10.3.6.1.1

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Question:
Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{1}{2x}+\frac{1}{3y}=2$$
$$\frac{1}{3x}+\frac{1}{2y}=\frac{13}{6}$$

Solution:
Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{x}=u \tag{1}$$
$$\frac{1}{y}=v \tag{2}$$

Then our equations become:

$$\frac{1}{2}u+\frac{1}{3}v=2 \tag{3}$$
$$\frac{1}{3}u+\frac{1}{2}v=\frac{13}{6} \tag{4}$$

$$3u+2v=12 \tag{5}$$
$$2u+3v=13 \tag{6}$$

This can be written in matrix form as:

$$\begin{pmatrix}3 & 2\\2 & 3\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}12\\13\end{pmatrix} \tag{7}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$\mathbf{Ax}=\mathbf{LUx}=\mathbf{b} \tag{8}$$

Factorization of LU:
Given a matrix $\mathbf{A}$ of size $n\times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

- Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L}=\mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.

- For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{9}$$

- For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{10}$$

By doing the following steps and solving we get :

$$\mathbf{U} = \begin{pmatrix} 3 & 2 \\ 0 & \frac{5}{3} \end{pmatrix} \tag{11}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \tag{12}$$

Now,

$$A = \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & \frac{5}{3} \end{pmatrix} \tag{13}$$

We can solve this using two steps:

$$L\mathbf{y} = \mathbf{b} \tag{14}$$

$$U\mathbf{x} = \mathbf{y} \tag{15}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} \tag{16}$$

This gives:

$$y_1 = 12 \tag{17}$$

$$\frac{2}{3}(12) + y_2 = 13 \tag{18}$$

$$y_2 = 5 \tag{19}$$

Now using back substitutio:

$$\begin{pmatrix} 3 & 2 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \tag{20}$$

This gives:

$$\frac{5}{3}v = 5 \tag{21}$$

$$\implies v = 3 \tag{22}$$

$$3u + 2\,(3) = 12 \tag{23}$$

$$u = 2 \tag{24}$$

Therefore:

$$\frac{1}{x} = 2 \implies x = \frac{1}{2} \tag{25}$$

$$\frac{1}{y} = 3 \implies y = \frac{1}{3} \tag{26}$$

The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \tag{27}$$

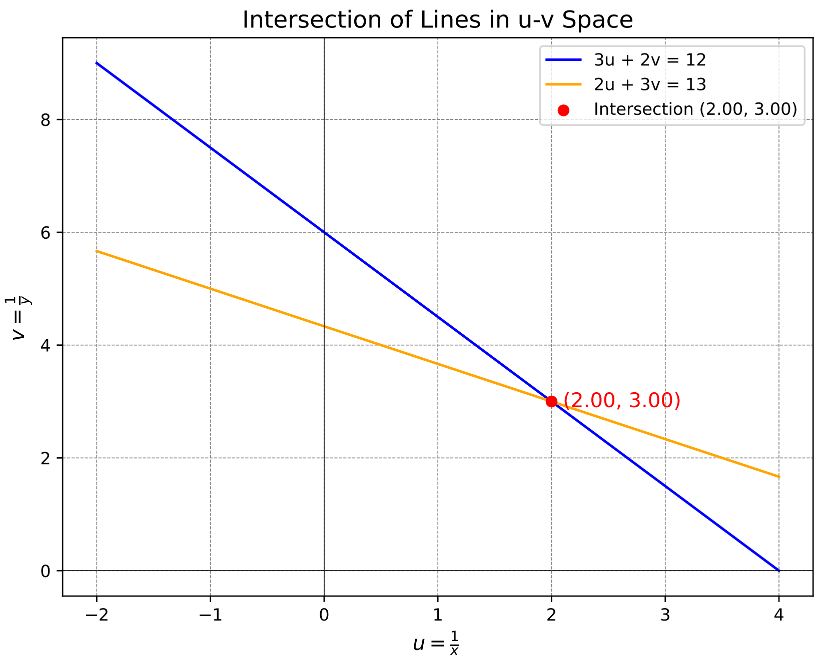

Fig. 1: Graph of the solution