NCERT-10.3.4.1.2

EE24BTECH11042 - SRUJANA

QUESTION:
Solve the following pair of linear equations:

3x + 4y = 10
2x - 2y = 2

Theoretical Solution:

$$x = 1 + y \tag{1}$$
$$3(1 + y) + 4y = 10 \tag{2}$$
$$7y = 7 \tag{3}$$
$$y = 1, \quad x = 2 \tag{4}$$

LU Decomposition Method

LU decomposition is a mathematical method used to solve systems of linear equations $Ax = b$, where A is a square matrix, x is an unknown vector, and b is a constant vector. It is especially useful for efficiently solving multiple systems of equations with the same coefficient matrix.

In this method, we decompose A into the product of two matrices L and U, where L is a lower triangular matrix and U is an upper triangular matrix.

This decomposition allows us to solve the system in two steps:

1) Solve $Ly = b$
2) Solve $Ux = y$

Given the system of equations:

$$3x + 4y = 10 \tag{5}$$

$$2x - 2y = 2 \tag{6}$$

We represent the system in matrix form:

$$A = \begin{bmatrix} 3 & 4 \\ 2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 2 \end{bmatrix} \tag{7}$$

Step 1: LU Decomposition

$$A = LU \tag{8}$$
$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \tag{9}$$

From $LU = A$:

$$u_{11} = 3, \quad u_{12} = 4 \tag{10}$$

$$l_{21} = \frac{2}{3}, \quad u_{22} = -2 - \frac{2}{3} \times 4 = -\frac{14}{3} \tag{11}$$

Thus:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 4 \\ 0 & -\frac{14}{3} \end{bmatrix} \tag{12}$$

Step 2: Solve $Ly = b$

$$\begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix} \tag{13}$$

$$y_1 = 10 \tag{14}$$

$$\frac{2}{3} y_1 + y_2 = 2 \quad \Rightarrow \quad y_2 = 2 - \frac{2}{3} \times 10 = -\frac{14}{3} \tag{15}$$

Step 3: Solve $Ux = y$

$$\begin{bmatrix} 3 & 4 \\ 0 & -\frac{14}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -\frac{14}{3} \end{bmatrix} \tag{16}$$

$$3x_1 + 4x_2 = 10 \tag{17}$$

$$-\frac{14}{3} x_2 = -\frac{14}{3} \quad \Rightarrow \quad x_2 = 1 \tag{18}$$

$$3x_1 = 6 \quad \Rightarrow \quad x_1 = 2 \tag{19}$$

Final Solution

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{20}$$

LU Decomposition using Doolittle's algorithm

The LU decomposition can be efficiently compted using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:
Elements of the U Matrix:
For each column j:

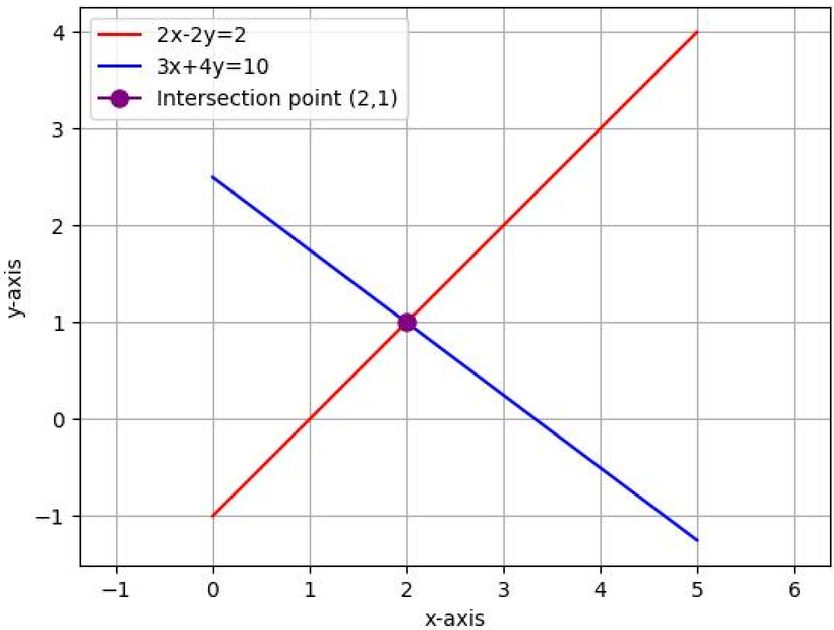
$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{2.1}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{2.2}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{2.3}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{2.4}$$


2x-2y=2
3x+4y=10
Intersection point (2,1)
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
y-axis
x-axis