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Resource-Efficient Linear Optical Quantum Computation

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We introduce a scheme for linear optics quantum computation, that makes no use of teleported gates, and requires stable interferometry over only the coherence length of the photons. We achieve a much greater degree of efficiency and a simpler implementation than previous proposals. We follow the "cluster state" measurement based quantum computational approach, and show how cluster states may be efficiently generated from pairs of maximally polarization entangled photons using linear optical elements. We demonstrate the universality and usefulness of generic parity measurements, as well as introducing the use of redundant encoding of qubits to enable utilization of destructive measurements—both features of use in a more general context.

Our understanding of the requirements for quantum computation has been greatly enhanced by Knill, Laflamme and Milburn's (KLM) discovery [1] that measurement induced nonlinearity suffices for efficient quantum computation. Specifically, they showed that linear optical elements (beam splitters, phase shifters, etc.) combined with single photon sources and single photon detectors can, in principle, be used for efficient quantum computation. In practice, even given these resources, significant obstacles stand in the way of making the KLM scheme a feasible technology for quantum computation. These include: (i) the sheer number of optical elements required, (ii) a need for extremely good, and very large, quantum memory, and (iii) a requirement of keeping what is essentially a giant interferometer phase stable to within a photon wavelength.

In this Letter we present a theoretical proposal for quantum computation with photons and linear optics which, in addition to a considerable number of other advantages, either overcomes or greatly alleviates all these key issues. Our proposal moves away from the use of teleportation to boost nondeterministic gates within a quantum circuit. Rather, we introduce two "fusion" mechanisms, which allow for the construction of entangled photonic states, known as cluster states. These states, introduced by Briegel and Raussendorf [2], allow for universal quantum computation by performing single-qubit measurements [3]. Since arbitrary single-qubit measurements are easy to perform on photonic qubits, it follows that our construction enables efficient quantum computation.

One key advantage of using cluster states is that the quantum gates are implemented with unit probability, rather than the "asymptotically unit" probability of the original KLM scheme. Other proposals to avoid this feature of the KLM scheme were presented by Yoran and Reznik [4] and Nielsen [5]; the latter also made use of cluster states. However, both of these proposals utilize the

same fundamental teleportation primitives introduced by KLM, and thus suffer similar problematic features. In contrast, our proposal overcomes the issues of nondeterministic gate operations by introducing the use of "qubit fusion" and "redundant encoding" of qubits.

In addition to overall smaller resource requirements in terms of the number of single photons, linear optical elements, and measurements required (we estimate factors of several orders of magnitude over Nielsen and many orders of magnitude over KLM, since the entangled resource states they require are generated via several or many low probability nondeterministic operations), our proposal has several other advantages. First, if we are prepared to accept a small (constant factor) overhead in resources, a simple extension of our basic proposal also has the significant advantage that photon-number-discriminating detectors are *not* required for its implementation. Moreover, there is no requirement for elaborate interferometers containing multiple beam splitters in series, which greatly reduces the complexity of mode-matching issues in an experimental implementation. More dramatically, it also removes the requirement of maintaining the phase stability of an extremely large and complex interferometer. The nondeterministic gates introduced by KLM, which are also the basis of [4,5], rely on Mach-Zehnder-type interference, which is sensitive to path length phase instabilities on the order of the photon's wavelength, i.e., around a micrometer for infrared light. In contrast, the interference we make use of is of the simple Hong-Ou-Mandel "coincidence" form, and thus only requires stability over the coherence length of the photons, a much larger distance. Recent down-conversion experiments [6] have obtained coherence lengths on the order of 10^{-4} m and in quantum dot experiments [7] coherence lengths several orders of magnitude greater than this have been reported. Thus the basic component of our scheme is at least 3 orders of
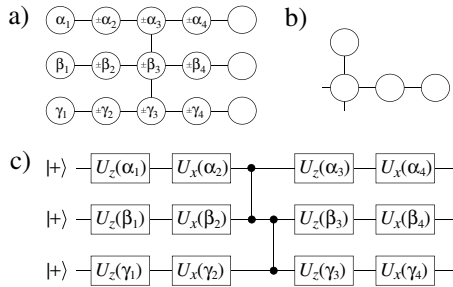
FIG. 1. The measurement pattern (a) simulates the quantum network (c). In this Letter we use a graphical representation of cluster states, where each vertex represents a qubit prepared in the state $|0\rangle + |1\rangle$ and each line ("bond") represents a controlled-σ_z (CZ) having been applied between the two connected qubits. To simulate the network in (c), the observable $\cos(\theta)\sigma_x + \sin(\theta)\sigma_y$ is measured on each cluster qubit, with the angle given each time by the symbols inside the circle. The sign of the measurement angle in all but the first column depend upon the outcome of measurements to the left of the qubit. Larger circuits can be simulated by larger cluster states with extensions of this pattern. Such layouts can be generated by tiling repeated 3-bond units of the "L shape" shown (b). Note that any additional Pauli transformations on cluster state qubits create an equivalent resource state for quantum computation since they can always be accounted for in the choice of measurement bases.

magnitude less sensitive to phase instability than these previous proposals.

For concreteness we phrase our proposal in terms of qubits encoded in horizontal (H) and vertical (V) photon polarization; however, the techniques we introduce are of widespread applicability. The primary resource we will make use of (and define resource usage with respect to) are two photon, polarization entangled Bell states. These can be obtained via linear optics and photodetection with probability 3/16 from four single photons [8]. In fact, since any nontrivial nondeterministic gate will create some entanglement, which can then be purified if necessary [9], a wide variety of options for creating this initial resource exist. Alternatively, it is also quite feasible that nonlinear optical processes be used to create the initial entanglement [10].

The key idea of cluster state quantum computation [3] is that single-qubit measurements on a cluster state of appropriate size and layout can simulate efficiently any quantum circuit. A more detailed description is provided in the caption of Fig. 1 and also, for example, in [3]. To simulate a quantum network made up of arbitrary rotations and controlled-phase gates, the cluster state layout in Fig. 1 (suggested by Nielsen in [5], although his scheme cannot actually realize its most compact form) is sufficient, and requires far fewer interqubit bonds than the original proposals [2,3]. As such, we concentrate on generating cluster states with this more compact layout.

We start by describing a "qubit fusion" operation. This *parity-check* [9,11] operation is implemented by mixing
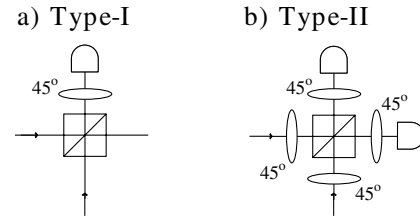

FIG. 2. The two qubit fusion nondeterministic gates: (a) Type-I consists of a mixing the two spatial modes on a PBS, which reflects vertically polarized light only, and measuring one of the output modes with a polarization discriminating photon counter after a 45° polarization rotation. (b) Type-II is obtained from Type-I by adding both 45° rotations to each input mode and measuring the output modes in the rotated basis. The 4 polarization rotators and PBS could be replaced by a PBS rotated at 45°.

the two modes on a polarizing beam splitter (PBS), rotating one of the output modes by 45° before measuring it with a polarization discriminating photon counter [see Fig. 2(a)]. Since we introduce a second fusion operation later, we refer to this as Type-I fusion [12].

The effect of this operation on input photons which form separate qubits of a cluster state depends upon the outcome of the measurement. When one and only one (H or V) polarized photon is detected (which occurs with probability 50%), the initially separate cluster qubits become a single "fused" cluster qubit which inherits all the cluster state bonds of the two qubits which were input (see Fig. 3). Thus, if the Type-I fusion is applied to the end qubits of linear (i.e., one-dimensional) clusters of lengths n and m, successful outcomes generate a linear cluster of length $(n + m - 1)$ [Fig. 3(a)]. Note that the two successful outcomes generate equivalent cluster states.

The Type-I fusion operation fails when either zero or two photons of either polarization are detected. The failure outcomes have the effect of measuring both input qubits in the σ_z eigenbasis (the computational basis). Measuring a cluster state qubit in the computational basis leaves the remaining qubits in a cluster state of the same layout as before the measurement, but now with all the bonds connected to the measured qubit severed [see Fig. 4(a)].
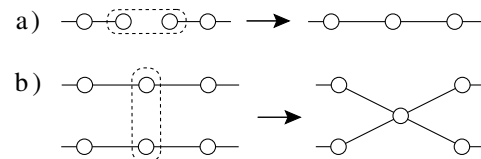

FIG. 3. Two examples of how clusters states can be joined together by fusing qubits from each. (a) Fusing the end qubits of two linear clusters of length n and m creates a cluster of length $(n + m - 1)$. (b) The midqubits of two linear clusters are fused to create a two-dimensional cluster with a crosslike layout. In this way nontrivial cluster layouts can be created.
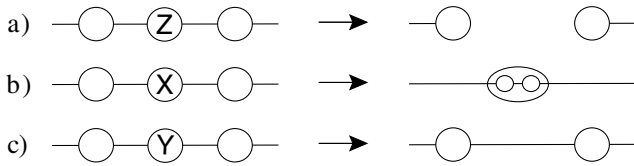
FIG. 4. Certain measurements on a cluster qubit will leave the remaining qubits in a new cluster state with a different layout: (a) A σ_z eigenbasis measurement removes the qubit from the cluster and breaks all bonds between that qubit and the rest of the cluster. (b) A σ_x measurement on a linear cluster removes the measured qubit and causes the neighboring qubits to be joined such that they now represent a single logical qubit with logical basis $|00\rangle$, $|11\rangle$. (c) A σ_y measurement removes the qubit from the linear cluster but links the neighboring qubits. These gain an extra $\pi/2$ rotation around the z axis which is accounted for when they are measured.

Starting from a supply of polarization Bell states (which are equivalent to a 2-qubit cluster state $|HH\rangle + |VH\rangle + |HV\rangle - |VV\rangle$), the Type-I fusion operation allows us to efficiently generate arbitrarily long linear cluster states. In the simplest case, a single successful Type-I fusion combines two Bell pairs into a 3-qubit cluster state, [which is also a Greenberger-Horne- Zeilinger (GHZ) state]. Since, on average, one must attempt this whole procedure twice before the desired three-qubit cluster is generated, the expected number of Bell states used to generate the 3-qubit cluster state is 4. A reasonably efficient method to create long linear clusters is to next generate 5-qubit clusters by Type-I fusing these 3-qubit clusters. We can then attempt to fuse such 5-qubit clusters one by one to any larger linear cluster. Each time, with probability 1/2, the cluster grows in length by 4 qubits, or, equally likely, loses a qubit. If one "recycles" the cluster states created by failed fusion attempts, the mean resources needed with this method are 6.5 Bell pairs per qubit added to the linear cluster. We do not know the optimal procedure for generating the linear clusters by Type-I fusion.

One-dimensional clusters are not, however, sufficient for universal quantum computation, as their geometry does not permit the implementation of 2-qubit gates. We thus need to create two-dimensional clusters, which can also be done by fusion, as depicted in Fig. 3(b). More precisely, we envisage fusing together qubits in linear clusters, as is illustrated in Fig. 5, which shows how the layout from Fig. 1 can be achieved. Type-I fusion is not appropriate
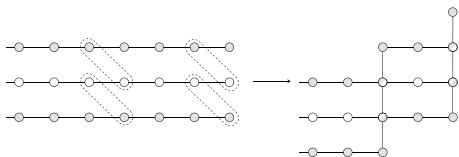

FIG. 5. If qubits from a linear cluster are fused according to the above pattern, a cluster state with the desired layout is generated.

for carrying out these fusions, since its failure outcome is a measurement in the computational (σ_z) basis, which would split the linear clusters in two [Fig. 4(a)].

Another approach to fusion is clearly necessary. In this alternate approach we introduce the use of *redundant encoding*. A single qubit in the cluster may be represented by multiple photons, such that a generic cluster state $|\phi_0\rangle|0\rangle + |\phi_1\rangle|1\rangle$ could be encoded $|\phi_0\rangle|H\rangle^{\otimes n} + |\phi_1\rangle|V\rangle^{\otimes n}$, where we have singled out from the rest of the cluster the particular qubit which is redundantly encoded with n photons. Note that a σ_x measurement (projection onto $|H\rangle \pm |V\rangle$) on one of the redundant photons does not destroy the cluster state, it removes one photon from the redundant encoding and perhaps adds an inconsequential phase. A σ_x measurement also has an interesting effect when performed on a qubit in a linear cluster; it does not split the cluster, rather it combines the adjacent qubits into a single redundantly encoded (by two photons) qubit, retaining the bonds attached to each, as shown in Fig. 4(b).

To utilize these features of σ_x measurements, we make use of the "Type-II" fusion gate depicted in Fig. 2(b). If this gate is successfully applied (as heralded by the detection of a photon in each output mode) to a single photon of each of a pair of logical qubits in a redundant n-photon encoding, it will lead to a projection onto a maximally entangled state. A failure outcome (signaled by detecting no photons in one of the modes) effectively performs a projective measurement of σ_x on each of the photons, thereby removing one photon from each qubit's redundant encoding. The gate could be reattempted, as long as sufficient photons remained in each qubit's redundant encod-
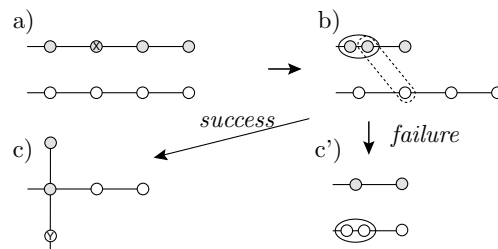

FIG. 6. Here we illustrate the construction of the L shape: (a) A σ_x measurement causes the neighboring qubits to be joined into a single logical qubit in the redundant encoding. (b) Type-II fusion is now attempted between this logical qubit and a qubit in the lower cluster. The fusion succeeds with probability 1/2. (c) If the fusion succeeds, a single further σ_y measurement creates the desired L shape [see Fig. 4(c)]. (c') If it fails, the σ_x measurements reduce the redundancy of the upper qubit and create a redundantly encoded qubit on the lower cluster. The qubits are now in a pattern similar to step (b), so with the addition of two further qubits another Type-II fusion can be attempted. These steps are repeated until a successful fusion is accomplished. On average, creating the L shape uses up 8 bonds from the linear clusters involved.

ing. Note that the Type-II fusion does not require the discrimination between different photon numbers.

To apply this to the fusing of linear clusters (see Fig. 6), we note that when two photons, each belonging to a different redundantly encoded logical qubit, are projected onto a maximally entangled state, the operation is logically equivalent to a fusion operation on the logical qubits, as long as one of these is encoded in at least two physical qubits. Figure 6 illustrates how one can use Type-II fusion to connect two linear clusters into a two-dimensional structure.

Cluster states with the layout illustrated in Fig. 1 can be generated by combining the two processes outlined above, i.e., first generating linear clusters by Type-I fusion, and then fusing their qubits by Type-II fusion to form the desired 2-dimensional cluster. We can quantify the resources required to build the cluster by recognizing that the layout of Fig. 1 can be broken down into the L-shaped units illustrated in Fig. 1(b). The L shape can be constructed from two linear clusters via a single (successful) Type-II fusion (illustrated in Fig. 6). On average, two Type-II fusion attempts are required and 8 qubit bonds from the linear clusters involved are used up. Note, that unlike in [5], there is no back propagation of errors here into the already generated cluster, meaning that the cluster qubits can be measured as soon as the next adjacent L shape has been completed. Since constructing the linear clusters requires on average no more than 6.5 Bell pairs for each qubit in the cluster, construction of the L shape requires on average no more than 52 Bell pairs. This is a great improvement compared with other linear optics-based quantum computation schemes [1,4,5].

For instance, the most efficient scheme so far is Nielsen's approach in [5]. Remember that each attempt of the implementation of a KLM $CZ_{n^2/(n+1)^2}$ gate requires a $4n$ photon entangled state for its implementation. Nielsen calculates that 24 successful $CZ_{4/9}$ gates are required per implemented two-qubit logical gate. Considering the number of times that a gate with success probability $(4/9)$ must be repeated, we see that in Nielsen's scheme $24 \times \frac{9}{4} = 54$ 8-photon entangled states are consumed per two-qubit gate. These 8-photon entangled states must be generated via a very complicated nondeterministic procedure involving multiple beam splitters and nondeterministic gates (see [1]).

We have made minimal use of the redundant encoding introduced for Type-II fusion. In fact, by using a redundant encoding for all qubits in a cluster it is possible to use only the parity gate of Fig. 2(b) for *all* gate operations. This has the considerable advantage that the gate can be implemented *without* photon-number-discriminating detectors, and naturally detects photon absorption errors. Since, in this case, two photons would be measured in each fusion, Bell states would not be a sufficient initial resource, and

one would have to use three-photon cluster states instead, which increases the resource requirements by a factor of 4. The nature of such a redundant encoding also allows for a single qubit to simultaneously be involved in bonding operations with multiple (possibly widely separated) other qubits. Incidentally, CZ gates (as opposed to fusion operations) between redundantly encoded qubits, can be directly implemented via the gate of Fig. 2(b), with an extra 45° rotation on one input mode.

Although we have phrased our results in terms of photon polarization, parity measurements are a natural 2-qubit measurement in bosonic systems. In fact, there has been much interest in the general question of when measurements can replace (all or part) of the processes of the standard circuit model. Our results can be interpreted as contributing to this effort by providing a proof that parity measurements (even nondeterministic ones), combined with single-qubit transformations or measurements, are universal for quantum computation.

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