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PHYSICAL AND QUANTUM OPTICS

Quantum Measurements of the Parameters of the Gell-Mann Optical Field with an SU(3) Interferometer

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Abstract—The SU(3) polarization theory in the Hilbert space is developed for quantum Bose systems with Gell-Mann symmetry. The degrees of polarization and isopolarization are determined. An original SU(3) interferometer for measuring the phase-dependent Gell-Mann parameters of an optical field is considered, and the signal-to-noise measurements limiting in quantum noises and based on the use of entangled and squeezed states of light at the entrance of the optical system are analyzed. A new type of quantum (helical) states of optical radiation for which the correlation between the Hermitian quadratures and isospin operators plays a crucial role is revealed for one of these modes in the classical-field approximation. © *2004 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

In recent years, quantum polarization states of an optical field have been the subject of extensive theoretical and experimental investigations in quantum optics [1–7]. Although the practical use of nonclassical polarization states for solving various problems of teleportation [1, 2], quantum information processing [3–5], and limiting measurements in ellipsometry [7–11] has already been widely covered in the literature, the physical nature of these states and the specific features of the generation of light with nonclassical polarization are still not clearly understood.

The universally accepted method for describing the quantum polarization states of an optical field is based on the use of Stokes parameters satisfying the SU(2) algebra. These nonclassical states include polarization-squeezed states of light with suppressed fluctuations of one of the Stokes parameters (i.e., with fluctuations that are smaller than those for coherent light) [7–9] and states of light with hidden [10] and scalar [5, 6] polarizations. The nonclassicality of these states of an optical field is associated primarily with the behavior of the moments of the fourth (and higher) order in field operators and their correlators. Moreover, there exist schemes of quantum detection of the Stokes parameters on the basis of their nondemolition (indirect) [8, 9] or parallel [11] measurement with the use of quantum polarometry. Of special interest is the polarization structure of multimode optical fields [12, 13], for which there can arise qualitatively new effects governed by quantum correlations of different modes; however, in the quantum case, this area has not been adequately investigated (see also [14]).

In this paper, we will analyze the quantum properties of polarization for three-mode optical fields in the representation of the SU(3) algebra (compare with [3, 4]). This is a standard approach to the description of quantum phenomena in chromodynamics and elementary-particle physics [15]. In optics, polarization states with SU(3) symmetry can be generated during the quantum parametric decay of a photon of a pump wave into two photons, namely, an idler wave and a signal wave (with inclusion of the energy exchange between all waves), upon the formation of quantum states of an optical field with SU(3) symmetry in an anisotropic cubically nonlinear medium [16]. Moreover, these states can arise in problems of analyzing the surface layers of materials and nanoobjects when the optical field has a longitudinal polarization component [17].

In atomic optics, this approach can be used both in describing the quantum dynamics of atomic three-level systems, for example, Bose–Einstein condensates [18], and in solving the problem of the interaction of atoms with an external quantized electromagnetic field [19–23].

2. QUANTUM DESCRIPTION OF THE SU(3) POLARIZATION IN BOSE SYSTEMS

In the Schwinger representation, a quantum three-mode Bose system can be described in terms of the SU(3) algebra by the Hermitian Gell-Mann operators λ_j ($j = 0$–8) (compare with [21–24]):[1]

$$\lambda_0 = a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3, \tag{1a}$$

$$\lambda_1 = a_1^+ a_2 + a_2^+ a_1, \quad \lambda_2 = i(a_2^+ a_1 - a_1^+ a_2), \quad \lambda_3 = a_1^+ a_1 - a_2^+ a_2, \tag{1b}$$

[1] The introduction of the SU(N) symmetry group specifying a vector field with dimension $N^2 - 1$, for which the requirements of invariance with respect to local phase transformations are satisfied, turns out to be useful in describing the relevant interaction in physics.

$$\lambda_4 = \overset{+}{a}_1 a_3 + \overset{+}{a}_3 a_1, \quad \lambda_5 = i(\overset{+}{a}_3 a_1 - \overset{+}{a}_1 a_3), \tag{1c}$$

$$\lambda_6 = \overset{+}{a}_2 a_3 + \overset{+}{a}_3 a_2, \quad \lambda_7 = i(\overset{+}{a}_3 a_2 - \overset{+}{a}_2 a_3), \tag{1d}$$

$$\lambda_8 = \frac{1}{\sqrt{3}}(\overset{+}{a}_1 a_1 + \overset{+}{a}_2 a_2 - 2\overset{+}{a}_3 a_3). \tag{1e}$$

Here, the creation (annihilation) operators $a_j(\overset{+}{a}_j)$ (j = 1–3) and the operators λ_j (j = 1–8) obey the following commutation relations for Bose systems in terms of the SU(3) algebra:

$$[a_i; \overset{+}{a}_j] = \delta_{ij}, \quad [a_i; a_j] = 0, \tag{2}$$

$$\begin{gathered}[\lambda_j; \lambda_k] = i\varepsilon_{jkm}\lambda_m, \quad j, k, m = 1, 2, 3; \\ [\lambda_4; \lambda_5] = i(\lambda_3 + \sqrt{3}\lambda_8), \quad [\lambda_6; \lambda_7] = i(\sqrt{3}\lambda_8 - \lambda_3), \\ [\lambda_0; \lambda_i] = 0, \quad i = 1, \ldots, 8.\end{gathered} \tag{3}$$

The completely antisymmetric structural coefficients ε_{jkm} are as follows (compare with [24]):

$$\begin{gathered}\varepsilon_{123} = 2, \quad \varepsilon_{584} = \varepsilon_{678} = \sqrt{3}, \\ \varepsilon_{147} = \varepsilon_{246} = \varepsilon_{257} = \varepsilon_{345} = \varepsilon_{516} = \varepsilon_{637} = 1.\end{gathered} \tag{4}$$

The operator λ_0 in expression (1a) accounts for the total number of photons in modes, and the operators λ_1–λ_3 form the SU(2) subgroup of the SU(3) algebra. In quantum optics, this subgroup is similar to the polarization Stokes parameters S_1–S_3 of an optical field, where modes 1 and 2 form a linear (circular) basis set of polarization. In the presence of the third mode a_3, by analogy with elementary-particle physics, we deal with the isospin (isopolarization) of the optical field. Within this formalism, the operators λ_4 and λ_5 in relationship (1c) and the operators λ_6 and λ_7 in relationship (1d) belong to the two other SU(2) subgroups, which, as a rule, are related to the U and V spins, respectively. The operator λ_8 in relationship (1e) is a combination of the numbers of photons of all three modes of the quantum field.

The polarization unit vector **e** for the three-mode system is determined by the relationship

$$\mathbf{e}a = \mathbf{e}_1 a_1 + \mathbf{e}_2 a_2 + \mathbf{e}_3 a_3, \tag{5}$$

where a is the annihilation operator for the three-mode field and $\mathbf{e}_j$ (j = 1–3) are the orthogonal vectors satisfying the condition

$$\sum_{j=1}^{3} |\mathbf{e}_j|^2 = 1. \tag{6}$$

Then, relationship (5) can be rewritten in the form

$$a = e_1^* a_1 + e_2^* a_2 + e_3^* a_3, \tag{7}$$

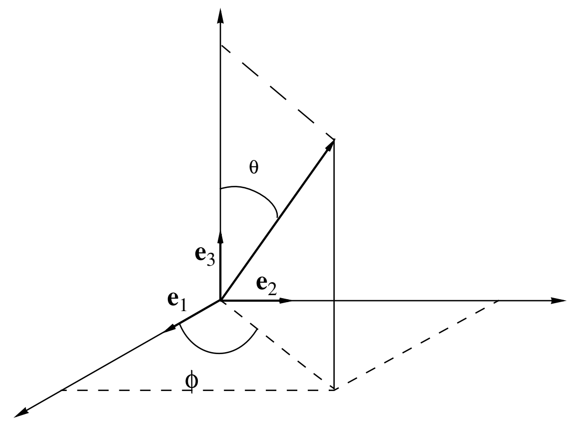

Fig. 1. Geometric representation of the SU(3) polarization. Designations: $\mathbf{e}_j$ are the basis vectors, and θ and ϕ are the phase angles. $\psi_1 = \psi_2 = 0$.

where $e_j^* = \mathbf{e}^*\mathbf{e}_j$ are the projections of the polarization vector. Actually, expressions (5)–(7) are the generalization of the standard formulas used for describing the polarization of the optical field in the basis set of two orthogonally polarized components of the polarization in quantum optics (see, for example, [25]). However, in the SU(3) algebra, the quantities e_j are determined by the four parameters θ, ϕ, ψ_1, and ψ_2 through the following relationships [24]:

$$\begin{gathered}e_1 = e^{i\psi_1}\sin\theta\cos\phi, \quad e_2 = e^{i\psi_2}\sin\theta\sin\phi, \quad e_3 = \cos\theta, \\ 0 < \theta, \quad \phi \le \frac{\pi}{2}, \quad 0 \le \psi_{1,2} < 2\pi.\end{gathered} \tag{8}$$

The geometric representation of the SU(3) polarization is given in Fig. 1. The isopolarization state in this case corresponds to conventional polarization of a two-mode field for which there exists a relation of the parameters θ, ϕ, ψ_1, and ψ_2 to the parameters measured in ellipsometry, namely, the ellipticity and the azimuthal angle of polarization (see, for example, [8–11]). However, in the quantum case, the procedure of measuring the parameters θ, ϕ, ψ_1, and ψ_2 calls for special analysis (see Section 4).

Let us consider the SU(3) polarization of an optical field in a coherent state, i.e., under the conditions

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad a_j|\alpha_j\rangle = \alpha_j|\alpha_j\rangle, \quad j = 1, 2, 3, \tag{9}$$

where $|\alpha\rangle = |\alpha_1\rangle|\alpha_2\rangle|\alpha_3\rangle$ are the coherent states of the three-mode field. From relationships (7)–(9), we obtain

$$\alpha = \sum_{j=1}^{3} e_j^*\alpha_j, \quad \alpha_j = e_j\alpha, \quad j = 1, 2, 3. \tag{10}$$

The second expression in relationships (10) results from the orthogonality of the basis vectors $\mathbf{e}_j$. In this case, we have

$$\sum_{j=1}^{3}|\alpha_j|^2 = |\alpha|^2 \equiv N. \qquad (11)$$

Next, we consider the variances $\langle(\Delta\lambda_j)^2\rangle$ of the Gell-Mann parameters (1) for the optical field in the coherent state $|\alpha\rangle$. These variances determine fluctuations in the system. Taking into account relationships (8)–(10), we obtain the following variances:

$$\langle(\Delta\lambda_0)^2\rangle_{\text{coh}} = N, \qquad (12a)$$

$$\langle(\Delta\lambda_{1,2,3})^2\rangle_{\text{coh}} = N\sin^2\theta, \qquad (12b)$$

$$\langle(\Delta\lambda_{4,5})^2\rangle_{\text{coh}} = N(\cos^2\theta + \sin^2\theta\cos^2\phi), \qquad (12c)$$

$$\langle(\Delta\lambda_{6,7})^2\rangle_{\text{coh}} = N(\cos^2\theta + \sin^2\theta\sin^2\phi), \qquad (12d)$$

$$\langle(\Delta\lambda_8)^2\rangle_{\text{coh}} = \frac{N}{3}(1 + 3\cos^2\theta). \qquad (12e)$$

An important distinguishing feature of expressions (12b)–(12d) for the variances $\langle(\Delta\lambda_j)^2\rangle$ (j = 1–7) is that they can vanish for specific phase parameters θ and ϕ. This is associated with the possibility of redistributing (transferring) the energy among all the modes of the optical field. For the isospin parameters, we have $\langle(\Delta\lambda_{1,2,3})^2\rangle_{\text{coh}} = 0$ at $\theta = \pi m$ (m = 0, 1, 2, …) and the average values $\langle\lambda_{1,2,3}\rangle$, which are measurable, also become zero. This situation is similar to the case with orbital angular momenta in quantum mechanics.

For atomically coherent states, in the second quantization operator representation, we have

$$|\Psi\rangle_N = \frac{1}{\sqrt{N!}}(e_1a_1^+ + e_2a_2^+ + e_3a_3^+)^N|0\rangle, \qquad (13)$$

where $|0\rangle \equiv |0\rangle_1|0\rangle_2|0\rangle_3$ corresponds to the vacuum state and $N = \langle\lambda_0\rangle$ is the total average number of particles. Unlike the optical coherent states (9), the quantum state (13) is entangled. In atomic optics, this state is used to describe the Bose–Einstein condensation effects [18]. In the microscopic limit (at N = 1), the quantum state (13) transforms into an entangled state of the quantum three-level system, the so-called qutrit state,

$$\begin{aligned}|\Psi\rangle_q &= (e_1a_1^+ + e_2a_2^+ + e_3a_3^+)|0\rangle \\ &= e_1|1\rangle_1|0\rangle_2|0\rangle_3 + e_2|0\rangle_1|1\rangle_2|0\rangle_3 + e_3|0\rangle_1|0\rangle_2|1\rangle_3,\end{aligned} \qquad (14)$$

which plays a crucial role in the modern quantum theory of information [26]. In the Appendix, we will consider the problem of generating SU(3)-squeezed states of the optical field in a cubically nonlinear medium.

3. DEGREE OF POLARIZATION OF LIGHT WITH SU(3) SYMMETRY

In optics, the polarization state of an optical field, as a rule, is quantitatively described by the degree of polarization. In the case where the optical radiation involves two orthogonal (linear or circular) polarization components, the classical definition of the measured degree of polarization can be represented in the form (see, for example, [25])

$$P = \frac{(\langle S_1\rangle^2 + \langle S_2\rangle^2 + \langle S_3\rangle^2)^{1/2}}{\langle S_0\rangle}, \qquad (15)$$

where S_j (j = 1–3) are the Stokes parameters of the optical field (in our case, these are the Gell-Mann operators λ_1–λ_3). The Stokes parameter $\langle S_0\rangle = \langle a_1^+a_1\rangle + \langle a_2^+a_2\rangle$ determines the total number of photons in polarized modes 1 and 2.

An important property of definition (15) is that the quantity P can be expressed through the scalar invariant $\mathrm{Tr}(J_{12}^2)$ of the coherence matrix J_{12}; that is,

$$P_{12} = \frac{(\mathrm{Tr}(J_{12}^2) - 2\det(J_{12}))^{1/2}}{\mathrm{Tr}(J_{12})}. \qquad (16)$$

Here, the coherence matrix J_{12} of the optical field has the form (compare with [25])

$$J_{12} = \begin{pmatrix} \langle a_1^+a_1\rangle & \langle a_1^+a_2\rangle \\ \langle a_2^+a_1\rangle & \langle a_2^+a_2\rangle \end{pmatrix}. \qquad (17)$$

Making allowance for all three modes of the optical field, definitions (16) and (17) should be revised. The coherence matrix in this situation can be written in the form

$$J = \begin{pmatrix} \langle a_1^+a_1\rangle & \langle a_1^+a_2\rangle & \langle a_1^+a_3\rangle \\ \langle a_2^+a_1\rangle & \langle a_2^+a_2\rangle & \langle a_2^+a_3\rangle \\ \langle a_3^+a_1\rangle & \langle a_3^+a_2\rangle & \langle a_3^+a_3\rangle \end{pmatrix}. \qquad (18)$$

Furthermore, the optical field with SU(3) symmetry has two scalar invariants: $\mathrm{Tr}(J^3)$ and $\mathrm{Tr}(J^2)$. Each invariant can be related to its own degree of polarization. In particular, from relationships (1) and (18), for the invariant $\mathrm{Tr}(J^3)$, we obtain

$$\mathrm{Tr}(J^3) = P_1^2(\mathrm{Tr}(J))^3 + 3\det(J), \qquad (19)$$

where the degree of polarization is given by the formula

$$P_1 = \frac{\sqrt{3}}{2}\frac{\left(\sum_{j=1}^{8}\langle\lambda_j\rangle^2\right)^{1/2}}{\langle\lambda_0\rangle}. \qquad (20)$$

The definition of the degree of polarization P_1 according to relationship (20) is the generalization of the standard expression (15) to the case with three modes of the optical field (compare with [12, 13]).

For $\det(J) = 0$, the light can be considered to be polarized. However, this assumption is not a sufficient condition. With the use of relationships (1) and (19), the second degree of polarization P_2 can be defined as

$$P_2 = \left(1 - \frac{2(\det(J_{12}) + \det(J_{13}) + \det(J_{23}))}{(\mathrm{Tr}(J))^2}\right)^{1/2}, \quad (21)$$

where the matrix J_{12} is determined by formula (17) and the coherence matrices J_{13} and J_{23} allow for the correlations of the modes taken in pairs and have the form

$$J_{13} = \begin{pmatrix} \langle a_1^+ a_1\rangle & \langle a_1^+ a_3\rangle \\ \langle a_3^+ a_1\rangle & \langle a_3^+ a_3\rangle \end{pmatrix}, \quad J_{23} = \begin{pmatrix} \langle a_2^+ a_2\rangle & \langle a_2^+ a_3\rangle \\ \langle a_3^+ a_2\rangle & \langle a_3^+ a_3\rangle \end{pmatrix}. \quad (22)$$

Therefore, the second, sufficient condition for the polarized light with SU(3) symmetry can be represented in the form

$$\det(J_{ij}) = 0. \quad (23)$$

The physical meaning of condition (23) can be easily elucidated when, in addition to definition (16), the degrees of polarization for the other modes taken in pairs are written in the following form:

$$P_{\mathrm{UP}} \equiv P_{13} = \frac{(\mathrm{Tr}(J_{13}^2) - 2\det(J_{13}))^{1/2}}{\mathrm{Tr}(J_{13})}, \quad P_{\mathrm{VP}} \equiv P_{23} = \frac{(\mathrm{Tr}(J_{23}^2) - 2\det(J_{23}))^{1/2}}{\mathrm{Tr}(J_{23})}, \quad (24)$$

where the condition $\mathrm{Tr}(J_{ij}) \neq 0$ must be satisfied.

Therefore, the second parameter of the degree of polarization P_2 (expression (21)) is completely determined by the correlation properties of the modes taken in pairs. These modes correspond to the degree of isopolarization $P_{\mathrm{IP}} \equiv P_{12}$, the degree of U-polarization P_{UP}, and the degree of V-polarization P_{VP}. Consequently, condition (23) at which $P_2 = 1$ takes the form

$$P_{\mathrm{IP}} = P_{\mathrm{UP}} = P_{\mathrm{VP}} = 1. \quad (25)$$

The parameters of the degree of polarization P_1 and P_2 are related by a simple expression. Actually, from relationships (18)–(22), we obtain

$$P_1^2 = P_2^2 - \frac{\det(J_{12}) + \det(J_{13}) + \det(J_{23})}{(\mathrm{Tr}(J))^2}. \quad (26)$$

Hence, it follows that, in the general case, the inequality $P_1 \le P_2$ is satisfied.

Condition (25) and the equality $P_1 = P_2 = 1$ hold true for the coherent (relationship (9)) and entangled (relationship (13)) three-mode states. This implies that the light in these states is completely polarized.

For arbitrary states of the three-mode field, the parameters of the degree of polarization defined by expressions (20) and (21) differ from each other. Indeed, let us assume for simplicity that the third mode is in the vacuum state. Then, from expressions (20) and (21), we obtain

$$P_1^2 = P_{\mathrm{IP}}^2 + \frac{\langle a_1^+ a_1\rangle\langle a_2^+ a_2\rangle - \langle a_1^+ a_2\rangle\langle a_2^+ a_1\rangle}{N^2}, \quad P_2^2 = P_{\mathrm{IP}}^2 + \frac{2(\langle a_1^+ a_1\rangle\langle a_2^+ a_2\rangle - \langle a_1^+ a_2\rangle\langle a_2^+ a_1\rangle)}{N^2}. \quad (27)$$

The term additional to P_{IP}^2 characterizes the difference in the behavior of the degree of polarization of two-mode and three-mode systems. As follows from relationships (27), the difference between the quantities P_1, P_2, and P_{IP} is maximum for states that are not eigenstates of the operators a_1 and a_2. For example, for such states with identical numbers of particles, we have $\langle a_1^+ a_1\rangle = \langle a_2^+ a_2\rangle$ and

$$P_{\mathrm{IP}} = 0, \quad P_1 = 1/2, \quad P_2 = 1/\sqrt{2}. \quad (28)$$

Thus, we can make the inference that the quantum state with SU(3) symmetry is polarized to a greater extent than the quantum state with SU(2) symmetry because $P_{1,2} \ge P_{\mathrm{IP}}$.

When the mode a_3 is in the coherent state $|\alpha_3\rangle$, from formula (20), we find

$$P_1^2 = 1 - \frac{3(\langle a_1^+ a_1\rangle\langle a_2^+ a_2\rangle - \langle a_2^+ a_2\rangle\langle a_1^+ a_2\rangle + |\alpha_3|^2(\langle a_1^+ a_1\rangle + \langle a_2^+ a_2\rangle - \langle a_1^+\rangle\langle a_1\rangle - \langle a_2^+\rangle\langle a_2\rangle))}{(\langle a_1^+ a_1\rangle + \langle a_2^+ a_2\rangle + |\alpha_3|^2)^2}. \quad (29)$$

In the approximation $|\alpha_3|^2 \gg \langle a_{1,2}^+ a_{1,2}\rangle$ (the mode a_3 corresponds to a strong field as compared to modes $a_{1,2}$), expression (29) leads to the obvious equality $P_1 = P_2 \simeq 1$.

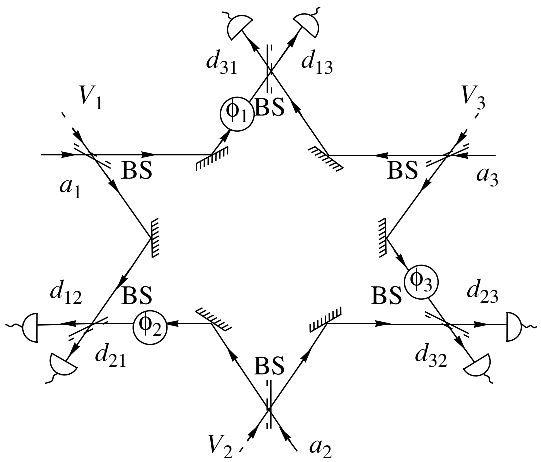

Fig. 2. Schematic diagram of the SU(3) interferometer for parallel (operational) measurements of Gell-Mann parameters in optics. The quantum modes a_j and vacuum modes V_j (j = 1–3) are fed to the entrance. Designations: ϕ_j are the phase shifts produced in the interferometer arms by linear optical elements, and BS stands for semitransparent beam splitters.

4. THE SU(3) INTERFEROMETER FOR DETECTING POLARIZATION STATES OF AN OPTICAL FIELD: HELICITY IN OPTICS

The measurement of the degree of polarization and the quantum properties of an optical system with SU(3) symmetry requires schemes and procedures different from those used to measure the polarization characteristics of optical systems with states corresponding to the SU(2) subgroup [1–9, 11].

Figure 2 presents a schematic diagram of the SU(3) interferometer for operational (parallel) measurements of the Gell-Mann parameters, through which the elements of the coherence matrix J (relationship (18)) can be expressed. The modes a_j (j = 1–3) and the vacuum modes V_j are fed to the interferometer entrance. The procedure of balanced homodyne detection [11] is used to measure the numbers of photons $N_{ij} = d_{ij}^{+} d_{ij}$ (i, j = 1–3; $i \neq j$), where d_{ij} (d_{ij}^{+}) are the annihilation (creation) operators at the exit of the interferometer.

After standard mathematical manipulations for the operators a_j of the quantum modes in the interferometer, we obtain the following expressions for the operators of observables, which are the differences between the numbers of photons at the interferometer exit:

$$\begin{aligned} N_{12}^{(-)} &= N_{12} - N_{21} = \frac{1}{2}\lambda_{12} + V_{12}, \\ N_{13}^{(-)} &= N_{13} - N_{31} = \frac{1}{2}\lambda_{13} + V_{13}, \\ N_{23}^{(-)} &= N_{23} - N_{32} = \frac{1}{2}\lambda_{23} + V_{23}. \end{aligned} \tag{30}$$

Here, the measured Gell-Mann parameters λ_{ij} are represented in the form

$$\lambda_{12} = a_1^{+} a_2 e^{i\phi_2} + a_2^{+} a_1 e^{-i\phi_2}, \tag{31a}$$

$$\lambda_{13} = a_1^{+} a_3 e^{-i\phi_1} + a_3^{+} a_1 e^{i\phi_1}, \tag{31b}$$

$$\lambda_{23} = a_2^{+} a_3 e^{i\phi_3} + a_3^{+} a_2 e^{-i\phi_3}. \tag{31c}$$

In relationships (30), the normally ordered operators V_{ij} depend on the annihilation (creation) operators of the vacuum modes at the interferometer entrance (Fig. 2). According to relationships (30) and (31), the measured average differences between the numbers of photons $\langle N_{ij}^{(-)} \rangle$ can be written as

$$\langle N_{12}^{(-)} \rangle = \frac{1}{2}\langle \lambda_{12} \rangle, \tag{32a}$$

$$\langle N_{13}^{(-)} \rangle = \frac{1}{2}\langle \lambda_{13} \rangle, \tag{32b}$$

$$\langle N_{23}^{(-)} \rangle = \frac{1}{2}\langle \lambda_{23} \rangle. \tag{32c}$$

Therefore, at specific phase differences ϕ_j in the interferometer arms, it is possible to perform parallel (operational) measurements of the Gell-Mann parameters $\lambda_{1,2}$, $\lambda_{4,5}$, and $\lambda_{6,7}$ (relationships (1b)–(1d)). These measurements also provide information on the phase parameters ψ_1 and ψ_2 in much the same manner as operational measurements of the ellipticity and the azimuthal angle of polarization for the quantum two-mode field [11].

The variances $\langle (\Delta N_{ij}^{(-)})^2 \rangle$ of the observables can be represented in the form

$$\begin{gathered} \langle (\Delta N_{ij}^{(-)})^2 \rangle = \frac{1}{4}\langle (\Delta\lambda_{ij})^2 \rangle + \frac{1}{4}(\langle a_i^{+} a_i \rangle + \langle a_j^{+} a_j \rangle), \\ i = 1, 2; \quad j = 2, 3; \quad i \neq j. \end{gathered} \tag{33}$$

The second term in expression (33) is associated with the vacuum fluctuations of the modes V_j at the interferometer entrance and determines the quantum limit of operational measurement of the Gell-Mann parameters according to the schematic diagram presented in Fig. 2.

Now, we examine the limiting (in quantum noise) measurements for the interferometer shown in Fig. 2. For this purpose, we determine the signal-to-noise ratio for the detected differences between the numbers of photons [14]; that is,

$$F_{ij} = \frac{|\langle N_{ij}^{(-)} \rangle|}{\langle (\Delta N_{ij}^{(-)})^2 \rangle}, \quad i = 1, 2; \quad j = 2, 3; \quad i \neq j. \tag{34}$$

For the variances $\langle(\Delta N_{ij}^{(-)})^2\rangle$, when they become zero, expression (34) is indeterminate, so that the behavior of the quantity F_{ij} calls for special analysis.

If the light at the interferometer entrance is in the coherent state described by expressions (9) and (10), from relationship (34), we obtain

$$F_{12,\,\mathrm{SQL}} = |\sin 2\phi \cos\xi_1|,$$
$$F_{13,\,\mathrm{SQL}} = \frac{|\sin 2\theta \cos\phi \cos\xi_2|}{\cos^2\theta + \sin^2\theta\cos^2\phi}, \tag{35}$$
$$F_{23,\,\mathrm{SQL}} = \frac{|\sin 2\theta \sin\phi \cos\xi_3|}{\cos^2\theta + \sin^2\theta\sin^2\phi},$$

where $\xi_1 = \phi_2 + \psi_2 - \psi_1$, $\xi_2 = \phi_1 + \psi_1$, and $\xi_3 = \phi_3 - \psi_2$ are the combinations of phases in the interferometer arms.

Expressions (35) determine the standard quantum limit for measurement of the signal-to-noise ratio for the SU(3) interferometer in Fig. 2. The quantities $F_{ij,\,\mathrm{SQL}}$ vary in the range $0 \le F_{ij,\,\mathrm{SQL}} \le 1$.

However, the coherent states (9) are not optimum from the standpoint of the limiting accuracy of the measurements performed. Actually, the maximum quantities F_{ij} can be achieved in the case where, at the interferometer entrance, we have nonclassical states of light for which the variances of the Gell-Mann parameters satisfy the condition

$$\langle(\Delta\lambda_{ij})^2\rangle \ll \langle a_i^+ a_i\rangle + \langle a_j^+ a_j\rangle, \tag{36}$$
$$i = 1, 2; \quad j = 2, 3; \quad i \neq j.$$

Then, the quantities F_{ij} defined by expressions (34) can be rewritten in the form

$$F_{ij} = \frac{2|\langle\lambda_{ij}\rangle|}{\langle a_i^+ a_i\rangle + \langle a_j^+ a_j\rangle}, \quad i = 1, 2;\ j = 2, 3;\ i \neq j. \tag{37}$$

Relationships (37) determine the limiting (in quantum noise) accuracy in measuring the quantities F_{ij} with the use of the interferometer depicted in Fig. 2. This accuracy is governed by the fundamental constraints imposed by quantum fluctuations of the optical fields V_j, which are in vacuum states at the entrance to the system.

Figure 3 shows the dependences of the signal-to-noise ratios F_{ij} on the phase parameter θ for the coherent (expression (9)), entangled (expression (13)), and squeezed (see the Appendix) states. The phases ξ_j for the squeezed states are optimized so that relationships (A4) for the variances of the Gell-Mann parameters are satisfied. It can be seen from these dependences that the use of the nonclassical states at the interferometer entrance is necessary for increasing the signal-to-noise ratio as compared to the corresponding ratio for light in the coherent state. Note that the appropriate choice of the input phase parameters θ and ϕ (SU(3) polarization

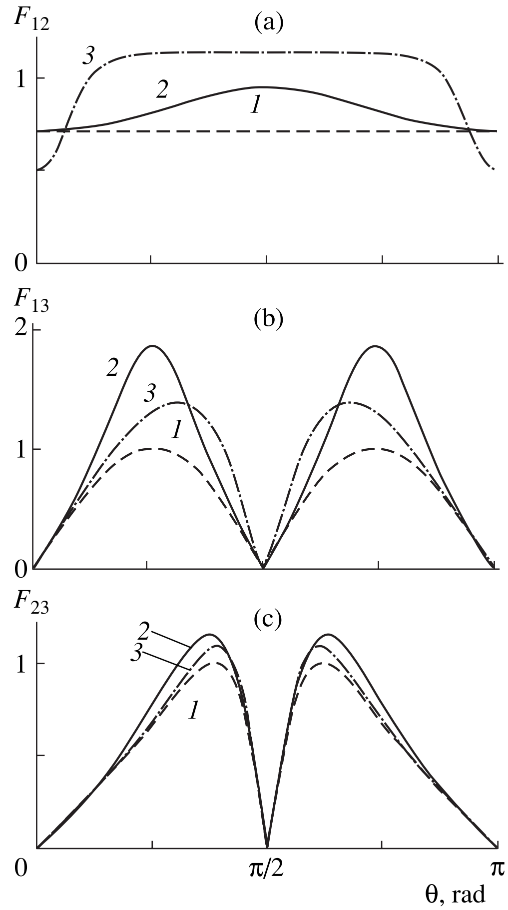

Fig. 3. Dependences of the signal-to-noise ratios (a) F_{12}, (b) F_{13}, and (c) F_{23} on the phase parameter θ for (*1*) coherent, (*2*) entangled, and (*3*) squeezed states. The parameters are as follows: $\kappa = \pi$, $\phi = \pi/8$, $r_2 = 20$, and $r_3 = 50$. $\xi_j = 0$ ($j = 1$–3) for coherent (relationship (9)) and entangled (relationship (13)) states. The phases ξ_j for the squeezed states are optimized so that the variances $\langle(\Delta\lambda_{ij})^2\rangle$ of the Gell-Mann parameters are minimum (see relationships (A4)).

states) and the fulfillment of the phase relationships for the parameters ψ_j and ϕ_j for the interferometer arms make it possible to reach the maximum ratio $F_{ij} = 2$.

Let us now consider the important (for balanced homodyne detection) limiting case of measurement according to the schematic diagram depicted in Fig. 2. We assume that the mode a_3 is a classical pump field described by the complex amplitude: $\alpha_3 = |\alpha_3|e^{i\varphi}$, where φ is a phase. Then, the operators λ_j ($j = 4$–8) defined by relationships (1b)–(1e) can be represented in the form

$$\lambda_{4,6} = |\alpha_3| Q_{1,2}, \tag{38a}$$

$$\lambda_{5,7} = -|\alpha_3| p_{1,2}, \tag{38b}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} N_{12}^{(+)} - \frac{2}{\sqrt{3}}|\alpha_3|^2. \tag{38c}$$

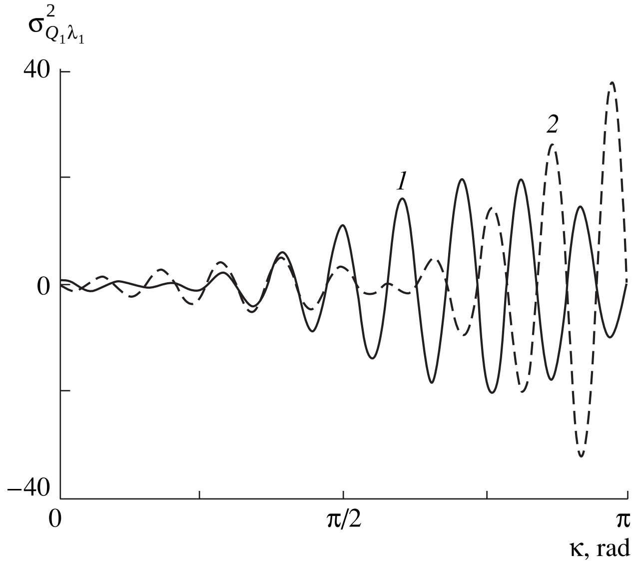

Fig. 4. Dependences of the normalized variance $\sigma^2_{Q_1\lambda_1}$ on the nonlinear phase incursion $\kappa = \gamma_1|\alpha|^2$ for φ = (*1*) 0 and (*2*) $\pi/2$. The complex amplitudes are $\alpha_{1,2} = |\alpha_{1,2}|e^{i\varphi_{1,2}} = \alpha$ for $\varphi_1 = \varphi_2 = 0$ and $r_2 = 20$.

Here, $N^+_{12} = a^+_1 a_1 + a^+_2 a_2$ is the total number of particles in modes 1 and 2 and Q_i and p_i (j = 1, 2) are the Hermitian quadratures of these modes, respectively; that is,

$$Q_j = a_j e^{-i\varphi} + a^+_j e^{i\varphi}, \quad p_j = i(a^+_j e^{i\varphi} - a_j e^{-i\varphi}), \quad j = 1, 2. \tag{39}$$

The measured average differences between the numbers of photons $\langle N^-_{13}\rangle$ and $\langle N^-_{23}\rangle$ are determined by expressions (31b), (31c), (32b), and (32c) and have the form

$$\begin{aligned}\langle N^{(-)}_{13}\rangle &= \frac{1}{2}|\alpha_3|(\langle Q_1\rangle\cos(\phi_1) - \langle p_1\rangle\sin(\phi_1)),\\ \langle N^{(-)}_{23}\rangle &= \frac{1}{2}|\alpha_3|(\langle Q_2\rangle\cos(\phi_3) + \langle p_2\rangle\sin(\phi_3)).\end{aligned} \tag{40}$$

It is easily seen from a comparison of formulas (39) and (40) that, by fixing the phase parameters φ, ϕ_1, and ϕ_3, it is possible to obtain information on the Hermitian quadratures of modes 1 and 2. Therefore, the classical pump field a_3 in the SU(3) interferometer (Fig. 2) plays the role of a reference generator field that is traditionally used in the scheme of balanced homodyne detection for recording quadrature-squeezed light (compare with [25]).

In the approximation under consideration for operators (1b) and (39), the commutation relations (3) are transformed into the following form:

$$\begin{aligned}&[Q_{1,2}, \lambda_1] = ip_{2,1}, \quad [\lambda_1, p_{1,2}] = iQ_{2,1},\\ &[Q_{1,2}, \lambda_2] = \mp iQ_{2,1}, \quad [\lambda_2, P_{1,2}] = \pm ip_{2,1},\\ &[Q_{1,2}, \lambda_3] = \pm ip_{1,2}, \quad [\lambda_2, p_{1,2}] = \pm iQ_{1,2}.\end{aligned} \tag{41}$$

Expressions (41) demonstrate that the operators (Stokes parameters) λ_j (j = 1–3) and the Hermitian quadratures Q_i and p_i (i = 1, 2) cannot be measured simultaneously and precisely. Of special physical interest can be quantum correlations of these quantities. These correlations are described by the variances (covariances) $\sigma^2_{Q_i\lambda_j}$ and $\sigma^2_{p_i\lambda_j}$ (i = 1, 2; j = 1–3); that is,

$$\begin{aligned}\sigma^2_{Q_i\lambda_j} &= \frac{1}{2}(\langle Q_i\lambda_j\rangle + \langle\lambda_j Q_i\rangle - 2\langle\lambda_j\rangle\langle Q_i\rangle),\\ \sigma^2_{p_i\lambda_j} &= \frac{1}{2}(\langle p_i\lambda_j\rangle + \langle\lambda_j p_i\rangle - 2\langle\lambda_j\rangle\langle p_i\rangle).\end{aligned} \tag{42}$$

In the quantum theory, the spin projection on the direction of the angular momentum of a particle is referred to as the helicity, which is a conservative quantity (see, for example, [27]). In this respect, the correlations between the quantities Q_i (p_i) and the isopolarization parameters λ_j can be considered an analogue of helical states in quantum optics. These states can be experimentally observed using the scheme shown in Fig. 2 by recording correlations of photocurrents at the interferometer exit.

The calculated dependences of the normalized variance (covariance) $\sigma^2_{Q_1\lambda_1} = \sigma^2_{Q_i\lambda_j}/|\alpha_2|$ ($i = 1, j = 1$) on the nonlinear phase incursion κ for the optical field at the exit of the cubically nonlinear anisotropic medium (see expression (A2)) are plotted in Fig. 4. The initial value of $\sigma^2_{Q_1\lambda_1}$ at $\kappa = 0$ corresponds to the light in the coherent state. It can be seen from Fig. 4 that an increase in the incursion κ can lead to an enhancement of correlations between the corresponding quadrature and the Gell-Mann parameter. This fact is responsible for the nonclassical behavior of the states under investigation. Note that the pump phase $\varphi = \pi/2$ determines the correlation between the isopolarization parameter λ_1 and the quadrature p_1.

5. CONCLUSIONS

Thus, we considered the quantum theory of polarization for optical and atomic systems with SU(3) symmetry. The occurrence of three modes of an optical field is of fundamental importance. This situation radically differs from the case with SU(3) symmetry discussed in the literature for a two-mode (biphoton) field that can be in three different quantum states (referred to as optical quarks [3, 4]). These states characterize the possible manifestations of a posteriori polarization of light. Moreover, these polarization states of an optical system are similar, in a sense, to macroscopic (stationary) polarization states such as the Schrödinger-cat states [28], which can also be assigned to the class of a posteriori polarization states of light. Undeniably, these states of an optical field are important for problems of quantum information [3, 4]. However, they can be revealed only by measuring the polarization of light, i.e., after realization of the hypothetical postulate for a specific quantum system.

It should also be noted that the obtained parameters of the degree of polarization (expressions (20), (21)) do not account for fluctuations of polarization of optical radiation, even though the inclusion of these fluctuations must necessarily be accomplished in quantum optics. In an earlier work [29], we took into account the higher moments of the Stokes parameters and their variances with the aim of refining the degree of polarization (relationship (15)) of two-mode optical fields. An alternative approach to this problem through the determination of the polarization Q-function for the Stokes parameters was recently proposed by Luis [30]. It is evident that a similar problem can also be solved for light with SU(3) symmetry. This problem is of particular interest and calls for a separate consideration.

Within the proposed approach to the description of optical systems with SU(3) symmetry, it is appropriate to use a different classification of nonclassical (entangled) states of light. This classification should be similar to that used for elementary particles in quantum physics [15]. It is also expedient to use the equivalent Elliott representation for this symmetry group with orbital angular momentum operators and quadrupole operator components, which corresponds to spatial symmetry of the system under investigation and is determined by the formalism of the problem of a three-dimensional oscillator in the quantum theory. The latter approach in quantum and atomic optics for optical fields and three-mode Bose condensates of atoms was applied in a number of works (see, for example, [31]) and showed considerable promise for analysis of new dynamic states.

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APPENDIX

Generation of Nonclassical States of Light with SU(3) Symmetry in a Cubically Nonlinear Anisotropic Medium

Let us consider the generation of squeezed states of an optical field in a cubically nonlinear anisotropic medium. In optics, such media can be represented by three-core optical fibers and waveguides [16]. In atomic optics, analogues of these systems are three-level condensate atoms [18].

The interaction Hamiltonian for our three-mode problem described by the creation (annihilation) operators a_j^+ (a_j) ($j = 1, 2, 3$) can be written in the form

$$H = \hbar(\chi_1 a_1^{+2} a_1^2 + \chi_2 a_2^{+2} a_2^2 + \chi_3 a_3^{+2} a_3^2), \qquad \text{(A1)}$$

where χ_j ($j = 1$–3) are the nonlinear coefficients of self-interaction of modes due to the cubic nonlinearity of the medium.

The solutions of the Heisenberg equations for the annihilation operators a_j have the form

$$b_j \equiv a_j(t) = e^{-i\gamma_j a_j^+ a_j} a_j, \qquad \text{(A2)}$$

where $\gamma_j \equiv 2\chi_j t$, and $a_j \equiv a_j(t = 0)$ are the operators at the initial instant of time (at the entrance of the nonlinear medium) provided that the numbers of photons in the modes are retained; that is,

$$b_j^+ b_j = a_j^+ a_j. \qquad \text{(A3)}$$

It is assumed that, at the initial instants of time, the operators a_j ($j = 1$–3) correspond to the coherent states (9). With due regard for expression (A2), the extremum (minimum) variances $\langle(\Delta\lambda_{ij})^2\rangle$ of the phase-dependent Gell-Mann parameters are determined by the relationships

$$\begin{aligned}
\langle(\Delta\lambda_{12})^2\rangle &= \langle(\Delta\lambda_{12})^2\rangle_{\text{coh}} \\
&- \frac{1}{2}N\kappa\sin^4(\theta)\sin^2(2\phi)\left(\sqrt{\kappa^2 M_{12}^2 + (1-r_2)^2} - \kappa M_{12}\right), \\
\langle(\Delta\lambda_{13})^2\rangle &= \langle(\Delta\lambda_{13})^2\rangle_{\text{coh}} \\
&- \frac{1}{2}N\kappa\sin^2(2\theta)\cos^2(\phi)\left(\sqrt{\kappa^2 M_{13}^2 + (1-r_3)^2} - \kappa M_{13}\right), \\
\langle(\Delta\lambda_{23})^2\rangle &= \langle(\Delta\lambda_{23})^2\rangle_{\text{coh}} \\
&- \frac{1}{2}N\kappa\sin^2(2\theta)\sin^2(\phi)\left(\sqrt{\kappa^2 M_{23}^2 + (r_2-r_3)^2} - \kappa M_{23}\right),
\end{aligned} \qquad \text{(A4)}$$

where we used the designations $\kappa = N\gamma_1$, $r_2 = \gamma_2/\gamma_1$, $r_3 = \gamma_3/\gamma_1$, $M_{12} = |e_1|^2 + r_2^2|e_2|^2$, $M_{13} = |e_1|^2 + r_3^2|e_3|^2$, and $M_{23} = |e_2|^2 r_2^2 + |e_3|^2 r_3^2$. The second terms in relationships (A4) are characteristic of nonclassical states of the optical field at the exit of the nonlinear medium.

Therefore, the suppression of fluctuations of the Gell-Mann parameters is determined by the nonlinear phase incursion κ and depends on the degree of anisotropy of the system, namely, the parameters r_2 and r_3. For the variances $\langle(\Delta\lambda_{3,8})^2\rangle$ of the Gell-Mann parameters (relationships (1b), (1e)), the suppression is absent (see expression (A3)) and the fluctuation amplitude corresponds to the fluctuation amplitude for the light in the coherent state (see formulas (12)).

REFERENCES

1. W. P. Bowen, N. Treps, R. Schnabel, and P. K. Lam, Phys. Rev. Lett. **89**, 253601 (2002).
2. N. Korolkova, G. Leuchs, R. Loudon, T. C. Ralph, and S. Silberhorn, Phys. Rev. A **65**, 052306 (2002).
3. A. V. Burlakov and D. N. Klyshko, Pis'ma Zh. Éksp. Teor. Fiz. **69**, 795 (1999) [JETP Lett. **69**, 839 (1999)].
4. A. V. Burlakov, M. V. Chekhova, O. A. Karabutova, D. N. Klyshko, and S. P. Kulik, Phys. Rev. A **60**, R4209 (1999).
5. V. P. Karasev and A. V. Masalov, Opt. Spektrosk. **74**, 928 (1993) [Opt. Spectrosc. **74**, 551 (1993)].
6. P. A. Bushev, V. P. Karasev, A. V. Masalov, and A. A. Putilin, Opt. Spektrosk. **91**, 558 (2001) [Opt. Spectrosc. **91**, 526 (2001)].
7. A. S. Chirkin, A. A. Orlov, and D. Yu. Parashchuk, Kvantovaya Élektron. (Moscow) **20**, 999 (1993).
8. A. P. Alodjants, S. M. Arakelian, and A. S. Chirkin, Quantum Semiclassic. Opt. **9**, 311 (1997).
9. A. P. Alodzhants and S. M. Arakelyan, Zh. Éksp. Teor. Fiz. **113**, 1235 (1998) [JETP **86**, 672 (1998)].
10. D. N. Klyshko, Zh. Éksp. Teor. Fiz. **111**, 1955 (1997) [JETP **84**, 1065 (1997)].
11. A. P. Alodjants and S. M. Arakelian, J. Mod. Opt. **46**, 475 (1999).
12. R. Barakat, Opt. Commun. **23**, 147 (1977).
13. C. Brosseau, *Fundamentals of Polarized Light: Statistical Optics Approach* (Wiley, New York, 1998).
14. S. N. Bagaev, A. P. Alodzhants, and S. M. Arakelyan, Dokl. Akad. Nauk **395** (2004) (in press).
15. K. Huang, *Quarks, Leptons, and Gauge Fields* (World Sci., Singapore, 1982; Mir, Moscow, 1985).
16. A. A. Mayer, Kvantovaya Élektron. (Moscow) **18**, 1264 (1991).
17. O. Keller, J. Chem. Phys. **112**, 7856 (2000).
18. C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).
19. A. S. Shumovsky, J. Phys. A **32**, 6589 (1999).
20. J. Delgado, E. C. Justas, L. L. Sanchez-Soto, and A. B. Klimov, Phys. Rev. A **63**, 063801 (2001).
21. F. T. Hioe, Phys. Rev. A **28**, 879 (1983).
22. A. V. Prokhorov, A. P. Alodjants, and S. M. Arakelian, Opt. Spektrosk. **94**, 55 (2003) [Opt. Spectrosc. **94**, 50 (2003)].
23. A. P. Alodzhants, A. V. Prokhorov, and S. M. Arakelyan, Pis'ma Fiz. Élem. Chastits At. Yadra **116**, 66 (2003).
24. G. Khanna, S. Mukhopadhyay, R. Simon, and N. Mukunda, Ann. Phys. (N.Y.) **253**, 55 (1997).
25. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge Univ. Press, Cambridge, 1995; Nauka, Moscow, 2000).
26. C. M. Caves and G. J. Milburn, Opt. Commun. **179**, 439 (2000).
27. V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Quantum Electrodynamics*, 3rd ed. (Nauka, Moscow, 1989; Pergamon, Oxford, 1982).
28. A. P. Alodjants, A. Yu. Leksin, A. V. Prokhorov, and S. M. Arakelian, Laser Phys. **10**, 603 (2000).
29. A. S. Chirkin, A. P. Alodzhants, and S. M. Arakelyan, Opt. Spektrosk. **82**, 1001 (1997) [Opt. Spectrosc. **82**, 919 (1997)].
30. A. Luis, Phys. Rev. A **66**, 013806 (2002).
31. Special issue of J. Opt. B: Quantum Semiclassic. Opt. **4** (2002).

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