10.3.6.1.8

EE24BTECH11060-Sruthi bijili

Question:

Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{1}{3x+y}+\frac{1}{3x-y}=\frac{3}{4}$$
$$\frac{1}{2\left(3x+y\right)}-\frac{1}{2\left(3x-y\right)}=\frac{-1}{8}$$

Solution:

Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{3x+y}=u \tag{1}$$
$$\frac{1}{3y-y}=v \tag{2}$$

Then our equations become:

$$u+v=\frac{3}{4} \tag{3}$$
$$\frac{1}{2}u-\frac{1}{2}v=\frac{-1}{8} \tag{4}$$

$$4u+4v=3 \tag{5}$$
$$4u-4v=-1 \tag{6}$$

This can be written in matrix form as:

$$\begin{pmatrix}4 & 4\\ 4 & -4\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}3\\ -1\end{pmatrix} \tag{7}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$\mathbf{Ax}=\mathbf{LUx}=\mathbf{b} \tag{8}$$

Factorization of LU:

Given a matrix $\mathbf{A}$ of size $n\times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

- Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L}=\mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.

- For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{9}$$

- For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{10}$$

By doing the following steps and solving we get :

$$\mathbf{U} = \begin{pmatrix} 4 & 4 \\ 0 & -8 \end{pmatrix} \tag{11}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{12}$$

Now,

$$A = \begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 0 & -8 \end{pmatrix} \tag{13}$$

We can solve this using two steps:

$$L\mathbf{y} = \mathbf{b} \tag{14}$$

$$U\mathbf{x} = \mathbf{y} \tag{15}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{16}$$

This gives:

$$y_1 = 3 \tag{17}$$

$$3 + y_2 = -1 \tag{18}$$

$$y_2 = -4 \tag{19}$$

Now using back substitution:

$$\begin{pmatrix} 4 & 4 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{20}$$

This gives:

$$\implies v = \frac{1}{2} \tag{21}$$

$$4u + 2 = 3 \tag{22}$$

$$u = \frac{1}{4} \tag{23}$$

Therefore:

$$\frac{1}{3x+y} = \frac{1}{4} \implies 3x + y = 4 \tag{24}$$

$$\frac{1}{3x-y} = \frac{1}{2} \implies 3x - y = 2 \tag{25}$$

$$\tag{26}$$

again by LU decomposition for

$$\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{27}$$

By doing the following factorization we get:

$$\mathbf{U} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} \tag{28}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{29}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{30}$$

$$\implies y_1 = 4 \tag{31}$$

$$\implies y_2 = -2 \tag{32}$$

Now using back substitution:

$$\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{33}$$

This gives:

$$\implies y = 1 \tag{34}$$

$$3x + 1 = 4 \tag{35}$$

$$x = 1 \tag{36}$$

The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{37}$$

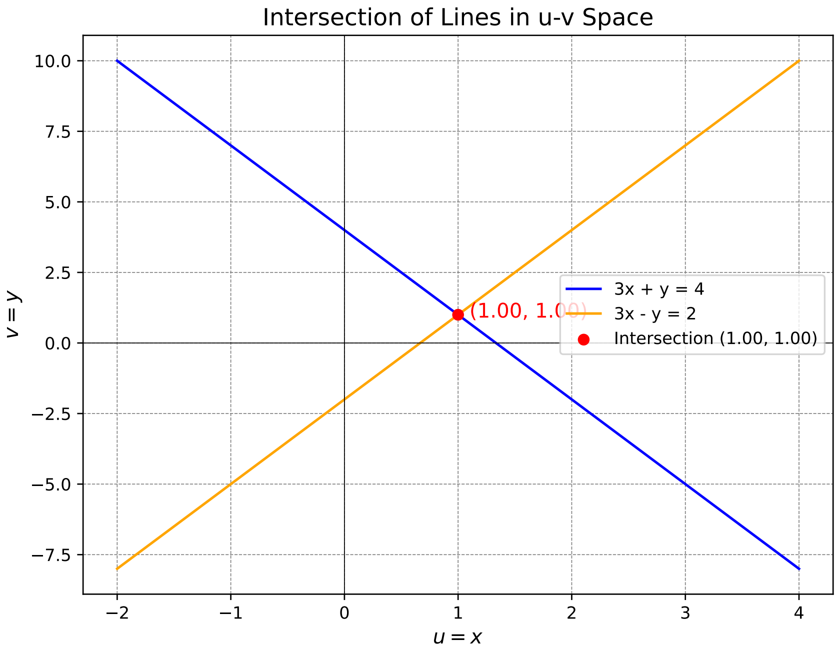

Fig. 1: Graph of the solution