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A discrete competitive facility location model with variable attractiveness

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We consider the discrete version of the competitive facility location problem in which new facilities have to be located by a new market entrant firm to compete against already existing facilities that may belong to one or more competitors. The demand is assumed to be aggregated at certain points in the plane and the new facilities can be located at predetermined candidate sites. We employ Huff's gravity-based rule in modelling the behaviour of the customers where the probability that customers at a demand point patronize a certain facility is proportional to the facility attractiveness and inversely proportional to the distance between the facility site and demand point. The objective of the firm is to determine the locations of the new facilities and their attractiveness levels so as to maximize the profit, which is calculated as the revenue from the customers less the fixed cost of opening the facilities and variable cost of setting their attractiveness levels. We formulate a mixed-integer nonlinear programming model for this problem and propose three methods for its solution: a Lagrangean heuristic, a branch-and-bound method with Lagrangean relaxation, and another branch-and-bound method with nonlinear programming relaxation. Computational results obtained on a set of randomly generated instances show that the last method outperforms the others in terms of accuracy and efficiency and can provide an optimal solution in a reasonable amount of time.

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1. Introduction

In competitive facility location (CFL) problems, a firm is concerned with installing new facilities to serve customers in a market where existing facilities with known locations and attractiveness levels compete for increasing their market share and profit. In some cases, the firm may be a new entrant with no already existing facilities, while in others the firm may own one or more existing facilities. The choice of the customers as to which facility to visit can be modelled using different approaches. For example, models can be formulated in which customers do not choose a facility solely on the basis of their proximity to its location, but they also take into account some of the characteristics of the facility. The first paper on CFL was by Hotelling (1929), in which he developed a model with two equally attractive ice cream sellers along a beach strip where customers patronize the closest one. This very first model was extended later for unequally attractive facilities, which is a more realistic assumption given the current situation in the market.

It is possible to divide the CFL models into two categories: deterministic utility models and random utility models. In both categories, the attractiveness level of a facility is determined by a function of its attributes, and customers' attraction is modelled by a utility function. The main difference between the two types of models is that in the deterministic utility models the customers patronize only the facility with the highest utility for them, whereas in random utility models customers visit each facility with a certain probability. Among the examples of deterministic utility models, we can mention the work by Drezner (1994), and Plastria and Carrizosa (2004). Although both papers consider a single facility in continuous space, the former focuses only on the determination of an optimal location, whereas the latter addresses the best location as well as the best quality of the single facility simultaneously.

The most widely used random utility model in the CFL literature is the gravity-based model introduced by Huff (1964, 1966). In this model, the probability that a customer patronizes a facility is proportional to the attractiveness of the facility and inversely proportional to a function of the distance between the customer and the facility. A variety of attributes can be used as a proxy for the attractiveness of the facility. For example, when the facility considered is

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a shopping mall, the number of stores, the size of the parking place, food court availability, or the proximity to the public transportation can be the attributes determining the overall attractiveness. In the multiplicative competitive interaction (MCI) model of Nakanishi and Cooper (1974), different attributes of the facility were used together by taking their product after raising each to a power. Achabal *et al* (1982) used the MCI model to determine the best location and design of a number of new stores in discrete space under the presence of a certain number of existing stores so that the market share is maximized. The design of a store is to be selected from a set of possible designs.

All the studies cited in the following apply the gravity-based rule unless mentioned otherwise. First, we review the papers in which facility attractivenesses are parameters and the decisions to be made consist of facility locations only. Drezner and Drezner (2006) consider two models in which they opt for optimally locating p facilities. In the first one, the aim is to minimize the distance travelled by the customers as is the case in the well-known p-median problem, whereas in the second one a balance is sought among the facilities so that the variance of the demand served by the facilities is minimized. The authors also combine the objective functions of the two models to obtain a new multi-objective model. The papers by Drezner and Drezner (2004) and Drezner *et al* (2002) differ from each other with respect to the number of facilities that are located in the continuous space. The former paper examines the case with a single facility, while the second one addresses a multi-facility case. Drezner and Drezner (2008) consider an extension of the CFL problem where the probability of customers' unwillingness to visit a facility, which ultimately affects the demand seen by the facility, is computed. They still use the gravity-based model, but utilities are computed using an exponential decay function of the facility attractiveness represented by a parameter and the distance between the facility and customer sites. The decay parameter increases as the facility attractiveness decreases. Benati and Hansen (2002) employ a utility function whose deterministic part is a linear function of the facility attractiveness and the distance, while the random part is assumed to follow the Gumble distribution. Aboolian *et al* (2007a) and Berman and Krass (2002) develop a spatial interaction model with variable expenditures where demand cannibalization and market expansion are taken into account. The objective is again to optimally locate new facilities in the discrete space.

There are also papers in the literature in which decisions are made not only about the locations of the facilities but also about their attractiveness levels. The first one is the paper by Achabal *et al* (1982). Aboolian *et al* (2007b) formulate a similar model to Aboolian *et al* (2007a), but they also incorporate the design characteristics of the facilities (ie, the attractiveness levels) into the model. Although they state that the attractiveness of each facility is a continuous decision variable, they employ discrete design scenarios in the solution of the model and one of a finite number of available designs is determined for each open facility. Fernández *et al* (2004, 2007) and Tóth *et al* (2009) analyse similar models in which the aim is to optimally locate new facilities in continuous space. They find out the attractiveness levels when some of the existing facilities belong to the firm's own chain. Only one new facility is opened in Fernández *et al* (2004), whereas two new facilities are located in Tóth *et al* (2009). The difference of Fernández *et al* (2007) is the development of a bi-objective model in which maximizing the profit and minimizing the cannibalization are targeted simultaneously. An interesting study is Drezner and Drezner (2002) in which no facility location is considered. Instead, the authors make use of the past data on the preference of the customers and determine the attractiveness of existing facilities using the gravity-based rule.

In this paper, we address a CFL problem in the discrete space with the objective of maximizing the profit where attractiveness of new facilities is a continuous decision variable as opposed to the case of a discrete design scenario. To the best of our knowledge, the CFL problem with a discrete set of candidate facility sites and continuous attractiveness is not addressed before. We develop a mixed-integer nonlinear programming formulation, and try to solve it using three different methods: a Lagrangean heuristic, a branch-and-bound method based on Lagrangean relaxation, and a branch-and-bound method based on nonlinear programming relaxation. The last two methods are exact methods and they are capable of finding an optimal solution of the formulated model when a sufficient amount of time is allotted.

An important aspect of our model is that the reaction of the competitors is omitted. Namely, the competitors do not open new facilities or close existing ones or change the attractiveness of their facilities as a reaction to the market entering firm. This assumption is realistic in those situations where there exists a static competition between players. For example, consider the competition in a small district of a city that takes place between the existing convenience stores and a supermarket chain aiming at opening new stores. When a big supermarket chain (such as Migros, Real, and Carrefour in Turkey) opens a gigantic store, the existing convenience stores owned by independent entrepreneurs cannot react most of the time even though they know that a new supermarket in the area will reduce their profit considerably. The reason for the lack of the competitive reaction lies in the fact that supermarkets with large sales volumes have the option of offering low prices to customers for a variety of goods compared to convenience stores whose sales volumes are much lower. As a result, customers begin to make their purchases in the supermarkets rather than in the convenience stores, which ultimately leads to the closure of some of these stores.

Certainly, there are cases where they succeed in their efforts to survive and these businesses continue their existence.

The remainder of the paper is structured as follows. We present our model and its properties in Section 2. Section 3 explains the solution methods. Computational results are given in Section 4 where a sensitivity analysis is carried out on some model parameters. Finally, we conclude in Section 5 with possible extensions of our model.

2. Model description

The aim of the model is to determine the optimal location and attractiveness of the new facilities to be opened by a firm in a market where there are r existing facilities that belong to a competitor or several competitors. The objective of the firm is to maximize its profit.

We assume that customers are aggregated at n (demand) points and the number of candidate facility sites is m. First, we define the parameters and decision variables. The points are indexed by $j=1, 2, \ldots, n$, the candidate facility sites by $i=1, 2, \ldots, m$, and the existing facilities by $k=1, 2, \ldots, r$.

Parameters:

- a_j annual buying power at point j
- c_i unit attractiveness cost at site i
- f_i annualized fixed cost of opening and operating a facility at site i
- d_{ij} Euclidean distance between site i and point j
- b_j total utility of existing facilities depending on their attractiveness and distance from point j
- u_i maximum attractiveness level of a facility to be opened at site i
- q_k attractiveness of existing facility k

Decision variables:

- Q_i attractiveness of the facility opened at site i
- X_i binary variable that is equal to one if a facility is opened at site i, and zero otherwise

When a facility is opened at site i with attractiveness Q_i, the utility of this facility for customers at point j is given by Q_i/d_{ij}^2 using Huff's gravity-based rule. The total utility of existing facilities for customers at point j is given by parameter $b_j = \sum_{k=1}^{r} q_k/d_{kj}^2$, where d_{kj} is the Euclidean distance between demand point j and existing facility k. As a result, the probability P_{ij} that customers at point j patronize a new facility at site i is expressed as

$$P_{ij} = \frac{Q_i/d_{ij}^2}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + b_j}. \tag{1}$$

Note that $\sum_{j=1}^{n} P_{ij}$ can be seen as the market share corresponding to facility at site i and its revenue is obtained by the expression $\sum_{j=1}^{n} a_j P_{ij}$. Hence, the total revenue captured by the new facilities is given as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_j P_{ij} = \sum_{j=1}^{n} a_j \sum_{i=1}^{m} P_{ij} = \sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right)}{\sum_{i=1}^{m}\left(Q_i/d_{ij}^2\right) + b_j}. \tag{2}$$

Now, we can formulate our model as the following mixed-integer nonlinear programme:

$$\text{P:}\quad \max z = \sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i=1}^{m}(Q_i/d_{ij}^2) + b_j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i \tag{3}$$

s.t.

$$Q_i \leqslant u_i X_i, \qquad i = 1, \ldots, m \tag{4}$$

$$X_i \in \{0, 1\}, \qquad i = 1, \ldots, m \tag{5}$$

$$Q_i \geqslant 0, \qquad i = 1, \ldots, m. \tag{6}$$

The objective function (3) consists of three terms. The revenue collected by the new facilities is represented by the first term, while the cost associated with opening and operating them is expressed by the last two terms, respectively. The first cost component is the annualized fixed cost of opening and operating the facilities, and the second one is the annualized variable cost of opening a facility at a certain attractiveness level Q_i. Constraint (4) along with constraint (6) ensure that if no facility is opened at site i, then the attractiveness Q_i of the facility is zero. On the other hand, when a facility is opened at site i, then its attractiveness cannot exceed an upper bound u_i. Constraints (5) and (6) are, respectively, the binary and nonnegativity restrictions on the location variables X_i and the attractiveness variables Q_i. We note that the number of facilities to be located is not fixed; its value is to be determined by the solution of the model.

3. Solution methods

We propose three methods for the solution of the mixed-integer nonlinear model P. The first one is a Lagrangean heuristic; the second and third ones are branch-and-bound (BB) methods. The difference between the two BB methods is that at each node of the BB tree a Lagrangean relaxation (LR) of the original problem P is solved in the former, while a nonlinear programming relaxation is solved in the latter. All of them exploit a property of the objective function, namely, its concavity in the attractiveness Q_i for $Q_i \geqslant 0$. We show this property with the following propositions before providing the detailed explanations of the solution methods.

Proposition 1 *$\sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j}$ is concave in $\mathbf{Q}=(Q_1, Q_2, \ldots, Q_m)^T$ for $\mathbf{Q} \geqslant \mathbf{0}$.*

Proof Since the sum of concave functions is a concave function, it suffices to show that $a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j}$ is concave in $\mathbf{Q} \geqslant \mathbf{0}$ for every $j=1,2,\ldots,n$. Let us define $g_j(\mathbf{Q}) = a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i}^{m}(Q_i/d_{ij}^2)+b_j}$. We now show that $g_j(\mathbf{Q})$ is concave in $\mathbf{Q} \geqslant \mathbf{0}$. The first and second order derivatives of $g_j(\mathbf{Q})$ are given as follows:

$$\frac{\partial g_j(\mathbf{Q})}{\partial Q_k} = a_j \frac{(1/d_{kj}^2)b_j}{\left[\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j\right]^2}$$

and

$$\frac{\partial g_j(\mathbf{Q})}{\partial Q_l \partial Q_k} = -2a_j \frac{(1/d_{kj}^2)(1/d_{lj}^2)b_j}{\left[\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j\right]^3}.$$

Then the Hessian matrix of $g_j(\mathbf{Q})$, denoted by $H_j(\mathbf{Q})$, becomes

$$H_j(\mathbf{Q}) = -r_j \begin{pmatrix} \frac{1}{d_{1j}^4} & \frac{1}{d_{1j}^2 d_{2j}^2} & \cdots & \frac{1}{d_{1j}^2 d_{mj}^2} \\ \frac{1}{d_{1j}^2 d_{2j}^2} & \frac{1}{d_{2j}^4} & \cdots & \frac{1}{d_{2j}^2 d_{mj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d_{1j}^2 d_{mj}^2} & \frac{1}{d_{2j}^2 d_{mj}^2} & \cdots & \frac{1}{d_{mj}^4} \end{pmatrix}, \tag{7}$$

where $r_j = 2a_j \frac{b_j}{\left[\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j\right]^3}$. We remark that $r_j \geqslant 0$ for $\mathbf{Q} \geqslant \mathbf{0}$ since $a_j \geqslant 0$ and $b_j \geqslant 0$. Also note that $g_j(\mathbf{Q})$ is concave if and only if $H_j(\mathbf{Q})$ is negative semidefinite for $\mathbf{Q} \geqslant \mathbf{0}$. To show the latter, we consider $\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V}$ for $\mathbf{V} \in \mathbb{R}^m$ which is expressed as

$$\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V} = -(V_1, V_2, \ldots, V_m)\, r_j \times \begin{pmatrix} \frac{1}{d_{1j}^4} & \frac{1}{d_{1j}^2 d_{2j}^2} & \cdots & \frac{1}{d_{1j}^2 d_{mj}^2} \\ \frac{1}{d_{1j}^2 d_{2j}^2} & \frac{1}{d_{2j}^4} & \cdots & \frac{1}{d_{2j}^2 d_{mj}^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d_{1j}^2 d_{mj}^2} & \frac{1}{d_{2j}^2 d_{mj}^2} & \cdots & \frac{1}{d_{mj}^4} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix} \tag{8}$$

$$= -(V_1, V_2, \ldots, V_m)\, r_j \times \left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \cdots + \frac{V_m}{d_{mj}^2}\right) \begin{pmatrix} 1/d_{1j}^2 \\ 1/d_{2j}^2 \\ \vdots \\ 1/d_{mj}^2 \end{pmatrix} \tag{9}$$

$$= -r_j \left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \cdots + \frac{V_m}{d_{mj}^2}\right)^2. \tag{10}$$

$r_j \geqslant 0$ and $\left(\frac{V_1}{d_{1j}^2} + \frac{V_2}{d_{2j}^2} + \cdots + \frac{V_m}{d_{mj}^2}\right)^2 \geqslant 0$ imply that $\mathbf{V}^T H_j(\mathbf{Q})\mathbf{V} \leqslant 0$ for $\mathbf{Q} \geqslant \mathbf{0}$. This means that $H_j(\mathbf{Q})$ is negative semidefinite, which proves the concavity of $g_j(\mathbf{Q})$ for every $j=1,2,\ldots,n$. Hence $\sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j}$ is concave in $\mathbf{Q}$ for $\mathbf{Q} \geqslant \mathbf{0}$. □

Proposition 2 *$\sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m}(Q_i/d_{ij}^2)}{\sum_{i=1}^{m}(Q_i/d_{ij}^2)+b_j} - \sum_{i=1}^{m} f_i X_i - \sum_{i=1}^{m} c_i Q_i$ is concave in $\mathbf{Q}$ for $\mathbf{Q} \geqslant 0$.*

Proof The first term is concave in $\mathbf{Q}$ as a consequence of Proposition 1, the second term is a constant, and the last term is a linear function of $\mathbf{Q}$. The result follows since the sum of concave functions is also concave. □

Regardless of the solution method that is employed, an important issue to be taken into consideration in our modelling framework is reverse fitting. This is related to determining for each facility the best values of the underlying attributes that collectively form the attractiveness of the facility. Using the function that gives the relationship between the attribute values and the resulting attractiveness level (eg, the MCI model of Nakanishi and Cooper (1974) where the product of the attributes raised by a power), one may generate the values of the attributes starting with the best attractiveness level obtained for the facility. The main difficulty in this process is the degrees of freedom associated with the individual attribute values, that is, different combinations of attribute values may result in the same attractiveness level. Since there is one equation and as many unknown values as the number of attributes, all attribute values but one can be fixed as desired and the remaining one can be determined from the equation. If the number of attributes is relatively low such as two or three, then even a trial-and-error procedure may help in finding suitable values for the attributes to approximately yield the optimal attractiveness. In some situations, however, it might be the case that the recommended attractiveness level for a facility cannot be achieved due to its unreasonably high value. In this case, a possible remedy would be to adjust the maximum attractiveness level for that facility, and resolve the model.

3.1. Lagrangean Heuristic

Lagrangean heuristics (LHs) have successfully been applied to various facility location problems (Beasley, 1993a). In this paper, we also devise an LH to solve the CFL problem.

To this end, we relax constraint (4) and put it into the objective function after multiplying with nonnegative Lagrange multipliers λ_i, $i=1,\ldots,m$. The Lagrangean subproblem then becomes

$$\text{P}':\quad z'(\boldsymbol{\lambda}) = \max \sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m} (Q_i/d_{ij}^2)}{\sum_{i=1}^{m} (Q_i/d_{ij}^2) + b_j} - \sum_{i=1}^{m} (f_i - \lambda_i u_i) X_i - \sum_{i=1}^{m} (c_i + \lambda_i) Q_i$$

subject to constraints (5) and (6). $z'(\boldsymbol{\lambda})$ provides an upper bound on the optimal objective value z^* of the original model P for any value of multiplier vector $\boldsymbol{\lambda} \geqslant \mathbf{0}$. P$'$ can be solved easily since it can be decomposed into two subproblems P1$'$ and P2$'$ with optimal objective values z_1' and z_2' as follows:

$$\text{P1}':\quad z_1'(\boldsymbol{\lambda}) = \max \sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m} (Q_i/d_{ij}^2)}{\sum_{i=1}^{m} (Q_i/d_{ij}^2) + b_j} - \sum_{i=1}^{m} (c_i + \lambda_i) Q_i$$

subject to constraint (6) and

$$\text{P2}':\quad z_2'(\boldsymbol{\lambda}) = \max - \sum_{i=1}^{m} (f_i - \lambda_i u_i) X_i$$

subject to constraint (5). In order to find the best (smallest) upper bound on z^* the so-called Lagrangean dual problem

$$\text{D}:\quad \min_{\boldsymbol{\lambda} \geqslant 0} z_1'(\boldsymbol{\lambda}) + z_2'(\boldsymbol{\lambda}) \tag{11}$$

is formulated and solved using the iterative 'subgradient optimization' procedure. At each iteration t of this procedure, a step size $\theta^{(t)}$ is computed and the Lagrange multipliers are updated using the subgradients according to the formula

$$\lambda_i^{(t+1)} = \max\{0,\ \lambda_i^{(t)} + \theta^{(t)}(u_i X_i - Q_i)\}. \tag{12}$$

The computation of $\theta^{(t)}$ requires a lower bound on the optimal objective value z^* of P, which is provided by a feasible solution to P. It can be derived from the solution of the Lagrangean subproblem P$'$ as will be explained subsequently. We use the step size formula that is commonly used in the literature (Held *et al*, 1974), that is, $\theta^{(t)} = \pi(UB^{(t)} - LB_{best}) / \sum_{i=1}^{m} ||u_i X_i - Q_i||^2$, where π is the step size parameter, $UB^{(t)}$ is the upper bound at iteration t, and LB_{best} is the best lower bound obtained until iteration t. As suggested by Beasley (1993b), the initial value of π is set to two and halved every 20 iterations without an improvement in the best upper bound. When π becomes less than 0.005, the algorithm is terminated, and the best (largest) lower bound generated throughout the iterations constitutes the solution of the Lagrangean heuristic.

Now we explain the solution procedure of subproblems P1$'$ and P2$'$. The solution of P2$'$ can easily be obtained by inspection. Namely, $X_i = 1$ if $(f_i - \lambda_i u_i) < 0$, $X_i = 0$ otherwise. To solve P1$'$ we make use of the concavity of its objective function $h(\mathbf{Q}) = \sum_{j=1}^{n} a_j \frac{\sum_{i=1}^{m} (Q_i/d_{ij}^2)}{\sum_{i=1}^{m} (Q_i/d_{ij}^2) + b_j} - \sum_{i=1}^{m} (c_i + \lambda_i) Q_i$ in terms of the attractiveness variables $\mathbf{Q}$, which is a direct consequence of Proposition 2. We add redundant constraints of the form $Q_i \leqslant u_i, i = 1, 2, \ldots, m$ to increase the quality of the upper bound. To find the solution of this concave maximization problem, we use the following optimality conditions: $\mathbf{Q}^*$ is a global optimal solution of P1$'$ if and only if

(i) $$\frac{\partial h(\mathbf{Q}^*)}{\partial Q_i} \leqslant 0 \quad \text{when } Q_i^* = 0,$$

(ii) $$\frac{\partial h(\mathbf{Q}^*)}{\partial Q_i} \geqslant 0 \quad \text{when } Q_i^* = u_i,$$

(iii) $$\frac{\partial h(\mathbf{Q}^*)}{\partial Q_i} = 0 \quad \text{when } 0 < Q_i^* < u_i.$$

These conditions can be derived easily using the optimality conditions for convex programming that can be found in any standard textbook on nonlinear programming (eg, Bertsekas, 1995). Hence, we apply a gradient ascent procedure to determine a global maximum of $h(\mathbf{Q})$. First, a small positive value is assigned to parameter ϵ that is used for the termination of the procedure. Then, initial values $\mathbf{Q}^{(0)}$ are assigned to variables $\mathbf{Q}$. After setting the iteration counter t to zero, a direction $\mathbf{d}^{(t)}$ and a step size $\alpha^{(t)} = \arg\max_\alpha h(\mathbf{Q}^{(t)} + \alpha\mathbf{d}^{(t)})$ is determined, and variables $\mathbf{Q}$ are updated as $\mathbf{Q}^{(t+1)} = \mathbf{Q}^{(t)} + \alpha^{(t)}\mathbf{d}^{(t)}$. The iteration counter is increased by one, and the procedure is repeated until the norm of the direction vector $||\mathbf{d}^{(t)}||$ is smaller than ϵ. In finding the optimal step size, we apply the golden section search (Press *et al*, 1986) with the initial interval $[0, \alpha_{max}]$, where α_{max} is the maximum possible value for the step size α to maintain the feasibility of $\mathbf{Q}$ with respect to its lower and upper bounds.

To generate a feasible solution for P we utilize the solution to the Lagrangean subproblem P2$'$. Since this solution gives us the set of facilities to be opened (and closed), that is, $X_i = 0$ and $X_i = 1$ for $i = 1, 2, \ldots, m$, we can fix them in the original problem P and solve the remaining problem in terms of $\mathbf{Q}$. Note that if $X_i^* = 0$, then $Q_i^* = 0$, and all the terms with a zero value for Q_i can be dropped from the problem. The latter can be expressed as

$$\max z = \sum_{j=1}^{n} a_j \frac{\sum_{i \in S} (Q_i/d_{ij}^2)}{\sum_{i \in S} (Q_i/d_{ij}^2) + b_j} - \sum_{i \in S} c_i Q_i \tag{13}$$

s.t.

$$Q_i \leqslant u_i, \qquad i \in S, \tag{14}$$

where $S=\{i\colon X_i=1\}$. Since we already know that the original problem P is concave in $\mathbf{Q}$ when the binary variables $\mathbf{X}$ are fixed, it can be solved optimally by the application of the same gradient ascent procedure that was used for the solution of P1′.

3.2. Branch-and-bound method using Lagrangean relaxation

The second solution procedure we propose is a BB method that is implemented using Lagrangean relaxation. We refer to this method as BB-LR in the sequel. In other words, at each node of the BB tree, subproblems are solved by LH to obtain upper bounds as well as lower bounds on the optimal objective value z^* of the original problem P. At any node k of the tree, some of the binary location variables X_i are fixed. Let $F_k^+=\{i=1,\ldots,m\colon X_i=1\}$ be the set of the sites with a facility and $F_k^-=\{i=1,\ldots,m\colon X_i=0\}$ be the set of the sites without a facility. Also let $G_k=\{i=1,2,\ldots,m\}\setminus(F_k^+\cup F_k^-)$ be the set of the sites without a decision with regard to opening a facility. Note that when $X_i=0$ for site i, then its corresponding Q_i must be equal to zero as well. Therefore, all location and attractiveness variables corresponding to the sites in F_k^- can be discarded from the model at node k. Furthermore, upper bound constraints for Q_i reduce to $Q_i\leqslant u_i$ for facility sites in F_k^+. As a result, the subproblem to be solved at node k of the BB tree can be formulated as follows:

$$\mathrm{P}_k\colon\quad \max\sum_{j=1}^{n}a_j\frac{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)}{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)+b_j} - \sum_{i\in G_k}f_iX_i-\sum_{i\in F_k^+}f_i-\sum_{i\in G_k\cup F_k^+}c_iQ_i \qquad (15)$$

subject to constraints (4) and (5) for $i\in G_k$, constraint (6) for $i\in G_k\cup F_k^+$, and the following

$$Q_i\leqslant u_i,\qquad i\in F_k^+\cup G_k. \qquad (16)$$

Note that constraint (16) for $i\in G_k$ are in fact redundant for the formulation because of constraint (4), but they will help to obtain a better upper bound for P_k. Constraint (4) for $i\in G_k$ can be dualized with nonnegative Lagrange multipliers γ_i, $i\in G_k$. Then the Lagrangean subproblem at node k becomes

$$\mathrm{P}_k'\colon\quad \max\sum_{j=1}^{n}a_j\frac{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)}{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)+b_j} - \sum_{i\in G_k}(f_i-\gamma_iu_i)X_i-\sum_{i\in F_k^+}f_i - \sum_{i\in G_k}(c_i+\gamma_i)Q_i-\sum_{i\in F_k^+}c_iQ_i$$

subject to constraint (5) for $i\in G_k$, constraint (6) for $i\in G_k\cup F_k^+$, and constraint (16). As was done in the previous subsection, it is possible to decompose P_k' into two problems as follows:

$$\mathrm{P1}_k'\colon\quad \max\sum_{j=1}^{n}a_j\frac{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)}{\sum_{i\in G_k\cup F_k^+}(Q_i/d_{ij}^2)+b_j} - \sum_{i\in G_k}(c_i+\gamma_i)Q_i-\sum_{i\in F_k^+}c_iQ_i \qquad (17)$$

subject to constraint (6) for $i\in G_k\cup F_k^+$ and constraint (16), and

$$\mathrm{P2}_k'\colon\quad \max-\sum_{i\in G_k}(f_i-\gamma_iu_i)X_i-\sum_{i\in F_k^+}f_i \qquad (18)$$

subject to constraint (5) for $i\in G_k$.

The sum of the optimal objective values to $\mathrm{P1}_k'$ and $\mathrm{P2}_k'$ provides an upper bound on P_k, which is the subproblem to be solved at node k. The Lagrange multipliers are updated using the subgradient optimization, at each iteration of which $\mathrm{P2}_k'$ is solved by inspection to find the optimal values of the location variables X_i for $i\in G_k$ and $\mathrm{P1}_k'$ is solved by the described gradient ascent procedure to obtain the optimal values of Q_i for $i\in G_k\cup F_k^+$ of this concave maximization problem. The lower bound, which is employed in updating the Lagrange multipliers, is found by generating a feasible solution to P_k. This is accomplished by making use of the solutions to the Lagrangean subproblems $\mathrm{P1}_k'$ and $\mathrm{P2}_k'$. We simply set $Q_i=0$ corresponding to $X_i=0$, and keep the values of Q_i corresponding to $X_i=1$. Then, we evaluate the objective function of P_k given in (15). After generating a lower bound and an upper bound at each subgradient optimization iteration, the best upper bound UB_k as well as the best lower bound LB_k obtained throughout the iterations are stored for node k of the BB tree. It must be emphasized that at the leaf nodes of the tree we do not employ the subgradient optimization procedure since all binary variables $\mathbf{X}$ are fixed, that is, $(F_k^+\cup F_k^-)=\{1,2,\ldots,m\}$ and $G_k=\varnothing$. At these nodes we simply apply the gradient ascent procedure to find lower bounds corresponding to the feasible solutions of P_k.

Now we want to explain two important issues regarding the implementation of the BB method: branching and pruning. Branching at node k is performed by considering the (feasible) solution providing the best lower bound at that node and selecting variable X_i, $i\in G_k$ for which $\gamma_i(u_iX_i-Q_i)$ is the largest. Two branches emanating from node k are obtained by setting the selected variable equal to one (left branch) and to zero (right branch), which implies that a binary search tree is generated. The rationale behind the above-mentioned selection can be explained by noting that a solution to P_k' is optimal for P_k if the relaxed constraint set $Q_i\leqslant u_iX_i$ and the complementary slackness condition $\gamma_i(u_iX_i - Q_i) = 0$ are satisfied by this solution. Here we apply a heuristic rule and choose to branch on the X_i variable that corresponds to the largest violation in the complementary slackness conditions. Pruning of the nodes in the BB tree is accomplished by comparing the upper bound UB_k at a node k with the current best lower bound LB_{best} obtained in the tree (the objective value of the best feasible solution). That is, node k is pruned when $UB_k \leqslant LB_{best}$.

It is also important to emphasize that we use a depth-first search strategy in the binary BB tree. Whenever a node is pruned, we backtrack and consider an unsolved node. Applying Lagrangean relaxation within a BB method is a computationally intensive approach as it involves subgradient optimization at every node. To reduce the computational burden, we apply the approach suggested in Beasley (1993b).

Namely, a large number of subgradient iterations are performed at the root node of the tree. This number is reduced to 30 whenever we branch on a new location variable in the tree, and it is doubled when we backtrack.

3.3. *Branch-and-bound method using nonlinear programming relaxation*

The last solution method we propose is also based on the principle of BB, but rather than using LR to solve P_k at node k of the BB tree we relax the binary restrictions of the location variables $X_i, i \in G_k$ in P_k. That is, $X_i \in \{0, 1\}$, $i \in G_k$ are replaced by $0 \leqslant X_i \leqslant 1$ and a continuous nonlinear programme P''_k is obtained at node k. Therefore, we call this method BB-NLP. It is clear that solving P''_k provides an upper bound for P_k. When in the relaxed solution to P''_k all X_i variables turn out to be zero or one, then we obtain a feasible solution of the original problem P, which provides a lower bound on the optimal objective value of P. We employ the MINOS solver (Murtagh *et al*, 2004) that is available within GAMS suite to solve the nonlinear programmes. Branching at node k is performed by considering the solution at that node and selecting the X_i variable whose value is the closest to 0.5. In other words, the most fractional variable is chosen as the variable to branch on. Pruning of the nodes is based on two conditions: node k is pruned if either a feasible solution to P is obtained with all X_i variables having binary values or the upper bound UB_k at that node is less than or equal to the best lower bound in the tree, that is, $UB_k \leqslant LB_{best}$.

4. Computational results

As there is no benchmark problem in the literature on CFL problems, we generate random test instances to assess the performance of the three methods proposed in the paper. We use nine data sets[1] where the number of demand points (n) and the number of candidate facility sites (m) are equal to each other, and take on values from the set $\{10, 15, 20, 25, 30, 35, 40, 45, 50\}$. For each data set, the number of existing facilities belonging to competitors, r, is assigned a value between one and five. As a result, we obtain 45 instances. The x and y-coordinates of the demand points, the candidate facility sites, and the existing facility sites are integer numbers generated from a uniform distribution defined in the interval $[0, 100]$. The distance d_{ij} between site i and demand point j is then calculated as the Euclidean distance. We pay attention not to overlap the demand points with either the candidate facility sites or the existing facility sites because of the fact that a facility coinciding with a demand point captures almost all of the available buying power at that point since the distance is zero. In fact, modelling the demand at discrete points is an approximation in which the point represents aggregations of customer demands that exist within an area. Since the actual distances between customers in the area and a facility will vary, Drezner and Drezner (1997) suggest to use a distance function where 24% of the area representing the customers' premises is added to the square of the distance to reflect an averaging of the distances involved. In particular, they suggest replacing the distance d_{ij} by $\sqrt{d_{ij}^2 + 0.24A}$, where A is the area in question.

The annualized buying power a_j of customers at point j, the unit attractiveness cost c_i at candidate site i and the attractiveness q_k of existing facility at site k are integer-valued parameters generated from uniform distributions as: $a_j \sim U(100, 10\,000)$, $c_i \sim U(1, 10)$, and $q_k \sim U(100, 1000)$. The fixed costs f_i which effectively determine the optimal number of new facilities to be opened are set to three different values as follows: $f_i = 100c_i$, $f_i = 1000c_i$, and $f_i = 10\,000c_i$. In other words, they are chosen as 100, 1000, and 10 000 times as large as the attractiveness costs. Upper bound u_i for the attractiveness of a facility at candidate site i is assigned a value of $u_i = 100c_i$. All three solution methods have been coded in C# and the computations have been performed on a server with Intel Xeon 3.16 GHz processor with 16 GB of RAM working under the Windows 2003 Server operating system.

4.1. *Comparison of the methods*

First, we compare the performance of the three methods in terms of accuracy and efficiency on the set of 45 instances. The results are presented in Tables 1–3. A time limit of 7200 s is allotted for each solution method. BB-NLP is the most efficient among the three approaches as it is able to obtain an optimal solution for every instance. The accuracy of the solutions given by LH and BB-LR is therefore measured by computing the percent deviation (PD) of the best lower bound LB_{best} obtained by these methods from

[1]They are available electronically on the website located at URL address http://www.ie.boun.edu.tr/~aras/hande/datasets.htm.

Table 1 Comparison of the solution methods when $f_i = 100c_i$

Instance	*BB-NLP*				*BB-LR*				*LH*			
(n, r)	*z**	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*Gap(%)*	*NF*
(10, 1)	41 653.0	1.2	3	4	0	325.8	757	4	0	6.7	0.26	4
(10, 2)	33 499.4	1.2	3	5	0	112.1	571	5	0	7200	1.76	5
(10, 3)	29 260.5	1.2	3	4	0	101.4	577	4	0	7200	2.01	4
(10, 4)	22 542.1	1.2	3	4	0	97.1	585	4	0	4.7	0.77	4
(10, 5)	19 500.0	1.3	3	4	0	91.1	575	4	0	6.8	0.75	4
(15, 1)	55 969.8	7.3	19	5	0	7200	9404	5	0	23.3	0.41	5
(15, 2)	52 346.4	8.8	23	6	0	7200	11 623	6	0	58.1	0.68	6
(15, 3)	47 213.9	2.9	7	7	0	4467.3	4914	7	0	19.6	0.32	7
(15, 4)	46 364.9	2.8	7	7	0	4754.5	6256	7	0	21.3	0.18	7
(15, 5)	44 908.9	2.8	7	7	0	4063.9	5190	7	0	32.4	0.02	7
(20, 1)	82 155.1	8.1	21	5	0.79	7200	1518	7	0	193.3	0.65	5
(20, 2)	67 557.8	10.7	27	7	0.06	7200	645	7	0	177.7	0.89	7
(20, 3)	55 354.3	15.2	39	7	0.01	7200	1461	8	0	155.8	0.5	7
(20, 4)	49 372.2	12.6	31	8	0.15	7200	1509	8	0	172.6	0.81	8
(20, 5)	45 441.3	9.6	25	8	0.8	7200	1332	8	0	165.9	0.78	8
(25, 1)	118 876.3	190.3	491	9	1.02	7200	132	13	0.07	1882.7	1.86	8
(25, 2)	105 115.5	143.1	373	10	0.82	7200	207	12	0	911	2.6	10
(25, 3)	89 590.0	103.3	263	12	1.32	7200	252	17	0.44	706.3	4.02	14
(25, 4)	86 162.3	82.0	211	13	0.49	7200	386	15	1	737.7	4.79	12
(25, 5)	83 629.0	82.5	213	13	0.77	7200	371	15	3.81	579.8	4.88	14
(30, 1)	118 770.6	25.3	65	11	0.73	7200	32	14	0	1716.6	1.41	11
(30, 2)	106 362.7	26.2	67	12	0.15	7200	98	13	0.09	1218.9	1.62	12
(30, 3)	93 515.8	34.8	89	13	0.08	7200	35	14	0	1483.3	1.93	13
(30, 4)	86 389.9	23.0	59	13	0.53	7200	60	13	0.09	777.7	1.77	14
(30, 5)	82 325.4	17.7	45	13	0.96	7200	133	16	0	943.5	1.89	13
(35, 1)	131 527.7	114.5	313	9	0.47	7200	52	12	0	1958.5	0.62	10
(35, 2)	121 441.5	40.5	111	11	0.48	7200	63	14	0	1982.1	0.53	11
(35, 3)	111 374.6	24.5	67	12	0.24	7200	65	14	0	1915.9	0.26	12
(35, 4)	104 418.4	9.7	21	13	0.004	7200	63	13	0	1836.9	0.23	13
(35, 5)	99 002.7	6.5	15	12	0.27	7200	64	14	0	1746.3	0.19	12
(40, 1)	177 402.1	67.2	181	10	1.39	7200	5	10	0	6955.1	22.67	10
(40, 2)	163 006.1	99.3	267	11	0.96	7200	7	14	0	2773.4	31.71	11
(40, 3)	146 687.2	53.3	143	14	0.55	7200	9	14	0.12	2833.6	31.56	14
(40, 4)	142 895.3	38.0	101	14	1.49	7200	8	14	0.13	1405.0	37.35	13
(40, 5)	140 349.3	28.2	75	15	0.66	7200	7	14	0.07	3609.2	25.19	14
(45, 1)	209 102.6	310.0	811	12	3.82	7200	1	8	0	7200	17.41	12
(45, 2)	181 111.1	266.0	705	15	7.04	7200	1	13	0.59	7200	28.66	17
(45, 3)	170 406.8	129.4	341	15	6.22	7200	1	16	0.29	7200	32.97	16
(45, 4)	164 095.8	96.0	251	15	4.74	7200	1	16	0.43	7200	31.35	18
(45, 5)	158 001.9	92.6	241	15	8.79	7200	1	17	0.1	7200	40.85	17
(50, 1)	209 621.6	325.7	817	13	NA	NA	NA	NA	1.18	7200	17.62	17
(50, 2)	180 258.3	1527.8	3787	16	NA	NA	NA	NA	0.66	7200	41.66	15
(50, 3)	168 533.4	3056.0	7479	17	NA	NA	NA	NA	0.36	7200	33.99	19
(50, 4)	157 999.1	2354.5	5771	18	NA	NA	NA	NA	1.08	7200	36.19	22
(50, 5)	148 419.3	1633.5	4007	19	NA	NA	NA	NA	1.01	7200	41.85	21
Average		246.4	613.4	10.7	1.2	6110.3	1224.3	10.9	0.3	2786.9	11.3	11.2

the optimal objective value z^* provided by BB-NLP. PD is expressed by the formula $100 \times (z^* - LB_{best})/z^*$.

The average percent deviation and average CPU time requirement computed over all the instances are given in the last rows of the tables. For all fixed cost values, LH performs better than BB-LR on the average in terms of accuracy. For example, at the lowest fixed cost level ($f_i = 100c_i$) the average PD is 0.3% for LH and 1.2% for BB-LR, while the CPU times are 2786.9 s for LH and 6110.3 s for BB-LR. When we consider the results for the medium level fixed costs presented in Table 2, a similar observation can be made where LH again performs better than BB-LR in terms of accuracy with an average percent deviation of 0.6% against 6.2%. LH beats BB-LR also in terms of efficiency: the CPU time requirement of LH is 2641.4 s whereas BB-LR spends 6413.3 s on the average. It is remarkable that BB-NLP solves the instances in this table in 464.9 s on the average. We observe that LH finds

Table 2 Comparison of the solution methods when $f_i = 1000c_i$

Instance	*BB-NLP*				*BB-LR*				*LH*			
(n, r)	*z**	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*Gap(%)*	*NF*
(10, 1)	34 142.7	3.8	9	2	0	188.2	1323	2	0	5.9	5.53	2
(10, 2)	24 360.4	3.0	7	2	0	132.3	917	2	0	6.4	3.2	2
(10, 3)	20 524.3	3.0	7	2	0	79.0	921	2	0	8.4	3.28	2
(10, 4)	13 734.3	3.8	9	2	0	97.0	777	2	0	4.7	5.32	2
(10, 5)	10 791.6	2.9	7	2	0	103.5	1027	2	0	4.8	8.46	2
(15, 1)	45 167.0	18.1	43	2	0	7200	9968	2	0	40.4	6.94	2
(15, 2)	39 642.7	23.8	57	3	0.66	7200	10 511	4	0	42.3	7.2	3
(15, 3)	33 036.4	22.3	53	3	0	7200	10 247	3	0	28.8	5.98	3
(15, 4)	31 989.6	13.9	33	3	0	7200	10 762	3	0	33.4	5.47	3
(15, 5)	30 251.1	11.4	27	3	0	7200	774	3	0	32.2	5.02	3
(20, 1)	69 794.6	51.9	123	4	0.34	7200	1481	5	0	256.9	4.49	4
(20, 2)	51 856.7	23.3	55	5	5.34	7200	1119	5	0	209.4	3.58	5
(20, 3)	36 322.0	14.1	33	5	12.97	7200	1550	7	0	178.9	4.27	5
(20, 4)	30 847.4	12.2	29	5	2.41	7200	1623	5	0	143.0	4.01	5
(20, 5)	27 188.2	11.3	27	5	8.6	7200	1613	5	0	126.4	3.82	5
(25, 1)	104 249.1	380.2	889	6	11.19	7200	113	6	1.23	1282.9	5.91	4
(25, 2)	83 780.4	295.4	695	7	4.15	7200	170	7	0.86	961.0	9.64	5
(25, 3)	62 782.3	435.0	1027	6	1.64	7200	225	6	0.26	932.8	14.54	7
(25, 4)	56 760.9	318.1	753	7	9.39	7200	325	7	0.43	648.9	17.81	6
(25, 5)	53 735.7	279.5	661	7	17.69	7200	311	7	0.6	692.9	18.46	6
(30, 1)	104 850.5	169.3	397	7	6.08	7200	29	8	3.58	2821.4	6.83	5
(30, 2)	88 375.3	152.3	357	8	6.45	7200	89	8	0.18	1287.7	4.54	6
(30, 3)	71 347.1	279.0	655	8	12.67	7200	33	11	1.08	1073.5	8.01	6
(30, 4)	61 971.6	155.7	363	8	11.81	7200	61	10	5.01	708.9	12.79	6
(30, 5)	56 824.7	146.8	345	8	15.48	7200	146	10	0	867.1	8.99	8
(35, 1)	116 892.7	145.6	319	5	1.34	7200	45	5	1.33	2565.4	4.05	4
(35, 2)	101 088.8	71.9	167	5	0.95	7200	63	6	0.09	1843.5	3.58	5
(35, 3)	87 153.4	51.2	119	6	1.09	7200	63	7	0.17	2080.1	3.96	6
(35, 4)	79 060.7	34.7	81	6	1.71	7200	56	6	0	1510.6	3.54	6
(35, 5)	72 875.3	28.9	67	6	1.61	7200	63	8	0	1705.1	3.31	6
(40, 1)	157 685.0	266.7	611	6	0.62	7200	7	6	0.05	4438.9	3.06	5
(40, 2)	137 842.0	263.6	603	6	4.21	7200	6	6	0.06	7200.0	16.11	7
(40, 3)	118 149.8	186.1	427	7	0.44	7200	7	7	0.49	3869.3	33.03	6
(40, 4)	113 466.0	237.3	547	7	6.03	7200	7	7	0.1	5188.2	33.25	7
(40, 5)	110 328.4	205.0	471	7	3.06	7200	7	8	0	4064.9	38.68	7
(45, 1)	188 660.0	734.2	1689	6	21.06	7200	2	4	0.17	7200.0	4.57	7
(45, 2)	152 066.4	1016.3	2349	8	9.9	7200	3	6	1.54	7200.0	20.85	8
(45, 3)	137 612.3	767.8	1771	8	6.68	7200	3	7	1.01	7200.0	23.94	11
(45, 4)	129 040.3	1222.4	2823	9	7.1	7200	3	7	0.61	7200.0	39.97	9
(45, 5)	121 796.2	1383.7	3183	8	13.2	7200	3	7	1.66	7200.0	26.09	10
(50, 1)	189 038.4	821.2	1327	8	8.1	7200	1	5	1.5	7200.0	7	8
(50, 2)	146 512.9	5570.1	10 175	10	16.85	7200	1	7	0.33	7200.0	32.87	10
(50, 3)	131 183.1	2536.0	5053	10	17.4	7200	1	11	1.69	7200.0	19.15	11
(50, 4)	117 765.9	1342.3	2839	11	12.17	7200	1	8	0.91	7200.0	47.94	13
(50, 5)	107 110.2	1205.7	2555	11	20.66	7200	1	10	1.49	7200.0	51.28	11
Average		464.9	974.2	6.0	6.2	6413.3	1254.6	6.0	0.6	2641.4	13.3	5.9

an optimal value for 19 instances, whereas BB-LR obtains the optimal solution for only nine instances. For the highest fixed cost values, the picture is not different.

The numbers in the 'NF' columns of the three tables indicate that the number of new facilities to be opened decreases as the fixed costs increases, which is an expected outcome. Results generated by BB-NLP for the high fixed cost level ($f_i = 10\,000c_i$) presented in Table 3 reveal that it is optimal not to open any new facilities in some instances. This is shown by '—' in the NF column with a corresponding z^* value equal to zero. In these instances, the fixed and variable costs of opening facilities do not justify the opening of new facilities.

We note that although BB-LR is an exact technique, it provides the worst solutions among the three methods. The reason lies in the fact that the allowed time limit of 7200 s is not sufficient for the BB tree to explore all the nodes. This can be seen in the columns of Tables 1–3 labelled as 'NN'.

Table 3 Comparison of the solution methods when $f_i = 10\,000c_i$

Instance	*BB-NLP*				*BB-LR*				*LH*			
(n, r)	z^*	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*NN*	*NF*	*PD(%)*	*CPU*	*Gap(%)*	*NF*
(10, 1)	10 930.0	3.9	11	1	0	314.2	1985	1	61.22	11.8	78.05	2
(10, 2)	0	3.2	9	—	0	193.0	2047	—	0	3.0	100	—
(10, 3)	0	2.5	7	—	0	183.9	2047	—	0	2.4	100	—
(10, 4)	0	1.1	3	—	0	170.0	2047	—	0	2.1	100	—
(10, 5)	0	1.1	3	—	0	152.8	2047	—	0	0.6	100	—
(15, 1)	10 217.5	14.6	41	1	NA	7200	8467	—	2.23	61.7	62.84	1
(15, 2)	1586.2	9.5	27	1	NA	7200	11 808	—	38.06	44.1	94.41	1
(15, 3)	0	7.5	21	—	0	7200	10 468	—	0	35.9	100	—
(15, 4)	0	6.0	17	—	0	7200	10 656	—	0	25.4	100	—
(15, 5)	0	5.3	15	—	0	7200	10 344	—	0	23.0	100	—
(20, 1)	29 258.0	41.3	119	1	31.61	7200	1184	1	31.61	370.7	52.13	1
(20, 2)	4413.6	27.9	71	1	0	7200	938	1	0	181.1	75.2	1
(20, 3)	0	7.5	21	—	0	7200	1382	—	0	136.2	100	—
(20, 4)	0	5.3	15	—	0	7200	1364	—	0	49.6	100	—
(20, 5)	0	5.5	15	—	0	7200	1471	—	0	44.8	100	—
(25, 1)	55 642.7	109.1	303	2	23.12	7200	81	2	3.93	855.1	28.85	2
(25, 2)	25 159.5	101.4	291	2	47.36	7200	125	1	5.81	545.0	51.01	2
(25, 3)	4567.0	59.1	163	2	40.52	7200	213	1	0	359.4	83.26	2
(25, 4)	2586.4	40.1	115	1	2206.52	7200	230	1	0	384.3	87.09	1
(25, 5)	1756.0	30.5	87	1	480.92	7200	305	1	0	347.4	89.13	1
(30, 1)	52 656.7	108.9	309	2	55.79	7200	37	1	0	1630.9	29.59	2
(30, 2)	28 399.7	49.7	141	2	100	7200	71	—	49.78	1177.4	71.16	1
(30, 3)	13 389.5	39.0	111	2	100	7200	40	—	0	687.9	60.27	2
(30, 4)	5371.7	23.0	65	1	100	7200	71	—	0	770.6	78.03	1
(30, 5)	1703.4	19.4	55	1	100	7200	156	—	0	663.2	91.93	1
(35, 1)	57 028.6	127.3	361	1	0	7200	33	1	0	3668.3	30.08	1
(35, 2)	30 528.8	86.5	245	1	55.06	7200	48	1	1.65	2612.2	43.62	1
(35, 3)	17 875.8	38.9	109	1	187.57	7200	42	1	0	2280.1	48.14	1
(35, 4)	12 014.2	18.8	53	1	170.24	7200	41	2	0	1946.2	53.66	1
(35, 5)	8681.5	11.9	33	1	197.2	7200	57	2	0	1412.7	60.09	1
(40, 1)	72 559.4	189.6	527	3	87.31	7200	3	3	4.38	3164.0	35.68	2
(40, 2)	39 978.0	226.0	623	3	8.22	7200	5	3	0	2311.9	49.28	3
(40, 3)	20 182.3	138.7	387	3	28.16	7200	6	1	14.66	2452.4	70.26	2
(40, 4)	16 228.6	120.6	337	2	33.64	7200	4	1	1.27	1778.9	70.04	2
(40, 5)	13 519.7	108.5	303	2	41.53	7200	6	1	25.02	1756.0	79.73	2
(45, 1)	116 801.8	401.5	1097	3	24.73	7200	2	1	16.87	2883.0	35.89	3
(45, 2)	57 344.7	521.2	1397	4	28.45	7200	3	3	11.43	3522.8	44.73	4
(45, 3)	38 496.8	409.4	1089	4	67.4	7200	3	2	0	2592.7	46.19	4
(45, 4)	32 622.3	311.8	831	3	104.9	7200	4	3	7.14	2049.3	51.66	4
(45, 5)	28 992.0	160.5	443	3	78.07	7200	4	1	7.61	2526.3	52.52	3
(50, 1)	114 507.8	364.3	889	3	NA	NA	NA	NA	10.37	4919.0	27.2	2
(50, 2)	47 511.9	383.7	951	3	NA	NA	NA	NA	34.6	4077.1	58.25	3
(50, 3)	30 740.2	156.4	387	2	NA	NA	NA	NA	51.32	4674.3	72.97	3
(50, 4)	16 413.4	94.8	259	2	100	7200	2	—	90.4	4825.3	95.94	3
(50, 5)	10 938.1	65.4	179	1	360.96	7200	2	1	96.26	3082.5	98.78	1
Average		103.5	278.6	1.5	115.0	6325.4	1746.1	0.9	12.6	1487.7	70.2	1.5

We observe that at each fixed cost level all the nodes of the BB tree can only be explored for problem instances with $n = 10$ customers, and furthermore for instances (15, 3), (15, 4), and (15, 5) at the lowest fixed cost level. In other words, BB-LR provides an optimal solution in these cases. However, it is clear from the results in Table 1 that only the root node is solved for problems with $n = 45$ (NN is equal to 1), and even the root node cannot be solved for problem instances with $n = 50$, which is shown by 'NA' in the corresponding cells of the table. Similar situations arise at the medium and high fixed cost levels too. BB-LR can only solve the root node for instances with $n = 50$ when $f_i = 1000c_i$ (see Table 2), and fails to do so for instances with $n = 50$ when $f_i = 10\,000c_i$ (see Table 3). We also would like to mention that there are several cases in Table 3 where BB-LR obtains a feasible solution with zero objective value in which no facilities are opened even though it is optimal to open some facilities. In these cases, the percent deviations are 100%. Finally, a PD value above 100 implies that the best feasible solution provided by BB-LR has a negative profit.

One may consider comparing the number of nodes explored under the two BB methods. When we do this, we can conclude that BB-NLP is much more efficient than BB-LR in pruning the nodes of the tree due to the fact that it can generate good feasible solutions so that many nodes can be pruned early in the algorithm. This can be observed in Tables 1–3 for problem instances up to $n=25$. For larger instances, one may wonder why the number of nodes solved under BB-NLP is more than that under BB-LR. It is due to the efficiency of BB-NLP in solving the problems at each node. To summarize, the overall effectiveness of the BB-NLP method can be attributed to both the speed at which the concave nonlinear problems can be solved at each node of the BB tree and the generation of good feasible solutions that helps to prune many nodes early in the method.

We also report the gap between the best lower and upper bounds for the Lagrangean heuristic LH, which is computed as $100 \times (UB_{best} - LB_{best})/UB_{best}$. The average gaps are 11.3%, 13.3%, and 70.2% for low, medium, and high fixed cost levels, respectively. The large gaps associated with the high fixed cost level can be attributed to the relatively small number of iterations that can be performed in the subgradient optimization when $f_i = 10\,000c_i$. It is equal to 260.6, whereas the average number of iterations is 349.2 when $f_i = 1000c_i$, and 351.1 when $f_i = 100c_i$. Therefore, for all the instances of the high fixed cost level, the step size parameter π is halved every five iterations instead of the suggested value of 30 (Beasley, 1993b).

To see the effectiveness of the recommended solution approach, that is, BB-NLP, we employ two commercial solvers for MINLP problems that are available within GAMS Suite. The first solver, DICOPT (Grossmann *et al*, 2004) is based on the outer approximation method in which a sequence of mixed-integer programmes and nonlinear programmes are solved. It is expected to perform better on models that have a significant and difficult combinatorial part. The second solver, OQNLP, is a multi-start heuristic algorithm designed to find global optima of constrained nonlinear programmes that are smooth. By 'multi-start' it is meant that the algorithm calls a nonlinear programming solver from multiple starting points that are determined by a scatter search implementation called OptQuest (Laguna and Martí, 2003). We choose the instances at the medium fixed cost level and solve them using these two solvers. The results are reported in Table 4 in terms of the percent deviation from the optimal objective value produced by BB-NLP. It is clear that DICOPT outperforms OQNLP with respect to both accuracy and efficiency. In fact, with the exception of two instances OQNLP provides only trivial solutions in which there is no opened facility and the corresponding profit is zero. Therefore, the resulting percent deviations are 100%.

Table 4 Comparative results with DICOPT and OQNLP solvers on the instances when $f_i = 1000c_i$

Instance	*DICOPT*		*OQNLP*	
(n, r)	*PD(%)*	*CPU*	*PD(%)*	*CPU*
(10, 1)	0	0.07	98	0.08
(10, 2)	0	0.12	100	0.06
(10, 3)	19.68	0.07	100	0.06
(10, 4)	0	0.08	100	0.06
(10, 5)	100	0.06	100	0.06
(15, 1)	0.36	0.11	100	0.13
(15, 2)	21.86	0.09	100	0.14
(15, 3)	27.2	0.06	100	0.14
(15, 4)	13.02	0.04	100	0.11
(15, 5)	26.48	0.04	100	0.13
(20, 1)	5.84	0.10	100	0.22
(20, 2)	25.07	0.03	100	0.20
(20, 3)	29.39	0.05	27.82	0.16
(20, 4)	17.48	0.04	100	0.17
(20, 5)	33.99	0.05	100	0.17
(25, 1)	7.61	0.05	100	0.34
(25, 2)	12.67	0.06	100	0.36
(25, 3)	3.79	0.12	100	0.33
(25, 4)	21.16	0.13	100	0.34
(30, 5)	18.4	0.12	100	0.39
(30, 1)	17.23	0.12	100	0.58
(30, 2)	27.31	0.09	100	0.59
(30, 3)	52.28	0.09	100	0.52
(30, 4)	26.86	0.11	100	0.52
(30, 5)	22.95	0.11	100	0.55
(35, 1)	2.85	0.13	100	0.88
(35, 2)	9.93	0.14	100	0.88
(35, 3)	13.73	0.12	100	0.84
(35, 4)	5.46	0.11	100	0.84
(35, 5)	8.31	0.09	100	0.84
(40, 1)	16.65	0.14	100	1.08
(40, 2)	2.93	0.12	100	1.14
(40, 3)	10.57	0.18	100	1.19
(40, 4)	16.02	0.11	100	1.08
(40, 5)	15.95	0.15	100	1.14
(45, 1)	6.63	0.21	100	1.56
(45, 2)	17.31	0.16	100	1.42
(45, 3)	9.84	0.12	100	1.45
(45, 4)	9.81	0.12	100	1.53
(45, 5)	13.82	0.14	100	1.50
(50, 1)	8.11	0.29	100	0.88
(50, 2)	11.1	0.26	100	0.75
(50, 3)	12.5	0.20	100	0.84
(50, 4)	11.49	0.20	100	0.77
(50, 5)	12.42	0.23	100	0.77
Average	16.58	0.12	98.35	0.62

DICOPT, on the other hand, yields solutions that are 16.58% worse than BB-NLP on the average.

4.2. Sensitivity analysis

To investigate the effect of the model parameters on the captured market share and the realized profit by the new facilities, we carry out further experiments by selecting instance (40, 3). We use BB-NLP as it is the most effective solution method and also can yield the optimal solution and the optimal objective value needed for the sensitivity analysis within a reasonable amount of computation time.

The parameters we consider are the number of existing facilities (r), the unit attractiveness cost (c_i), the fixed cost of new facilities (f_i), and the upper bound for attractiveness (u_i). When we analyse the market share and resulting profit values obtained by varying each of the first three parameters one at a time, we obtain results that validate logical expectations. These can be summarized as follows. First, both the market share and the profit decrease as r increases. Second, the market share as well as the profit decline with an increase in the unit attractiveness costs. This is an expected outcome because as c_i increases, either some of the facilities are not opened or the attractiveness levels of some open facilities are reduced resulting a loss in the market share and profit. Third, the profit is a monotonically decreasing function of the fixed cost. The market share, on the other hand, either decreases or remains the same with an increase in the fixed cost. We explain this pattern on the basis of an observation we make from the optimal solutions. Namely, as the fixed costs increase, the number of opened facilities is either reduced or remains the same. Furthermore, the opened facilities always have the same optimal attractiveness values and are at the same locations. When the number of facilities remains the same with an increase in the fixed costs, the market share is not affected since the attractiveness of neither the competitors nor the new facilities change. However, the resulting profit is reduced since a higher fixed cost is incurred. When the number of open facilities decreases, the market share is also affected in a negative way since the competitors can capture more of the customers' buying power. This results in diminishing revenue from the customers. The decrease in the profit can be attributed to the outcome that the reduction in the revenue outweighs the decrease in the fixed costs.

Now, we turn our focus on the sensitivity with respect to the parameter u_i. In the base case scenarios, u_i values were set equal to a multiplier times c_i where the multiplier is 100. When we conduct new experiments by varying this multiplier between 100 and 25 000, and plot the market share and profit values corresponding to the optimal solutions, we obtain Figure 1. The first part of the figure shows that the market share exhibits a slowly increasing trend with some drops and stabilizes after u_i reach very large values. When we examine the solutions, we observe that as the maximum attractiveness levels increase, the optimal attractiveness of some facilities are increased and that of some others are reduced. The facilities with increased attractiveness steal from the market share of the facilities with a reduced attractiveness, which implies that the market share of different facilities may increase or decrease. As a consequence, the overall market share exhibits a fluctuating pattern. It turns out that the resulting profit always shows an increasing trend that asymptotically converges to a limiting value.

To shed further light on the sensitivity analysis with respect to the parameter u_i and find out whether there persist unacceptably low optimal attractiveness levels for some facilities, we carry out the following experiment. Using the same (40, 3) instance with $f_i = 1000c_i$, we only change the maximum attractiveness levels of two facilities (facilities at sites 10 and 37) among seven opened ones. We assign the same values to the multipliers of u_{10} and u_{37} from the set $\{1000, 2000, 3000, 4000, \ldots, 30\,000\}$ and examine the optimal attractiveness levels of the two facilities. As is illustrated in Figure 2, the optimal

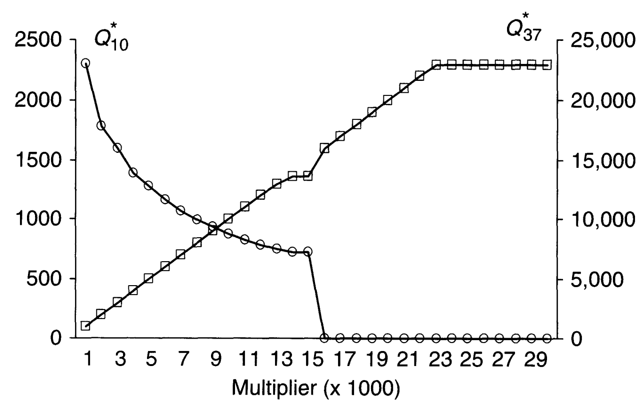

Figure 2 A closer look at Q^* values with varying u_i values.

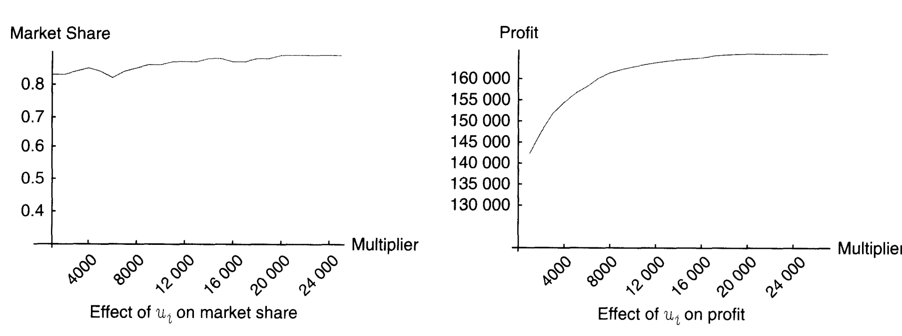

Figure 1 Sensitivity analysis with respect to u_i.

attractiveness Q_{10}^* of the facility at site 10 decreases and the optimal attractiveness Q_{37}^* of the facility at site 37 increases monotonically as the multiplier and hence the maximum attractiveness levels of the two facilities increase. Q_{10}^* is initially equal to 2303.1 when $u_{10} = 1000c_{10}$, and decreases down to 725.5 when $u_{10} = 15\,000c_{10}$. For larger values of the multiplier, no facility is opened at site 10. This means that unacceptably low optimal levels of the attractiveness are not likely to be seen. The reason is that the cost savings resulting from facility closures is not compensated by the additional revenue that the firm can earn by keeping the facility open at a low attractiveness level. On the other hand, the facility at site 37 has a nondecreasing optimal attractiveness level Q_{37}^* as a result of increasing multiplier values. When the multiplier hits 23 000, Q_{37}^* becomes 22 940.1 and this value remains constant for further increases in the multiplier value. The maximum attractiveness level is a parameter that is determined by the firm, and in order to prevent situations in which the optimal attractiveness level is regarded as unacceptably high, the firm should set reasonable upper bounds that can be achieved. Our experiments with other problem instances have shown that there might persist unacceptably high optimal attractiveness levels, but unacceptably low optimal levels do not occur.

5. Conclusion

In this paper we consider a variant of the discrete CFL problem in which both the locations as well as the attractiveness of new facilities have to be determined simultaneously to maximize the profit. To solve the problem, we formulate a mixed-integer nonlinear programming model and propose three solution methods. One of them is a heuristic based on the Lagrangean relaxation of the model (LH), while the others are exact procedures based on the branch-and-bound (BB) technique. The difference between the BB-based methods is that one relaxes the integrality restrictions on the binary variables and solves a nonlinear programme at each node of the BB tree (BB-NLP), whereas the other solves at each node the original problem relaxed in a Lagrangean fashion (BB-LR). All of the three solution procedures make use of the concavity of the objective function in terms of the attractiveness variables when the binary location variables are fixed.

The results obtained on a set of problem instances of varying sizes indicate that BB-NLP is the most efficient method and provides the optimal solution for all instances within the allowed time limit of 7200 s. The LH heuristic is also quite accurate in the sense that the average percent deviation of the solutions generated by LH is 0.3 and 0.6% away from the optimal objective values when the fixed cost levels are low and medium, respectively. We also make sensitivity analysis by changing the four main parameters of the model. An interesting finding of our experiments is that unacceptably low optimal attractiveness levels do not occur; the firm is better off when it does not open a facility rather than opening a facility with a low attractiveness. Unacceptably high attractiveness values, however, can be encountered especially when the maximum attractiveness levels are set to high values. But, since it is a parameter that is determined by the firm, it would never result in a situation where the firm has to open a facility with an unrealistic attractiveness level.

This model can be extended in several ways. For example, in calculating the probability of a customer to visit a facility, it is possible to incorporate some customer characteristics such as the income level in addition to the facility characteristics used in the current model, which are distance and attractiveness. Another direction which we want to pursue as a future study is to extend the model in a setting where the competitor reacts to the firm entering into the market by opening new facilities, changing the attractiveness levels of some existing facilities or closing them altogether.

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Appendix

We give below the algorithmic description of the three solution methods proposed in the study when the sense of optimization is maximization.

A1. Lagrangean heuristic

1. Introduce arbitrary Lagrangean multipliers $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, m$, and relax constraint (4) to obtain subproblems P1′ and P2′. Let $\pi = 2$, $UB_{best} = \infty$, and $LB_{best} = -\infty$.
2. Solve P2′ by inspection and P1′ using the gradient ascent algorithm. $z_1'(\lambda) + z_2'(\lambda)$ provides an upper bound $UB^{(t)}$ on z^*.
3. Fix the value of $\mathbf{X}$ obtained by the solution of P2′ in the original problem P, and solve it in terms of $\mathbf{Q}$ using the gradient ascent algorithm. This yields a feasible solution which provides a lower bound $LB^{(t)}$ on z^*.
4. Let $UB_{best} = \min\ \{UB^{(t)},\ UB_{best}\}$ and $LB_{best} = \max\ \{LB^{(t)},\ LB_{best}\}$.
5. Calculate $\theta^{(t)} = \pi(UB^{(t)} - LB_{best}) / \sum_{i=1}^{m} ||u_i X_i - Q_i||^2$.
6. Update the Lagrangean multipliers using the formula $\lambda_i^{(t+1)} = \max\{0,\ \lambda_i^{(t)} + \theta^{(t)}(u_i X_i - Q_i)\}$.
7. Update π if necessary.
8. If π is less than 0.005, then stop. Otherwise go to Step 2.

A2. Branch-and-bound with Lagrangean relaxation

1. Let $G_0 = \{1, 2, \ldots, m\}$, $LB_{best} = -\infty$. Apply the Lagrangean heuristic with no location variables X_i fixed. Let $LB_{best} = \max\ \{LB_{best},\ LB_0\}$.
2. If no active node exists, go to Step 5. Otherwise select an active node and generate two child nodes by branching on variable X_i, $i \in G_k$ of the (feasible) solution providing the best lower bound at that node such that $\gamma_i\,(u_i X_i - Q_i)$ is the largest.
3. Apply the Lagrangean heuristic with the imposed constraints coming from branching. Obtain LB_k and UB_k. Let $LB_{best} = \max\ \{LB_{best},\ LB_k\}$.
4. If $UB_k \leqslant LB_{best}$, prune node k and backtrack. Go to Step 2.
5. The feasible solution which yields LB_{best} is optimal.

A3. Branch-and-bound with NLP relaxation

1. Let $G_0 = \{1, 2, \ldots, m\}$, $LB_{best} = -\infty$. All X_i are relaxed between 0 and 1. Solve the relaxed model P″.
2. If no active node exists, go to Step 5. Otherwise select an active node and generate two child nodes by branching on variable X_i, $i \in G_k$ in the solution at node k whose value is the closest to 0.5.
3. Apply the NLP relaxation procedure. Obtain UB_k of the active node from the solution of the relaxed problem P″.
4. If a feasible solution is obtained, let $LB_k = UB_k$. Let $LB_{best} = \max\{LB_{best},\ LB_k\}$, prune that node and backtrack. Otherwise if $UB_k \leqslant LB_{best}$, prune that node and backtrack. Go to Step 2.
5. The feasible solution which yields LB_{best} is optimal.

A4. Step-by-step solution of a small problem by BB-NLP

In this sample problem there are $m = 4$ candidate facility sites $n = 4$ customers, and $r = 2$ existing facilities of the competitor(s). The attractiveness levels Q_1 and Q_2 of the existing facilities are 100 and 700, respectively. The unit attractiveness costs are given as $c_1 = 4$, $c_2 = 1$, $c_3 = 9$, $c_4 = 4$. The fixed costs f_i are set to $1000c_i$ and upper bounds for attractiveness levels u_i are assigned to $100c_i$. The annual buying powers at demand points are given as $a_1 = 1696$, $a_2 = 1245$, $a_3 = 3283$, $a_4 = 5766$. The matrix $\mathbf{D}_1$ below includes the squared Euclidean distances between customers and candidate facility sites while $\mathbf{D}_2$ contains the distances between customers and existing facilities. In these matrices, the rows represent the customers and the columns represent either the candidate facility sites or the existing facilities.

$$\mathbf{D}_1 = \begin{pmatrix} 185 & 2600 & 1945 & 2221 \\ 2920 & 257 & 9224 & 2708 \\ 5098 & 8389 & 8570 & 10\,730 \\ 661 & 4122 & 325 & 1669 \end{pmatrix}$$

$$\mathbf{D}_2 = \begin{pmatrix} 941 & 2234 \\ 1108 & 1089 \\ 6170 & 9477 \\ 2005 & 2756 \end{pmatrix}$$

The branch-and-bound tree produced by the BB-NLP method for this problem is illustrated in Figure A1, and the step-by-step procedure is explained below.

1. At root node 0, all X_i variables are relaxed between 0 and 1. Let $LB_{best}=-\infty$. We solve the relaxed problem using the nonlinear solver MINOS. The solution to this problem is $X_1=0.455$, $X_2=0$, $X_3=0$, $X_4=0$, and $Q_1=182.1$, $Q_2=0$, $Q_3=0$, $Q_4=0$. The upper bound UB_0 is 2609.05.
2. We branch on X_1 because it is the only fractional variable. Hence, nodes 1 and 2 are created with $X_1=1$ and $X_1=0$, respectively.
3. We first solve node 1 with $X_1=1$. The solution is $X_1=1$, $X_2=0$, $X_3=0$, $X_4=0$, and $Q_1=400$, $Q_2=0$, $Q_3=0$, $Q_4=0$. Since all X_i variables have binary values, a feasible solution is obtained. Thus, $LB_1=1380.50$. After updating the best lower bound of the tree as $LB_{best}=\max\{-\infty, 1380.50\}=1380.504$, we fathom node 1 and backtrack.
4. Now we solve node 2 with $X_1=0$. The location and attractiveness variables corresponding to candidate site 1 can be discarded from the problem. The solution to this problem is $X_1=0$, $X_2=0.165$, $X_3=0.112$, $X_4=0$, and $Q_1=0$, $Q_2=16.5$, $Q_3=101.2$, $Q_4=0$. The upper bound UB_2 is 1589.46. We branch on X_2 as it is X_i variable with the closest value to 0.5. Two new nodes are created.
5. We solve the partially relaxed problem at node 3 where $X_1=0$ and $X_2=1$. The solution is $X_1=0$, $X_2=1$, $X_3=0.1$, $X_4=0$, and $Q_1=0$, $Q_2=100$, $Q_3=89.8$, $Q_4=0$. The upper bound UB_3 is 1436.43. The X_i variable whose value is closest to 0.5 is X_3.
6. We solve the partially relaxed problem at node 4 where $X_1=0$, $X_2=1$, and $X_3=1$. The solution to this problem is $X_1=0$, $X_2=1$, $X_3=1$, $X_4=0$, and $Q_1=0$,

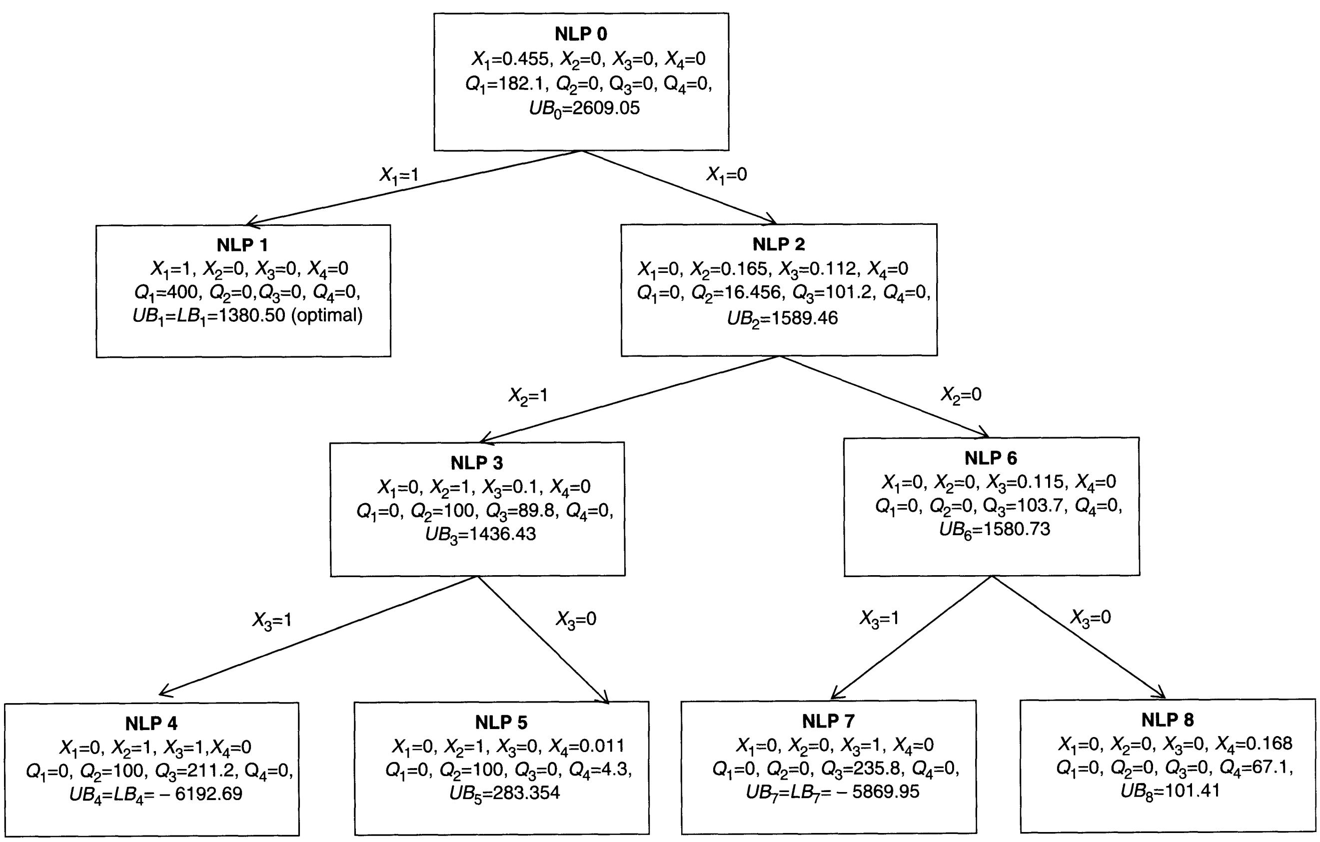

Figure A1 Branch-and-bound tree of the sample problem.

$Q_2 = 100$, $Q_3 = 211.2$, $Q_4 = 0$. Another feasible solution is obtained with $LB_4 = -6192.69$. There is no need to update the lower bound of the tree as $LB_4 < LB_{best}$. We fathom node 4 and backtrack.

7. We solve the partially relaxed problem at node 5 where $X_1 = 0$, $X_2 = 1$, and $X_3 = 0$. The solution is $X_1 = 0$, $X_2 = 1$, $X_3 = 0$, $X_4 = 0.011$, and $Q_1 = 0$, $Q_2 = 100$, $Q_3 = 0$, $Q_4 = 4.342$. The upper bound UB_5 is 283.35. Since $UB_5 < LB_{best}$, we fathom node 5 and backtrack.
8. We solve the partially relaxed problem at node 6 where $X_1 = 0$ and $X_2 = 0$. The solution is $X_1 = 0$, $X_2 = 0$, $X_3 = 0.115$, $X_4 = 0$, and $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 103.7$, $Q_4 = 0$. The upper bound UB_6 is 1580.73. The most fractional X_i variable is X_3, so we branch on it. Two new nodes are created.
9. We solve the partially relaxed problem at node 7 where $X_1 = 0$, $X_2 = 0$, and $X_3 = 1$. The solution to this problem is $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, $X_4 = 0$, and $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 235.8$, $Q_4 = 0$. A feasible solution with $LB_5 = -5689.95$ is obtained. There is no need to update the lower bound of the tree as $LB_5 < LB_{best}$. We fathom node 7 and backtrack.
10. We solve the partially relaxed problem at node 8 where $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$. The solution to this problem is $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0.168$, and $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = 67.1$. The upper bound UB_8 is 101.41. Since $UB_8 < LB_{best}$, we fathom node 8 and backtrack.
11. Since no active node exists, the feasible solution found at node 1 with $X_1 = 1$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0$, and $Q_1 = 400$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = 0$ is optimal and $LB_{best} = 1380.50$ is the optimal objective value.

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