INTEGRAL CALCULUS.

Evaluation of double and triple integration:

(1) $\displaystyle\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$\rightarrow$ Let $I = \displaystyle\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$= \int_{x=0}^{1} \left(x^2.y + \frac{y^3}{3}\right)_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[x^2(\sqrt{x}-x) + \frac{1}{3}\left((\sqrt{x})^3 - x^3\right)\right] dx$$

$$= \int_{x=0}^{1} \left[x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3}\right] dx$$

$$= \int_{x=0}^{1} \left[x^{5/2} - \frac{4}{3}x^3 + \frac{x^{3/2}}{3}\right] dx$$

$$= \left[\frac{x^{5/2+1}}{5/2+1} - \frac{4}{3}\frac{x^4}{4} + \frac{1}{3}\frac{x^{3/2+1}}{3/2+1}\right]_{x=0}^{1}$$

$$= \left[\frac{2}{7}x^{7/2} - \frac{1}{3}x^4 + \frac{2}{15}x^{5/2}\right]_{x=0}^{1}$$

$$= \frac{2}{7}(1-0) - \frac{1}{3}(1-0) + \frac{2}{15}(1-0) \Rightarrow \boxed{I = \frac{3}{35}}$$

(2) $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} x^3y\,\cancel{dy}\,dy$

$\rightarrow$ Let $I = \displaystyle\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x^3y\,dx\,dy$

$$= \int_{y=0}^{1} \left(\frac{x^4}{4}.y\right)_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4}\int_{y=0}^{1} y.\left((\sqrt{1-y^2})^4 - 0\right) dy$$

$(\sqrt{t})^4 = (t^{1/2})^4 = t^2$

$$= \frac{1}{4}\int_{y=0}^{1} y.(1-y^2)^2\,dy$$

$(a-b)^2$

$$= \frac{1}{4}\int_{y=0}^{1} y\left(1+y^4-2y^2\right) dy$$

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$$= \frac{1}{4}\int_{y=0}^{1}\left[y+y^5-2y^3\right]dy$$

$$= \frac{1}{4}\left[\frac{y^2}{2}+\frac{y^6}{6}-\cancel{2}\frac{y^4}{\cancel{4}_2}\right]_{y=0}^{1}$$

$$= \frac{1}{4}\left[\cancel{\frac{1}{2}}+\frac{1}{6}-\cancel{\frac{1}{2}}\right] \Rightarrow \boxed{I=\frac{1}{24}}$$

(3) $\int_0^1\int_x^{\sqrt{x}} xy\, dy\, dx$

$\rightarrow$ $I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy\, dy\, dx$

$$= \int_{x=0}^{1} x.\left[\frac{y^2}{2}\right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \frac{x}{2}\left[x-x^2\right]$$

$$= \int_{x=0}^{1} \frac{1}{2}\left[x^2-x^3\right] dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{12}\right]$$

$$\boxed{I = \frac{1}{24}}$$

***(4) $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$\rightarrow$ Let $\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$\Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\left[x.y+\frac{y^2}{2}+z.y\right]_{y=x-z}^{x+z}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[x\{\cancel{x}+z-\cancel{x}+z\}+\frac{1}{2}\{(x+z)^2-(x-z)^2\}\right.$$
$$\left.+z\{(\cancel{x}+z)-(\cancel{x}-z)\}\right]dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}x.2z+\frac{1}{2}\{\cancel{x^2}+\cancel{z^2}+2xz-(\cancel{x^2}+\cancel{z^2}-2xz)\}+z.2z\,dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[2xz+\frac{1}{\cancel{2}}(\cancel{4}^{2}xz)+2z^2\right]dx\,dz$$

$$=\int_{z=-1}^{1}\int_{x=0}^{z}\left[2xz+2xz+2z^2\right]dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[4xz+2z^2\right]dx\,dz$$

$$\int_{z=-1}^{1}\left[\cancel{4}^{2}\frac{x^2}{\cancel{2}}.z+2z^2.x\right]_0^z dz$$

$$\int_{z=-1}^{1}\left[2z[z^2-0]+2z^2[z-0]\right]dz$$

$$\int_{z=-1}^{1}4z^3\,dz$$

$$\int_{z=-1}^{1}=\left[\cancel{4}\times\frac{z^4}{\cancel{4}}\right]_{z=-1}^{1}$$

$$=(+1)^4-(-1)^4 \Rightarrow 1-1 \Rightarrow \boxed{I=0}$$

*** (5) $I=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

$\longrightarrow$ Let $I=\int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

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$$I = \int_{z=-c}^{c}\int_{y=-b}^{b} \left[\frac{x^3}{3} + y^2 \cdot x + z^2 \cdot x\right]_{x=-a}^{a} dy\, dz$$

$$= \int_{z=-c}^{c}\int_{y=-b}^{b} \left[\frac{1}{3}\left[a^3-(-a)^3\right] + y^2\left[a-(-a)\right] + z^2\left[a-(-a)\right]\right] dy\, dz$$

$$\int_{z=-c}^{c}\int_{y=-b}^{b} \left[\frac{1}{3}\cdot 2a^3 + y^2 \cdot 2a + z^2 \cdot 2a\right] dy\, dz$$

$$\int_{z=-c}^{c} \left[\frac{2a^3}{3}\cdot y + 2a\frac{y^3}{3} + 2az^2 \cdot y\right]_{y=-b}^{b} dz$$

$$\int_{z=-c}^{c} \left[\frac{2a^3}{3}\left(b-(-b)\right) + \frac{2a}{3}\left(b^3-(-b)^3\right) + 2az^2\left(b-(-b)\right)\right] dz$$

$$\int_{z=-c}^{c} \left[\frac{4a^3b}{3} + \frac{4ab^3}{3} + 4abz^2\right] dz$$

$$\int_{z=-c}^{c} = \left[\frac{4a^3b}{3}\cdot z + \frac{4ab^3}{3}\cdot z + 4ab \cdot \frac{z^3}{3}\right]_{z=-c}^{c}$$

$$= \frac{4a^3b}{3}\left[c-(-c)\right] + \frac{4ab^3}{3}\left[c-(-c)\right] + \frac{4ab}{3}\left[(c)^3-(-c)^3\right]$$

$$\boxed{I = \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3}} = \frac{8abc}{3}\left(a^2+b^2+c^2\right)$$

⑥ $$I = \int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

→ $$I = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y} \cdot e^z\, dz\, dy\, dx$$

$$\int_{x=0}^{a}\int_{y=0}^{x} \left[e^{x+y} \cdot \left(e^z\right)_{z=0}^{x+y}\right] dy\, dx$$

$$\int_{x=0}^{a}\int_{y=0}^{x} \left[e^{x+y}\left(e^{x+y} - e^0\right)\right] dy\, dx$$

$$\int_{x=0}^{a}\int_{y=0}^{x} \left[e^{2x+2y} - e^{x+y}\right] dy\, dx$$

$$\int_{x=0}^{a}\int_{y=0}^{x} \left[e^{2x} \cdot e^{2y} - e^x \cdot e^y\right] dy\, dx$$

$$\int_{x=0}^{a} \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^{x} \cdot e^{y} \right]_{y=0}^{x} dx$$

$$\int_{x=0}^{a} \left[\frac{1}{2} e^{2x} (e^{2x} - e^{0}) - e^{x} (e^{x} - e^{0}) \right] dx$$

$$\int_{x=0}^{a} \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^{x} \right] dx$$

$-1-\frac{1}{2} = -\frac{3}{2}$

$$\int_{x=0}^{a} \left[\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right] dx$$

$$= \left[\frac{1}{2} \cdot \frac{e^{4x}}{4} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^{x} \right]_{0}^{a}$$

$$= \left[\frac{1}{8} \left[e^{4a} - e^{0} \right] - \frac{3}{4} \left[e^{2a} - e^{0} \right] + \left[e^{a} - e^{0} \right] \right]$$

$$I = \frac{e^{4a}}{8} - \frac{1}{8} - \frac{3}{4} e^{2a} + \frac{3}{4} + e^{a} - 1$$

$$\boxed{I = \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}}$$

7) $$I = \int_{0}^{4} \int_{0}^{2\sqrt{z}} \int_{0}^{\sqrt{4z - x^2}} dz\, dy\, dx$$

→ Rewriting given integral ~~of~~ wrt proper limits

$$I = \int_{0}^{4} \int_{0}^{2\sqrt{z}} \int_{0}^{\sqrt{4z-x^2}} dy\, dx\, dz$$

$$= \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} [y]_{0}^{\sqrt{4z-x^2}} dx\, dz$$

$$= \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \sqrt{4z - x^2}\, dx\, dz$$

Let $4z = a^2 \Rightarrow a = \sqrt{4z} \Rightarrow a = 2\sqrt{z}$

$$I = \int_{z=0}^{a} \int_{x=0}^{a} \sqrt{a^2 - x^2}\, dx\, dz$$

Formula: $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \int_{z=0}^{4} \left[\frac{x}{2}\cancel{\sqrt{a^2-x^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^{a} dz$$

$$= \int_{z=0}^{4} \frac{a^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dz$$

$$= \int_{z=0}^{4} \frac{a^2}{2}\left(\frac{\pi}{2} - 0\right) dz$$

$$= \frac{\pi}{\cancel{4}} \int_{z=0}^{4} \cancel{4}z \, dz \qquad 4z = a^2$$

$$= \pi \left[\frac{z^2}{2}\right]_{z=0}^{4}$$

$$= \frac{\pi}{2}\left[4^2 - 0\right] \Rightarrow \frac{\pi}{2} \cdot 16 \Rightarrow \boxed{I = 8\pi}$$

(8) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$\rightarrow$ $$\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}} \, dz\,dy\,dx$$

Let $\sqrt{a^2-x^2-y^2} = K$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{0}^{K} \frac{1}{\sqrt{K^2-z^2}} \, dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left\{\sin^{-1}\left(\frac{z}{K}\right)\right\}_{z=0}^{K} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left\{\sin^{-1}(1) - \sin^{-1}(0)\right\} dy\,dx$$

$$= \frac{\pi}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 1 \cdot dy\,dx$$

$$I = \frac{\pi}{2}\int_{x=0}^{a}[y]_{y=0}^{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\pi}{2}\int_{x=0}^{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\cancel{\frac{x}{2}\sqrt{a^2-x^2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{\pi}{2}\cdot\frac{a^2}{2}\left\{\sin^{-1}(1) - \cancel{\sin^{-1}(0)}\right\}$$

$$= \frac{\pi a^2}{4}\cdot\frac{\pi}{2} \Rightarrow \boxed{I = \frac{\pi^2 a^2}{8}}$$

(9) $$\int_{-3}^{3}\int_{0}^{1}\int_{1}^{2}(x+y+z)\,dx\,dy\,dz = 12$$

(10) $$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{1}{48}$$

(9) Let $$I = \int_{z=-3}^{3}\int_{y=0}^{1}\int_{x=1}^{2}(x+y+z)\,dx\,dy\,dz$$

$$\int_{z=-3}^{3}\int_{y=0}^{1}\left[\frac{x^2}{2} + yx + zx\right]_{x=1}^{2} dy\,dz$$

$$\int_{z=-3}^{3}\int_{y=0}^{1}\left[\frac{(4-1)}{2} + y(2-1) + z(2-1)\right] dy\,dz$$

$$\int_{z=-3}^{3}\int_{y=0}^{1}\left[\frac{3}{2} + y + z\right] dy\,dz$$

$$\int_{z=-3}^{3}\left[\frac{3}{2}y + \frac{y^2}{2} + zy\right]_{y=0}^{1}$$

$$\int_{z=-3}^{3}\left[\frac{3}{2}(1-0) + \frac{(1^2-0)}{2} + z(1-0)\right] dz$$

$$\int_{z=-3}^{3} \left(\frac{3}{2} + \frac{1}{2} + z\right) dz$$

$$I = \left[\frac{3}{2}z + \frac{1}{2}z + \frac{z^2}{2}\right]_{z=-3}^{3}$$

$$I = \frac{3}{2}\left[3-(-3)\right] + \frac{1}{2}\left[3-(-3)\right] + \frac{1}{2}\left[3^2-(-3)^2\right]$$

$$I = \frac{3}{\cancel{2}} \times \cancel{6}^{3} + \frac{1}{\cancel{2}} \times \cancel{6}^{3} + 0 \Rightarrow \boxed{I = 12}$$

(10) Let $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x \cdot y \cdot \left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[x \cdot y \cdot \frac{1-x^2-y^2}{2}\right] dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(xy - x^3y - xy^3\right) dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \cancel{\int_{y=0}} \left[x \cdot \frac{y^2}{2} - x^3 \cdot \frac{y^2}{2} - x\frac{y^4}{4}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[\frac{x}{2}\left[1-x^2\right] - \frac{x^3}{2}\left[1-x^2\right] - \frac{x}{4}\left(1-x^2\right)^2\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4}\left(1+x^4-2x^2\right)\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[\frac{x}{2} - \frac{x^3}{2} - \cancel{\frac{x^3}{2}} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \cancel{\frac{x^3}{2}}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{12}-\frac{x^2}{8}-\frac{x^6}{24}\right]_{x=0}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4}-\frac{1}{8}+\frac{1}{12}-\frac{1}{8}-\frac{1}{24}\right]$$

$$= \frac{1}{2}\left(\frac{1}{24}\right)$$

$$\boxed{I = \frac{1}{48}}$$

Evaluation of double integral by changing the order of integration:

Evaluate the following integrals by changing the order of integration:

1.

$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$$

Let $I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy \, dy \, dx$

To identify the region of integration bounded by the curves $y=x$, $y=\sqrt{x}$ between the lines $x=0$, $x=1$

To find point of intersection:

Consider, $y=x$

put $x=0$; $y=0$

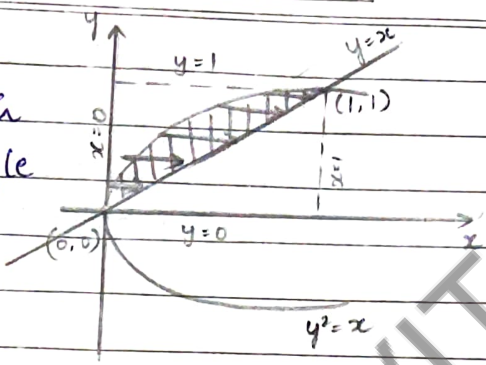
put $x=1$; $y=1$

$\therefore$ points of intersection are $(0,0)$ and $(1,1)$

↑↑↑ limits for y

⇉ limits for x

The curve $y=x$ represents a straight line passing through the origin making an angle 45° with x-axis. Also, the curve $y=\sqrt{x}$ or $y^2=x$ represents a parabola symmetrical about x-axis



By changing the order of integration we get,

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy$$

$$= \int_{y=0}^{1} \left(\frac{x^2}{2} \cdot y \right)_{x=y^2}^{y} dy$$

$$= \frac{1}{2} \int_{y=0}^{1} (y^2 - y^4) \, y \, dy$$

$$= \frac{1}{2} \int_{y=0}^{1} (y^3 - y^5) \, dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} \times \frac{1}{12} \Rightarrow \boxed{I = \frac{1}{24}}$$

② $\displaystyle\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

Let $\displaystyle I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

To find point of intersection:

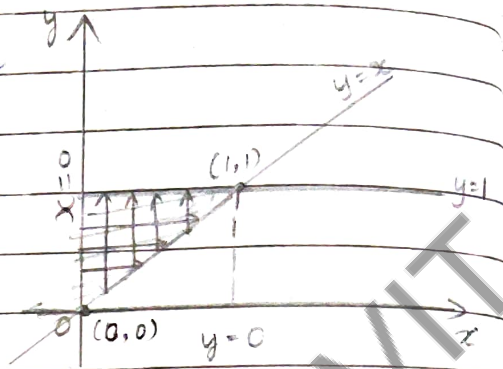
consider $y = x$

put $x = 0$; $y = 0$

put $x = 1$; $y = 1$

$\therefore$ points of intersection are $(0,0)$ and $(1,1)$

By changing the order of integration.



$$I = \int_{y=0}^{1}\int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy \quad \text{— (1)}$$

Let $x^2+y^2 = t$

diff wrt

$2x\,dx = dt$

$x\,dx = dt/2$

$$\therefore \int \frac{x}{\sqrt{x^2+y^2}} = \int \frac{dt/2}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = \frac{1}{2}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \frac{\cancel{1}}{\cancel{2}}\,\frac{\sqrt{t}}{\cancel{1/2}} \Rightarrow \sqrt{t} \Rightarrow \sqrt{x^2+y^2}$$

Sub in (1),

$$I = \int_{y=0}^{1}\left(\sqrt{x^2+y^2}\right)_{x=0}^{y} dy \longrightarrow \int_{y=0}^{1} \sqrt{y^2+y^2} - \sqrt{0+y^2}\,dy$$

$$I = \int_{y=0}^{1}\left[\sqrt{2y^2} - \sqrt{y^2}\right] dy$$

$$I = \int_{y=0}^{1}\left(\sqrt{2}\,y - y\right) dy$$

$$= \left(\sqrt{2}-1\right)\int_{y=0}^{1} y\,dy$$

$$= \left(\sqrt{2}-1\right)\left(\frac{y^2}{2}\right)_{y=0}^{1}$$

$$= \left(\sqrt{2}-1\right)\frac{1}{2} \Rightarrow \boxed{I = \frac{\sqrt{2}-1}{2}}$$

(3) $$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$$

Let $$I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx.$$

given : $x : 0$ to $4a$

$y : \frac{x^2}{4a}$ to $2\sqrt{ax}$

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$ represents a parabola symmetri-cal about y-axis

Also, $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ represents a parabola symmetrical about x-axis

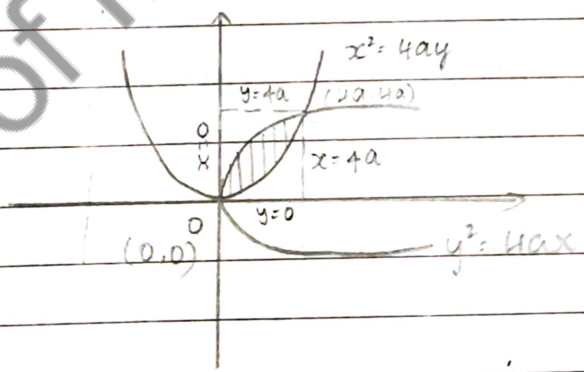
To find pt of intersection:

consider $y = \frac{x^2}{4a}$

put $x = 0$; $y = 0$

put $x = 4a$; $y = \frac{16a^2}{4a} = 4a$

$\therefore$ points of intersection are $(0,0)$ and $(4a, 4a)$



By changing the order of int, we get

$$I = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy\, dx\, dy$$

$$\int_{y=0}^{4a} y\left(\frac{x^2}{2}\right)_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$\int_{y=0}^{4a} \frac{y}{2}\left(4ay - \frac{y^4}{16a^2}\right) dy$$

$$\int_{y=0}^{4a} \left(2ay^2 - \frac{y^5}{32a^2}\right) dy$$

$$= \left[2a\frac{y^3}{3} - \frac{1}{32a^2}\frac{y^6}{6} \right]_{y=0}^{4a}$$

$$= \frac{2a}{3}(64a^3 - 0) - \frac{1}{192a^2}(4096a^6 - 0)$$

$$= \frac{128a^4}{3} - \frac{4096a^4}{192}$$

$$= \left(\frac{128}{3} - \frac{4096}{192}\right)a^4 \Rightarrow \boxed{I = \frac{64}{3}a^4}$$

(4) $$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

let $$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$$

given, y : 0 to a

x : y to a

$x=y$ is a straight line passing through the origin making an angle 45° to x-axis

Also, $x=a$ is a straight line parallel to Y-axis

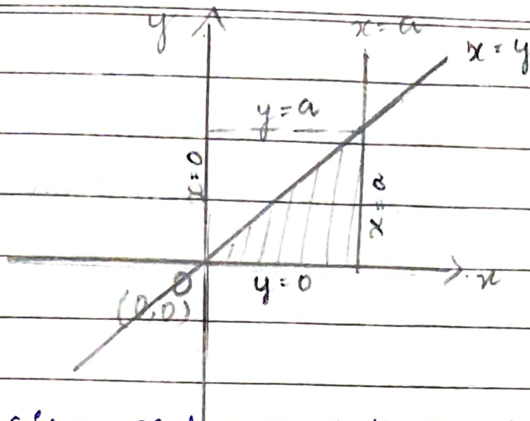
To find pts of intersection:

Consider $x=y$

put $y=0$; $x=0$

put $y=a$; $x=a$

points are $(0,0)$ and (a,a)



By changing order of integration we get,

$$I = \int_{x=0}^{a}\int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x\left\{\int_{y=0}^{x} \frac{1}{y^2+x^2}\, dy\right\} dx$$

$$= \int_{x=0}^{a} \cancel{x}\left\{\frac{1}{\cancel{x}}\cdot \tan^{-1}\left(\frac{y}{x}\right)\right\}_{y=0}^{x} dx$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$$= \int_{x=0}^{a} \left(\tan^{-1}(1) - \tan^{-1}(0)\right) dx$$

$$= \int_{x=0}^{a} \left(\frac{\pi}{4} - 0\right) dx$$

$$= \frac{\pi}{4}(x)_0^a \Rightarrow \boxed{I = \frac{\pi a}{4}}$$

5. $$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$$

let $I = \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$

given, $y = 0$ to b

$x = 0$ to $\frac{a}{b}\sqrt{b^2-y^2}$

consider, $x = \frac{a}{b}\sqrt{b^2-y^2}$

$\therefore x^2 = \frac{a^2}{b^2}(b^2-y^2)$

$\therefore \frac{x^2}{a^2} = \frac{1}{b^2}(b^2-y^2)$

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$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ represents ellipse

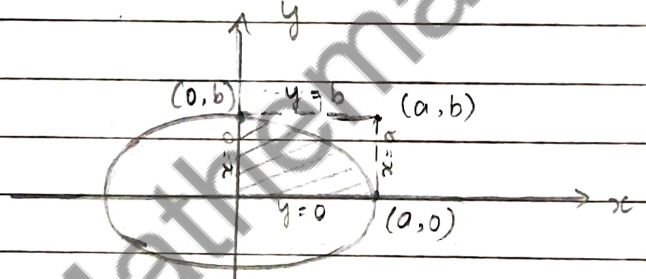
To find pts of intersection:

Consider $x = \frac{a}{b}\sqrt{b^2 - y^2}$

put ~~x~~ $y = 0$; $x = \frac{a}{b}\sqrt{b^2} = \frac{a}{b}\cdot b \Rightarrow a$

put $y = b$; $x = \frac{a}{b}\sqrt{b^2 - b^2} \Rightarrow 0$

pts of intersection are $(a, 0)$ and $(0, b)$.



By changing order of integration,

$$I = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy \, dy \, dx$$

$$= \int_{x=0}^{a} x \cdot \left(\frac{y^2}{2}\right)_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \frac{x}{2} \left\{\frac{b^2}{a^2}(a^2 - x^2)\right\} dx$$

$$= \int_{x=0}^{a} \frac{x}{2}\left(b^2 - \frac{b^2x^2}{a^2}\right) dx$$

$$= \int_{x=0}^{a} \left(\frac{b^2x}{2} - \frac{b^2x^3}{2a^2}\right) dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = b^2\left(\frac{a^2 - x^2}{a^2}\right)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \left[\frac{b^2}{2}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a}\left(\frac{x^4}{4}\right)_0^a \right]$$

$$= \left[\frac{b^2}{4}\left[a^2 - 0\right) - \frac{b^2}{8a^2}\left(a^4 - 0\right) \right]$$

$$= \frac{a^2b^2}{4} - \frac{a^2b^2}{8} \Rightarrow \boxed{I = \frac{a^2b^2}{8}}$$

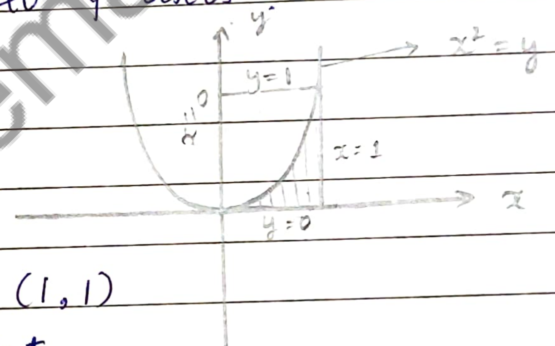
(6) $\int_0^1 \int_{\sqrt{y}}^1 dy\, dx$

given $\int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} 1\, dx\, dy$

given : $y = 0$ to 1 ; $x = \sqrt{y}$ to 1

consider $x^2 = y$ is a parabola and $x = 1$ is a straight line parallel to y-axis

pts of intersection:



$x^2 = y$

$x = 0$; $y = 0$

$x = 1$; $y = 1$

pts of intersection $(0,0)$ $(1,1)$

By changing order of int,

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} 1\, dy\, dx$$

$$= \int_{x=0}^{1} [y]_{y=0}^{x^2}\, dx$$

$$= \int_{x=0}^{1} x^2\, dx$$

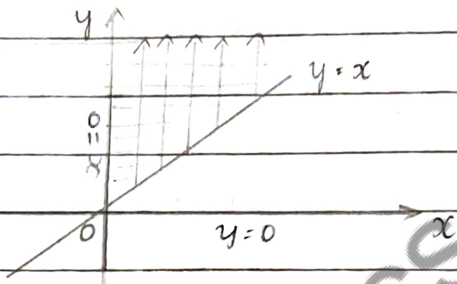
$$= \left[\frac{x^3}{3}\right]_{x=0}^{1}$$

$$\Rightarrow \boxed{I = \frac{1}{3}}$$

(7) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Let $I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$

$y=x$ is a straight line passing through origin making an angle 45° with x-axis.



By changing order of integration.

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \left[\frac{e^{-y}}{y} [x]_{x=0}^{y} \right] dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot (\cancel{y} - 0)\, dy$$

$$= \int_{y=0}^{\infty} e^{-y}\, dy \Rightarrow \int_{y=0}^{\infty} \left(\frac{e^{-y}}{-1}\right)_{y=0}^{\infty}$$

$$= \int_{y=0}^{\infty} -1\,(e^{-\infty} - e^{0})$$

$$= -1(0-1) \Rightarrow \boxed{I = 1}$$

(8) $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

→ given $x : 0$ to 1

$y : x^2$ to $2-x$

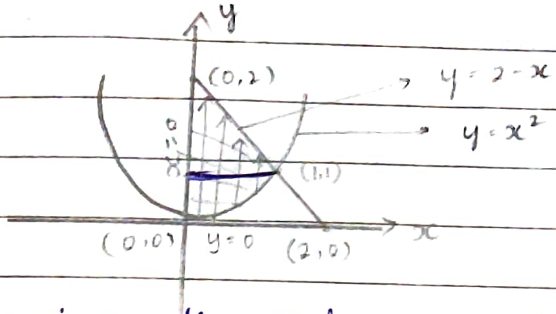
$y = x^2$ represents a parabola symmetrical about y-axis. Also, $y = 2-x$ (or) $x+y=2$ (or) $\frac{x}{2}+\frac{y}{2}=1$ represents a straight line passing through the points $(2,0)$ and $(0,2)$

To find pts of intersection:

consider $y = x^2$

put $x = 0$; $y = 0$ $\Rightarrow$ $(0,0)$ $(1,1)$

$x = 1$; $y = 1$



By changing the order of integration,

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=0}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy + \int_{y=0}^{2} y \left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy$$

$$= \int_{y=0}^{1} \frac{y}{2}(y)\, dy + \int_{y=1}^{2} \frac{y}{2}\left\{(2-y)^2\right\} dy$$

$$= \int_{y=0}^{1} \frac{y^2}{2}\, dy + \int_{y=1}^{2} \frac{y}{2}\left[4 + y^2 - 4y\right] dy$$

$$= \int_{y=0}^{1} \frac{1}{2}\frac{y^3}{3} + \int_{y=1}^{2} \left(2y + \frac{y^3}{2} - 2y^2\right) dy$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \int_{y=1}^{2} \left[2\frac{y^2}{2} + \frac{1}{2}\frac{y^4}{4} - 2\frac{y^3}{3}\right]_{y=1}^{2}$$

$$= \frac{1}{6} + \left[y^2 + \frac{1}{8}y^4 - \frac{2}{3}y^3\right]_{y=1}^{2}$$

$$+ \left[(4-1) + \frac{1}{8}(16-1) - \frac{2}{3}(8-1)\right]$$

$$+ \quad 3 + \frac{15}{8} - \frac{14}{3} \Rightarrow \frac{5}{24}$$

$$I = \frac{1}{6} + \frac{5}{24} \Rightarrow \boxed{I = \frac{3}{8}}$$

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Evaluation of double integral by changing into polar form (OR) Change of variables.

- Use polar form of substitution $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$; $\frac{y}{x} = \tan\theta$

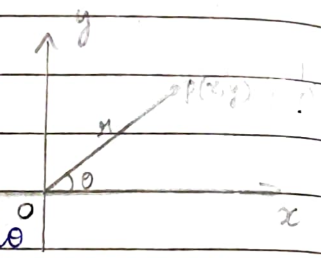
- $dx.dy = r\,dr\,d\theta$ (by using $dx\,dy = J\,dr\,d\theta$, where $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$)

- change the limits of integration to (r,θ) and evaluate.

Evaluate following integrals by changing into polar coordinates

(1) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

→ In polars,



$x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

Since x, y varies from 0 to ∞, r also varies from 0 to ∞. Also in the first quadrant 'θ' varies from 0 to $\pi/2$.

Converting to polar coordinates, we get

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta \quad \text{——— (1)}$$

put $r^2 = t$

$2r\,dr = dt \Rightarrow r\,dr = dt/2$

$$\therefore \int e^{-r^2} r\,dr = \int e^{-t}.\frac{dt}{2} = \frac{1}{2}\cdot\frac{e^{-t}}{-1} = -\frac{1}{2}e^{-r^2}$$

sub in (1),

$$I = \int_{\theta=0}^{\pi/2} -\frac{1}{2}\left(e^{-r^2}\right)_{r=0}^{\infty} d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2} \left(e^{-\infty} - e^{0}\right) d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2} (0-1)\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} 1 .\, d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) \Rightarrow \boxed{I = \frac{\pi}{4}}$$

(2) $$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

→ given $x : -a$ to a

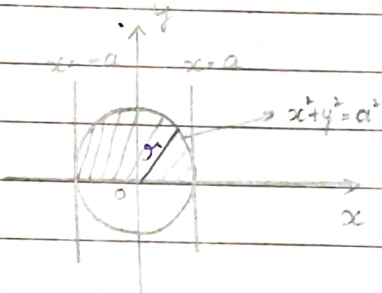
$y : 0$ to $\sqrt{a^2-x^2}$

$y = \sqrt{a^2-x^2} \Rightarrow y^2 = a^2-x^2 \Rightarrow x^2+y^2 = a^2$ (circle)

WKT, $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

$\Rightarrow a^2 = r^2 \Rightarrow \boxed{r = a}$



$\therefore r : 0$ to a

$\theta : 0$ to π

$\therefore$ given integral is, $x^2+y^2 = r^2$

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{r^2}\,.\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^3}{3}\right]_{r=0}^{a} d\theta$$

$$\int_{\theta=0}^{\pi} \frac{1}{3}\left(a^3 - 0\right) d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\pi} 1 \cdot d\theta$$

$$= \frac{a^3}{3} (\theta)_{\theta=0}^{\pi} \Rightarrow \boxed{I = \frac{\pi a^3}{3}}$$

(3) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$

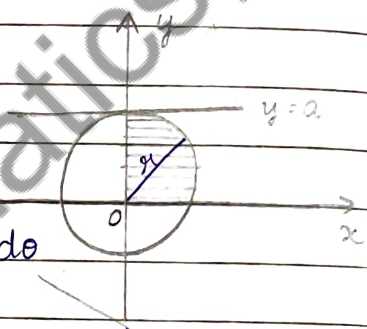
given, $x : 0$ to $\sqrt{a^2-y^2}$

$y : 0$ to a

Consider, $x = \sqrt{a^2-y^2} \Rightarrow x^2 = a^2 - y^2 \Rightarrow x^2+y^2 = a^2$ (circle)

$\therefore$ $r : 0$ to a

$\theta : 0$ to $\pi/2$



given integral is,

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\sin\theta \sqrt{r^2} \cdot r\, dr\, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

also $dx\,dy = r\,dr\,d\theta$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3 \sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^4}{4}\right)_{r=0}^{a} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{\sin\theta}{4} (a^4)\, d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{a^4}{4} (-\cos\theta)_{\theta=0}^{\pi/2}$$

$$= -\frac{a^4}{4} \left[\cos\frac{\pi}{2} - \cos(0)\right]$$

$$= -\frac{a^4}{4} (0-1) \Rightarrow \boxed{I = \frac{a^4}{4}}$$

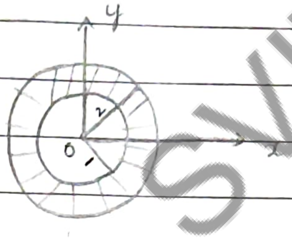
Region b/w 2 circles

(4) Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ where 'R' is ~~angular~~ annular region b/w circles $x^2+y^2=4$ and $x^2+y^2=1$ convert to polar coordinates and integrate

→ from the fig,

clearly, $r: 1$ to 2

$\theta: 0$ to 2π



Polar conversion:

$x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

Also $dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{r^2\cos^2\theta(r^2\sin^2\theta)}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^3\sin^2\theta\cos^2\theta\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi} \sin^2\theta\cos^2\theta\left(\frac{r^4}{4}\right)_{r=1}^{2} d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi} \sin^2\theta.\cos^2\theta\ \cancel{d\theta}\ (2^4-1^4)\,d\theta$$

$$= \frac{15}{4}\int_{\theta=0}^{2\pi} \sin^2\theta\cos^2\theta\,d\theta.$$

$$= \frac{15}{4}\int_{\theta=0}^{2\pi} (\sin\theta\cos\theta)^2\,d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \frac{\sin 2\theta}{2} = \sin\theta\cos\theta$

$$= \frac{15}{4}\int_{\theta=0}^{2\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{15}{16}\int_{\theta=0}^{2\pi} \sin^2 2\theta\,d\theta$$

WKT, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

$$\therefore I = \frac{15}{16}\int_{\theta=0}^{2\pi} \cancel{\theta - \sin}\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{15}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_{\theta=0}^{2\pi}$$ $\sin n\pi = 0 \; \forall n$

$$= \frac{15}{32}\left[2\pi - \cancel{\frac{\sin 8\pi}{4}}^{\,0}\right] \Rightarrow \boxed{I = \frac{15\pi}{16}}$$

(5) $$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2 \, dy \, dx$$

given : $x = 0$ to $2a$

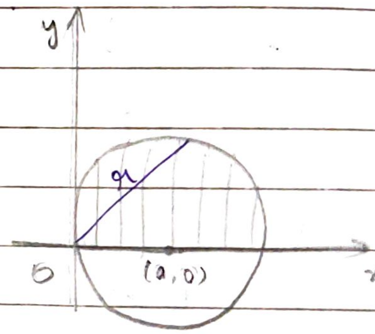
$y = 0$ to $\sqrt{2ax-x^2}$

consider, $y^2 = 2ax - x^2$

$\Rightarrow x^2+y^2 = 2ax$

$\Rightarrow x^2+y^2-2ax = 0$ (circle).

This represents the circle eq^n with centre at $(a,0)$ and radius $= a$.



clearly, θ: 0 to $\pi/2$

consider $x^2+y^2-2ax=0$

polar conversion:

$x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

Also $dx\,dy = r\,dr\,d\theta$

$\therefore r^2 - 2ar\cos\theta = 0$

$r(r-2a\cos\theta) = 0$

$\Rightarrow \boxed{r=0}$; $r - 2a\cos\theta$

$\Rightarrow \boxed{r = 2a\cos\theta}$

$\therefore r$: 0 to $2a\cos\theta$

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\cos^2\theta \; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^3 \cancel{\cos}^2\theta \, dr \, d\theta.$$

$$= \int_{\theta=0}^{\pi/2} \cos^2\theta \left[\frac{r^4}{4}\right]_{r=0}^{2a\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{\cos^2\theta}{\cancel{4}} \left[\cancel{16}a^4\cos^4\theta\right] d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(4a^4\cos^6\theta\right) d\theta$$

$$= 4a^4 \int_{\theta=0}^{\pi/2} \cos^6\theta \, d\theta$$

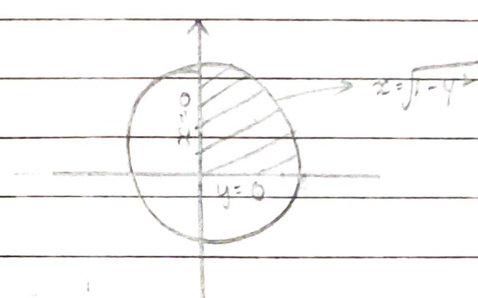
$$= \cancel{4}a^4 \left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] \Rightarrow \boxed{I = \frac{5\pi a^4}{8}}$$

Note: Reduction formula:

(i)
$$\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$
if n is even.

(ii)
$$\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}$$
if n is odd

⑥ $$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2) \, dx \, dy$$



given : $x = 0$ to $\sqrt{1-y^2}$

$y = 0$ to 1

Consider, $x = \sqrt{1-y^2}$

$x^2 = 1-y^2 \Rightarrow x^2+y^2 = 1$ (circle)

Polar conversion,

$x = r\sin\theta$; $y = r\cos\theta$; $x^2+y^2 = r^2$

Also, $dx\,dy = r\,dr\,d\theta$

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$\therefore \theta = 0$ to $\pi/2$

$r = 0$ to 1

By changing to polar form

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r \, r dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^3 \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{1} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{4} \, d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} 1 . d\theta \Rightarrow \frac{1}{4} [\theta]_{\theta=0}^{\pi/2} = \frac{1}{4}\left[\frac{\pi}{2}\right] \Rightarrow \boxed{I = \frac{\pi}{8}}$$

Applications to find area and volume by double integrals

Formulae:

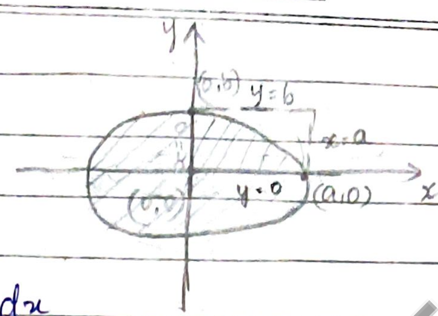
(i) $\iint_R dx \, dy$ = area of the region 'R' in cartesian form

(ii) $\iint_R r \, dr \, d\theta$ = area of the region capital 'R' in the polar form

(iii) $\iint_R z \, dy \, dx$ = volume of the solid in the cartesian form.

① Find the area of the ellipse by double integration.

→ Area $A = \iint_R dx\,dy$



we have $A = 4A_1$, where A_1 is the area in the 1st quadrant

$\therefore A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} 1\,dy\,dx$

$$A = 4\int_{x=0}^{a} [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$$

$$= 4\int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\cancel{\frac{x}{2}\sqrt{a^2-x^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{4b}{a}\cdot\frac{a^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$$

$$= 2ba\left[\frac{\pi}{2} - 0\right] \Rightarrow \boxed{A = \pi ab \text{ sq. units}}$$

we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

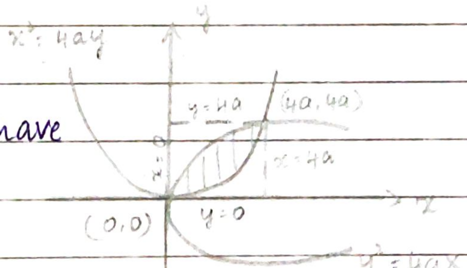
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

(2) Find the area bounded b/w parabolas $y^2 = 4ax$ and $x^2 = 4ay$



To find pts of intersection:

from the curve $y^2 = 4ax$, we have

$$x = \frac{y^2}{4a}$$

sub this x in $x^2 = 4ay$

we get, $\left(\frac{y^2}{4a}\right)^2 = 4ay$

$$\frac{y^4}{16a^2} = 4ay$$

$$y^4 = 64a^3y$$

$$y^4 - 64a^3y = 0$$

$$y(y^3 - 64a^3) = 0$$

$\Rightarrow \boxed{y=0}$ $\quad y^3 = 64a^3 \Rightarrow \boxed{y = 4a}$

Consider,

$x = \frac{y^2}{4a}$

put $y = 0$; $x = 0$

put $y = 4a$; $x = 4a$

pts are $(0,0)$ $(4a, 4a)$

$$\text{Area, } A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} 1 \, dy \, dx$$

$$= \int_{x=0}^{4a} (y)_{\frac{x^2}{4a}}^{2\sqrt{ax}} \, dx$$

$$= \int_{x=0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[2\sqrt{a} \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{4a} \frac{x^3}{3} \right]_{x=0}^{4a}$$

$$= \left[2\sqrt{a} \frac{2}{3} x^{3/2} - \frac{1}{4a} \frac{x^3}{3} \right]_{x=0}^{4a}$$

$$= \left[2\sqrt{a} \frac{2}{3} (4a)^{3/2} - 0 - \frac{1}{12a} 64a^3 \right]$$

$$= \left[\frac{4\sqrt{a}}{3} \sqrt{64}\, a^{3/2} - \frac{1}{12a} 64a^3 \right]$$

$$= \frac{4}{3} a^2 \cdot 8 - \frac{64}{12} a^2 \Rightarrow a^2 \left(\frac{32}{3} - \frac{64}{12} \right) \Rightarrow \boxed{I = \frac{16}{3} a^2}$$ sq units

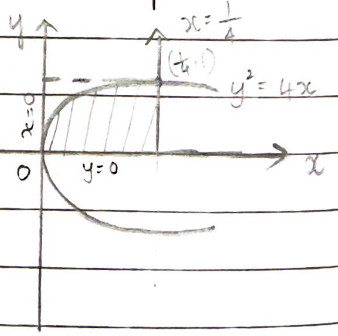
(3) Evaluate ~~ove~~ $\iint dx\, dy$ over the region bounded by the parabola $y^2 = 4x$ and line $x = \frac{1}{4}$

→ <u>To find pts of intersection:</u>

consider $y^2 = 4x$

put $x = \frac{1}{4}$; $y = 1$

pts is $\left(\frac{1}{4}, 1\right)$



$\therefore$ Area, $A = \int_{x=0}^{1/4} \int_{y=0}^{2\sqrt{x}} 1 .\, dy\, dx$

$\int_{x=0}^{1/4} [y]_{y=0}^{2\sqrt{x}}\, dx$

$= \int_{x=0}^{1/4} 2\sqrt{x}\, dx$

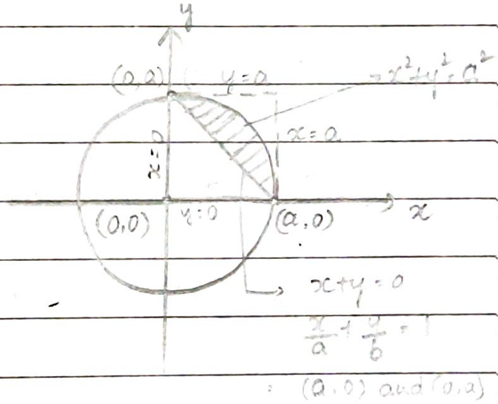
$= \int_{x=0}^{1/4} 2x^{1/2}\, dx$

$= \left[2\frac{x^{1/2+1}}{1/2+1} \right]_{x=0}^{4}$

$= \frac{4}{3}\left[x^{3/2}\right]_{x=0}^{1/4}$

$= \frac{4}{3}\left(\frac{1}{4}\right)^{3/2} = \frac{4}{3}\left(\frac{1}{\sqrt{64}}\right) = \frac{4}{3}\left(\frac{1}{8}\right) \Rightarrow \boxed{I = \frac{1}{6}}$

(4) Find by double integration, the area lying b/w the circle $x^2+y^2=a^2$ and the line $x+y=a$ in the 1st quadrant.



$\rightarrow$ $\therefore$ Area, $A = \iint_R dx\, dy$

$\int_{x=0}^{a} \int_{y=a-x}^{\sqrt{a^2-x^2}} 1 .\, dy\, dx$

$\int_{x=0}^{a} [y]_{a-x}^{\sqrt{a^2-x^2}}\, dx$

$\int_{x=0}^{a} \left[\left(\sqrt{a^2-x^2}\right) - (a-x)\right] dx$

$\int_{x=0}^{a} \left(\cancel{\frac{x}{2}\sqrt{a^2-x^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2}\right)_{x=0}^{a}$

$\int_{x=0}^{a} \frac{a^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] - a^2 + \frac{1}{2}a^2$

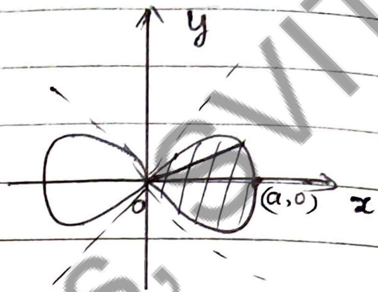
$\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a^2}{2} \Rightarrow \frac{\pi a^2}{4} - \frac{a^2}{2} \Rightarrow \boxed{I = \left(\frac{\pi}{4} - \frac{1}{2}\right)a^2 \text{ sq units}}$

(5) Find the area enclosed by following curve.

(i) one loop of the lemniscate $r^2 = a^2\cos 2\theta$

(ii) upper part of the cardioid $r = a(1+\cos\theta)$

(i) Leminiscate is as shown below:



area enclosed by one loop of this curve,

A = 2 x area bounded by the curve in first quadrant

In the first quadrant,

θ : 0 to $\pi/4$

r : 0 to $a\sqrt{\cos 2\theta}$

$$\therefore A = 2\times \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$A = 2\times \int_{\theta=0}^{\pi/4} \left(\frac{r^2}{2}\right)_{r=0}^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\times \int_{\theta=0}^{\pi/4} \frac{1}{2}\left(a^2\cos 2\theta - 0\right) d\theta$$

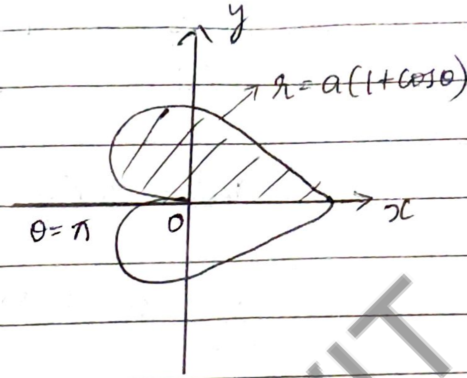
$$= \cancel{2}\times \frac{a^2}{\cancel{2}} \int_{\theta=0}^{\pi/4} \cos 2\theta\, d\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\pi/4}$$

$$= \frac{a^2}{2}\left[\sin\left(\frac{\pi}{2}\right) - \cancelto{0}{\sin(0)}\right]$$

$$\boxed{A = \frac{a^2}{2} \text{ sq units}}$$

(ii) Required area $= \iint_R r\,dr\,d\theta$



θ : 0 to π

r : 0 to $a(1+\cos\theta)$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$\int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_{r=0}^{a(1+\cos\theta)} d\theta$$

$$\int_{\theta=0}^{\pi}\frac{1}{2}\,a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2\left[1+\cos^2\theta+2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi}\left[1+\left(\frac{1+\cos2\theta}{2}\right)+2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}\theta+\frac{1}{2}\frac{\sin2\theta}{2}+2\sin\theta\right]_{\theta=0}^{\pi}$$

$$= \frac{a^2}{2}\left[\pi+\frac{\pi}{2}+\frac{1}{4}\left[\cancelto{0}{\sin2\pi-\sin0}\right]+2\left[\cancelto{0}{\sin\pi-\sin0}\right]\right]$$

$$= \frac{a^2}{2}\left[\frac{3\pi}{2}\right]$$

$$\boxed{A = \frac{3a^2\pi}{4}\text{ sq units}}$$

(6) A pyramid is bounded by 3 coordinate planes and the plane $x+y+3z=6$. compute the volume by double integration

→ volume by double integration is given by $V=\iint z\,dx\,dy$

consider $x+2y+3z=6$

$$\Rightarrow z = \frac{6-x-2y}{3} = 2-\frac{x}{3}-\frac{2}{3}y$$

If $z=0$, then $x+2y=6 \Rightarrow y=3-\frac{x}{2}$

If $z=0$ and $y=0$, then, $x=6$

$\therefore x: 0$ to 6

$y: 0$ to $3-\frac{x}{2}$

$$\therefore V=\int_{x=0}^{6}\int_{y=0}^{3-x/2}\left[2-\frac{x}{3}-\frac{2}{3}y\right]dy\,dx$$

$$V=\int_{x=0}^{6}\left[2y-\frac{x}{3}(y)-\frac{2}{3}\cdot\frac{y^2}{2}\right]_{y=0}^{3-x/2}dx$$

$$V=\int_{x=0}^{6}2\left(3-\frac{x}{2}\right)-\frac{x}{3}\left(3-\frac{x}{2}\right)-\frac{2}{6}\left(3-\frac{x}{2}\right)^2 dx$$

$$V=\int_{x=0}^{6}6-x-x+\frac{x^2}{6}-\frac{1}{3}\left(9+\frac{x^2}{4}-3x\right)dx$$

$$\int_{x=0}^{6}\left[6-2x+\frac{x^2}{6}-3-\frac{x^2}{12}+x\right]dx$$

$$=\int_{x=0}^{6}\left[\frac{x^2}{12}-x+3\right]dx$$

$$=\left[\frac{1}{12}\frac{x^3}{3}-\frac{x^2}{2}+3x\right]_{x=0}^{6}$$

$$=\frac{1}{36}[216]-\frac{1}{2}[36]+3(6)$$

$$=6-\cancel{18}+\cancel{18}\Rightarrow\boxed{A=6\text{ sq units}}$$

7) Find the volume of the tetrahedron by double integration bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\rightarrow$ given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}\Rightarrow z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

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put $z=0$; $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y=b\left(1-\frac{x}{a}\right)$

put $y=0$, $z=0$; $\frac{x}{a}=1 \Rightarrow \boxed{x=a}$

$\therefore$ x : 0 to a

y : 0 to $b\left(1-\frac{x}{a}\right)$

Thus, volume, $V=\iint z\,dy\,dx$

$$V=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_{x=0}^{a}\left[y-\frac{x}{a}(y)-\frac{1}{b}\frac{y^2}{2}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)}dx$$

$$=c\int_{x=0}^{a}\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$=c\int_{x=0}^{a}\left[b-\frac{bx}{a}-\frac{bx}{a}+\frac{bx^2}{a^2}-\frac{b}{2}\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\right]dx$$

$$=c\int_{x=0}^{a}\left[b-\frac{2bx}{a}+\frac{bx^2}{a^2}-\frac{b}{2}-\frac{bx^2}{2a^2}+\frac{\cancel{2}bx}{\cancel{2}a}\right]dx$$

$$=c\int_{x=0}^{a}\left[\frac{b}{2}-\frac{bx}{a}+\frac{1}{2}\frac{bx^2}{a^2}\right]dx$$

$$=c\left[\frac{b}{2}x-\frac{b}{a}\frac{x^2}{2}+\frac{b}{2a^2}\frac{x^3}{3}\right]_{x=0}^{a}$$

$$=c\left[\frac{b}{2}(a)-\frac{b}{2a}(a^2)+\frac{b}{6a^2}(a^3)\right]$$

$$=c\left[\cancel{\frac{ab}{2}}-\cancel{\frac{ab}{2}}+\frac{ab}{6}\right]$$

$$\boxed{A=\frac{abc}{6}\text{ sq units}}$$

Note: volume of triple integral is given by,

$V=\iiint dx\,dy\,dz$

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Beta and Gamma functions:

Def:

(i) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad (m,n>0)$ — ①

(ii) sub $x = \sin^2\theta$ and simplifying we get

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \text{ — ②}$$

(iii) $\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx \quad (n>0)$ — ③

(iv) sub $x = t^2$ and simplifying we get

$$\Gamma(n) = 2\int_0^\infty t^{2n-1} \cdot e^{-t^2}\, dt \text{ — ④}$$

Eq^n ① and ② are definitions of β function, eq^n ③ and ④ are definitions of Γ function

properties:

(i) $\beta(m,n) = \beta(n,m)$

(ii) $\Gamma(n+1) = n\,\Gamma(n) = n!$ for a +ve integer 'n'.

(iii) $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$ is the relation b/w β and Γ function.

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(v) $\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

(vi) from eq^n ②, sub $2m-1 = p$; $2n-1 = q$

$$\Rightarrow m = \frac{p+1}{2} \; ; \; n = \frac{q+1}{2}$$

eq^n ② $\Rightarrow$ $\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$

Relation b/w Beta and Gamma function:

with usual notations prove that

$$\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

proof: By definition $\Gamma(n) = 2\int_0^\infty x^{2n-1}.e^{-x^2}dx$ — (*)

$|||^{ly}$ $\Gamma(m) = 2\int_0^\infty y^{2m-1}.e^{-y^2}dy$

$\Gamma(m).\Gamma(n) = 2\int_0^\infty y^{2m-1}.e^{-y^2}dy \times 2\int_0^\infty x^{2n-1}.e^{-x^2}dx$

$\Gamma(m).\Gamma(n) = 4\int_0^\infty\int_0^\infty x^{2n-1}.y^{2m-1}.e^{-(x^2+y^2)}dx\,dy$ — (1)

In polars, $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

Also $dxdy = r\,dr\,d\theta$

$\therefore$ $\theta = 0$ to $\pi/2$

$r = 0$ to ∞

sub in eq^n (1)

$\Gamma(m).\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}(r\cos\theta)^{2n-1}.(r\sin\theta)^{2m-1}.e^{-r^2}.r\,dr\,d\theta$

$\Gamma(m).\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} r^{2n-1}.(\cos\theta)^{2n-1}.r^{2m-1}.\sin^{2m-1}\theta..e^{-r^2}.r\,dr\,d\theta$

$\Gamma(m).\Gamma(n) = \left(2\int_{\theta=0}^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta\,d\theta\right)\times\left(2\int_{r=0}^{\infty} r^{2(m+n)-1}.e^{-r^2}dr\right)$

$\Gamma(m).\Gamma(n) = \beta(m,n)\times\Gamma(m+n)$ using (*)

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

proof: By def^n $\Gamma(n) = 2\int_0^\infty x^{2n-1} \cdot e^{-x^2}\, dx$

put $n = 1/2$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty x^0 e^{-x^2}\, dx$$

i.e, $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx$

|||^{ly} $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy$

Multiplying,

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \times 2\int_0^\infty e^{-y^2} dy$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \quad \text{—— (1)}$$

In polars, $x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$

Also $dx\,dy = r\,dr\,d\theta$

θ : 0 to $\pi/2$

r : 0 to ∞

sub in eq^n (1),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta \quad \text{—— (2)}$$

put $r^2 = t$

$\Rightarrow 2r\,dr = dt \Rightarrow r\,dr = dt/2$

$\therefore \int e^{-r^2} r\,dr = \int e^{-t} \frac{dt}{2} = \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) = \frac{-1}{2} e^{-r^2}$

sub in eq^n (2),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\pi/2} \frac{-1}{2}\left(e^{-r^2}\right)_0^\infty d\theta$$

$$= -2\int_{\theta=0}^{\pi/2} \left(\underset{0}{e^{-\infty}} - \underset{1}{e^{0}}\right) d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} 1 \, d\theta$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\,[\theta]_0^{\pi/2}$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \cancel{2}\left[\frac{\pi}{\cancel{2}}\right]$$

$$\Rightarrow \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

problems on Beta and Gamma functions

Note: (i) For expressions of the form $(a-x^n)$, substitute $x^n = a\sin^2\theta$

(ii) For expressions of the form $(a+x^n)$; substitute $x^n = a\tan^2\theta$.

Evaluate the following:

** ① S.T $\displaystyle\int_0^\infty \sqrt{y}\,.\,e^{-y^2}dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}dy = \frac{\pi}{2\sqrt{2}}$

Let $I_1 = \displaystyle\int_0^\infty \sqrt{y}\,.\,e^{-y^2}dy = \int_0^\infty y^{1/2}.\,e^{-y^2}dy$ —— ①

$I_2 = \displaystyle\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}dy = \int_0^\infty y^{-1/2}\,e^{-y^2}dy$ —— ②

By def^n $\Gamma(n) = 2\displaystyle\int_0^\infty x^{2n-1}.\,e^{-x^2}dx$

$\Rightarrow \displaystyle\int_0^\infty x^{2n-1}.\,e^{-x^2}dx = \frac{1}{2}\Gamma(n)$ —— ③

comparing ① and ③,

$2n-1 = \frac{1}{2} \Rightarrow 2n = \frac{3}{2} \Rightarrow \boxed{n = \frac{3}{4}}$

$\therefore \boxed{I_1 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)}$

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Comparing ② and ③,

$2n-1 = -\frac{1}{2} \Rightarrow 2n = \frac{1}{2} \Rightarrow \boxed{n = \frac{1}{4}}$

$\therefore \boxed{I_2 = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)}$

Thus $I_1 \times I_2 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$ $\left(\Gamma\left(\frac{3}{4}\right)\times\Gamma\left(\frac{1}{4}\right) = \pi\sqrt{2}\right)$

$= \frac{1}{4}\pi\sqrt{2}$

$= \frac{\pi\cancel{\sqrt{2}}}{2\sqrt{2}\cancel{\sqrt{2}}} \Rightarrow \frac{\pi}{2\sqrt{2}}$ //

② ST $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Let $I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$= \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta\, d\theta$

(prop 6) Using $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Here, $p = -1/2$, $q = 1/2$

$\therefore I = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$\Rightarrow \frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1/4+3/4)}$ $\left[\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$

$= \frac{1}{2}\frac{(\pi\sqrt{2})}{\Gamma(1)}$

$= \frac{1}{2}\pi\sqrt{2}$ [WKT $\Gamma(n+1) = n!$
$\Gamma(1) = 0! = 1$]

$= \frac{\pi\cancel{\sqrt{2}}}{\sqrt{2}\cancel{\sqrt{2}}}$

$\boxed{I = \frac{\pi}{\sqrt{2}}}$

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③ ST $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

Let $I_1 = \int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^0\theta\, d\theta$ —— ①

$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta\, d\theta$ —— ②

WKT, $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

from ①, $p = -1/2$, $q = 0$

$\therefore I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

from ②, $p = 1/2$, $q = 0$

$\therefore I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$\therefore I_1 \times I_2 = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$= \frac{1}{4}\left[\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \times \frac{\Gamma(3/4) \cdot \Gamma(1/2)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\right]$

$= \frac{1}{4}\left[\frac{\Gamma(1/4) \cdot \sqrt{\pi}}{\cancel{\Gamma\left(\frac{3}{4}\right)}} \times \frac{\cancel{\Gamma(3/4)} \cdot \sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}\right]$

$= \frac{\pi}{4}\left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma(5/4)}\right]$

But $\Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$ $\left[\Gamma(n+1) = n\Gamma(n) = n! \quad \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\right]$

$I_1 \times I_2 = \frac{\pi}{\cancel{4}} \cdot \frac{\cancel{\Gamma(1/4)}}{\cancel{\frac{1}{4}}\cancel{\Gamma\left(\frac{1}{4}\right)}}$

$\Rightarrow \boxed{I_1 \times I_2 = \pi}$

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(4) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Let $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$= \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta\, d\theta$

Here $p = 1/2, \quad q = -1/2$

$I = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$

$= \frac{1}{2} \frac{\Gamma(3/4) \cdot \Gamma(1/4)}{\Gamma(3/4 + 1/4)}$

$= \frac{1}{2} \frac{\pi\sqrt{2}}{\Gamma(1)}$

$= \frac{1}{2}\pi\sqrt{2} \Rightarrow \frac{\pi\cancel{\sqrt{2}}}{\sqrt{2}\cancel{\sqrt{2}}} \Rightarrow \boxed{I = \frac{\pi}{\sqrt{2}}}$

(5) $\int_0^1 x^{3/2}(1-x)^{1/2}\, dx$

Let $I = \int_0^1 x^{3/2}(1-x)^{1/2}\, dx$

for expressions of the form $(a - x^n)$ we substitute $x^n = a\sin^2\theta$

here, $a = 1, \quad n = 1$

$\therefore x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$\therefore x^{3/2} = (\sin^2\theta)^{3/2} = \sin^3\theta$.

Also, $(1-x)^{1/2} = (1-\sin^2\theta)^{1/2} = (\cos^2\theta)^{1/2} = \cos\theta$

consider $x = \sin^2\theta$

put $x = 0$; $\quad 0 = \sin^2\theta \Rightarrow \boxed{\theta = 0}$

put $x = 1$; $\quad 1 = \sin^2\theta \Rightarrow \boxed{\theta = \pi/2}$

sub in (1),

$$I = \int_{\theta=0}^{\pi/2} \sin^3\theta \cdot \cos\theta \, 2\sin\theta\cos\theta \, d\theta .$$

$$I = \int_{\theta=0}^{\pi/2} 2\sin^4\theta \cdot \cos^2\theta \, d\theta .$$

WKT $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}}\beta\left(\frac{4+1}{2}, \frac{2+1}{2}\right) = \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$I = \frac{\Gamma(5/2) \cdot \Gamma(3/2)}{\Gamma(5/2 + 3/2)}$$

$$= \frac{3/2 \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$$

$\Gamma(n+1) = n!$

$$= \frac{3}{8} \frac{\sqrt{\pi}\sqrt{\pi}}{3!} \Rightarrow \boxed{I = \frac{\pi}{16}}$$

$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right)$
$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right)$
$= \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right)$
$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

6) $\int_0^2 (4-x^2)^{3/2} \, dx$

Let $I = \int_0^2 (4-x^2)^{3/2} \, dx$ — (1)

on comparing, $a = 4$, $n = 2$

$$\boxed{x^2 = 4\sin^2\theta}$$
$$\Rightarrow x = 2\sin\theta$$
$$\Rightarrow dx = 2\cos\theta \, d\theta .$$

consider, $(4-x^2)^{3/2}$

$= (4 - 4\sin^2\theta)^{3/2}$

$= 4^{3/2}(1-\sin^2\theta)^{3/2}$

$= \sqrt{64}\cos^3\theta \Rightarrow 8\cos^3\theta .$

consider, $x = 2\sin\theta$

put $x = 0$; $0 = 2\sin\theta \Rightarrow \boxed{\theta = 0}$

put $x = 2$; $2 = 2\sin\theta \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

Sub in ①,

$$I = \int_{\theta=0}^{\pi/2} 8\cos^3\theta \, . \, 2\cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} 16 \, \cancel{\cos} \sin^0\theta \, . \, \cos^4\theta \, d\theta$$

WKT, $\int_0^{\pi/2} \sin^p\theta \, . \, \cos^q\theta \, . \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$= 16 . \frac{1}{2}\beta\left(\frac{0+1}{2}, \frac{4+1}{2}\right) = 8\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= 8 . \frac{\Gamma(1/2) . \Gamma(5/2)}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)}$$

$$= 8 \times \frac{\sqrt{\pi} . \frac{3}{2} . \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$$

$$= \frac{\overset{2}{\cancel{8}} \times \sqrt{\pi} \times \frac{3}{\cancel{4}} \times \sqrt{\pi}}{2!} = \frac{\cancel{2} \times 3\sqrt{\pi}\sqrt{\pi}}{\cancel{2}} = \boxed{I = 3\pi}$$

⑦ $\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$

Let $I = \int_0^{\infty} x^{-1/2} . \frac{1}{(1+x)} dx$ — ①

For expressions of the form $(a+x^n)$, we sub $x^n = a\tan^2\theta$.

here $a = 1$ $n = 1$

$\therefore \boxed{x = \tan^2\theta}$

$\Rightarrow dx = 2\tan\theta \sec^2\theta \, d\theta$

consider $x^{-1/2} = (\tan^2\theta)^{-1/2} = \tan^{-1}\theta = \frac{\cos\theta}{\sin\theta}$

and $(1+x) = (1+\tan^2\theta) = \sec^2\theta = \frac{1}{\cos^2\theta}$

$$dx = 2\tan\theta \sec^2\theta \, d\theta = 2 \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos^2\theta} \, d\theta.$$

$$dx = \frac{2\sin\theta \, d\theta}{\cos^3\theta}$$

consider $x = \tan^2\theta$

put $x = 0$; $0 = \tan^2\theta \Rightarrow \boxed{\theta = 0}$

put $x = \infty$; $\infty = \tan^2\theta \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

Sub in ①,

$$I = \int_{\theta=0}^{\pi/2} \frac{\cos\theta}{\sin\theta} \cdot \cos^2\theta \cdot \frac{2\sin\theta}{\cos^3\theta} \, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} 1 \cdot d\theta$$

$$\cancel{= 2\,[\theta]_0^{\pi/2} \Rightarrow \boxed{I = \pi}}$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \, d\theta$$

WKT, $\displaystyle\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\therefore \cancel{2} \cdot \frac{1}{\cancel{2}} \beta\left(\frac{0+1}{2}, \frac{0+1}{2}\right) = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$I = \frac{\Gamma(1/2) \cdot \Gamma(1/2)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{\Gamma(1)}$$

$$= \frac{\pi}{0!} \Rightarrow \boxed{I = \pi}$$

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(8) ST $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Let $I = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ —— (1)

For expression of the form $(a+x^n)$, we sub $x^n = a\tan^2\theta$

here $a=1$, $n=1$

$\therefore \boxed{x = \tan^2\theta}$

$\Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$.

put $x=0$; $\tan^2\theta = 0 \Rightarrow \boxed{\theta=0}$

put $x=\infty$; $\tan^2\theta = \infty \Rightarrow \boxed{\theta = \cancel{\infty}\, \pi/2}$

Sub in (1),

$$I = \int_{\theta=0}^{\pi/2} \frac{(\tan^2\theta)^{m-1}}{(1+\tan^2\theta)^{m+n}} \cdot 2\tan\theta\sec^2\theta\, d\theta.$$

$$= \int_{\theta=0}^{\pi/2} \frac{\tan^{2m-2}\theta}{(\sec^2\theta)^{m+n}} \cdot 2\tan\theta\sec^2\theta\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \frac{\tan^{2m-2}\theta}{\sec^{2m+2n}\theta} \cdot \tan\theta\sec^2\theta\, d\theta.$$

$$= 2\int_{\theta=0}^{\pi/2} \tan^{2m-1}\theta \cdot \frac{1}{\sec^{2m+2n-2}\theta}\, d\theta.$$

$$= 2\int_{\theta=0}^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta} \cdot \cos^{2m+2n-2}\theta\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{\cancel{2m}+2n-2-\cancel{2m}+1}\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

$= \beta(m, n)$ [by defn]

Thus $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m,n)$

(8) $\int_0^\infty \frac{dx}{1+x^4}$

(9) $\int_0^1 \sqrt{\frac{1-x}{x}} dx$

→ Let $I = \int_0^\infty \frac{1}{1+x^4} dx$

For expression of the form $(a+x^n)$,
we sub $x^n = a \tan^2\theta$
here, $a=1$ $n=4$

$\Rightarrow x^4 = \tan^2\theta$

$\therefore x = \sqrt{\tan\theta}$

$\therefore dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$

$x = \sqrt{\tan\theta} \Rightarrow x^2 = \tan\theta$

put $x=0 \longrightarrow \tan\theta = 0 \Rightarrow \boxed{\theta = 0}$

$x = \infty \longrightarrow \tan\theta = \infty \Rightarrow \boxed{\theta = \pi/2}$

$I = \int_{\theta=0}^{\pi/2} \frac{1}{1+\tan^2\theta} \cdot \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$

$\frac{1}{2}\int_{\theta=0}^{\pi/2} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta \, d\theta$

By using formula,

$= \frac{\beta}{2}\frac{1}{2}\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right) = \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$= \frac{1}{4} \frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$

$= \frac{1}{4}\pi\sqrt{2} \Rightarrow \boxed{I = \frac{\pi}{2\sqrt{2}}}$

(9) →

$$\int_0^1 \sqrt{\frac{1-x}{x}}\, dx$$

$$\int_0^1 \frac{(1-x)^{1/2}}{x^{1/2}}\, dx$$

For expression of the form $(a \mp x^n)$,
we sub $x^n = a \sin^2\theta$.

Here, $a = 1$ $n = 1$

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x^{1/2} = (\sin^2\theta)^{1/2} \Rightarrow x^{1/2} = \sin\theta$

$(1-x)^{1/2} = (1-\sin^2\theta)^{1/2} = (\cos^2\theta)^{1/2} = \cos\theta$

put $x = 0$; $\boxed{\theta = 0}$

$x = 1$, $\boxed{\theta = \pi/2}$

$$I = \int_{\theta=0}^{\pi/2} \frac{\cos\theta}{\cancel{\sin\theta}} \cdot d\theta \cdot 2\,\cancel{\sin\theta}\cos\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \sin^0\theta \cdot \cos^2\theta \cdot d\theta$$

$$= \beta\left(\frac{0+1}{2}, \frac{2+1}{2}\right) = \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma(1/2)\cdot\Gamma(3/2)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}$$

$$= \frac{\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{1}$$

$$\Rightarrow \boxed{I = \frac{\pi}{2}}$$