VECTOR CALCULUS.

Gradient of a scalar field:

If $\phi(x,y,z)$ is a differentiable scalar function, then the gradient of ϕ is defined as

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

where i, j, k are unit vectors and $\nabla\phi$ is a vector quantity.

Note: (i) ∇ (del) is a vector differentiable operator given by $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$

(ii) $\frac{\nabla\phi}{|\nabla\phi|}$ is the unit normal vector to the given surface $\phi(x,y,z) = c$

Problems of gradient and to find unit normal vector:

① Find grad ϕ at $(2,1,-1)$ given $\phi = x^2y - 2xz + 2y^2z^4$

→ $$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2y - 2xz + 2y^2z^4)\hat{i} + \frac{\partial}{\partial y}(x^2y - 2xz + 2y^2z^4)\hat{j} + \frac{\partial}{\partial z}(x^2y - 2xz + 2y^2z^4)\hat{k}$$

$$\nabla\phi = (2xy - 2z + 0)\hat{i} + (x^2 - 0 + 4yz^4)\hat{j} + (0 - 2x + 8y^2z^3)\hat{k}$$

$$\nabla\phi_{(2,1,-1)} = [2(2)(+1) - 2(-1)]\hat{i} + [2^2 + 4(1)(-1)^4]\hat{j} + [-2(2) + 8(1)^2(-1)^3]\hat{k}$$

$$\nabla\phi_{(2,1,-1)} = 6\hat{i} + 8\hat{j} - 12\hat{k}$$

② Find grad ϕ at $(1,-2,-1)$ given $\phi = 3x^2y - y^3z^2$. Also find magnitude of grad ϕ

$\rightarrow$ $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} +$

$\frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$

$\nabla\phi = (6xy - 0)\hat{i} + (3x^2 - 3y^2z^2)\hat{j} + (0 - 2y^3z)\hat{k}$

$\nabla\phi_{(1,-2,-1)} = [6(1)(-2)]\hat{i} + [3(1)^2 - 3(-2)^2(-1)^2]\hat{j}$
$+ [-2(-2)^3(-1)]\hat{k}$

$= -12\hat{i} - 9\hat{j} - 16\hat{k}$

$|\nabla\phi| = \sqrt{(-12)^2 + (-9)^2 + (-16)^2} = \sqrt{481} = 21.9317$ //

(3) Find the unit normal vector to the surface $\phi = xy^3z^2 - 4$ at $(-1,-1,2)$

$\rightarrow$ $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = \frac{\partial}{\partial x}(xy^3z^2 - 4)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2 - 4)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2 - 4)\hat{k}$

$\nabla\phi = (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$

$\nabla\phi(-1,-1,2) = [(-1)^3(2)^2]\hat{i} + [3(-1)(-1)^2(2)^2]\hat{j} +$
$2[(-1)(-1)^3(2)]\hat{k}$

$= -4\hat{i} - 12\hat{j} + 4\hat{k}$

$|\nabla\phi| = \sqrt{(-4)^2 + (-12)^2 + 4^2} = \sqrt{176} = 13.2665$

$4\sqrt{11}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{4\sqrt{11}} = \frac{\cancel{4}(-\hat{i} - 3\hat{j} + \hat{k})}{\cancel{4}\sqrt{11}}$

$\Rightarrow$ $\hat{n} = \frac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$

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(4) Find the unit vector normal to the surface ~~at~~ $\phi = x^2y + y^2z + z^2x = 5$ at $(1,-1,2)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2y + y^2z + z^2x - 5)\hat{i} + \frac{\partial}{\partial y}(x^2y + y^2z + z^2x - 5)\hat{j}$$
$$+ \frac{\partial}{\partial z}(x^2y + y^2z + z^2x - 5)\hat{k}$$

$$\nabla\phi = (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2z x)\hat{k}$$
$$\nabla\phi_{(1,-1,2)} = [2(1)(-1) + 2^2]\hat{i} + [1^2 + 2(-1)(2)]\hat{j}$$
$$+ [(-1)^2 + 2(2)(1)]\hat{k}$$

$$\nabla\phi = 2\hat{i} - 3\hat{j} + 5\hat{k}$$
$$|\nabla\phi| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} - 3\hat{j} + 5\hat{k}}{\sqrt{38}}$$

Gradient of a scalar field:
$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Note: (i) The angle b/w 2 surfaces = the angle b/w their normals. If $\phi_1(x,y,z) = c_1$ and $\phi_2 = (x,y,z) = c_2$ be two surfaces then $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$, where θ is the angle b/w the normals.

(ii) If $\theta = \pi/2$, then the surfaces are said to intersect each other orthogonally. Hence $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ is the condition for surfaces to intersect orthogonally

Find the angle b/w the surfaces:

(5) $x^2+y^2-z^2=4$ and $z=x^2+y^2-13$ at $(2,1,2)$

Let $\phi_1 = x^2+y^2-z^2-4$

$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\hat{i} + \frac{\partial\phi_1}{\partial y}\hat{j} + \frac{\partial\phi_1}{\partial z}\hat{k}$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

$\nabla\phi_1(2,1,2) = 4\hat{i} + 2\hat{j} - 4\hat{k}$

$\therefore |\nabla\phi_1| = \sqrt{4^2+2^2+(-4)^2} = \sqrt{36} = 6$

Let $\phi_2 = z-x^2-y^2+13$

$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k}$

$\nabla\phi_2 = -2x\hat{i} - 2y\hat{j} + 1\hat{k}$

$\nabla\phi_2(2,1,2) = -4\hat{i} - 2\hat{j} + 1\hat{k}$

$\therefore |\nabla\phi_2| = \sqrt{(-4)^2+(-2)^2+1^2} = \sqrt{21}$

Angle b/w two surfaces,

$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$= \frac{(4\hat{i}+2\hat{j}-4\hat{k})(-4\hat{i}-2\hat{j}+1\hat{k})}{6\sqrt{21}}$

$= \frac{-16-4-4}{6\sqrt{21}} = \frac{-24}{6\sqrt{21}} = \frac{-4}{6\sqrt{21}}$

$|\cos\theta| = \cancel{\cos} \left|\frac{-4}{6\sqrt{21}}\right|$

$$\boxed{\theta = \cos^{-1}\left(\frac{4}{6\sqrt{21}}\right)}$$

(6) Find the angle b/w the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

→ Let $\phi = xy - z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$\nabla\phi(4,1,2) = \hat{i} + 4\hat{j} - 4\hat{k} = \nabla\phi_1$$

$$\nabla\phi(3,3,-3) = 3\hat{i} + 3\hat{j} + 6\hat{k} = \nabla\phi_2$$

$$|\nabla\phi_1| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{33}$$

$$|\nabla\phi_2| = \sqrt{3^2 + 3^2 + 6^2} = 3\sqrt{6}$$

Angle b/w two surfaces,

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| . |\nabla\phi_2|}$$

$$= \frac{(\hat{i} + 4\hat{j} - 4\hat{k})(3\hat{i} + 3\hat{j} + 6\hat{k})}{\sqrt{33} \times 3\sqrt{6}}$$

$$= \frac{3 + 12 - 24}{3\sqrt{198}} = \frac{-9}{3 \times 3\sqrt{22}} = \left| -\frac{1}{\sqrt{22}} \right|$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)}$$

(7) Find the constants a and b such that the 2 surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at a point $(1, -1, 2)$

→ Sub $(1, -1, 2)$ into the surface $ax^2 - byz = (a+2)x$ we get,

$$ax^2 - byz = (a+2)x$$

$$(\cancel{a} + 2b) = \cancel{a} + 2$$

$$2b = 2 \Rightarrow \boxed{b = 1}$$

Rewriting we get $ax^2 - yz = (a+2)x$

Let $\phi_1 = ax^2 - yz - ax - 2x$; $\phi_2 = 4x^2y + z^3 - 4$

To find 'a' using the orthogonality condition

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$ —— ①

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\hat{i} + \frac{\partial\phi_1}{\partial y}\hat{j} + \frac{\partial\phi_1}{\partial z}\hat{k}$$

$$\nabla\phi_1 = (2ax - a - 2)\hat{i} - z\hat{j} - y\hat{k}$$

$$\nabla\phi_1(1,-1,2) = (2a - a - 2)\hat{i} - 2\hat{j} + \hat{k}$$

$$= (a-2)\hat{i} - 2\hat{j} + \hat{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k}$$

$$\nabla\phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

$$\nabla\phi_2(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

Sub in eqⁿ ①,

$$[(a-2)\hat{i} - 2\hat{j} + \hat{k}] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$= -8(a-2) - 8 + 12 = 0$$

$$= -8a + 16 - 8 + 12 \Rightarrow -8a = -20 \Rightarrow \boxed{a = \frac{5}{2}}$$

Thus, required values are $a = \frac{5}{2}$, $b = 1$ //

Directional derivative :

Directional derivative of $\phi(x,y,z)$ along a given direction $\vec{d}$ is $\nabla\phi \cdot n$ where $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$

noting that $\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$.

Note: Directional derivative is maximum along the normal vector i.e grad ϕ.

(8) Find the directional derivative of $4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2\hat{y} - 3\hat{j} + 6\hat{k}$

$\rightarrow$ Directional derivative $= \nabla\phi \cdot \hat{n}$ —— (1)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$\nabla\phi_{(2,-1,2)} = (32-24)\hat{i} + (+48)\hat{j} + (96-12)\hat{k}$$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k}.$$

Let $\vec{d} = 2\hat{y} - 3\hat{j} + 6\hat{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

Sub in (1),

$$D.D = (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{(2\hat{i} - 3\hat{j} + 6\hat{k})}{7}$$

$$= \frac{16 - 144 + 504}{7} \Rightarrow \frac{376}{7}$$

(2) Find the directional derivative of $\phi = \frac{xz}{x^2+y^2}$ at the point $(1,-1,1)$ at the direction of $i - 2j + k$.

Directional derivative $= \nabla\phi \cdot \hat{n}$ —— (1)

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\hat{k}$$

$$= z\left\{\frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}\right\}\hat{i} + xz\left\{\frac{-1}{(x^2+y^2)^2}\cdot 2y\right\}\hat{j} + \frac{x}{x^2+y^2}\hat{k}$$

$$\nabla\phi_{(1,-1,1)} = 1\left\{\frac{2-2}{4}\right\}\hat{i} + \frac{2}{4}\hat{j} + \frac{1}{2}\hat{k} \Rightarrow 0\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Let $\vec{d} = \hat{i} - 2\hat{j} + \hat{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$= \frac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{6}}$$

Sub in ①,

$$D.D = \left(0\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) \cdot \frac{(\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{6}}$$

$$= \frac{0 - 1 + 1/2}{\sqrt{6}} \Rightarrow \frac{-1}{2\sqrt{6}} \text{ //}$$

(10) In which direction the D.D of x^2yz^3 is max^m at $(2,1,-1)$ and find the magnitude of this maximum.

→ WKT, the directional derivative is max^m along the normal vector i.e. $\nabla\phi$.

Let $\phi = x^2yz^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

$\nabla\phi\,(2,1,-1) = -4\hat{i} - 4\hat{j} + 12\hat{k}$ is the required direction in which D.D is max^m.

Also, $|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + 144} = 4\sqrt{11}$ //

(11) Find $\nabla\phi$ and $|\nabla\phi|$ of $\phi = x^3 + y^3 + z^3 - 3xy$ at $(1,-2,-1)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \cancel{3x^2\hat{i} + z}$$

$$= (3x^2 - 3y)\hat{i} + (3y^2 - 3x)\hat{j} + 3z^2\hat{k}$$

$$\nabla\phi(1,-2,-1) = 9\hat{i} + 9\hat{j} + 3\hat{k}$$

$|\nabla\phi| = \sqrt{9^2+9^2+3^2} = 3\sqrt{19}$

(12) Find the angle b/w surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$

→ Let $\phi_1 = x^2+y^2+z^2-9$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_1(2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$

$|\nabla\phi_1| = \sqrt{4^2+(-2)^2+4^2}$

$= \sqrt{36} = 6$

Let $\phi_2 = z - x^2 - y^2 + 3$

$\nabla\phi_2 = -2x\hat{i} - 2y\hat{j} + \hat{k}$

$\nabla\phi_2(2,-1,2) = -4\hat{i} + 2\hat{j} + \hat{k}$

$|\nabla\phi_2| = \sqrt{(-4)^2+2^2+1^2}$

$= \sqrt{21}$

Angle b/w two surfaces is given by,

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$\cos\theta = \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(-4\hat{i}+2\hat{j}+\hat{k})}{6\sqrt{21}}$$

$$= \frac{-16-4+4}{6\sqrt{21}} = \frac{-16}{6\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

(13) Find D.D of $x^2+y^2+2z^2$ at $(1,2,3)$ in direction of the line $\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

→ Directional derivative $= \nabla\phi \cdot \hat{n}$ — (1)

$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$

$\nabla\phi(1,2,3) = 2\hat{i} + 4\hat{j} + 12\hat{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{4^2+(-2)^2+1^2}} = \frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}}$$

Sub in (1),

$$D.D = (2\hat{i}+4\hat{j}+12\hat{k})\cdot\frac{(4\hat{i}-2\hat{j}+\hat{k})}{\sqrt{21}}$$

$$= \frac{8-8+12}{\sqrt{21}} \Rightarrow \frac{12}{\sqrt{21}}$$

Divergence of a vector field:

If $\vec{F} = F_1 i + F_2 j + F_3 k$ is a continuosly differentiable vector function, then divergence of $\vec{F}$ is given by

$$\text{div}\,\vec{F} = \nabla . \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) . \left(F_1 i + F_2 j + F_3 k\right)$$

i.e $\text{div}\,\vec{F} = \nabla . \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

where F_1, F_2, F_3 are functions of x, y, z

Clearly $\text{div}\,\vec{F}$ is a scalar quantity.

Note: If $\text{div}\,\vec{F} = 0$, then $\vec{F}$ is said to be solenoidal

Curl of a vector field:

If $\vec{F} = F_1 i + F_2 j + F_3 k$ is a continuosly differentiable vector factor, the curl of $\vec{F}$ is defined as

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

where F_1, F_2, F_3 are all functions of x, y, z

clearly $\text{curl}\,\vec{F}$ is a vector quantity.

Note: If $\text{curl}\,\vec{F} = 0$, then $\vec{F}$ is said to be irrotational. Irrotational vector field is also called as conservative or potential field

Finding scalar potential ϕ:

If $\vec{F}$ is irrotational, then there always exists a scalar point function ϕ such that $\nabla \phi = \vec{F}$

Then ϕ is called a scalar potential of $\vec{F}$

Problems on divergence and curl

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~~Irrotational vectors:~~

~~Finding scalar potential ϕ:~~

① Find divergence and curl of the vector $\vec{V} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2,-1,1)$

→ To find div $\vec{V}$:

Let $\vec{V} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$

$= V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

where, $V_1 = xyz$ $\quad V_2 = 3x^2y$ $\quad V_3 = xz^2 - y^2z$

$\therefore$ div $\vec{V} = \nabla \cdot \vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$\text{div}\,\vec{V}(2,-1,1) = -1 + 12 + 4 - 1 \Rightarrow 14$ //

$$\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$\text{Curl}\,\vec{V} = \hat{i}[-2yz-0] - \hat{j}[z^2-xy] + \hat{k}[6xy-xz]$

$\text{Curl}\,\vec{V}_{(2,-1,1)} = \underline{\underline{2\hat{i} - 3\hat{j}}} - 14\hat{k}$

② PYQ

Find divergence $\vec{F}$ and curl $\vec{F}$ if $\vec{F} = \text{grad}(xy^3z^2)$

Let $\phi = xy^3z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla\phi = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\vec{F} = \nabla\phi = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \quad (\text{say})$$

where, $F_1 = y^3z^2$, $F_2 = 3xy^2z^2$ $F_3 = 2xy^3z$

$\therefore$ $\text{div}\,\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$= \underline{2xy(3z^2 + y^2)}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xy^3z) - \frac{\partial}{\partial z}(3xy^2z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2xy^3z) - \frac{\partial}{\partial z}(y^3z^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3xy^2z^2) + \frac{\partial}{\partial y}(y^3z^2)\right]$$

$$= [6xy^2z - 6xy^2z]\hat{i} - [2y^3z - 2y^3z]\hat{j} + [3y^2z^2 - 3y^2z^2]\hat{k}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

$\therefore \vec{F}$ is irrotational

(3) PYQ ****

Find divergence $\vec{F}$ and curl $\vec{F}$ if $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Let $\phi = x^3+y^3+z^3-3xyz$

$$\nabla\phi = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla\phi = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

$$\vec{F} = \nabla\phi = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

$$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \quad (\text{say})$$

where, $F_1 = (3x^2-3yz)$; $F_2 = (3y^2-3xz)$; $F_3 = 3z^2-3xy$

$\therefore \text{div}\,\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z$$

$$= 6(x+y+z)$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$$

$$\hat{i}\left[\frac{\partial}{\partial y}(3z^2-3xy)-\frac{\partial}{\partial z}(3y^2-3xz)\right]-\hat{j}\left[\frac{\partial}{\partial x}(3z^2-3xy)-\frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3y^2-3xz)-\frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$=[-3x+3x]\hat{i}-\hat{j}[-3y+3y]+\hat{k}[-3z+3z]$$

$$=0\hat{i}-0\hat{j}+0\hat{k}$$

$$=\vec{0}$$

$\therefore \vec{F}$ is irrotational

(4) If $\vec{F}=(x+y+1)\hat{i}+\hat{j}-(x+y)\hat{k}$. Prove that $\vec{F}\cdot \text{curl}\,\vec{F}=0$

Let $F_1=x+y+z$, $F_2=1$, $F_3=-x-y$

$$\text{curl}\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+1 & 1 & -x-y\end{vmatrix}$$

$$\hat{i}\left[\frac{\partial}{\partial y}(-x-y)-\frac{\partial}{\partial z}(1)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-x-y)-\frac{\partial}{\partial z}(x+y+1)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(1)-\frac{\partial}{\partial y}(x+y+1)\right]$$

$$=-\hat{i}+1\hat{j}-\hat{k}$$

$$\therefore \vec{F}\cdot \text{curl}\,\vec{F}=\left[(x+y+1)\hat{i}+\hat{j}-(x+y)\hat{k}\right]\cdot\left[-\hat{i}+1\hat{j}-\hat{k}\right]$$

$$=-(x+y+1)\hat{i}+1\hat{j}+(x+y)\hat{k}$$

$$=\cancel{-x}\cancel{-y}\cancel{-1}\cancel{+1}\cancel{+x}\cancel{+y}$$

$$=\underline{\underline{0}}$$

(5) Find divergence of grad ϕ given $\phi=2x^3y^2z^4$

(6) Given $F=3xyz^2\,\hat{i}+4x^3y\,\hat{j}+xy^2z\,\hat{k}$. Find $\nabla(\nabla\cdot\vec{F})$ at point $(-1,2,1)$

(5) $\phi = 2x^3y^2z^4$

$$\nabla\phi = \frac{\partial}{\partial x}(2x^3y^2z^4)\hat{i} + \frac{\partial}{\partial y}(2x^3y^2z^4)\hat{j} + \frac{\partial}{\partial z}(2x^3y^2z^4)\hat{k}$$

$$\nabla\phi = 6x^2y^2z^4 + 4x^3yz^4 + 8x^3y^2z^3$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \;//$$

(6) Here,

$F_1 = 3xyz^2$, $F_2 = 4x^3y$, $F_3 = -xy^2z$,

$$\text{div.}\vec{F} = \nabla\vec{F} = \frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}$$

$$\frac{\partial}{\partial x}(3yxz^2)\hat{i} + \frac{\partial}{\partial y}(4x^3y) + \frac{\partial}{\partial z}(-xy^2z)\hat{k}$$

$$\nabla\vec{F} = 3yz^2 + 4x^3 - xy^2$$

$$\nabla(\text{div}\vec{F}) = \frac{\partial\Delta\vec{F}}{\partial x}\hat{i} + \frac{\partial\Delta\vec{F}}{\partial y}\hat{j} + \frac{\partial\Delta\vec{F}}{\partial z}\hat{k}$$

$$= (12x^2 - y^2)\hat{i} + (3z^2 - 2xy)\hat{j} + 6yz\hat{k}$$

$$\nabla(\text{div.}\vec{F})_{(-1,2,1)} = 8\hat{i} + 7\hat{j} + 12\hat{k}$$

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(7) Show that $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is both solenoidal and irrotational

→ Let $\vec{F} = \frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j$

$F_1 = \frac{x}{x^2+y^2}$; $F_2 = \frac{y}{x^2+y^2}$; $F_3 = 0$

(i) $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial z}(0)$

$= \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} + \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2} \Rightarrow 0$ //

$\therefore \nabla \cdot \vec{F} = 0 \Rightarrow \vec{F}$ is solenoidal

(ii) curl $\vec{F} = \nabla \times \vec{F}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]$

$= 0\hat{i} + 0\hat{j} + \hat{k}\left[y\left\{\frac{-1}{(x^2+y^2)^2} 2x\right\} - x\left\{\frac{-1}{(x^2+y^2)^2} \cdot 2y\right\}\right]$

$= 0\hat{i} + 0\hat{j} + \hat{k}\left[\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right]$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$ //

(8) Find the constant A such that $\vec{F} = (x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)\hat{k}$ is solenoidal

→ Since $\vec{F}$ is solenoidal we must have $\nabla\cdot\vec{F} = 0$

$$\Rightarrow \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

Here $F_1 = x+3y$, $F_2 = y-2x$, $F_3 = x+az$

$$\therefore \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(x+az) = 0$$

$$= 1+1+a = 0 \Rightarrow \boxed{a=-2}$$

(9) If $\vec{F} = (x^2-y^2+x)i - (2xy+y)j$. Show that $\vec{F}$ is irrotational

$F_1 = x^2-y^2+x$; $F_2 = -2xy-y$; $F_3 = 0$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2+x & -2xy-y & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy-y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2+x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xy-y) - \frac{\partial}{\partial y}(x^2-y^2+x)\right]$$

$$= 0\hat{i} + 0\hat{j} + \hat{k}\left[-2y+2y\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} \Rightarrow 0$$

(10) Show that $\vec{F} = (y+z)i + (z+x)j + (x+y)k$ is irrotational. Also find scalar function ϕ such that $\nabla\phi = \vec{F}$

→ $F_1 = y+z$; $F_2 = z+x$; $F_3 = x+y$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - j\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right]$

$+ k\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$

$\nabla \times \vec{F} = i[1-1] - j[1-1] + k[1-1]$

$= 0i - 0j + 0\hat{k} \Rightarrow \vec{0}$ //

$\therefore \vec{F}$ is irrotational.

<u>To find ϕ such that $\nabla\phi = \vec{F}$:</u>

consider $\nabla\phi = \vec{F}$

$\Rightarrow \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

Equating,

$\frac{\partial\phi}{\partial x} = y+z$; $\frac{\partial}{\partial y} = z+x$; $\frac{\partial\phi}{\partial z} = x+y$

$\phi = \int(y+z)dx + f_1(y,z)$; $\phi = \int(z+x)\,dy + f_2(x,z)$;

$\phi = \int(x+y)\,dz + f_3(x,y)$

$\Rightarrow$ ~~$\phi = \int(y+z)\,dx + f_1(y,z)$~~

$\phi = (y+z)x + f_1(y,z)$; $\phi = (z+x)y + f_2(x,z)$; $\phi = (x+y)z + f_3(x,y)$

$\phi = xy + xz + f_1(y,z)$; $\phi = yz + yx + f_2(x,z)$; $\phi = xz + yz + f_3(x,y)$

choosing, $f_1(y,z) = yz$

$f_2(x,z) = xz$

$f_3(x,y) = xy$

$\therefore$ Unique answer for ϕ is,

$\phi = \underline{xy + xz + yz}$

(9) Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + zx + 2yz^2)j + (2y^2z + xy)\hat{k}$ is conservative force field. Hence find the scalar potential field.

$\rightarrow$ $F_1 = 2xy^2 + yz$; $F_2 = 2x^2y + zx + 2yz^2$; $F_3 = 2y^2z + xy$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + yz & 2x^2y + zx + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(2y^2z + xy) - \frac{\partial}{\partial z}(2x^2y + zx + 2yz^2)\right]$$

$$- j\left[\frac{\partial}{\partial x}(2y^2z + xy) - \frac{\partial}{\partial z}(2xy^2 + yz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2x^2y + zx + 2yz^2) - \frac{\partial}{\partial y}(2xy^2 + yz)\right]$$

$$\nabla \times \vec{F} = i[(4yz + x) - (x + 4yz)] - \hat{j}[y - y] + k[(4xy + z) - (4xy + z)]$$

$$= 0i + 0j + 0k \Rightarrow 0$$

$\therefore$ $\vec{F}$ is irrotational.

<u>To find ϕ such that $\nabla\phi = \vec{F}$:</u>

consider $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = [2xy^2 + yz]\hat{i} + [2x^2y + zx + 2yz^2]\hat{j} + [2y^2z + xy]\hat{k}$$

Equating,

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy^2 + yz \ ; \ \frac{\partial\phi}{\partial y} = [2x^2y + zx + 2yz^2] \ ; \ \frac{\partial\phi}{\partial z} = 2y^2z + xy$$

$$\phi = \int(2xy^2 + yz)\,dx + f_1(y, z) \ ; \ \phi = \int(2x^2y + zx + 2yz^2)\,dy + f_2(x, z)$$

$$\phi = \int(2y^2z + xy)\,dz + f_3(x, y)$$

$$= \phi = 2y^2 \cdot \frac{x^2}{2} + yzx + f_1(y, z) \ ; \ \phi = 2x^2 \cdot \frac{y^2}{2} + zxy + 2z^2\frac{y^2}{2} + f_2(x, z)$$

$$\phi = 2y^2 \cdot \frac{z^2}{2} + xyz + f_3(x, y)$$

$\phi = x^2y^2 + xyz + f_1(y,z)$; $\phi = x^2y^2 + xyz + y^2z^2 + f_2(x,z)$;
$\phi = y^2z^2 + xyz + f_3(x,y)$

Choosing, $f_1(y,z) = y^2z^2$
$f_2(x,z) = \cancel{xyz}\ 0$
$f_3(x,y) = x^2y^2$

$\therefore$ Unique ans for ϕ is $xyz + x^2y^2 + y^2z^2$

(10) Find the constants a, b, c if $\vec{F} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)\hat{k}$ is irrotational. Hence find the scalar potential ϕ such that $\nabla\phi = \vec{F}$.

Given $\vec{F}$ is irrotational i.e. curl $\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

where $F_1 = x+y+az$; $F_2 = bx+2y-z$; $F_3 = x+cy+2z$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix} = 0$$

$\therefore \hat{i}[c+1] - \hat{j}[1-a] + \hat{k}[b-1] = 0\hat{i} + 0\hat{j} + 0\hat{k}$.

$c+1 = 0$; $-1+a = 0$; $b-1 = 0$

$\boxed{c=-1}$ $\boxed{a=1}$ $\boxed{b=1}$

$\therefore \vec{F} = (x+y+z)\hat{i} + (x+2y-z)\hat{j} + (x-y+2z)\hat{k}$

To find ϕ such that $\nabla\phi = \vec{F}$:

consider $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}k = (x+y+z)i + (x+2y-z)j + (x-y+2z)\hat{k}$$

$$\frac{\partial \phi}{\partial x} = x+y+z \; ; \quad \frac{\partial \phi}{\partial y} = x+2y-z \; ; \quad \frac{\partial \phi}{\partial z} = x-y+2z$$

$$\phi = \int (x+y+z)\,dx + f_1(y,z) \; ; \quad \phi = \int (x+2y-z)\,dy + f_2(x,z) \; ;$$
$$\phi = \int (x-y+2z)\,dz + f_3(x,y)$$

$$= \phi = \frac{x^2}{2} + yx + zx + f_1(y,z) \; ; \quad \phi = xy + \frac{y^2}{\cancel{2}}\times\cancel{2} - zy + f_2(x,z)$$

$$\phi = xz - yz + \frac{\cancel{2}z^2}{\cancel{2}} + f_3(x,y)$$

i.e, $\phi = \frac{x^2}{2} + xy + xz + f_1(y,z)$;

$$\phi = xy + y^2 - yz + f_2(x,z) \; ;$$
$$\phi = xz - yz + z^2 + f_3(x,y)$$

Choosing
$$f_1(y,z) = y^2 - yz + z^2$$
$$f_2(x,z) = \frac{x^2}{2} + xz + z^2$$
$$f_3(x,y) = \frac{x^2}{2} + xy + y^2$$

we get unique exp. for ϕ as :
$$\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2$$

(11) Find the constants a and b. $\vec{F} = (ax^2y + z^3)\hat{i} + (3x^3 - z)\hat{j} + (bxz^2 - y)\hat{k}$ is irrotational. Hence find scalar potential ϕ such that $\nabla\phi = \vec{F}$.

$\rightarrow$ given $\vec{F}$ is irrotational i.e, curl $\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

where $F_1 = axy + z^3 \quad F_2 = 3x^3 - z \quad F_3 = bxz^2 - y$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2y+z^3 & 3x^3-z & bxz^2-y \end{vmatrix} = 0$$

$= i[-1+1] - j[bz^2-3z^2] + \hat{k}[9x^2-ax^2] = 0\hat{i}+0\hat{j}+0\hat{k}$

$= -bz^2+3z^2 = 0$; $9x^2-ax^2=0$

$(-b+3)z^2 = 0$; $x^2(9-a)=0$

$-b+3=0$; $9-a=0$

$\Rightarrow \boxed{b=3}$; $\Rightarrow \boxed{a=9}$

$\therefore \vec{F} = (9x^2y+z^3)\hat{i} + (3x^3-z)\hat{j} + (3xz^2-y)\hat{k}$

To find ϕ such that $\nabla\phi = \vec{F}$:

$\nabla\phi = \vec{F}$

$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (9x^2y+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$

$\frac{\partial\phi}{\partial x} = 9x^2y+z^3$; $\frac{\partial\phi}{\partial y} = 3x^3-z$; $\frac{\partial\phi}{\partial z} = 3xz^2-y$

$\int\phi = \int 9x^2y+z^3 + f_1(y,z)$; $\int\phi = \int 3x^3-z+f_2(x,z)$

$\int\phi = \int 3xz^2-y + f_3(x,y)$

$\phi = 9\frac{x^3}{3}y + xz^3 + f_1(y,z)$; $\phi = 3x^3y - zy + f_2(x,z)$

$\phi = 3x\frac{z^3}{3} - yz + f_3(x,y)$

i.e, $\phi = 3x^3y + xz^3 + f_1(y,z)$

$\phi = 3x^3y - yz + f_2(x,z)$

$\phi = xz^3 - zy + f_3(x,y)$

Choosing, $f_1(y,z) = -\cancel{xy}\, yz$

$f_2(x,z) = xz^3$

$f_3(x,y) = 3x^3y$

we get unique exp for ϕ as : $3x^3y + xz^3 - yz$

Curvilinear Coordinates:

Let (x, y, z) be the coordinates of the point P, expressible in terms of new coordinates (u_1, u_2, u_3). If it is possible to express u_1, u_2, u_3 in terms of x, y, z then the coordinates u_1, u_2, u_3 are known as curvilinear coordinates of point P.

Note: (i) The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ are called coordinate surfaces where c_1, c_2, c_3 are constants

(ii) The intersection of a pair of coordinate surface give rise to a curve called as coordinate curve

Orthogonal curvilinear coordinates:

A system of curvilinear coordinates is said to be orthogonal if at each, the tangents to the coordinates curve are mutually $\perp$lar.

Unit vectors, scale factors and Orthogonality condn

Suppose $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ are called the tangent vectors to coordinate curves and unit tangent vectors in same dirn are respectively,

$$\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \Big/ \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \Big/ \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

The quantities $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$

are called scale factors

for orthogonality condⁿ, we must have.

$\hat{e}_1 \cdot \hat{e}_2 = 0$, $\hat{e}_2 \cdot \hat{e}_3 = 0$, $\hat{e}_3 \cdot \hat{e}_1 = 0$

[III'ar to $\hat{i} \cdot \hat{j} = 0$, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$]

Note: (i) $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$, $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$

(ii) If $\vec{A}$ is any vector in the orthogonal curvilinear coordinate system, then $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ where A_1, A_2, A_3 are scalar fns of u_1, u_2, u_3.

* cyclindrical polar coordinates (ρ, ϕ, z) given by the transformation: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z=$

* Spherical polar coordinates (r, θ, ϕ) given by transformat $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Scalar factors for the cartesian, cyclindrical and spherical system.

cartesian ~~form~~ system : $h_1 = 1$, $h_2 = 1$, $h_3 = 1$

Cyclindrical system : $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

Spherical system : $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Prove that cyclindrical system is orthogonal.

proof: for cyclindrical system,

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ becomes,

$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$.

let $\hat{e}_\rho$. $\hat{e}_\phi$. $\hat{e}_z$

$\therefore \hat{e}_\rho$

$\therefore \hat{e}_\rho = \dfrac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|} = \dfrac{1}{h_1} \dfrac{\partial \vec{r}}{\partial \rho}$

$$\boxed{\hat{e}_\rho = \frac{1}{1}\left[\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\hat{k}\right]}$$

$\hat{e}_\phi = \dfrac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \dfrac{1}{h_2} \dfrac{\partial \vec{r}}{\partial \phi}$

$\hat{e}_\phi = \frac{1}{\rho}\left[-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k\right]$

$$\boxed{\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

$\hat{e}_z = \dfrac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \dfrac{1}{h_3} \dfrac{\partial \vec{r}}{\partial z}$

$$\boxed{\hat{e}_z = 1\left[0\hat{i} + 0\hat{j} + 1\hat{k}\right]}$$

$\hat{e}_\rho \cdot \hat{e}_\phi = \sin\phi\cdot\cos\phi + \sin\phi\cdot\cos\phi = 0$

$\hat{e}_\rho \cdot \hat{e}_z = 0$

$\hat{e}_\phi \cdot \hat{e}_z = 0$.

Thus, the cyclindrical system is orthogonal.

P.T spherical system is orthogonal.

For spherical system,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\therefore$ position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ becomes,

$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

let $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ be the basic unit vectors.

$$\therefore \hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r}$$

$$\boxed{\hat{e}_r = [\sin\theta\cos\phi\,\hat{i} + \sin\theta . \sin\phi\,\hat{j} + \cos\theta\,\hat{k}]}$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta}$$

$$\hat{e}_\theta = \frac{1}{r}[r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}]$$

$$\boxed{\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}}$$

$$\hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi}$$

$$\hat{e}_\phi = \frac{1}{r\sin\theta}[-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\,\hat{k}]$$

$$\boxed{\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

$$\begin{aligned}\therefore \hat{e}_r . \hat{e}_\theta &= \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta \\ &= \sin\theta\cos\theta[\cos^2\phi + \sin^2\phi] - \sin\theta\cos\theta \\ &= \sin\theta\cos\theta - \sin\theta\cos\theta \\ &= 0.\end{aligned}$$

$$\begin{aligned}\hat{e}_r . \hat{e}_\phi &= -\sin\theta\sin\phi\cos\phi + \sin\theta\sin\phi . \cos\phi \\ &= 0\end{aligned}$$

$$\begin{aligned}\hat{e}_\theta . \hat{e}_\phi &= -\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi \\ &= 0\end{aligned}$$

Thus, spherical system is orthogonal.

Bafna Gold
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Note:

Representation of a vector in u_1, u_2, u_3 system:

In cyclindrical system,

WKT $\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{e}_z = 0\hat{k}$.

we have, $i = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi$

$j = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$

$k = \hat{e}_z$

In spherical system,

WKT $\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\hat{k}$

we have,

$i = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$

$j = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$

$k = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta + 0\,\hat{e}_\phi$

(1) Express the vector $\vec{F} = zi - 2xj + yk$ in cyclindrical polar coordinates. Hence find F_ρ, F_ϕ, F_z.

→ In cyclindrical polar system,

$x = \rho\cos\phi \quad y = \rho\sin\phi, \; z = z$

$i = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi$

$j = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$

$k = \hat{e}_z$

Consider $\vec{F} = z\hat{i} - 2x\hat{j} + y\hat{k}$

$\vec{F} = z(\cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi) - 2\rho\cos\phi(\sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi)$
$+ \rho\sin\phi\,(\hat{e}_z)$

$\vec{F} = (z\cos\phi - 2\rho\sin\phi\cos\phi)\,\hat{e}_\rho + (-z\sin\phi - 2\rho\cos^2\phi)\,\hat{e}_\phi$
$+ (\rho\sin\phi)\,\hat{e}_z$

coefficients of $\hat{e}_\rho$, $\hat{e}_\phi$, $\hat{e}_z$ are resp F_ρ, F_ϕ, F_z

i.e, $F_\rho = z\cos\phi - 2\rho\sin\phi\cos\phi$

$F_\phi = -z\sin\phi - 2\rho\cos^2\phi$

$F_z = \rho\sin\phi$

(2) Express the vector $\vec{F} = 2x\hat{i} - 3y^2\hat{j} + xz\hat{k}$ in cylindrical polar coordinates. Hence find F_ρ, F_ϕ, F_z

→ In cylindrical polar system,

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$i = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi$

$j = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$

$z = \hat{e}_z$

Consider $\vec{F} = 2x\hat{i} - 3y^2\hat{j} + xz\hat{k}$

$\vec{F} = 2\rho\cos\phi(\cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi) - 3\rho^2\sin^2\phi(\sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi) + \rho z\cos\phi(\hat{e}_z)$

$\vec{F} = (2\rho\cos^2\phi - 3\rho^2\sin^3\phi)\hat{e}_\rho + (-2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi)\hat{e}_\phi + \rho z\cos\phi\,\hat{e}_z$

Coefficients of $\hat{e}_\rho$, $\hat{e}_\phi$, $\hat{e}_z$ are resp F_ρ, F_ϕ, F_z

i.e, $F_\rho = 2\rho\cos^2\phi - 3\rho^2\sin^3\phi$

$F_\phi = -2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi$

$F_z = \rho z\cos\phi$

③ Represent $\vec{F} = yi - zj + xk$ in spherical polar coordinates, Hence find F_r, F_θ, F_ϕ

→ In spherical polar system,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$i = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$

$j = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\phi + \cos\phi\,\hat{e}_\phi$

$k = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

Consider $\vec{F} = yi - zj + xk$

$\vec{F} = r\sin\theta\sin\phi(\sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi)$
$\quad - r\cos\theta(\sin\theta\sin\phi\,\hat{e}_r + \cos\theta.\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi)$
$\quad + r\sin\theta\cos\phi(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$

$\hat{e}_r$

$\vec{F} = (r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + r\sin\theta\cos\theta\cos\phi)\hat{e}_r$
$+ (r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - r\sin^2\theta\cos\phi)\hat{e}_\theta$
$+ (-r\sin\theta\sin^2\phi - r\cos\theta\cos\phi)\,\hat{e}_\phi$

Coefficients of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are F_r, F_θ, F_ϕ

ie, $F_r = r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + r\sin\theta\cos\theta\cos\phi$

$F_\theta = r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - r\sin^2\theta\cos\phi$

$F_\phi = -r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$.