MODULE - 5 - QUANTUM COMPUTING

Moore's law - limitation of VLSI, Classical vs Quantum Computation, bit, Qubit and its properties, Bloch Sphere, Dirac notation, Brief discussion on types of qubit, Superconducting qubits, Harmonic oscillator (qualitative) – Need for anharmonicity, Charge qubit, Quantum Gates – Pauli Gates, Phase gate (S, T), Hadamard Gate, Two qubit gates – CNOT gate, Predicting the outputs of various combinations of single and two-qubit gates, Numerical Problems.

Module - 5 Blow-up

Subtopics	Topics to be covered	Duration
Moore's Law & Limitations	Physical limits of miniaturization, motivation for quantum computation	½
Classical vs Quantum Computation	Bits vs Qubits, advantages of quantum computing (parallelism, entanglement)	½
Qubit Properties & Bloch Sphere	Properties of Qubits : Superposition states, entanglement, Measurement(Collapse of states), Quantum Interference. Reversibility. Bloch sphere representation – Diagram and Explanation	1
Dirac Notation and Qubit Types	superconducting qubits, harmonic oscillator model(qualitative), need for anharmonicity, Charge Qubit Dirac notation (|0>, |1>),	1 ½
Operators and Operations (Matrix form)	Unitary and Hermitian operators, inner product, orthogonality, orthonormality	1
Quantum gates-Single-Qubit Gates	Matrix representation of Pauli (X,Y,Z), Phase (S,T), Hadamard gates(Circuit, Matrix, Explanation, Truth Table)	1
Two-Qubit Gates, Entanglement, Bell States	Controlled-NOT gate (Circuit, Matrix, Explanation, Truth Table), Entanglement, Bell states	1
Numerical Problems on Gates	Predictions of circuits with single/two-qubit gates, unitary matrices, matrix operations	1 ½

Introduction:

Today's computers—Both in theory and practice (PCs, HPCs, laptops, tablets, smartphone) are based on classical physics. They are limited by locality (operations have only local effects) and by the classical fact that systems can be in only one state at the time. However, modern quantum physics tells us that the world behaves quite differently. A quantum system can be in a superposition of many different states at the same time, and can exhibit interference effects during the course of its evolution. Moreover, spatially separated quantum systems may be entangled with each other and operations may have "non-local" effects because of this.

Quantum computation is the field that investigates the computational power and other properties of computers based on quantum-mechanical principles. It combines two of the most important strands of 20th-century science: quantum mechanics (developed by Planck, Einstein, Bohr, Heisenberg, Schrödinger and others in the period 1900–1925) and computer science . An important objective is to find quantum algorithms that are significantly faster than any classical algorithm solving the same problem.

Quantum computing is a type of computation whose operations can harness the phenomena of quantum mechanics, such as superposition, interference, and entanglement. To store and manipulate the information, they use their own quantum bits also called 'Qubits' unlike other classical computers which are based on classical computing that uses binary bits 0 and 1 individually.

The computers using such type of computing are known as 'Quantum Computers'. In such small computers, circuits with transistors, logic gates, and Integrated Circuits are not possible. Hence, it uses the subatomic particles like atoms, electrons, photons, and ions as their bits along with their information of spins and states. Instead of 0 and 1, Quantum computers can choose two electronics states of an atom or two different polarization orientations of light for the two states They can be superposed and can give more combinations.

The Essential Elements of Quantum Theory

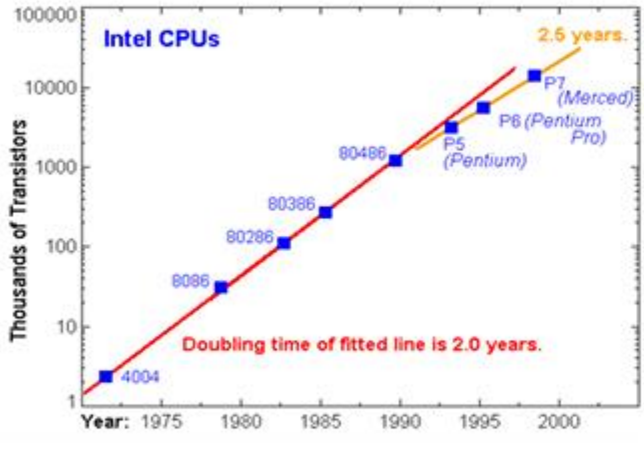
- Energy values are discrete units.
- Elementary particles may behave like particles or waves.
- The movement of elementary particles is inherently random and, thus, unpredictable.
- The simultaneous measurement of two complementary values - such as the position and momentum of a particle - is imperfect. The more precisely one value is measured, the more flawed the measurement of the other value will be.

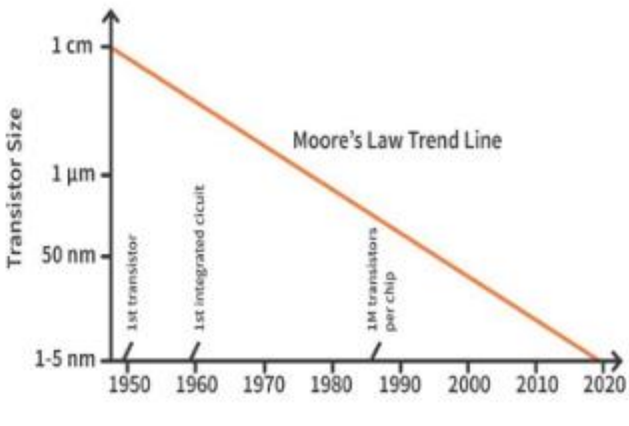
Moore's law & its end:

In 1965, Gordon E. Moore—co-founder of Intel—postulated, "the number of transistors that can be packed into a given unit of space will double about every eighteen months". This is also known as Moore's Law. Moore's law is an observation and projection of a historical trend. Rather than a law of physics, it is an empirical relationship linked to gains from experience in production.

Gordon Moore did not call his observation as "Moore's Law," nor did he set out to create a "law". Moore made this statement based on noticing emerging trends in chip manufacturing at the semiconductor industry. Eventually, Moore's insight became a prediction, which in turn became the golden rule known as Moore's Law. Moore's Law implies that computers, machines that run on computers, and computing power all become smaller, faster, and cheaper with time, as transistors on integrated circuits become more efficient.

Here is a graphic representation for microprocessors.





Is Moore's Law Coming to an End?

The electronic industry for computers grows hand-in-hand with the decrease in size of the integrated circuits. This miniaturization is necessary to increase computational power, that is, the number of floating-point operations per second (FLOPS) a computer can perform. In 1950's, electronic computers were capable of performing approximately 10^3 FLOPS while present supercomputers have a power greater than 10^{13} FLOPS. According to Moore's law, the number of transistors that may be placed on a single integrated circuit chip doubles approximately every 18 – 24 months. The present limit is approximately 10^8 transistors per chip and the typical size of circuit components is of the order of 100 Nano meters. That means, we have reached the atomic size for storing a single bit of information and quantum effects have become unavoidably dominant. This is nothing but an end of Moore's law. We need some alternative and is one such alternative is quantum computing. Quantum computers are based on quantum bits (qubits) and use quantum effects like superposition and entanglement to their benefit, hence overcoming the miniaturization problems of classical computing.

Comparison of Classical and Quantum Computing:

Comparison key	Classical computer	Quantum computer
Basis of computing	Large scale multipurpose computer Quantum computer based on classical physics	High speed computer based on quantum mechanics
Information storage	Bit-based information storage using voltage/ charge	Quantum bit-based information storage using electron spin or polarization
Bit values	Bits having a value of either 0 or 1 can have a single value at any instant	Qubits have a value of 0, 1 or sometimes linear combination of both, (a property known as superposition)
Number of possible states	The number of possible states is 2 which is either 0 or 1	The number of possible states is infinite since it can hold combinations of 0 or 1 along with some complex information
Output	Deterministic (repetition of computation on the same input gives the same output)	Probabilistic (repetition of computation on superposed states gives probabilistic answer)
Gates used for processing	Logic gates (AND, OR, NOT, etc.)	Quantum gates (X, Y, Z, H, CNOT etc.)
Operations	Operations use Boolean Algebra.	Operations use linear algebra and are represented with unitary matrices.
Circuit implementation	Circuit implemented in macroscopic technologies	Circuits implemented in microscopic technologies.
Data processing	Data processing is carried out by logic and in sequential order.	Data processing is carried out by quantum logic at parallel instances.

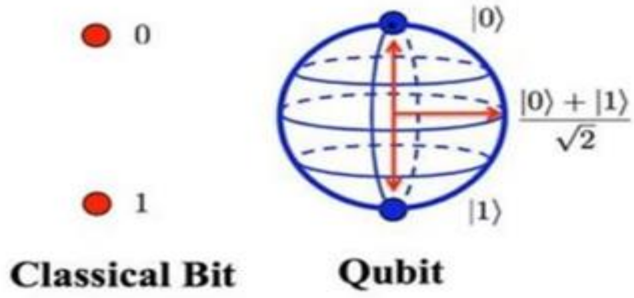
Classical Bits: It's a single unit of information that has a value of either 0 or 1 (off or on, false or true, low or high).

Quantum Bits: In quantum computing, a qubit or quantum bit is the basic unit of quantum information. A quantum system like atom or electrons can exist in states as 0 and 1 or simultaneously both as 0 and 1.

Concept of qubit and its properties:

From bits to qubit

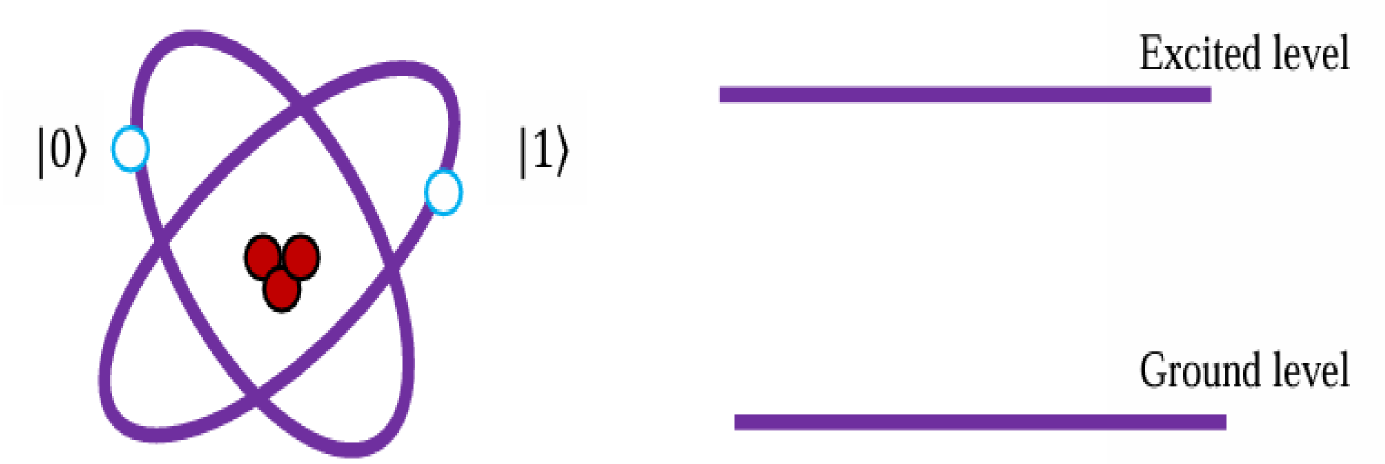
Bit: A digital computer stores and processes information using bits, which can be either 0 or 1. In modern computing and communications, bits are represented by the absence or presence of an electrical signal, encoding "0" and "1" respectively.



Qubit is the physical carrier of quantum information. It is the quantum version of a bit, and its quantum state can be written in terms of two levels, labelled $|0\rangle$ and $|1\rangle$. $|\ \rangle$ this notation is known as 'ket' notation and $\langle\ |$ is known as 'brac' notation. Both are together called as Dirac notations 'Ket' is analogous to a column vector. They are also called basis vectors and represented by two-dimensional column vectors as follows

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The qubit can be in any one of the two states as well as in the superposed state simultaneously. In quantum computation two distinguishable states of a system are needed to represent a bit of data. For example, two states of an electron orbiting a single atom is shown in the following figure. Spin up is taken as $|1\rangle$ and spin down is taken as $|0\rangle$. Similarly ground state energy level is $|0\rangle$ and excited state level is $|1\rangle$



Qubit represented by two electronic levels in an atom

<u>Advantages of Quantum Computing:</u>

Quantum Computing offers many advantages that promise to revolutionise various industries and domains. Below are the key advantages of Quantum Computing elaborated:

1) Rapid calculations at unprecedented speeds:

Quantum Computers can perform calculations exponentially faster than classical computers. They harness the principles of quantum mechanics to process information in parallel, allowing for rapid execution of complex algorithms. This capability enables Quantum Computers to tackle problems currently infeasible for classical computers due to their computational limitations.

2) Efficient data storage and retrieval capabilities:

Quantum Computing offers efficient data storage and retrieval capabilities through quantum memory systems. Quantum bits, or qubits, can store and manipulate vast amounts of data in quantum states, allowing for high-density data storage and faster access times than classical storage systems. This enables Quantum Computers to handle large datasets more effectively, facilitating data-intensive tasks such as machine learning and data analysis.

3) Mastery in resolving intricate problems:

Quantum Computing solves intricate problems involving complex mathematical calculations, optimisation, and simulation. Quantum algorithms utilises the unique properties of quantum mechanics to find optimal solutions to optimisation problems, simulate quantum systems with high accuracy, and solve cryptographic challenges efficiently. This mastery over intricate problems opens new avenues for scientific research, engineering, and innovation.

4) Accelerated computational speeds:

Quantum Computers offer accelerated computational speeds, enabling organisations to perform computations at unprecedented rates. This speed advantage translates into faster decision-making, reduced time-to-market for products and services, and increased productivity across various industries. Quantum Computers can solve computational tasks in minutes or hours that would take classical computers days, weeks, or even years to complete.

5) Revolutionising Google searches:

Quantum Computing has the potential to revolutionise search algorithms and information retrieval systems, including Google searches. Quantum algorithms can process large amounts of data more efficiently, providing users with more accurate and relevant search results. This could enhance user experiences, improve search engine performance, and enable new functionalities such as personalised recommendations and predictive search.

6) Pioneering new technological frontiers:

Quantum Computing is paving the way for the development of new technological frontiers and innovations. It enables researchers and engineers to explore previously uncharted territories in materials science, drug discovery, artificial intelligence, and quantum communication. Quantum Computers are strong machines that help us discover new things and go beyond what we currently know and can do.

7) Elevated privacy standards:

Quantum Computing offers elevated privacy standards through advanced encryption and cryptographic techniques. Quantum cryptography uses the rules of quantum mechanics to create safe ways to send messages that are very hard for others to spy on or hack into. This enhances privacy and data security in digital communication networks, safeguarding sensitive information from unauthorised access and interception.

8) Conducting complex simulations with ease:

Quantum Computing facilitates the simulation of complex systems and phenomena with ease. Quantum simulators can accurately model quantum systems, chemical reactions, biological processes, and physical phenomena that are challenging to simulate using classical computers. This capability enables researchers to gain insights into complex systems, predict their behaviour, and optimise their performance for real-world applications.

Disadvantages of Quantum Computing:

Quantum Computing holds immense promise but has several challenges and disadvantages that must be addressed. Below are the key disadvantages of Quantum Computing explained:

1) Formulating algorithms with quantum precision:

Developing algorithms for Quantum Computers requires a deep understanding of quantum mechanics and specialised expertise. Quantum algorithms must be formulated with precision to effectively leverage qubits' unique properties effectively. This is a significant challenge for researchers and developers, as Quantum Computing concepts can be complex and abstract. Additionally, debugging and optimising quantum algorithms can be challenging due to the non-intuitive behaviour of quantum systems.

2) Requirement for extremely low temperatures:

Quantum Computers operate using qubits, which are highly sensitive to external disturbances such as temperature fluctuations and electromagnetic interference. To maintain the intricate quantum states of qubits, Quantum Computers require extremely low temperatures close to absolute zero (-273.15°C or 0 Kelvin). Achieving and maintaining these ultra-low temperatures using cryogenic systems adds complexity and cost to Quantum Computing infrastructure, limiting scalability and accessibility.

3) Limited accessibility to the public:

Quantum Computing resources are limited and primarily available to academic institutions, research laboratories, and large technology companies. Accessing Quantum Computers for experimentation and research purposes can be challenging for individuals and smaller organisations due to cost constraints and limited availability. This lack of accessibility impedes widespread adoption and hinders innovation in Quantum Computing.

4) Navigating challenges in Internet security:

Quantum Computing potentially threatens existing cryptographic systems used to secure digital communication and data storage. Quantum algorithms like Shor's can factor large prime numbers efficiently, compromising widely-used encryption methods like RSA and ECC. Addressing this challenge requires the development of quantum-resistant cryptographic algorithms and deploying quantum-safe encryption technologies to ensure cybersecurity in the quantum era.

5) Addressing heat-related concerns:

Quantum Computing systems generate significant heat during operation due to the energy dissipation associated with qubit manipulation and control. Managing heat dissipation and thermal management in Quantum Computers is essential to prevent qubit decoherence and ensure reliable operation. Heat issues create technical problems and can stop Quantum Computing systems from becoming larger and more efficient.

6) Overcoming the complexity in construction:

Building and scaling Quantum Computing hardware is a highly complex and resource-intensive process. Quantum systems require precise control over qubits and sophisticated error correction and fault-tolerance mechanisms to mitigate de-coherence and errors. Designing, fabricating, and calibrating quantum processors involves intricate engineering and manufacturing processes, making achieving reliable and scalable Quantum Computing systems challenging.

Quantum Parallelism:

Quantum parallelism is the ability of a quantum computer to perform calculations on all possible input values simultaneously by leveraging the quantum property of superposition.

Unlike classical computers that process tasks sequentially or split them across multiple processors, quantum computers use superposition to have qubits exist in multiple states (0 and 1) at the same time. This allows a quantum computer to explore and process a vast number of possibilities simultaneously. Because of this ability to explore many paths at once, a quantum algorithm can perform computations that would take a classical computer an infeasible amount of time.

A quantum computer can check many potential solutions at once, drastically reducing the number of steps needed compared to a classical computer that would have to check them one by one.

Superposition of two states:

The difference between qubits and classical bits is that a qubit can be in a linear combination (superposition) of the two states $|0\rangle$ *and* $|1\rangle$. For ex, if $\propto$ and β are the probability amplitudes of electron in ground state (ie, in $|0\rangle$ state) and in excited state (ie, in $|1\rangle$ state) then the linear combination of two states is $|\psi\rangle = \alpha\,|0\rangle + \beta|1\rangle$

The numbers α and β are complex but due to normalization conditions $|\alpha|^2 + |\beta|^2 = 1$. Here

$|\alpha|^2$ is the probability of finding $|\,\psi\,\rangle$ in state $|0\rangle$ and

$|\beta|^2$ is the probability of finding $|\,\psi\,\rangle$ in state $|1\rangle$.

So, that when a qubit is measured, it only gives either '0' or '1' as the measurement result probabilistically. Consider the following example of qubit representation

$$|\,\psi\,\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

$$\therefore\ \alpha = \frac{1}{\sqrt{2}}\ and\ \beta = \frac{1}{\sqrt{2}}$$

$|\alpha|^2 = |\beta|^2 = \frac{1}{\sqrt{2}}$

This means that with 50% probability the qubit will be found in $|0\rangle$ state as well as in $|1\rangle$ state. The superposed states are also called as space states where as $|0\rangle$ and $|1\rangle$ are called basis states.

Properties of qubits:

Qubits make use of discrete energy state particles such as electrons and photons. Qubits exists in two quantum state $|0\rangle$ and $|1\rangle$ or in a linear combination of both states. This is known as superposition. This property allows for exponentially many logical states at once (and no classical computer can achieve it)

Unlike classical bits, qubit can work with the overlap of both 0 & 1 states. For ex, a 4-bit register can store one number from 0 to 15 (because of $2^n = 2^4=16$), but 4-qubit register can store all 16 numbers

When the qubit is measured, it collapses to one of the two basis states $|0\rangle$ or $|1\rangle$
Quantum entanglement and quantum tunnelling are two exclusive properties of qubit.

(In other words, Two Qubits can strongly correlated with each other. Changing state of one of the qubits will instantaneously change the state of the other one in predictable way. This happens even if they are separated by very long distances).

Entanglement:

Quantum entanglement is one of the foundational properties of quantum mechanics and quantum technology. It is a phenomenon that explains how two subatomic particles can be intimately linked to each other so that two particles behave like a single unified unit, irrespective of their distance....

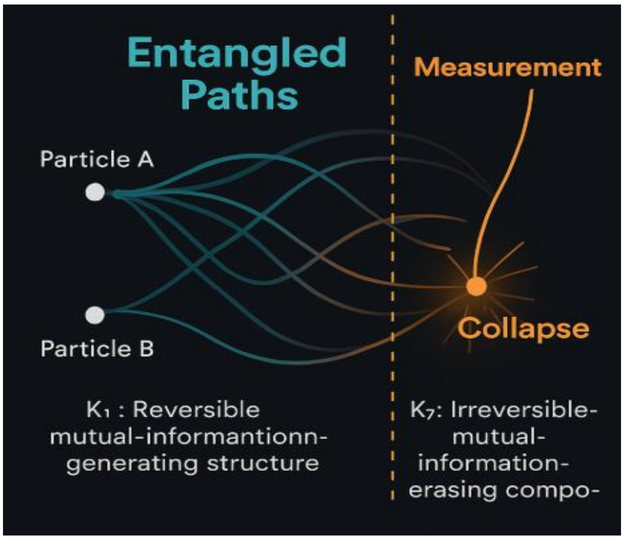
For example, consider a pair of entangled photons with random polarisations.

- When the polarization of one photon is measured and found to be vertical, the entanglement ensures that the polarization of the other photon will also be vertical.
- When an operation is performed on one of the entangled particles, there is an instantaneous reaction on the other. Hence, measurements of one state can affect the other....
- Read more at: https://vajiramandravi.com/upsc-exam/quantum-entanglement

Measurement of collapse of states:

When one particle in an entangled pair is measured, its quantum state "collapses" into a definite state, and because of the entanglement, the state of the other particle instantly collapses too, regardless of the distance between them.

Entanglement describes a special correlation between multiple quantum systems, like two particles. The state of the combined system cannot be described as two independent, separate states. This measurement instantly determines the state of the other particle. If one particle is measured to be spin-up, the other is instantly known to be spin-down (in a standard Bell state example)



Quantum interference in quantum computing:

It is the phenomenon where the probability amplitudes of qubits combine, either reinforcing each other (constructive interference) or canceling each other out (destructive interference). This process is used to manipulate quantum states, amplify correct answers, and suppress incorrect ones, a key principle for improving computational performance and speed for certain problems. This is called quantum interference.

Quantum Reversibility:

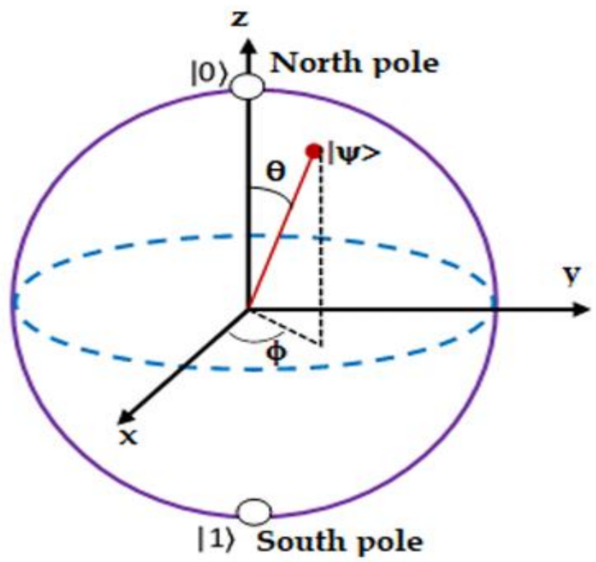
It is a fundamental principle in quantum mechanics, meaning that every quantum operation can be undone; this is crucial because it preserves quantum information and prevents the loss of control over interference effects needed for accurate calculations.

Together, these two concepts are essential for quantum algorithms to work: interference helps to amplify correct answers, while reversibility ensures that the process is controllable and that the necessary quantum information is not lost to the environment.

State of the qubits is represented using Bloch sphere.

After studying the physics of qubits it is now time to look at the mathematics of qubits. Let us start with the representation of qubit using Bloch sphere in a vector space. Later on we proceed towards single qubit, multi qubit, tensor operation, operators and matrix representation.

Bloch Sphere:



Bloch sphere is an imaginary sphere which is used to represent pure single-qubit states as a point on its surface. It has unit radius. Its North Pole and South Pole are selected to represent the basis states namely $|0\rangle$ and $|1\rangle$. North Pole represents $|0\rangle$ (say spin up↑) and South Pole represents $|1\rangle$ (say spin down ↓). All other points on the sphere represent superposed states (ie, state space). Bloch sphere allows the state of a qubit to be represented in spherical coordinates (ie, r, θ and ϕ). It is as follows

The state qubit $|\psi\rangle$ on the Bloch sphere makes an angle θ with z-axis and its projection (azimuth) makes angle ϕ with x-axis as shown. It is clear from the fig that $0 < \theta < \pi$ and $0 < \phi < 2\pi$. $|\psi\rangle$ is represented as

$$|\psi\rangle = \alpha\,|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle ----------(1)$$

Using this equation we can represent $|\psi\rangle$ for different θ and ϕ as follows

Case-1: $let\ \theta = 0\ and\ \phi = 0$, then eq (1) becomes

$$|\Psi\rangle = cos0\,|0\rangle + e^{i\phi}sin0\,|1\rangle = |0\rangle + 0$$

$$\therefore |\Psi\rangle = |0\rangle$$

Case-2: $let\ \theta = \pi\, and\ \phi = 0$, then eq (1) becomes

$$|\psi\rangle = cos\frac{\pi}{2}\,|0\rangle + e^{i\phi}\sin\frac{\pi}{2}\,|1\rangle = 0 + |1\rangle$$

$$\therefore |\Psi\rangle = |1\rangle$$

Case-3: $let\ \theta = \pi/2\ and\ \phi = 0$, then eq (1) becomes

$$|\psi\rangle = cos\frac{\pi}{4}\,|0\rangle + e^{i.0}\sin\frac{\pi}{4}\,|1\rangle$$

$$|\,\psi\,\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Case-4: $let\ \theta = \pi/2\ and\ \phi = \pi$, then eq (1) becomes

$$|\psi\rangle = cos\frac{\pi}{4}\ |0\rangle + e^{i.\pi} \sin\frac{\pi}{4}\ |1\rangle$$

$$|\ \psi\ \rangle = \frac{1}{\sqrt{2}}\ |0\rangle - \frac{1}{\sqrt{2}}\ |1\rangle$$

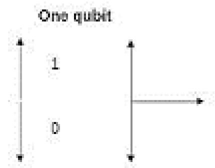
$$|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

In the above discussion we have represented only single qubit state. Bloch sphere is a nice visualization of single qubit states.

Single and Two qubits and Extension to N qubits:

A Single Qubit has two computational basis states $|0\rangle$ and $|1\rangle$. In single qubit the state of qubit will be either in $|0\rangle$ and $|1\rangle$

where $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$



Above basis vectors are written as column matrix in the form of identity matrix.

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$Which represents the single qubit state. The pictorial representation of the single qubit is as follows. $\alpha\ |0\rangle\ +\ \beta\ |1\rangle$.

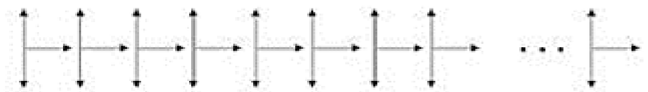
A two-qubit system has 4 computational basis states denoted as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. The matrix representation of two qubit state is $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

The pictorial representation of two qubit is as follows. $\alpha|00\rangle\ + \beta|01\rangle\ + \gamma|10\rangle\ + \delta|11\rangle$

Extension to N qubits:

A multi-qubit system of N qubits has 2^N computational basis states.

For example, a state with 3 qubits has 2^3 computational basis states. Thus for N qubits the computational basis states are denoted as $|00\ \cdots\ 00\rangle, |00\ \cdots\ 01\rangle, |00\ \cdots\ 10\rangle, |00\ \cdots\ 11\rangle\ \cdots\ |11\ \cdots\ 11\rangle$. The block diagram of representation of N qubits is as follows



Harmonic oscillator:

A system vibrating about an equilibrium position or vibrating with equilibrium configuration is known as harmonic oscillator. In harmonic oscillator there is a linear relation between restoring force and the displacement. Simple pendulum, an object supported by spring system or floating in a fluid, atoms m crystal, a diatomic molecule etc. are the examples for harmonic oscillator. Or any oscillator with small amplitudes of oscillation can be treated as harmonic oscillator.

These oscillators are classified into classical and quantum harmonic oscillators. Simple pendulum, an object supported by spring system or floating in a fluid are classical whereas electrons in the atom, atoms in crystal, a diatomic molecule are quantum mechanical oscillators. There are many differences between them. Two main differences are

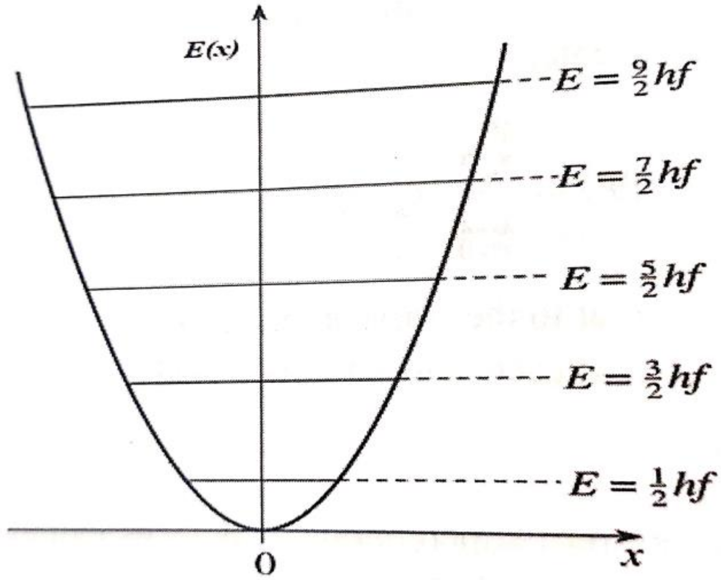
i. The probability of finding the classical oscillator is maximum at the extreme position and minimum at the equilibrium position. It is just opposite for quantum mechanical oscillator
ii. The energy of the classical oscillator is continuous whereas that of quantum mechanical oscillator is discrete

Using Schrodinger wave equation, we can show that the energy of an oscillator is

$$E_n = \left(n + \frac{1}{2}\right) hf$$

Where, h—Planck's constant, f—frequency of the oscillation and n = 0,1,2,3…

According to this equation energy levels of harmonic oscillator are discrete and equally spaced as shown in fig The difference between any two successive levels is $\frac{1}{2}hv$.



When particle bound to such a system makes a transition from a higher-energy state to a lower-energy state, a photon of energy hf is emitted. Similarly, when the particle makes a transition from a lower-energy state to a higher-energy state, the smallest-energy quantum hf is absorbed by the particle. There are infinite levels and we can use any two of them (say n=1 and n= 2) to build qubits. However unfortunately there are a few obstacles.

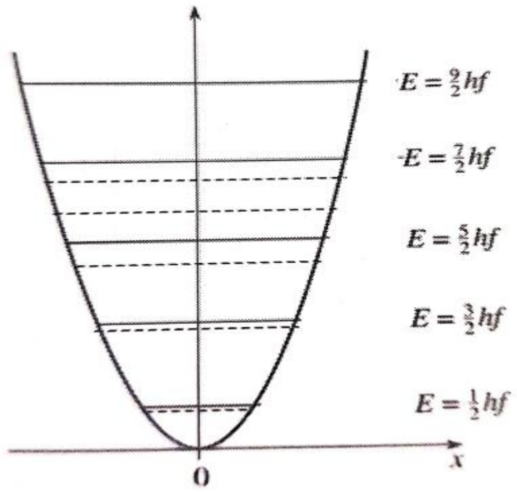
But Josephson junction acts as superconducting LC circuit and due to tunnelling of Cooper pairs superconducting current can flow through the element. Here, quantum mechanically, we are able to obtain a situation where the two lowest energy levels are separated fairly well from the rest of the energy spectrum and therefore the corresponding subsystem can approximately be treated as a two-level harmonic oscillator system, i. e. a qubit

Anharmonicity:

If the restoring force is not linearly dependent on the displacement then the oscillation is called as anharmonic. In anharmonic oscillation, the amplitude is large and energies are not equally spaced. The spacing between the levels becomes less at higher energy values (ie, they are closely spaced at higher energy values). In the following fig dotted lines are anharmonic oscillator levels whereas solid lines are for harmonic.

At higher temperature, the molecules vibrate with larger amplitude and hence oscillations are anharmonic.

The concept of anharmonicity is useful



- to understand molecular spectroscopy
- to explain thermal expansion
- to study non-linear optics
- in astronomical studies
- in quantum technologies

Need for anharmonicity in quantum technologies:

i. Since the energy difference between any two successive levels in harmonic oscillator is same and it is not possible to distinguish the levels and selectively excite. For example, you designate ground state as $|0\rangle$, 1st excited state as$|1\rangle$, the second excited state as $|2\rangle$ and So on. Now you try to excite $|0\rangle \rightarrow |1\rangle$ there is every possibility of exciting $|1\rangle \rightarrow |2\rangle$or $|2\rangle \rightarrow |3\rangle$.

Hence it is impossible to isolate two level system for qubit. Anharmonicity removes the problem of isolate because the energy levels are not not uniformly spaced. The excitation of $|0\rangle \rightarrow |1\rangle$ *does not excie* $|1\rangle \rightarrow |2\rangle$ either accidentally or intentionally.

ii. By tuning into precise frequency, it is possible to operate on only one particular qubit without spilling (leaking) into other qubits i.e., a particular qubit can be targeted.

iii. Higher order anharmonicity makes the qubit more sensitive to charge or flux noise. This increases the quantum gate fidelity and leads to more accurate quantum operations.

DIRAC REPRESENTATION AND MATRIX OPERATIONS:

Linear Algebra:

Linear Algebra is the study of vector spaces and operations on vector spaces. The Standard quantum mechanical notation for a quantum state ψ in a vector space is $|\psi\rangle$. The notation $|\ \rangle$ is known as 'ket' notation and $\langle\ |$ is known as 'bra' notation. Both are together called as Dirac notations.
The examples of ket vectors are $|\psi\rangle, |\phi\rangle$ and $|u\rangle$ etc.

Matrices:

Conjugate matrices:

If the elements in a matrix A are complex numbers, then the matrix obtained by the corresponding conjugate complex elements is called the conjugate of A and is denoted by A^*. For example

If $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ *then its conjugate* $A^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Example (ii) , $A = \begin{bmatrix} 1 & 2i \\ 2 & 3i \end{bmatrix}$ *then its conjugate* $A^* = \begin{bmatrix} 1 & -2i \\ 2 & -3i \end{bmatrix}$

Example (iii) , $A = \begin{bmatrix} 1 & 4+2i \\ 2-i & 4 \end{bmatrix}$ *then its conjugate* $A^* = \begin{bmatrix} 1 & 4-2i \\ 2+i & 4 \end{bmatrix}$

Transpose of matrix:

The transpose of a matrix is found by interchanging its rows into columns or columns into rows. The Transpose of a matrix A is denoted by using the superscript asAT. Consider a matrix A as given below.

If $A = \begin{bmatrix} i & 3+2i \\ 4 & 2 \end{bmatrix}$ then its transpose of a matrix is $A^T = \begin{bmatrix} i & 4 \\ 3+2i & 2 \end{bmatrix}$

Example 2: $A = \begin{bmatrix} 2+i & 7i \\ 4i+2 & 8 \end{bmatrix}$ then its transpose of a matrix is $A^T = \begin{bmatrix} 2+i & 4i+2 \\ 7i & 8 \end{bmatrix}$

Thus A^T is the Transpose of A.

The Conjugate Transpose of a Matrix:

The complex conjugate transpose of a matrix interchanges the row and column index for each element, reflecting the elements across the main diagonal. The operation also negates the imaginary part of any complex numbers. It is denoted by a '†' symbol as a super script.

$$A = \begin{bmatrix} 1 & 4+2i \\ 2-i & 4 \end{bmatrix}$$

Step-1 $\quad$ $taking\ conjugate\ of\ matrix, A^* = \begin{bmatrix} 1 & 4-2i \\ 2+i & 4 \end{bmatrix}$

Step-2 $\quad$ $taking\ transpose\ of\ a\ matrix, A^\dagger = (A^*)^T = \begin{bmatrix} 1 & 2+i \\ 4-2i & 4 \end{bmatrix}$

Thus $A^\dagger$ is the Conjugate-Transpose of A.

Hermitian matrix:

The matrix that is equal to its conjugate-transpose is called Hermitian. Thus If $A^\dagger = A$ then it is called Hermitian or Self-Adjoint matrix

$$A = \begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix}$$

The complex conjugate of A* is given by $A^* = \begin{bmatrix} 3 & 3-i \\ 3+i & 2 \end{bmatrix}$

The transpose of A^* is given by $A^\dagger = (A^*)^T = \begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix}$ $\quad$ Hence $A^\dagger$= A

Unitary matrix:

Matrix A is said to be unitary if it produces an identity matrix I when multiplied by its conjugate transpose

Or A matrix is said to be Unitary if the condition $AA^\dagger = I\ or\ \ UU^\dagger = I$ is satisfied.

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$$

$$U^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$U^\dagger = (U^*)^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{-i}{2}+\frac{i}{2} \\ \frac{i}{2}-\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Hence U is unitary

Column and Row Matrices and their inner product:

The Column Vectors are called ket Vectors denoted by $|\Psi\rangle$ and are represented by Column Matrices. The Row Vectors are called Bra Vectors denoted by $\langle\phi|$ and are represented by Row Matrices.

Let us consider a ket vector represented in the form of a column matrix. $|\Psi\rangle = \begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix}$

The Row Matrix is represented as $\langle\Psi| = [\alpha_1^* \quad \beta_1^*]$

Here, $\begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix}^\dagger = [\alpha_1^* \quad \beta_1^*]$

Thus the Bra is the complex conjugate of ket and vice versa. $\begin{bmatrix}1 \\ i\end{bmatrix}^\dagger = [1 \quad -i]$

Flipping between kets and bras is called "Taking the Dual".
Thus for $|0\rangle$ state the corresponding $\langle 0|$ is given by

$|0\rangle = \begin{bmatrix}1 \\ 0\end{bmatrix}$ and $\langle 0| = [1 \quad 0]$

And similarly for and $|1\rangle$ states we have $\langle 1|$ as follows.

$|1\rangle = \begin{bmatrix}0 \\ 1\end{bmatrix}$ and $\langle 1| = [0 \quad 1]$

Inner Product - Multiplication of Row and Column Matrices:

Let us consider two states $|\psi\rangle$ *and* $|\phi\rangle$ as follows

$$|\psi\rangle = \begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix} \; and \; |\phi\rangle = \begin{bmatrix}\alpha_2 \\ \beta_2\end{bmatrix}$$

$$\langle\Psi| = [\alpha_1^* \quad \beta_1^*]$$

The multiplication of the $|\psi\rangle$ *and* $|\phi\rangle$ is possible only by taking the inner product and is given by $\langle\psi|\phi\rangle$,

$$\langle\psi|\phi\rangle = [\alpha_1^* \quad \beta_1^*] \begin{bmatrix}\alpha_2 \\ \beta_2\end{bmatrix}$$

$$\langle\psi|\phi\rangle = \alpha_1^*\alpha_2 + \beta_1^*\beta_2$$

The inner product always results in a scalar product.

Probability:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

The above equation represents the Quantum Superposition of states $|0\rangle$ *and* $|1\rangle$

$$|\psi\rangle = \alpha \begin{bmatrix}1 \\ 0\end{bmatrix} + \beta \begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

The inner product $\langle\psi|\psi\rangle$ is given by

$\langle\psi|\psi\rangle = [\alpha^* \quad \beta^*] \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha\alpha^* + \beta\beta^*$

Thus, $\alpha\alpha^* + \beta\beta^* = |\alpha|^2 + |\beta|^2$

This could also be written as $|\Psi|^2 = \Psi\Psi^*$

Thus the above equation represents Probability Density. As per the principle of Normalization.

$$|\Psi|^2 = \Psi\Psi^* = \langle\psi|\psi\rangle = 1 = |\alpha|^2 + |\beta|^2$$

Thus it implies $|\psi\rangle$ is normalized.

Orthogonality:

Two states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if their inner product is Zero.
Mathematically $\langle\phi|\psi\rangle = 0$

The two states are orthogonal means they are mutually exclusive. Like Spin Up and Spin Down of an electron.

Consider $\langle 0|1\rangle$ $\quad \langle 0|1\rangle = [1 \quad 0] \begin{bmatrix}0 \\ 1\end{bmatrix} = (0 + 0) = 0$

Consider $\langle 1|0\rangle$

$$\langle 1|0\rangle = [0 \quad 1]\begin{bmatrix}1\\0\end{bmatrix} = (0+0) = 0$$

Thus the states $|1\rangle$ *and* $|0\rangle$ are Orthogonal.

Orthonormality:

If each element of a set of vectors is normalized and the elements are orthogonal with respect to each other, we say the set is orthonormal (ortho + normalization = ortho-normalization)

Consider the set $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ *and* $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$$\langle 0|0\rangle = [1 \quad 0]\begin{bmatrix}1\\0\end{bmatrix} = 1+0 = 1 \ldots\ldots\ldots Normalized$$

$$\langle 1|1\rangle = [0 \quad 1]\begin{bmatrix}0\\1\end{bmatrix} = 0+1 = 1 \ldots\ldots\ldots Normalized$$

$$\langle 0|1\rangle = [1 \quad 0]\begin{bmatrix}0\\1\end{bmatrix} = (0+0) = 0 \ldots\ldots\ldots Orthogonal$$

$$\langle 1|0\rangle = [0 \quad 1]\begin{bmatrix}1\\0\end{bmatrix} = (0+0) = 0 \ \ldots\ldots\ldots Orthogonal$$

Hence set of $|0\rangle$ and $|1\rangle$ is orthonormal.

Matrix representation of 0 and 1 states:

The wave function could be expressed in ket notation as $|\psi\rangle$ (ket Vector), ψ is the wave function. The $|\psi\rangle = \begin{bmatrix}\alpha 1\\ \alpha 2\end{bmatrix}$. The matrix for of the states $|0\rangle$ and $|1\rangle$. $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ *and* $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

The wave function could be expressed in ket notation as $|\psi\rangle$ (ket Vector), is the wave function. Hence, any arbitrary state can be represented as

$$|\psi\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \ or \ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The 'ket' vector typically represented as a column vector and 'bra' vector typically represented as a row vector. The matrix for of the states $|0\rangle$ *and* $|1\rangle$ as follows;

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \ and \ |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} ------ ket\ notation.$$

$$\langle 0| = [1 \quad 0] \ and \ \langle 1| = [0 \quad 1] ------ bra\ notation.$$

Identity Operator:

The operator of type $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is called identity operator. When an identity operator acts on a state vector its keeps the state intact. By analogy we study identity operator as an identity matrix.

Let us consider the operation of Identity operator on $|0\rangle$ and $|1\rangle$ states. As per the principle of identity operation $I|0\rangle = |0\rangle$ *and* $I|1\rangle = |1\rangle$

$$I|0\rangle = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$$

$$\therefore I|0\rangle = |0\rangle$$

$$I|1\rangle = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$$

$$\therefore I|1\rangle = |1\rangle$$

Thus, the operation of identity matrix (operator) on $|0\rangle$ *and* $|1\rangle$ leaves the states unchanged.

Identity matrix acts as number 1. It is always a square matrix.

Pauli Matrices and Their operation on $|0\rangle$ and $|1\rangle$ States:

There are four extremely useful matrices called Pauli Matrices that are often used in quantum computers. The Pauli matrices of the following form,

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is an identity matrix.

$$\sigma_1 = \sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 = \sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Here X,Y,Z are known as Pauli Matrices.

Operation of Pauli Matrices on 0 and 1 states:

Three Pauli matrices X, Y and Z operates on states $|0\rangle$ and $|1\rangle$ as follows

(1) $\sigma_0|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

$\sigma_0|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

(2) $\sigma_X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

$\sigma_X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

Since X inverts each input (ie, $|0\rangle$ becomes $|1\rangle$ and $|1\rangle$ becomes $|0\rangle$) it is also called as bit-flip gate.

(3) $\sigma_Y|0\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle$

$\sigma_Y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i|0\rangle$

(4) $\sigma_z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

$\sigma_z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|1\rangle$

Properties of Pauli matrices:

- Square Pauli matrices gives identity matrix I

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $Y^2 = 1$ and $Z^2 = 1$

- Pauli matrices are unitary matrix.

$$XX^\dagger = 1, YY^\dagger = 1 \; and \; ZZ^\dagger = 1$$

- Pauli matrices are Hermitian:

Let A be a matrix, A* be its complex conjugate and $A^\dagger$ is its transpose. If $A = A^\dagger$ then the matrix is Hermitian.

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$1^{st}\ step\ (taking\ conjugate) --- \ A^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$2^{nd}\ step\ (Taking\ transpose\ ) --- A^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\therefore A = A^\dagger$$

Quantum Gates:

A quantum gate is a very simple computing device that performs quantum operation on qubits. Quantum gates are one of the essential parts of a quantum computer and are the building blocks of all quantum algorithms. Quantum gates are mathematically represented as transformation matrices which operate on inputs to give outputs.

Single-Qubit Gates:

Single qubit inputs are $|0\rangle$ to $|1\rangle$ and can be represented by matrix forms as

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Single Qubit Gates are X- gate, Y-gate, Z- gate, H –gate, S-gate, T-gate.

1. **X – Gate or Quantum Not Gate**

   X-gate is the single qubit input gate and is also called as Pauli X – gate or quantum NOT gate.

   The Matrix form of X is given by

$$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

   Action of the X-gate on inputs: When X gate operates of inputs $|0\rangle$, $|1\rangle$, $|\psi\rangle$;

   When X operates on $|0\rangle$ and $|1\rangle$ the output will be inverted (ie, $|0\rangle$ becomes $|1\rangle$ and $|1\rangle$ becomes $|0\rangle$)

$$X|0\rangle = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$$

$$X|1\rangle = |0\rangle$$

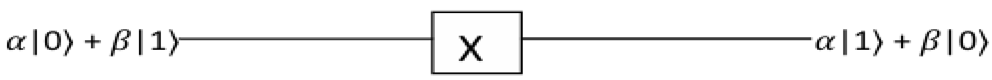
Since X inverts each input it is also called as bit-flip gate.

If a superposed qubit goes through X gate, the result will be

$$X|\psi\rangle = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}\beta\\\alpha\end{bmatrix} = \alpha|1\rangle + \beta|0\rangle$$

$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$$

Gate representation is



Truth table is

X -gate	
Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$

2. **Y – Gate** : Y-gate is the single qubit input gate. This is also called as Pauli Y – gate.

The matrix form of Y gate is $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Action of Y gate on inputs : When Y operates on $|0\rangle$ and $|1\rangle$

$$Y|0\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ i+0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle$$

$$Y|0\rangle = i|1\rangle$$

$$Y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-i \\ 0+0 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i|0\rangle$$

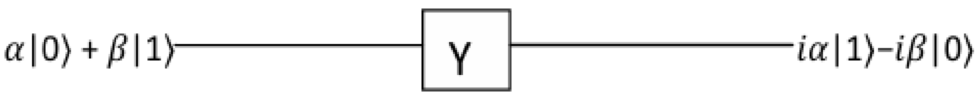
$$Y|1\rangle = -i|0\rangle$$

If a superposed qubit goes through Y gate, the result will be

$$Y|\psi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix} = -i\beta|0\rangle + i\alpha|1\rangle$$

$$Y|\psi\rangle = -i\beta|0\rangle + i\alpha|1\rangle$$

Gate representation is



Truth table is

Y -gate	
Input	Output
$\|0\rangle$	$i\|1\rangle$
$\|1\rangle$	$-i\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$i\alpha\|1\rangle - i\beta\|0\rangle$

3. **Z – Gate** : Z-gate is the single qubit input gate. This is also called as Pauli Z – gate.

The matrix form of Z gate is

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Action of Z gate on inputs: When Z operates on $|0\rangle$ and $|1\rangle$ the phase will change. Hence this is also called as phase-flip gate

$$Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|1\rangle$$

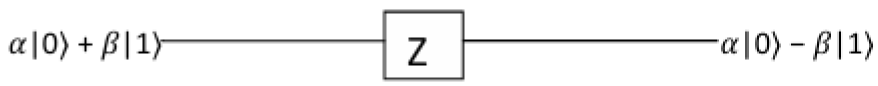
$$Z|1\rangle = -|1\rangle$$

If a superposed qubit goes through Y gate, the result will be

$$Z|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = \alpha|0\rangle - \beta|1\rangle$$

$$Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$$

Gate representation is

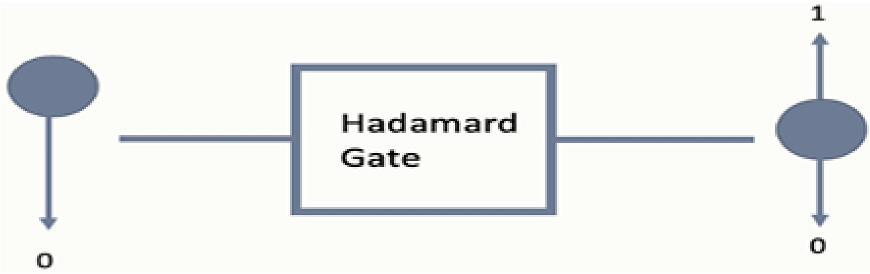


The truth tables for X, Y and Z gates are as follows;

Z-gate	
Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$-\|1\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$

4. **Hadamard Gate (H-gate)** – It is a single qubit gate and is also gate to superposition. H gate acts on single qubit input and produce superposition state output. Matrix form of H gate and its symbol is

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Action of H gate on inputs:

Let us find out what happens when Hadamard gate operates on a qubit that is in the $|0\rangle$ state

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Let us find out what happens when Hadamard gate operates on a qubit that is in the $|1\rangle$ state.

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Let us find out what happens when Hadamard gate operates on a qubit that is in the $|1\rangle$ state.

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If a superposed qubit goes through H gate, the result will be

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha + \beta\\\alpha - \beta\end{bmatrix} = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle$$

$$H|\psi\rangle = \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Gate representation is

The truth table is as follows

Input	Output
$\vert 0\rangle$	$\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}}$
$\vert 1\rangle$	$\frac{\vert 0\rangle - \vert 1\rangle}{\sqrt{2}}$
$\alpha\vert 0\rangle + \beta\vert 1\rangle$	$\alpha\left(\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{\vert 0\rangle - \vert 1\rangle}{\sqrt{2}}\right)$

Note: Difference between X, Y, Z and H gates is that in X, Y and Z gates, output is in single state whereas in H gate output is superposed state.

5. Phase Gate (S Gate):

It is a single qubit gate. The Phase gate or S gate is a gate that transfers $|0\rangle$ into $|0\rangle$ and $|1\rangle$ into $i|1\rangle$. The matrix form of S gate is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Action of S gate on inputs:

Consider S gate apply to a state $|0\rangle$ it will remain same

$$S|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$S|0\rangle = |0\rangle$$

If S gate apply to a state $|1\rangle$ it will be transformed into $i|1\rangle$

$$S|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle$$

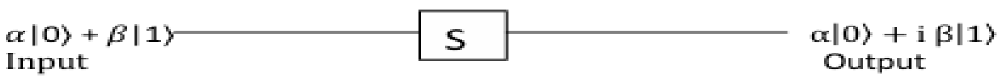
$$S|1\rangle = i|1\rangle$$

S gate apply to the state $\alpha|0\rangle + \beta|1\rangle$ it transforms to the state $\alpha|0\rangle + i\beta|1\rangle$

$$S|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ i\beta \end{bmatrix}$$

$$S|\psi\rangle = \alpha|0\rangle + i\,\beta|1\rangle$$

The S gate representation is as follows



The truth table is as follows

Input	Output
$\vert 0\rangle$	$\vert 0\rangle$
$\vert 1\rangle$	$i\vert 1\rangle$
$\alpha\vert 0\rangle + \beta\vert 1\rangle$	$\alpha\vert 0\rangle + i\,\beta\vert 1\rangle$

6. **T- Gate** : It is a single qubit gate and it is also called π/8 gate.

Its matrix form is

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Action of T gate on inputs:

If T gate operates on input is $|0\rangle$ then the output is also $|0\rangle$

$$T|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$T|0\rangle = |0\rangle$$

If T gate operates on input is $|1\rangle$ then the output is also $e^{i\pi/4}|1\rangle$

$$T|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{i\pi/4} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{i\pi/4}|1\rangle$$

$$T|0\rangle = e^{i\pi/4}|1\rangle$$

If T gate operates on superposed state $\alpha|0\rangle + \beta|1\rangle$, It transforms to $\alpha|0\rangle + e^{i\pi/4}$ $\beta|1\rangle$

$$T|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta e^{\frac{i\pi}{4}} \end{bmatrix} = \alpha|0\rangle + e^{i\pi/4}\,\beta|1\rangle$$

$$T|\psi\rangle = \alpha|0\rangle + e^{i\pi/4}\,\beta|1\rangle$$

T Gate representation is

$$\alpha|0\rangle + \beta|1\rangle \;\text{———}\boxed{T}\text{———}\; \alpha|0\rangle + e^{i\pi/4}\,\beta|1\rangle\rangle$$

The truth table is as follows

Input	**Output**
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$e^{i\pi/4}\|1\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|0\rangle + e^{i\pi/4}\,\beta\|1\rangle$

Multiple Qubit gates: Quantum gates operating on multiple qubits are called as multiple qubit gates. Multiple Qubit Gates operate on Two or More input Qubits. Multiple qubit consists of control gate and target gate. The action of gate as follows;

i) The Target qubit is altered only when the control qubit is $|1\rangle$, and

ii) The control qubit remains unaltered during the transformations.

For two qubits, inputs qubits are $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$

For three qubits, inputs qubits are $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$ and $|111\rangle$

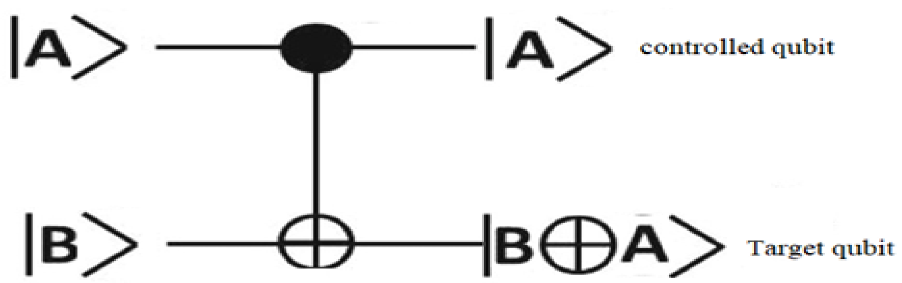
In multiple qubit gate, the input qubit applied in the form like $|AB\rangle$, first term $|A\rangle$ goes to control qubit and the second term $|B\rangle$ goes to target qubit.

Some of the multiple qubit gate are as follows; Controlled gate (CNOT gate), Swap gate, Controlled Z gate and Toffoli gate (CCNOT gate)

1. Controlled Gate (CNOT)

The CNOT gate is a two-qubit operation, where the first qubit is referred as the control qubit $|A\rangle$ and the second qubit as the target qubit $|B\rangle$. If the control qubit is $|1\rangle$ then it will flip the target qubit state from $|0\rangle$ to $|1\rangle$ or from $|1\rangle$ to $|0\rangle$. When the control qubit is in state $|0\rangle$ then the target qubit remains unchanged.

The symbolic representation is as follows. The upper line represents control qubit and bottom line represents target qubit.



NOTE: In diagram the control qubit is represented by ● and target is represented by ⊕

In the combined qubit, first term is control qubit and the second term is target qubit. For ex, in $|AB\rangle$, A is control qubit and B is target qubit

Matrix form of **CNOT** Gate is given by

$$\mathbf{CNOT} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

The Transformation could be expressed as $|A, B\rangle \rightarrow |A, B \oplus A\rangle$

Action of the gate: Consider the operations of CNOT gate on the four inputs $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$.

i) Operation of CNOT Gate for input $|00\rangle$: $|00\rangle \rightarrow |00\rangle$

Here, control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|0\rangle$.

ii) Operation of CNOT Gate for input $|01\rangle$: $|01\rangle \rightarrow |01\rangle$

Here, control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|1\rangle$

iii) Operation of CNOT Gate for input $|10\rangle$: $|10\rangle \rightarrow |11\rangle$

Here, control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|0\rangle$ to $|1\rangle$.

iv) Operation of CNOT Gate for input $|11\rangle$: $|11\rangle \rightarrow |10\rangle$

Here control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|1\rangle$ to $|0\rangle$.

Input		Output	
$\|A\rangle$	$\|B\rangle$	$\|A\rangle$	$\|B \oplus A\rangle$
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

The Truth Table of operation of CNOT gate is as follows.

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Bell states:

Let's consider a slightly more complicated circuit, shown in Figure , which has a Hadamard gate followed by a CNOT, and transforms the four computational basis states according to the table given. As an explicit example, the Hadamard gate takes the input $|00\rangle$ to $\frac{(|0\rangle+|1\rangle)|0\rangle}{\sqrt{2}}$, and then the CNOT gives the output state $\frac{(|00\rangle+|01\rangle)}{\sqrt{2}}$. Note how this works: first, the Hadamard transform puts the top qubit in a superposition; this then acts as a control input to the CNOT, and the target gets inverted only when the control is 1. The output states

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \ldots\ldots\ldots (1)$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \ldots\ldots\ldots (2)$$

$$|\beta_{10}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \ldots\ldots\ldots (3)$$

$$|\beta_{11}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \ldots\ldots\ldots (4)$$

are known as the Bell states.

The mnemonic notation $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle$ & $|\beta_{11}\rangle$ may be understood via the equations.

$$|\beta_{xy}\rangle \equiv \frac{|0, y\rangle + (-1)^x|1, \bar{y}\rangle}{\sqrt{2}}$$

Where, $\bar{y}$ is the negation of y.

In	Out
$\|00\rangle$	$(\|00\rangle + \|11\rangle)/\sqrt{2} \equiv \|\beta_{00}\rangle$
$\|01\rangle$	$(\|01\rangle + \|10\rangle)/\sqrt{2} \equiv \|\beta_{01}\rangle$
$\|10\rangle$	$(\|00\rangle - \|11\rangle)/\sqrt{2} \equiv \|\beta_{10}\rangle$
$\|11\rangle$	$(\|01\rangle - \|10\rangle)/\sqrt{2} \equiv \|\beta_{11}\rangle$

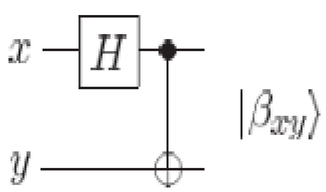


Figure - Quantum circuit to create Bell states, and its input–output quantum 'truth table'.

Questions from VTU Model and previous year Questions

1. Explain the representation of qubit using Block sphere.
2. Explain Moore's law and its end.
3. Define bit and qubit. Explain the properties of qubit.
4. Elucidate the difference between classical computing and quantum computing.
5. Explain the matrix representation of 0 and 1 states and apply identity operator to $|0\rangle$ and $|1\rangle$.
6. State Pauli Matrices. Apply Pauli matrices on the states $|0\rangle$ *and* $|1\rangle$.
7. Explain Hadamard Gate. Apply Hadamard Gate on the states$|0\rangle$ *and* $|1\rangle$.
8. Discuss the CNOT gate and its operation of four different input states.
9. Explain single qubit gate and multiple qubit gate with an example for each.
10. Discuss the working of Phase gate mentioning the matrix representation and truth table.
11. Discuss the working of Swap gate mentioning the matrix representation and truth table.
12. Discuss the working of Toffoli gate mentioning the matrix representation and truth table.
13. Explain orthogonality and orthonormality with an example for each.
14. Find the inner product of states $|11\rangle$ *and* $|10\rangle$ draw the conclusion the result.
15. Given prove that $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, prove\ that\ A^{\dagger} = A$
16. Given $|\Psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ *and* $|\phi\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$, $Prove\ that\ \langle\Psi|\phi\rangle = \langle\phi|\Psi\rangle^*$