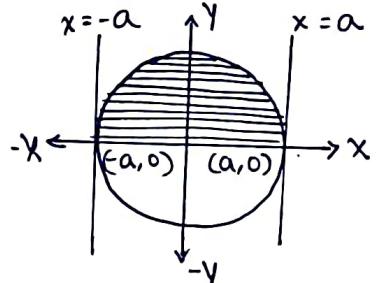
$$\int_0^{\pi} \int_0^a r^2 dr d\theta$$

$$\Rightarrow \int_0^{\pi} \left[\frac{r^3}{3} \right]_0^a d\theta$$

$$\Rightarrow \frac{1}{3} \int_0^{\pi} a^3 d\theta$$

$$\Rightarrow \frac{1}{3} \left[a^3 \right]_0^{\pi}$$

$$\Rightarrow \underline{\underline{\frac{\pi a^3}{3}}}$$



SUB CODE: BMATS201 / C201 / E201 • CURVILINEAR.MODEL QUESTION PAPER SOLUTIONS → MODULE-02

(1) Find $\nabla\phi$, if $\phi = x^3 + y^3 + z^3 - 3xyz$ at the point $(1, -1, 2)$

:- Given $\phi = x^3 + y^3 + z^3 - 3xyz$

To find $\nabla\phi$

$$\text{WKT } \nabla\phi = \frac{\partial\phi}{\partial x} \hat{i} + \frac{\partial\phi}{\partial y} \hat{j} + \frac{\partial\phi}{\partial z} \hat{k}$$

$$= (3x^2 - 3yz) \hat{i} + (3y^2 - 3xz) \hat{j} + (3z^2 - 3xy) \hat{k}$$

at $(1, -1, 2)$

$$\nabla\phi = (3+6) \hat{i} + (3-6) \hat{j} + (12+3) \hat{k}$$

$$= 9 \hat{i} + (-3) \hat{j} + 15 \hat{k}$$

$$= 3(3\hat{i} - \hat{j} + 5\hat{k}) //$$

(2) If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find $\text{div } \vec{F}$ and $\text{curl } \vec{F}$

:- Given $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

$$\vec{F} = (3x^2 - 3yz) \hat{i} + (3y^2 - 3xz) \hat{j} + (3z^2 - 3xy) \hat{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$= \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

$$\nabla \cdot \vec{F} = 6x + 6y + 6z$$

$$\operatorname{div} \vec{F} = 6(x+y+z) //$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3yx \end{vmatrix}$$

$$= \hat{i}(-3x+3x) - \hat{j}(-3y+3y) + \hat{k}(-3z+3z)$$

$$\operatorname{curl} \vec{F} = \vec{0} //$$

(3) If $\vec{F} = \nabla(xy^3z^2)$, find $\operatorname{div} \vec{F}$ and $\operatorname{curl} \vec{F}$ at $(1, -1, 1)$

Given $\vec{F} = \nabla(xy^3z^2)$

$$\vec{F} = y^3z^2 \hat{i} + 3xy^2z^2 \hat{j} + 2xy^3z \hat{k}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \frac{\partial}{\partial x} y^3z^2 + \frac{\partial}{\partial y} 3xy^2z^2 + \frac{\partial}{\partial z} 2xy^3z$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$\operatorname{div} \vec{F} = 2xy(y+z^2) // \text{ at } (1, -1, 1)$$

$$\operatorname{div} \vec{F} = -8 //$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{i}(6xy^2z - 6xy^2z) - \hat{j}(2y^3z - 2y^3z) + \hat{k}(3y^2z^2 - 3y^2z^2)$$

$$\operatorname{curl} \vec{F} = \vec{0}$$

At $(1, -1, 0)$, $\operatorname{curl} \vec{F} = \vec{0} \parallel$

~~(4) Find the Directional derivative of~~
~~Repeated~~
 $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction
of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

:- w.r.t Directional derivative = $\nabla \phi \cdot \hat{n}$

Given, $\phi = x^2yz + 4xz^2$, $(1, -2, -1)$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$= (2xy + 4z^2) \hat{i} + (x^2z) \hat{j} + (x^2y + 8xz) \hat{k}$$

$$\text{at } (1, -2, -1) \quad \nabla \phi = 0\hat{i} - \hat{j} - 10\hat{k}$$

$$n^{NKT} = \frac{\vec{a}}{|\vec{a}|}, \quad \text{Given } \vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$\hat{n} = \frac{1}{3} (2\hat{i} - \hat{j} - 2\hat{k})$$

$$\begin{aligned}\therefore |\vec{a}| &= \sqrt{4+1+4} \\ &= \sqrt{9} = 3\end{aligned}$$

$$\begin{aligned}\therefore D.D &= \nabla \phi \cdot \hat{n} \\ &= (0\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{1}{3} (2\hat{i} - \hat{j} - 2\hat{k}) \\ &= 0 + \frac{1}{3} + \frac{20}{3} \\ &= \frac{21}{3} = 7\end{aligned}$$

(5) Find the angle between the surfaces

$$x^2 + y^2 - z^2 = 4 \text{ and } z = x^2 + y^2 - 13 \text{ at } (2, 1, 2)$$

:- Given,

$$\phi_1 = x^2 + y^2 - z^2 - 4 \quad \text{&} \quad \phi_2 = z - x^2 - y^2 + 13.$$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \hat{i} + \frac{\partial \phi_1}{\partial y} \hat{j} + \frac{\partial \phi_1}{\partial z} \hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$\text{at } (2, 1, 2)$$

$$\nabla \phi_1 = 4\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \hat{i} + \frac{\partial \phi_2}{\partial y} \hat{j} + \frac{\partial \phi_2}{\partial z} \hat{k}$$

$$= -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\text{at } (2, 1, 2)$$

$$\nabla \phi_2 = -4\hat{i} - 2\hat{j} + \hat{k}$$

NKT,

Angle b/w the surfaces.

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

$$= \frac{(4\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (-4\hat{i} - 2\hat{j} + \hat{k})}{(\sqrt{36}) (\sqrt{21})}$$

$$= \frac{-16 - 4 - 4}{6\sqrt{21}}$$

$$= \frac{-24}{6\sqrt{21}}$$

$$\cos\theta = \frac{-4}{\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{\sqrt{21}}\right) //$$

(6) Show that the vector $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$

is both solenoidal and irrotational

:- Given $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) + 0$$

$$= \frac{(x^2+y^2)1-x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}$$

$$= -\frac{x^2+y^2+x^2-y^2}{(x^2+y^2)^2}$$

$\operatorname{div} \vec{F} = 0_{//} \Rightarrow \vec{F}$ is Solenoidal.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= \vec{0}$$

$\Rightarrow \vec{F}$ is irrotational.

(7) Using Mathematical tools, write a code
to find curl of $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

- PROGRAM:

from sympy import *

```
from sympy import symbols  
N = CoordSys3D('N')
```

```
x, y, z = symbols('x y z')
```

```
A = N.x*N.y**2*N.z + 2*N.x**2*N.y*N.z  
    * N.j - 3*N.y*N.z**2*N.k
```

```
delop = Del()
```

```
curl A = delop.cross(A)
```

```
print(delop(A))
```

```
print(f"\n curl of {A} is \n")
```

```
print(curl(A))
```

(8) Using Mathematical tools, write the code to
find gradient of $\phi = x^2yz$

:- PROGRAM:

```
from sympy.vector import *
```

```
from sympy import symbols
```

```
N = CoordSys3D('N')
```

```
x, y, z = symbols('x y z')
```

```
A = N.x**2*N.y*N.z
```

```
delop = Del()
```

```
print(delop(A))
```

```
grad(A) = gradient(A)
```

```
print("the gradient is ", grad(A))
```

(g) Show that the Spherical coordinate system is orthogonal.

∴ For the Spherical system, we have

$$\vec{r} = r \sin\theta \cos\phi \hat{i} + r \sin\theta \sin\phi \hat{j} + r \cos\theta \hat{k}$$

Let $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ be the basic vectors

$$\text{we have, } h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta$$

Consider

$$\hat{e}_r = \frac{1}{h_1} \frac{\partial \vec{r}}{\partial r} = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}$$

$$\hat{e}_\theta = \frac{1}{h_2} \frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r} (\cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} + r(-\sin\theta) \hat{k})$$

$$\hat{e}_\phi = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}$$

$$\hat{e}_\phi = \frac{1}{h_3} \frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r \sin\theta} (-r \sin\theta \sin\phi \hat{i} + r \sin\theta \cos\phi \hat{j} + 0 \hat{k})$$

$$\hat{e}_\phi = -\sin\phi \hat{i} + \cos\phi \hat{j}$$

Consider,

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta \cos\theta (\cos^2\phi + \sin^2\phi) - \sin\theta \cos\theta = 0$$

$$\text{Hence } \hat{e}_\theta \cdot \hat{e}_\phi = 0 \quad \& \quad \hat{e}_\phi \cdot \hat{e}_r = 0$$

Thus the Spherical co-ordinate system is orthogonal

(10) Show that cylindrical co-ordinate system is orthogonal.

:- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a position vector.

Cylindrical co-ordinate system,

$$x = \rho \cos\phi, y = \rho \sin\phi, z = z$$

$$\therefore \vec{r} = \rho \cos\phi \hat{i} + \rho \sin\phi \hat{j} + z \hat{k}$$

Let $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z$ be the basic vectors.

$$\therefore \hat{e}_\rho = \frac{1}{h_1} \frac{\partial \vec{r}}{\partial \rho} = \cos\phi \hat{i} + \sin\phi \hat{j} + 0\hat{k} \quad \because h_1 = 1$$

$$\hat{e}_\phi = \frac{1}{h_2} \frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho} (-\rho \sin\phi \hat{i} + \rho \cos\phi \hat{j} + 0\hat{k}) \quad \because h_2 = \rho$$

$$\hat{e}_z = \frac{1}{h_3} \frac{\partial \vec{r}}{\partial z} = 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k} \quad \therefore h_3 = 1$$

Consider,

$$\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi \sin\phi + \sin\phi \cos\phi = 0$$

$$\hat{e}_\phi \cdot \hat{e}_z = 0$$

$$\hat{e}_z \cdot \hat{e}_\rho = 0$$

\therefore Cylindrical system is orthogonal.

(II) Express the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical co-ordinates.

\therefore let $\vec{A} = A_1 \hat{e}_\rho + A_2 \hat{e}_\phi + A_3 \hat{e}_z \quad \text{--- (1)}$

where A_1, A_2, A_3 are to be determined

from (1), $A_1 = \vec{A} \cdot \hat{e}_\rho ; A_2 = \vec{A} \cdot \hat{e}_\phi ; A_3 = \vec{A} \cdot \hat{e}_z \quad \text{--- (*)}$

In cylindrical, $x = \rho \cos\phi, y = \rho \sin\phi, z = z$

$$\therefore \vec{A} = z\hat{i} - 2\rho \cos\phi \hat{j} + \rho \sin\phi \hat{k} \quad \text{--- (2)}$$

Also we've

$$\hat{e}_\rho = \frac{1}{h_1} \frac{\partial \vec{r}}{\partial \rho}, \quad \hat{e}_\phi = \frac{1}{h_2} \frac{\partial \vec{r}}{\partial \phi}, \quad \hat{e}_z = \frac{1}{h_3} \frac{\partial \vec{r}}{\partial z}$$

where $h_1 = 1, h_2 = \rho, h_3 = 1$ & $\vec{r} = \rho \cos\phi \hat{i} + \rho \sin\phi \hat{j} + z\hat{k}$

Hence, $\hat{e}_\rho = \rho \cos\phi \hat{i} + \sin\phi \hat{j}$. }
 $\hat{e}_\phi = -\sin\phi \hat{i} + \cos\phi \hat{j}$. }
 $\hat{e}_z = k$. } $\quad \text{--- (3)}$

using (2) & (3) in (*)

$$\therefore A_1 = z \cos\phi - 2\rho \cos\phi \sin\phi = z \cos\phi - \rho \sin 2\phi$$

$$A_2 = -z \sin\phi - 2\rho \cos^2\phi$$

$$A_3 = \rho \sin\phi.$$

Thus. $\vec{A} = A_1 \hat{e}_\rho + A_2 \hat{e}_\phi + A_3 \hat{e}_z //$

MODULE - 03 : Vector Space and Linear Transform

- 10m

Model Question paper Solution

SUBCODE - BMATS201

1. prove that the subset $W = \{(x, y, z) / x - 3y + 4z = 0\}$ of the vector space \mathbb{R}^3 is a Subspace of \mathbb{R}^3 .

:- Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$ be any two elements of W , $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$

Then $a x_1 + b y_1 + c z_1 = 0$ & $a x_2 + b y_2 + c z_2 = 0$

$$\Rightarrow \underbrace{x_1 - 3y_1 + 4z_1}_{u} = 0 \text{ and } \underbrace{x_2 - 3y_2 + 4z_2}_{v} = 0$$

$$\begin{aligned} \text{For } \alpha \in \mathbb{R}, \quad \alpha u + v &= \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2) \\ &= (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2) \\ &= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \end{aligned}$$

where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2 \in \mathbb{R}$

Now, we have $x - 3y + 4z = 0$

$$\begin{aligned} 1(\alpha x_1 + x_2) - 3(\alpha y_1 + y_2) + 4(\alpha z_1 + z_2) \\ &= \alpha(x_1 - 3y_1 + 4z_1) + (x_2 - 3y_2 + 4z_2) \\ &= \alpha(0) + 0 \quad (0 = \alpha(u) + v) \\ &= 0 \end{aligned}$$

Hence W is a Subspace of \mathbb{R}^3

(Q) Let $V = \mathbb{R}^3$ be a vector space and consider the subset W of V consisting of vectors of the form (a, a^2, b) , where the second component is the square of the first. Is W a subspace of V ?

\therefore Let $V = \mathbb{R}^3 = \{(a, b, c) / a, b, c \in \mathbb{R}\}$

Subspace $W = \{(a, a^2, b) / a, b \in \mathbb{R}\}$

Cond'n, (i) $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

(ii) $\forall \alpha \in F, w \in W \Rightarrow \alpha w \in W$

$\because \alpha \in F$, let $\alpha = -1 \in \mathbb{R}$ ($\because \mathbb{R}$ is a field)

Let $a=1, b=0 \therefore w = (1, 1, 0) \in W$

Consider, $\alpha \cdot w = -1(1, 1, 0)$

$$= (-1, -1, 0) \notin W$$

Scalar Multiplication fails.

$\Rightarrow W$ is not a subspace of V .

(3) Let $f(x) = 2x^2 - 5$ and $g(x) = x + 1$. Show that
the function $h(x) = 4x^2 + 3x - 7$ lies in the subspace
 $\text{Span}\{f, g\}$ of P_2 .

:-

(4) Find the basis and the dimension of the subspace spanned by the vectors

$\{(2, 4, 2), (1, -1, 0), (1, 2, 1), (0, 3, 1)\}$ in $V_3(\mathbb{R})$

$\therefore \text{Let } A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

$$R_1 \rightarrow \frac{1}{2} R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 ; R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of A are

$\therefore \{(1, 2, 1), (0, -3, -1)\}$
are linearly independent
vectors
 \therefore Basis of A is
 $\{(1, 2, 1), (0, -3, -1)\}$

\therefore WKT Dimensions is No of Non-zero rows
 $\therefore \text{Dimension}(A) = \dim[A] = 2$

(5) Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, in the vector space M_{22} of 2×2 matrices.

\therefore Let,

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \beta \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta & -3\beta + \gamma \\ 2\alpha + 2\gamma & \alpha + 2\beta \end{bmatrix} \quad (1)$$

\therefore On equating we get

$$\begin{array}{l} \alpha + 2\beta = -1 \quad (1) \\ -3\beta + \gamma = 7 \quad (2) \\ 2\alpha + 2\gamma = 8 \quad (3) \\ \alpha + 2\beta = -1 \quad (4) \end{array}$$

$$\alpha + 2\beta = -1 \quad (1)$$

$$2\alpha + 2\beta = 8 \quad (3)$$

$$4\beta = 8 \quad (2)$$

$$2\beta = 2 \quad (2)$$

$$-3\beta + \gamma = 7 \quad (2)$$

$$\frac{-\beta = 2}{-\beta = 2} \Rightarrow \boxed{\beta = -2}$$

$$\therefore (1) \Rightarrow \alpha + 2(-2) = 1$$

$$\boxed{\alpha = 5}$$

$$(2) \Rightarrow -3\beta + \gamma = 7$$

$$-3(-2) + \gamma = 7 \Rightarrow \boxed{\gamma = 1}$$

$$(1) \Rightarrow \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 7 \\ 12 & 1 \end{bmatrix}$$

which is not equal

The matrix is not a linear combination of the given vectors.

(6) Show that the set $S = \{(1, 2, 4), (1, 0, 0), (0, 1, 0)$
 $(0, 0, 1)\}$ is linearly dependent.

:- Let

$$(1, 2, 4) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

$$(1, 2, 4) = (a, b, c)$$

$$\Rightarrow a = 1, b = 2, c = 4$$

$$\therefore (1, 2, 4) = 1(1, 0, 0) + 2(0, 1, 0) + 4(0, 0, 1)$$

$$= (1, 2, 4)$$

∴ The set is linearly dependent

because one of the element in set S can be expressed as linear combination of others vectors.

(#) prove that the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x_1, y) = (3x_1, x_1 + y)$ is linear. Find the images of the vectors $(1, 3)$ and $(-1, 2)$ under this transformation.

$$\text{:- Let } u = (x_1, y_1) \Rightarrow T(u) = T(x_1, y_1) = (3x_1, x_1 + y_1)$$

$$v = (x_2, y_2) \Rightarrow T(v) = T(x_2, y_2) = (3x_2, x_2 + y_2)$$

$$(i) T(u+v) = T(x_1+x_2, y_1+y_2)$$

$$= (3(x_1+x_2), x_1+x_2+y_1+y_2)$$

$$= (3x_1+3x_2, x_1+x_2+y_1+y_2)$$

$$= (3x_1, x_1+y_1) + (3x_2, x_2+y_2)$$

$$= T(u) + T(v)$$

$$(ii) T(\alpha u) = T(\alpha x_1, \alpha y_1)$$

$$= (\alpha 3x_1, \alpha (x_1+y_1))$$

$$= \alpha (3x_1, x_1+y_1)$$

$$= \alpha T(u)$$

$\Rightarrow T(x_1, y)$ is linear.

$$\therefore \text{Image of } (1, 3), T(1, 3) = (3, 4)$$

$$\text{Image of } (-1, 2), T(-1, 2) = (3(-1), -1+2) = (-3, 1)$$

(8) Find the Kernel and range of the linear operator $T(x,y,z) = (x+y, z)$ of $\mathbb{R}^3 \rightarrow \mathbb{R}^2$

- Linear operator means Domain and Co-domain must be same and also same dimension

ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$$

here,

$$\dim [\text{Domain } (\text{or } \mathbb{R}^3(\mathbb{R}))] = 3$$

$$\dim [\text{Co-domain } (\text{or } \mathbb{R}^2(\mathbb{R}))] = 2$$

Null Space

$$\text{Ker}(T) = N(T) = \{v \in \mathbb{R}^3(\mathbb{R}) \mid T(v) = 0\}$$

Let $(x,y,z) \in N(T)$

$$T(x,y,z) = 0 = (0,0)$$

$$\Rightarrow (x+y, z) = (0,0)$$

$$x = -y, \quad z = 0$$

$$\therefore N(T) = \{(x,y,z) = \{(-x, x, 0)\}\}$$

x is non-trivial entry.

$$\text{hence } \dim(N(T)) = 1 //$$

Range Space

Let $\{(1,0,0), (0,1,0), (0,0,1)\}$ be stand Basis.

for $\mathbb{R}^3(\mathbb{R})$

Their image $T(1,0,0) = (1,0)$, $T(0,1,0) = (0,1)$
 $T(0,0,1) = (1,0)$

$$R(T) = \{x(1,0) + y(1,0), z(0,1) / x, y, z \in \mathbb{R}\}$$

$$= \{x+y, z / x, y, z \in \mathbb{R}\}$$

$$\text{Let } (1,0) = 1(1,0) + 0(0,1)$$

$$= (1,0) //$$

$\Rightarrow (1,0)$ can be written as linear combination of remaining two vectors \therefore It is L.T

$$\dim(R(T)) = \text{rank} = f(T) = 2$$

$\therefore \text{Rank} + \text{Nullity} = \text{dimension of Domain}$

$$2+1 = 3$$

$$3 = 3 // \text{ verified.}$$

(q) Find the matrix of the linear transformation $T: V_2(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ given $T(-1,1) = (-1,0,2)$ and

$$T(2,1) = (1,2,1)$$

$$\therefore \text{Let } (x,y) = \alpha(-1,1) + \beta(2,1)$$

$$= (-\alpha + 2\beta, \alpha + \beta)$$

$$\Rightarrow -\alpha + 2\beta = x ; \alpha + \beta = y$$

$$\therefore -\alpha + 2\beta = x$$

$$\alpha + \beta = y$$

$$\underline{-3\beta = x+y} \quad \beta = \frac{x+y}{3}$$

$$\therefore \alpha + \beta = y$$

$$\alpha + \left(\frac{\alpha+y}{3}\right) = y$$

$$\alpha = \frac{3y - \alpha - y}{3}$$

$$\alpha = \frac{2y - \alpha}{3}$$

Now,

$$(\alpha, y) = \left(\frac{2y - \alpha}{3}\right)(-1, 1) + \left(\frac{\alpha + y}{3}\right)(2, 1)$$

$$T(\alpha, y) = \left(\frac{2y - \alpha}{3}\right) T(-1, 1) + \left(\frac{\alpha + y}{3}\right) T(2, 1)$$

$$= \left(\frac{2y - \alpha}{3}\right) (-1, 0, 2) + \left(\frac{\alpha + y}{3}\right) (1, 2, 1)$$

$$= \left(-1 \left(\frac{2y - \alpha}{3}\right) + \left(\frac{\alpha + y}{3}\right), 0 + 2 \left(\frac{\alpha + y}{3}\right)\right)$$

$$2 \left(\frac{2y - \alpha}{3}\right) + \left(\frac{\alpha + y}{3}\right)$$

$$T(\alpha, y) = \left(\frac{2x - y}{3}, \frac{2(x+y)}{3}, \frac{5y - x}{3}\right)$$

$\therefore T$ is linear transformation.

(10) Verify the Rank-Nullity theorem for the linear transformation

$$T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R}) \text{ defn by } T(x, y, z) = (y-x, y-z)$$

\therefore Let $T: \mathbb{R}^3(\mathbb{R}) \rightarrow \mathbb{R}^2(\mathbb{R})$

$$\text{such that } T(x, y, z) = (y-x, y-z)$$

clearly T is linear transformation

$$V_3(\mathbb{R}) = \mathbb{R}^3(\mathbb{R}) \text{ and } V_2(\mathbb{R}) = \mathbb{R}^2(\mathbb{R}), \dim(\mathbb{R}^3(\mathbb{R})) = 3 \text{ and } \dim(\mathbb{R}^2(\mathbb{R}))$$

Null Space

$$\text{Ker}(T) (\text{or } N(T)) = \{v \in V_3(\mathbb{R}) / T(v) = 0\}$$

Let $(x, y, z) \in N(T)$

$$T(x, y, z) = 0 = (0, 0)$$

$$\Rightarrow (y-x, y-z) = (0, 0)$$

$$y-x=0 \text{ and } y-z=0$$

$$\Rightarrow y=x \quad y=z$$

$$\Rightarrow x=y=z=0$$

$$\therefore N(T) = \{(x, y, z)\} = \{(0, 0, 0)\} \text{ is zero.}$$

$$\dim(N(T)) = \eta(T) = \text{nullity} = 0 //$$

Range Space:

We have $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, & be the

standard basis of $V_3(\mathbb{R})$

$$\therefore \text{Their images } T(1, 0, 0) = (-1, 0), T(0, 0, 1) = (0, -1)$$

$$T(0, 1, 0) = (1, 1)$$

\therefore Images of basis of domain spans Co-domain
 $\therefore \{(-1, 0), (1, 1), (0, -1)\} \text{ span } R(T)$

$$\therefore R(T) = \left\{ x(-1, 0) + y(1, 1) + z(0, -1) / x, y, z \in \mathbb{R} \right\}$$
$$= \left\{ -x + y, y - z / x, y, z \in \mathbb{R} \right\}$$

Hence $(-1, 0), (1, 1), (0, -1)$ spans $R(T) \& LT$

$$\therefore \dim(\text{Range Space}) = \text{rank} = f(t) = 3$$

$\therefore \text{Rank} + \text{Nullity} = \text{Dimension Domain}$

$$3+0 = 3$$

$$3 = 3 //$$

(ii) Let P_n be the vector space of real polyⁿ func of degree $\leq n$ S.T the transformation $T: P_2 \rightarrow P_1$ defn by $T(ax^2+bx+c) = (a+b)x + c$ is linear

$$\therefore \text{Vector Space} = \left\{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n / a_i \in \mathbb{R} \right\}$$

Consider,

$$T(\alpha p_1(x) + \beta p_2(x)) = T(h(x))$$

$$\begin{aligned} h(x) &= \alpha p_1(x) + \beta p_2(x) = T(ax^2 + bx + c) \\ &= \alpha p_1(ax^2 + bx + c) + \beta p_2(ax^2 + bx + c) \\ &= \alpha T(p_1(x)) + \beta T(p_2(x)) \end{aligned}$$

Clearly T is linear transformation.

(12) S.T the function $f(x) = 3x - 2$ and $g(x) = x$
are orthogonal in P_3 with inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx$$

$$\therefore v_1 = f(x) = 3x - 2 ; v_2 = g(x) = x$$

S.T $\langle v_1, v_2 \rangle = 0$

here $\int_0^1 f(x) g(x) dx$ is the inner product func

$$\therefore \langle v_1, v_2 \rangle = \int_0^1 f(x) g(x) dx$$

$$= \int_0^1 (3x - 2)x dx$$

$$= \left[\frac{3x^3}{3} - \frac{2x^2}{2} \right]_0^1$$

$$= 1 - 1$$

$$= 0$$

$\therefore v_1$ and v_2 are orthogonal.

MODULE-04 - NUMERICAL METHOD-01

SUBCODE: BMATS201/C201/E201/M201

MODEL QUESTION PAPER - SOLUTIONS → MODULE-4

(1) Find an approximate soln of the root of the equation $x e^x = 3$, using the Regula-Falsi method, carry out three iterations.

:- Given that

$$y(x) = x e^x - 3 \quad (\text{or } f(x) = x e^x - 3)$$

HKT Regula-Falsi formula,

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

To find (a, b)

$$\text{Let } x=0, f(0) = -3 < 0 \Rightarrow a=0$$

$$x=1, f(1) = e-3 = -0.28 < 0 \Rightarrow a=1$$

$$x=2, f(2) = 2e^2 - 3 = 11.77 > 0 \Rightarrow b=2$$

The root may lie between (1, 2) = (a, b)

1st approximation (a, b) = (1, 2)

$$x_1 = \frac{1(11.77) - 2(-0.28)}{(11.77) - (-0.28)} = 1.0232$$

$$\therefore f(1.0232) = -0.1533 \Rightarrow a=1.0232$$

2nd approximation (1.0232, 2)

$$x_2 = \frac{1.0357(11.77) - 2(-0.1533)}{11.77 - (-0.1533)} = 1.0357$$

$$\therefore f(1.0357) = -0.0823 < 0 \Rightarrow a = 1.0357$$

3rd approximation $(a, b) = (1.0357, 2)$

$$x_3 = \frac{1.0357(11.77) - 2(-0.0823)}{11.77 - (-0.0823)} = 1.0423 //$$

Thus the required root is $x = 1.0423 //$

(Q) Find an approximate root of the equation $x^3 - 3x + 4 = 0$ using the method of false position, correct to three decimal places between -3 & -2 .

:- Given that,

$$f(x) = x^3 - 3x + 4 \text{ and } (a, b) = (-2, -3)$$

WKT Regula-falsi formula,

$$x_1 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

$f(-2) = 2$
$f(-3) = -14$

1st approximation:

$$x_1 = \frac{-2(-14) - (-3)(2)}{-14 - (2)} = -2.125$$

$$\therefore f(-2.125) = (-2.125)^3 - 3(-2.125) + 4 = 0.7792$$

$$\Rightarrow a = -2.125$$

2nd approximation: $(a, b) = (-2.125, -3)$

$$x_2 = \frac{-2.125(-14) - (-3)(0.2794)}{-14 - (0.2794)} = -2.1711$$

$$\therefore f(-2.1711) = 0.2794$$

$$\Rightarrow a = -2.1711$$

3rd approximation: $(a, b) = (-2.1711, -3)$

$$x_3 = \frac{(-2.1711)(-14) - (-3)(0.2794)}{-14 - 0.2794} = -2.1873 //$$

Thus the required root is $x = -2.1873 //$

(3) Find the real root of the equation $3x = \cos x + 1$
correct to three decimal places using Newton's
Raphson method.

:- Given that

$$f(x) = 3x - \cos x - 1 \quad \& \quad f'(x) = 3 + \sin x$$

To find (a, b)

$$x=0, \quad f(0) = 0 - 1 - 1 = -2 < 0$$

$$x=1, \quad f(1) = 3 - 0.9998 - 1 = 1.0002 > 0$$

\therefore The root lies between $(a, b) = (0, 1)$

WKT Newton's Raphson's formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

1st approximation ; $x_0 = 0$

$$x_1 = 0 - \frac{(-2)}{3} = 0.6666$$

$$\therefore f(0.6666) = -0.0001$$

$$f'(0.6666) = 3.0116$$

2nd approximation ; $x_1 = 0.6666$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6666 - \frac{(-0.0001)}{3.0116}$$

$$x_2 = 0.66663 //$$

thus required root is $x = 0.6666 //$

(4) Find the real root of the equation $\cos x = xe^x$ which is nearer to $x = 0.5$ by the Newton-Raphson method, correct to 3 decimal places.

:- Given that,

$$f(x) = \cos x - xe^x, x_0 = 0.5$$

$$f'(x) = -\sin x - (xe^x + e^x)$$

$$= -\sin x - e^x(x+1)$$

I&KT,

Newton-Raphson method,

1st approximation; $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$f(0.5) = 0.1756$$
$$f'(0.5) = -2.4818$$

$$= 0.5 - \left[\frac{0.1756}{-2.4818} \right]$$

$$x_1 = 0.5707$$

2nd approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$f(0.5707) = -0.0099$$
$$f'(0.5707) = -2.7893$$

$$= 0.5707 - \left[\frac{-0.0099}{-2.7893} \right]$$

$$x_2 = 0.5671$$

3rd approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$f(0.5671) = 0.00017$$
$$f'(0.5671) = -2.7729$$

$$= 0.5671 - \frac{(0.00017)}{-2.7729}$$

$x_3 = 0.5671$, the required root is 0.5671

(5) Given, $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$,
 $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$. Find $\sin 48^\circ$
using Newton's forward interpolation formula

:- Need to find $\sin 48^\circ$

i.e $x = 48^\circ$ near to x_0

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	<u>0.7071</u>	<u>$\Delta y_0 = 0.0589$</u>	<u>$\Delta^2 y_0 = -5.7 \times 10^{-3}$</u>	
50°	0.7660	$\Delta y_1 = 0.0532$	<u>$\Delta^2 y_1 = -6.4 \times 10^{-3}$</u>	<u>$\Delta^3 y_0 = -7 \times 10^{-4}$</u>
55°	0.8192			
60°	0.8660	$\Delta y_2 = 0.0468$		

$$h = 5, x = 48, n = \frac{x - x_0}{h} = \frac{48 - 45}{5} = 0.6$$

Newton's forward interpolation formula,

$$y = y_0 + n \Delta y_0 + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0$$

$$y = 0.7071 + (0.6)(0.0589) + \frac{0.6(0.6-1)}{2} (-5.7 \times 10^{-3})$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)}{6} (-7 \times 10^{-4})$$

$$= 0.74308 // \quad \text{i.e } \sin 48^\circ = 0.7430 //$$

(6) Using Newton's appropriate interpolation formula, find the values of y at $x=8$ and at $x=22$ from the following table,

x	0	5	10	15	20	25
y	7	11	14	18	24	32

y at

:- Here we need to find $\hat{x}=8$ and at $x=22$
i.e $x=8$ near to x_0 & $x=22$ near to x_n

x	y	1 st Diff	2 nd Diff	3 rd Diff	4 th Diff	5 th Diff
0	$y_0 = 7$	$\Delta y_0 = 4$	$\Delta^2 y_0 = -1$	$\Delta^3 y_0 = 2$	$\Delta^4 y_0 = -1$	
5	11	3	1	1		
10	14	4	2			
15	18	6				
20	24					
25	$y_n = 32$	$\nabla y_n = 8$	$\nabla^2 y_n = 2$	$\nabla^3 y_n = 0$	$\nabla^4 y_n = -1$	0

To find $y(8)$;

Newton Forward interpolation formula,

$$n = \frac{x-x_0}{h} = \frac{8-0}{5} = 1.6$$

$$y = y_0 + n \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 +$$

$$\frac{n(n-1)(n-2)(n-3)}{4!} \Delta^4 y_0$$

$$y = 7 + 1.6(4) + \frac{(1.6)(0.6)(-1)}{2} + \frac{(1.6)(0.6)(-0.4)}{6} 2$$

$$+ \frac{(1.6)(0.6)(-0.4)(-1.4)(-1)}{24}$$

$$\begin{aligned} y(8) &= 7 + 6.4 + (-0.48) - 0.128 - 0.0224 \\ &= 12.4696 // \end{aligned}$$

To find $y(22)$, Newton's Backward interpolation formula

$$x = 22, \quad n = \frac{x - x_0}{h} = \frac{22 - 25}{5} = -0.6$$

$$y = y_n + n \nabla y_n + \frac{n(n+1)}{2!} \nabla^2 y_n + \dots$$

$$= 32 + (-0.6)8 + \frac{(0.6)(0.4)}{2} 2 + \frac{(-0.6)(0.4)(1.4)(0)}{6} +$$

$$\frac{(-0.6)(0.4)(1.4)(2.4)}{24} (-1)$$

$$= 32 - 4.8 - 0.24 + 0.0336$$

$$y = 26.9936 //$$

(7) Using Newton's Divided difference formula, evaluate $f(8)$ from the following

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

:- WKT, Newton's Divided difference formula,

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + \dots + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots$$

x	$f(x)$	1 st D.D	2 nd D.D	3 rd D.D	4 th D.D
4	48	$\frac{100-48}{5-4} = 52$			
5	100	97	$\frac{97-52}{7-4} = 15$	<u>1</u>	
7	294	202	21	1	<u>0</u>
10	900	310	27	1	0
11	1210	409	33		
13	2028				

$$f(8) = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)1 + 0$$

$$= 448 //$$

(8) Find y at $x=5$ if $y(1)=-3$, $y(3)=9$,
 $y(4)=30$, $y(6)=132$ using Lagrange's interpolation formula.

\therefore Given that

$$\begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ 1 & 3 & 4 & 6 \end{array}$$

$$\begin{array}{cccc} y & -3 & 9 & 30 \end{array}$$

Hence, Lagrange's interpolation formula,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

To find y at $x=5$,

$$y = \frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)} (-3) + \frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)} 9 +$$

$$\frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)} 30 + \frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)} 132.$$

$$= -0.2 + (-6) + 40 - 35.2$$

$$y = -1.4 //$$

(g) Evaluate $\int_0^{\pi/2} \sqrt{\cos x} dx$ by using the trapezoidal rule by taking 7 ordinates,

$$:- f(x) = \sqrt{\cos x}$$

$$a=0, b=\frac{\pi}{2}, n+1=7 \\ \Rightarrow n=6.$$

$$h = \frac{\frac{\pi}{2} - 0}{6} \Rightarrow h = \frac{\pi}{12}.$$

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	0.9999	0.999	0.9999	0.9999	0.9998	0.9998

Trapezoidal rule,

$$y = y \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_n)]$$

$$= \frac{\pi}{24} \left[1 + 2(0.9999 + 0.9999 + 0.9999 + 0.9999 + 0.9998 + 0.9998) \right]$$

$$y = 1.5706 //$$

(10) Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ using the Trapezoidal rule by taking 6 divisions,

$$\therefore a=0, b=1, n=6$$

$$\therefore h = \frac{b-a}{n} = \frac{1}{6} = 0.166$$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	1
y	1	0.9729	0.9	0.8	0.6923	0.5901	0.5

Trapezoidal rule,

$$\begin{aligned}
 I &= \frac{h}{2} \left[y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n \right] \\
 &= \frac{0.166}{2} \left[1 + 2(0.9729 + 0.9 + 0.8 + 0.6923 + 0.5901) \right. \\
 &\quad \left. + 0.5 \right] \\
 &= 0.083 \left[1 + 4.9106 + 0.5 \right]
 \end{aligned}$$

$$I = 0.4810 //$$

(II) Evaluate $\int_0^3 \frac{1}{4x+5} dx$ by using Simpson's $\frac{1}{3}$ rd rule by taking 7 ordinates.

$$\therefore f(x) = \frac{1}{4x+5}, \quad a=0, \quad b=3, \quad n=6.$$

$$h = \frac{3-0}{6} = 0.5$$

x	0	0.5	1	1.5	2	2.5	3
y	0.2	0.142	0.111	0.09	0.076	0.06	0.058

Simpson's $\frac{1}{3}$ rd rule,

$$I = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= \frac{0.5}{3} \left[(0.2 + 0.058) + 4(0.142 + 0.09 + 0.06) + 2(0.111 + 0.076) \right]$$

$$= 0.1666 (0.258 + 1.168 + 0.374)$$

$$I = 0.2998 //$$

(12) Evaluate $\int_0^1 \frac{1}{1+x} dx$ by taking 7 ordinates
and by using Simpson's $\frac{3}{8}$ th rule.

$$\therefore f(x) = \frac{1}{1+x}, \quad a=0, \quad b=1, \quad n=6$$

$$h = \frac{1}{6}.$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
y	1	0.8571	0.75	0.666	0.6	0.5454	0.5

Simpson's $\frac{3}{8}$ th rule

$$I = \frac{3h}{8} \left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3 + y_5) \right]$$

$$= \frac{3}{8(6)} \left[(1 + 0.5) + 3(0.8571 + 0.75 + 0.6 + 0.5454) + 2(0.666) \right]$$

$$= 0.0625 \left[1.5 + 8.2575 + 1.332 \right]$$

$$I = 0.6930 //$$

SUBCODE: BMATS201/C201/M201/E201

MODEL QUESTION PAPER SOLUTIONS: MODULE-05

(Numerical Methods - 02)

1. Employ Taylor's Series method to obtain approx
-imate value of y when $x=0.2$ given that
 $\frac{dy}{dx} = xy$ and $y=1$ when $x=0$.

:- WKT Taylor's Series method,

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \dots$$

Given that

$$y'(x) = xy, x_0 = 0, y_0 = 1$$

$$\therefore \text{At } x=0, y'(0) = 0+1 = 1$$

$$\text{At } x=0, y''(x) = 1+y' \Rightarrow y''(0) = 2$$

$$\text{At } x=0, y'''(x) = y'' \Rightarrow y'''(0) = 2$$

$$\text{At } x=0, y^{IV}(x) = y''' \Rightarrow y^{IV}(0) = 2$$

∴ Taylor's Series method,

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) \\ + \frac{(x-x_0)^4}{4!}y^{IV}(x_0)$$

$$y(0.2) = 1 + (0.2)1 + \frac{(0.2)^2}{2}2 + \frac{(0.2)^3}{6}2 + \frac{(0.2)^4}{24}2$$

$$= 1.2428 //$$

(Q) By Taylor's series method, find the value of y at $x=0.1$ and $x=0.2$ to 5 places of decimals

from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

\therefore Given, $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

$$y(x_0) = y_0 \Rightarrow y(0) = 1$$

$$y' = x^2y - 1, \text{ at } x=0, \boxed{y'(0) = -1}$$

$$y'' = x^2y' + 2xy, \text{ at } x=0, \boxed{y''(0) = 0}$$

$$\begin{aligned} y''' &= x^2y'' + 2xy' + 2xy' + 2y \\ &= x^2y'' + 4xy' + 2y, \text{ at } x=0, \boxed{y'''(0) = 2} \end{aligned}$$

$$\begin{aligned} y^{iv} &= x^2y''' + 2xy'' + 4xy'' + 4y' + 2y \\ &= x^2y''' + 6xy'' + 6y', \text{ at } x=0, \boxed{y^{iv}(0) = -6} \end{aligned}$$

$$\begin{aligned} y^v &= x^2y^{iv} + 2xy''' + 6xy'' + 6y'' + 6y' \\ &= x^2y^{iv} + 8xy''' + 12y'', \text{ at } x=0, \boxed{y^v(0) = 0} \end{aligned}$$

Taylor's Series Method,

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \dots$$

At $x=0.1$

$$y(0.1) = 1 + (0.1)(-1) + \cancel{\frac{(0.1)^2}{2}(0)} + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(6) + 0$$

$$y(0.1) = 1 + (-0.1) + 0.00033 - 0.000025 \\ = 0.900305 //$$

$$y(0.2) = 1 + (0.2)(-1) + 0 + \frac{(0.2)^3}{6} 2 + \frac{(0.2)^4}{24} (-6) + 0 \\ = 1 - 0.2 + 0.00266 - 0.0004 \\ = 0.80226 //$$

(3) Solve using Modified Euler's Method

$$y'(x) = 3x + \frac{y}{2}, \quad y(0) = 1 \quad \text{then find } y(0.2) \text{ with}$$

$$h = 0.2$$

$$\therefore \text{Given } y' = 3x + \frac{y}{2}, \quad y(0) = 1$$

from Euler method,

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_1 = 1 + (0.2) \left[0 + \frac{1}{2} \right] = 1.1$$

Modified Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] \\ = 1 + \frac{(0.2)}{2} \left[\left(0 + \frac{1}{2} \right) + \left(3(0.2) + \frac{1.1}{2} \right) \right]$$

$$y_1^{(1)} = 1.165$$

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 1 + \frac{0.2}{2} \left[0.5 + \left(3(0.2) + \frac{1.165}{2} \right) \right] \\ = 1.1682 //$$

$$y_1^{(3)} = 1 + \frac{0.2}{2} \left[0.5 + \left(3(0.2) + \frac{1.1682}{2} \right) \right] \\ = 1.1684 //$$

The required value is 1.1684 //

(4) Using the Modified Euler's method, find $y(0.1)$
given that $\frac{dy}{dx} = x^2 + y$ and $y(0) = 1$ take step
 $h = 0.05$ and perform two modification in each stage

Given that

$$\frac{dy}{dx} = x^2 + y, \quad y_0 = 1, \quad x_0 = 0, \quad h = 0.05$$

To find $y(0.1) = ?$

Ist stage

$$x_1 = 0.05, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.05$$

Euler's method,

$$y_1 = y_0 + h f(x_0, y_0) \\ = 1 + 0.05(1) = 1.05 //$$

Modified Euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 1 + 0.025 \left[1 + (0.0025 + 1.05) \right] \\ = 1.0513.$$

$$y_1^{(2)} = y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \\ = 1 + 0.025 \left[1 + (0.0025 + 1.0513) \right] \\ = 1.0513 //$$

IInd stage

$$x_2 = 0.1 , h = 0.05 , x_1 = 0.05 , y_1 = 1.0513.$$

Euler's formula

$$y_2 = y_1 + h f(x_1, y_1) \\ = 1.0513 + 0.05 (1.0538) \\ = 1.1039$$

Modified Euler's Method

$$y_2^{(1)} = y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] \\ = 1.0513 + 0.025 \left[1.0538 + 1.1139 \right] \\ = 1.1054$$

$$y_2^{(2)} = y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_2, y_2^{(1)}) \right] \\ = 1.0513 + 0.025 \left[1.0538 + 1.1154 \right] \\ = 1.1055 //$$

The required root is 1.1055 //

(5) Apply Runge-Kutta method of fourth order to find an approximate value of y when $x=0.2$ given that

$$\frac{dy}{dx} = x+y \text{ and } y(0)=1$$

Given that, $y' = x+y$, $x_0=0$, $y_0=1$, $h=0.2$
W.R.T Runge-Kutta 4th order formula,

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

Where

$$\begin{aligned} k_1 &= h f(x_0, y_0) \\ &= 0.2 [0+1] \end{aligned}$$

$$k_1 = 0.2$$

$$\begin{aligned} k_2 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ &= 0.2 f(0.1, 1.1) \\ &= 0.2 [0.1+1.1] = 0.24 \end{aligned}$$

$$\begin{aligned} k_3 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) & k_4 &= h f(x_0 + h, y_0 + k_3) \\ &= 0.2 f(0.1, 1.12) & &= 0.2 f(0.2, 1.244) \\ & & &= 0.2888 \\ & & & \\ & & & \\ & & & \end{aligned}$$

$$\therefore y_1 = 1 + \frac{1}{6} [0.2 + 2(0.24) + 2(0.244) + 0.2888]$$

$$y_1 = 1.2428 \quad \text{at } x_1 = 0.2$$

(6) Using Runge-Kutta Method of fourth order, find $y(0.1)$ given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0)=0$ taking

$$h=0.1$$

$$\therefore \text{Given, } \frac{dy}{dx} = 3e^x + 2y, x_0=0, y_0=0, h=0.1$$

WKT Runge-Kutta 4th order formula,

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

where,

$$k_1 = hf(x_0, y_0)$$

$$= 0.1 f(0, 0)$$

$$= 0.1 \times 3$$

$$= 0.3$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f(0.05, 0.15)$$

$$= 0.1 (2.4538)$$

$$= 0.3453$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 \left[f(0.05, 0.172) \right]$$

$$= 0.3497$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1 f(0.1, 0.3497)$$

$$= 0.4014$$

$$y_1 = 0 + \frac{1}{6} [0.3 + 2(0.3453) + 2(0.3497) + 0.4014]$$

$$= 0.3490 //$$

(7) Using the Runge-Kutta method of fourth order
find $y(0.2)$ given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0.1) = 1.0912$

taking $h = 0.1$

$$\therefore \text{Given } \frac{dy}{dx} = \frac{y-x}{y+x}, \quad x_0 = 0.1, \quad y_0 = 1.0912$$
$$h = 0.1$$

WKT, Runge-Kutta 4th order method,

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = hf(x_0, y_0)$$

$$= 0.1f(0.1, 1.0912)$$

$$= 0.0832$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.15, 1.1328)$$

$$= 0.0766$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1f(0.15, 1.1295)$$

$$= 0.0765$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0.2, 1.1675)$$

$$= 0.0707$$

$$y_1 = 1.0912 + \frac{1}{6} [0.0832 + 2(0.0766) + 2(0.0765) + 0.0707]$$

$$= 1.1678 //$$

(8) using the Runge-Kutta method of fourth order, solve $y' = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, at $x = 0.2$

:- Given that

$$y' = \frac{y^2 - x^2}{y^2 + x^2}, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.2$$

WKT Runge-Kutta 4th order method,

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = h f(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f(0.1, 1.1)$$

$$= 0.1967$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f(0.1, 1.0983)$$

$$= 0.2 (0.9835)$$

$$= 0.1967$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= 0.2 f(0.2, 1.1967)$$

$$= 0.1891$$

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1 + \frac{1}{6} [0.2 + 2(0.1967) + 2(0.1967) + 0.1891]$$

$$= 1.1959 //$$

(g) Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$,

$y(0.2) = 1.2773$, $y(0.3) = 1.5049$ compute $y(0.4)$

using Milne's method

Given $y' = xy + y^2$, $x_0 = 0$ $x_1 = 0.1$ $x_2 = 0.2$ $x_3 = 0.3$

$$y_0 = 1 \quad y_1 = 1.1169 \quad y_2 = 1.2773 \quad y_3 = 1.5049$$

To find $y_4 = ?$ at $x_4 = 0.4$

x	y	$y' = xy + y^2$
$x_0 = 0.0$	$y_0 = 1$	$y'_0 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$y'_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$y'_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$y'_3 = 2.7161$
$x_4 = 0.4$	$y_4 = ?$	$y'_4 = ?$

Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3} [2y'_1 - y'_2 + 2y'_3]$$

$$= 1 + \frac{4(0.1)}{3} [2(1.3591) - (1.8869) + 2(2.7161)]$$

$$y_4^{(P)} = 1.8330$$

$$\therefore y'_4 = (0.4)(1.8330) + (1.8330)^2 \quad (\because y' = xy + y^2)$$

$$y'_4 = 4.0930$$

corrector formula

$$y_4^C = y_2 + \frac{h}{3} [y'_2 + 4y'_3 + y'^{(P)}_4]$$

$$= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7161) + 4.0930]$$

$$y_4^C = 1.8387 //$$

thus reqd soln $y_4 = 1.838$

$$y'_4 = 4.0930 //$$

(10) Given $\frac{dy}{dx} = x - y^2$, data $y(0) = 0$, $y(0.2) = 0.02$

$y(0.4) = 0.0795$, $y(0.6) = 0.1762$ Compute y at $x = 0.8$
by applying Milne's Method.

$$\therefore y' = x - y^2,$$

$$x \quad y \quad y' = x - y^2$$

$$x_0 = 0 \quad y_0 = 0 \quad y'_0 = 0$$

$$x_1 = 0.2 \quad y_1 = 0.02 \quad y'_1 = 0.1996$$

$$x_2 = 0.4 \quad y_2 = 0.0795 \quad y'_2 = 0.3936$$

$$x_3 = 0.6 \quad y_3 = 0.1762 \quad y'_3 = 0.5689$$

$$x_4 = 0.8 \quad y_4 = \quad y'_4 =$$

Milne's predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3} [2y_1' - y_2' + 2y_3'] \\ = 0 + \frac{4(0.2)}{3} [2(0.1996) - (0.3936) + 2(0.5689)]$$

$$y_4^{(P)} = 0.266 [1.1434] = 0.3041$$

$$\therefore y_4'^{(P)} = 0.8 - (0.3041)^2 = 0.4075$$

corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3} [y_2' + 4y_3' + y_4'^{(P)}] \\ = 0.0795 + \left(\frac{0.2}{3}\right) [0.3936 + 4(0.5689) + 0.4075] = 0.3046$$

The mean value $y_4 = 0.3046$.

(1) Using Mathematical tools, write the code to find the soln of $\frac{dy}{dx} = 1 + \frac{y}{x}$ at $y(2)$ taking $h=0.2$ given that $y(1)=2$ by Runge-Kutta 4th order method.

:- from sympy import *

import numpy as np

def RungeKutta (g, x0, h, y0, xm):

$x, y = \text{symbols}('x y')$

$f = \text{lambdify}([x, y], g)$

$xt = x0 + h$

$y = [y0]$

while $xt <= xm$:

$K1 = h * f(x0, y0)$

$K2 = h * f(x0 + h/2, y0 + K1/2)$

$K3 = h * f(x0 + h/2, y0 + K2/2)$

$K4 = h * f(x0 + h, y0 + K3)$

$y1 = y0 + (1/6) * (K1 + 2 * K2 + 2 * K3 + K4)$

$y.append(y1)$

$x0 = xt$

$y0 = y1$

$xt = xt + h$

return np.round(y, 2)

RungeKutta('1+(y/x)', 1, 0.2, 2, 2)

(12) Write a program to solve D.E $y' = 2y + 3e^x$ with $y(0) = 0$ using the Taylor's series at $x = 0.1(0.1)0.3$.

- From numpy import array

```
def taylor(deriv, x, y, xstop, h):
```

```
    x = []
```

```
    y = []
```

```
    x.append(x)
```

```
    y.append(y)
```

```
    while x < xstop:
```

```
        D = deriv(x, y)
```

```
        H = 1.0
```

```
        for j in range(3):
```

```
            H = H * h / (j + 1)
```

```
            y = y + D[j] * H
```

```
            x = x + h
```

```
            x.append(x)
```

```
            y.append(y)
```

```
    return array(x), array(y)
```

```
def deriv(x, y):
```

```
    D = zeros((4, 1))
```

```
    D[0] = [2 * y[0] + 3 * exp(x)]
```

```
    D[1] = [4 * y[0] + 9 * exp(x)]
```

```
    D[2] = [8 * y[0] + 21 * exp(x)]
```

```
    D[3] = [16 * y[0] + 45 * exp(x)]
```

```
    return D
```

$$x = 0.0$$

$$xStop = 0.3$$

$$y = \text{array}([0.0])$$

$$h = 0.1$$

`x, y = taylor(deiv, x, y, xStop, h)`

`point("The required value are: at`

`x = 0.2f, y = 0.5f")` $(x[0], y[0], x[1], y[1],$
 $, x[2], y[2], x[3], y[3]))$