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2 authors:

Pratima Rai
University of Delhi

Kapil k Sharma
South Asian University

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NUMERICAL METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS WITH TURNING POINT

Pratima Rai[1], Kapil K. Sharma[2 §]

1, 2Department of Mathematics
Center for Advance Study in Mathematics
Panjab University
Chandigarh, 160014, INDIA

Abstract: In this paper, we present a numerical method to solve boundary value problems for singularly perturbed differential-difference equations with turning point. A singularly perturbed differential-difference equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving at least one delay term . The points of the domain where the coefficient of the convection term in the singularly perturbed differential equation vanishes are known as the turning points. The solution of such type of differential equations exhibits boundary layer(s) or interior layer(s) depending upon the nature of the convection and the reaction term. In the development of numerical scheme for singularly perturbed differential-difference equations with turning point, we use a scheme based on El-Mistikawy Werle exponential finite difference scheme [21]. Some a priori estimates have been established to prove the convergence and stability of the proposed scheme.

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§Correspondence author

1. Introduction

We consider singularly perturbed differential-difference equations with turning point. A singularly perturbed differential-difference equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving at least one delay term. The arguments for small delay problems are found throughout the literature on epidemics and population where these small shifts play important role in the mathematical modeling of various practical phenomena, for example, in the study of variational problems in control theory where problems are complicated by the effect of time delays in signal transmission [7], in the study of bistable devices [13], evolutionary biology [19], description of human pupil light reflex [2], a variety of models of physiological processes or diseases [9] and [12]. They are also satisfied by the moments of the time of first exit of temporally homogeneous Markov processes [9] governing such phenomena as the time between impulses of a nerve cell and the persistence times of populations with large random fluctuations. Turning point is a point of the domain where the coefficient of the convection term vanishes. The solution of such type of differential equations exhibits boundary layer(s) or interior layer(s) depending upon the nature of the convection and the reaction term.

Lange and Miura initiate the asymptotic study of boundary value problems for singularly perturbed differential-difference equations [3], [5], [6] and [4] and discuss the case of small as well as large delay. Kadalbajoo and Sharma [14], [15], [16], [17] and [18] initiate the numerical approach to study further the effect of small and large delay as well as advance on the layer behavior of the solution. Patidar and Sharma [10] consider second order linear singularly perturbed differential-difference equations having delay in the differentiated term where the coefficient of the first order term is either positive or negative throughout the domain. But, there is no literature till date for the case when $a(x)$ vanishes or changes sign in the domain for such type of differential equations. In this paper, we initiate the numerical study of singularly perturbed differential- difference equation with turning points which can result in boundary or interior layer depending upon the value of coefficients.

We consider the singularly perturbed differential-difference equation having an isolated turning point at $x = 0$:

$$-\varepsilon y''(x) - a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad \text{on } \Omega = (-1, 1) \tag{1.1}$$

$$\begin{aligned} y(x) &= \phi(x), \quad \text{on} \quad -\delta - 1 \leq x \leq -1, \\ y(1) &= \gamma. \end{aligned} \tag{1.2}$$

where ε is a constant such that $0 < \varepsilon \ll 1$, $a(x)$ is assumed to be in $C^2[-1,1]$, $b(x)$ and $f(x)$ are required to be in $C^1[-1,1]$, $\delta = o(\varepsilon)$ such that $(\varepsilon - \delta a(x)) > 0$ for all $x \in [-1,1]$, γ is a positive constant and

$$a(0) = 0, \quad a'(0) > 0. \tag{1.3}$$

In order to exclude the so called resonance cases we require

$$b(x) \geq b_0 > 0 \quad \forall\, x \in [-1,1]. \tag{1.4}$$

We also impose following restriction so as to ensure that there are no other turning points in the interval $[-1,1]$

$$|a'(x)| \geq |a'(0)/2| \quad \text{for } -1 \leq x \leq 1. \tag{1.5}$$

Under condition (1.3)-(1.5) the solution of (1.1)-(1.2) has an internal layer at $x = 0$. The bounds on the behavior of $y(x)$ near the turning point depend specifically on ε and on the characteristic parameter β

$$\beta = b(x)/a'(x)\,|_{x=0} \tag{1.6}$$

and there exists constants $\beta_l,\ \beta_s$ such that

$$\beta_l \leq |\beta| \leq \beta_s. \tag{1.7}$$

If $\beta < 0$, $y(x)$ is "smooth" near $x = 0$; on the other hand if $\beta > 0$, then, there is in general an "interior layer" at $x = 0$, nature of which depends upon the value of β.

Abrahamsson [11] proves estimates on the analytical solution and discusses the nature of the solution of the singularly perturbed turning point problems. Further, the author proves results concerning non-uniform convergence of the difference schemes for such type of problems. Berger et al. [1] prove analytical results for the bounds on the derivatives of such problems where β is an integer in addition to the non -integral case. Using these bounds it is shown that modified version of El Mistikawy Werle scheme give by the author is uniformly convergent of the order $h^{\min(\beta,1)}$. Farrell [20] gives necessary and sufficient condition for uniform convergence of a difference scheme for such problems.

There are two kind of ε-uniform finite difference schemes which have small truncation error in the layer region(s). The first are the fitted operator methods which comprise specially designed finite difference operators on standard meshes; the second are the fitted mesh methods which comprises standard finite difference operators on specially designed meshes. In this paper, we will adopt fitted operator approach.

The objective of this paper is to present a numerical approach to solve singularly perturbed differential-difference equations with turning point. In this approach, we first approximate the shifted term by Taylor series and then, apply a difference scheme, provided shifts are of $o(\varepsilon)$. The scheme used here is based on El Mistikawy Werle exponential finite

2. Some a Priori Estimates

Taking Taylor series expansion of the term $y'(x-\delta)$ in equation (1.1), we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x). \tag{2.1}$$

Using this approximation in (1.1), we get

$$\begin{aligned} L_\varepsilon(u) &\equiv -(\varepsilon - \delta a(x))u''(x) - a(x)u'(x) + b(x)u(x) = f(x), \\ u(-1) &= \phi(-1), \quad u(1) = \gamma. \end{aligned} \tag{2.2}$$

Here u is approximation to $y(x)$. Now, we establish some a priori estimates about the solution and its derivatives for the singularly perturbed differential-difference equations with turning point. For a given function $g(x) \in C^k[-1,1]$, $||g||_k$ denote $\Sigma_{i=0}^{k} \max_{-1\leq x\leq 1}|g^i(x)|$, where $g^i(x)$ denote ith derivative of g. Let $C_\varepsilon(x) = \varepsilon - \delta a(x)$. Hence after we will use C_ε as constant part of $C_\varepsilon(x)$ when $a(x)$ depends on x (since δ is of $o(\varepsilon)$ and $a(x)$ is bounded therefore $C_\varepsilon = o(\varepsilon)$).

Lemma 2.1. (Continuous Maximum Principle) *Let $\Psi(x)$ be any sufficiently smooth function satisfying $\Psi(-1) \geq 0$ and $\Psi(1) \geq 0$. Then, $L_\varepsilon \Psi(x) \geq 0$ for all $x \in (-1,1)$ implies that $\Psi(x) \geq 0$ for all $x \in [-1,1]$.*

Proof. Let q be such that $\Psi(q) = \min_{x\in[-1,1]}\Psi(x)$. Let us assume that $\Psi(q) < 0$. Clearly $q \notin \{-1,1\}$. Since, q is point of minima therefore $\Psi'(q) = 0$ and $\Psi''(q) > 0$.

Now,

$$\begin{aligned} L_\varepsilon \Psi(q) &= -(\varepsilon - \delta a(q))\Psi''(q) - a(q)\Psi'(q) + b(q)\Psi(q), \\ &= -(\varepsilon - \delta a(q))\Psi''(q) + b(q)\Psi(q), \\ &< 0. \end{aligned}$$

which is a contradiction. This follows that $\Psi(q) \geq 0$ and therefore $\Psi(x) \geq 0 \ \forall x \in [-1,1]$. $\square$

Lemma 2.2. *Let $u(x)$ be solution of the problem* (2.2) *then we have*

$$||u|| \leq ||f||/b_0 + max(|\phi(-1)|, |\gamma|)$$

.

Proof. Let us define $\Psi^{\pm}(x) = ||f||/b_0 + \max(|\phi(-1)|, |\gamma|) \pm u(x)$.
Then, we have

$$\begin{aligned}
\Psi^{\pm}(-1) &= ||f||/b_0 + \max(|\phi(-1)|, |\gamma|) \pm u(-1) \\
&= ||f||/b_0 + \max(|\phi(-1)|, |\gamma|) \pm \phi(-1) \\
&\geq 0, \\
\Psi^{\pm}(1) &= ||f||/b_0 + \max(|\phi(-1)|, |\gamma|) \pm u(1) \\
&= ||f||/b_0 + \max(|\phi(-1)|, |\gamma|) \pm \gamma \\
&\geq 0 \\
\text{and} & \\
L_\varepsilon \Psi^{\pm}(x) &= -(\varepsilon - \delta a(x))(\Psi^{\pm}(x))'' - a(x)(\Psi^{\pm}(x))' + b(x)\Psi^{\pm}(x) \\
&= b(x)(||f||/b_0 + \max(|\phi(-1)|, |\gamma|)) \pm L_\varepsilon u(x) \\
&= b(x)(||f||/b_0 + \max(|\phi(-1)|, |\gamma|)) \pm f(x) \\
&\geq ||f|| \pm f(x) + b(x)\max(|\phi(-1)|, |\gamma|)(\text{as } b(x) \geq b_0 > 0) \\
&\geq 0.
\end{aligned}$$

Therefore from Lemma 2.1, we obtain $\Psi^{\pm}(x) \geq 0$ for all $x \in [-1, 1]$ which gives the required estimates. □

Remark 1. Lemma 2.1 implies that the solution is unique and since the problem under consideration is linear, the existence of the solution is implied by its uniqueness. Further, the boundedness of the solution is implied by Lemma 2.2. Suppose $[p, q]$ is a subinterval of $[-1, 1]$ which do not contain the turning point. Then, Lemma 2.2 provide a bound for $u(p)$ and $u(q)$. Then following theorem gives bound for u on $[p, q]$:

Theorem 2.3. *Let $u(x)$ be solution of* (2.2). *Suppose $a,\ b,\ f \in C^j[p, q],\ j > 0$, $|a(x)| \geq \eta$(η a positive constant), $||a||_\infty = M$ for $p \leq x \leq q$ and $S_1(j)$ denote the set $\{||a||_j,\ ||b||_j,\ ||f||_j,\ \beta,\ p-q,\ u(p),\ u(q),\ j\}$. Then, there exist a positive constant C depending only on $S_1(j)$ such that if $a(x) < 0$ on $[p, q]$, then*

$$|u^{(k)}(x)| \leq C(1 + (\varepsilon + \delta\eta)^{-k} exp\,(-\eta(q - x)/(\varepsilon + \delta M))), \quad \text{for } k = 1, \ldots j + 1, x \in [p, q].$$

Proof. Putting $H(x, u) = f(x) - b(x)u$ in the equation (2.2), we get

$$-(\varepsilon - \delta a(x))u'' - a(x)u' = H(x, u), \tag{2.3}$$

putting $a(x) = -k(x)$ in the above equation and integrating it twice using integrating factor, we obtain

$$u(x) = u_b(x) + K_1 + K_2 \int_x^q \exp\left(-\int_t^q \frac{k(\alpha)}{\varepsilon + \delta k(\alpha)} d\alpha\right) dt, \quad x \le t, \tag{2.4}$$

where

$$u_b(x) = -\int_x^q z(t)dt, \quad z(x) = \int_x^q \frac{H(t,u)}{\varepsilon + \delta k(t)} \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon + \delta k(\alpha)} d\alpha\right) dt,$$

here constants K_1 and K_2 may depend upon ε. Now

$$0 < \eta < k(\alpha) \le ||k|| = M \quad \text{for } \alpha \in (p,q)$$

$$\Rightarrow \varepsilon < \varepsilon + \delta\eta < \varepsilon + \delta k(\alpha) \le \varepsilon + \delta M \quad (\text{because } k(x) > 0)$$

$$\Rightarrow \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon} d\alpha\right) < \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon + \delta\eta} d\alpha\right)$$

$$< \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon + \delta k(\alpha)} d\alpha\right) \le \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon + \delta M} d\alpha\right)$$

Let $K(x) = \int_p^x a(t)dt$. Then, bounds on u implied by Lemma 2.2 leads to

$$\begin{aligned} |z(x)| &\le \tfrac{C}{\varepsilon+\delta\eta} \textstyle\int_x^q \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon+\delta k(\alpha)} d\alpha\right) dt \\ &\le \tfrac{C}{\varepsilon+\delta\eta} \textstyle\int_x^q \exp\left(-\int_x^t \frac{k(\alpha)}{\varepsilon+\delta M} d\alpha\right) dt \\ &= \tfrac{C}{\varepsilon+\delta\eta} \textstyle\int_x^q \exp\left(-\frac{1}{\varepsilon+\delta M}(K(t)-K(x))\right) dt, \end{aligned}$$

applying the inequality

$$\exp\left[-(\varepsilon + \delta M)^{-1}(K(t) - K(x))\right] \le \exp\left[-\eta(t-x)(\varepsilon + \delta M)^{-1}\right] \quad x \le t$$

we obtain

$$|z(x)| \le C(\varepsilon + \delta\eta)^{-1} \int_x^q \exp\left(-(\varepsilon + \delta M)^{-1}\eta(t-x)\right) dt \le C. \tag{2.5}$$

Hence, $|u_b(p)| \le C$. The boundary condition $u(q) = d_2$ implies that $K_1 = d_2$. One can also see that $u'(q) = -K_2$. Now $u(p) = d_1$ gives

$$K_2 \int_p^q \exp\left(-\int_t^q \frac{k(\alpha)}{\varepsilon + \delta k(\alpha)} d\alpha\right) dt = -u_b(p) + d_1 - d_2 \tag{2.6}$$

and we have

$$\begin{aligned}\int_p^q \exp\left(-\int_t^q \frac{k(\alpha)}{\varepsilon+\delta k(\alpha)}d\alpha\right)dt &\geq \int_p^q \exp\left(-\int_t^q \frac{k(\alpha)}{\varepsilon+\delta\eta}d\alpha\right)dt\\ &\geq \int_p^q \exp\left(-\frac{K(q)-K(t)}{\varepsilon+\delta\eta}\right)dt\\ &\geq \int_p^q \exp\left(-\frac{M(q-t)}{\varepsilon+\delta\eta}\right)dt\\ &\geq C(\varepsilon+\delta\eta)\end{aligned}$$

Using this in (2.6), we have $|K_2| \leq C(\varepsilon+\delta\eta)^{-1}$.

Now

$$u'(x) = z(x) - K_2 \exp\left(-\int_x^q \frac{k(\alpha)}{\varepsilon+\delta k(\alpha)}d\alpha\right)$$

implies that

$$\begin{aligned}|u'(x)| &\leq |z(x)| + K_2\left|\exp\left(-\int_x^q \frac{k(\alpha)}{\varepsilon+\delta k(\alpha)}d\alpha\right)\right|\\ &\leq C\left(1+(\varepsilon+\delta\eta)^{-1}\exp\left(-\int_x^q \frac{k(\alpha)}{\varepsilon+\delta k(\alpha)}d\alpha\right)\right)\\ &\leq C\left(1+(\varepsilon+\delta\eta)^{-1}\exp\left(-\frac{1}{\varepsilon+\delta M}\int_x^q k(\alpha)d\alpha\right)\right) \qquad (2.7)\\ &\leq C\left(1+(\varepsilon+\delta\eta)^{-1}\exp\left(-\frac{1}{\varepsilon+\delta M}[K(q)-K(x)]\right)\right)\\ &\leq C\left(1+(\varepsilon+\delta\eta)^{-1}\exp\left(-\frac{\eta(q-x)}{\varepsilon+\delta M}\right)\right)\end{aligned}$$

The bound for $u^{(i)}(x)$ for $i > 1$ follows by induction on i and repeated differentiation. □

Thus, for $a(x) < 0$ we have boundary layer at the right end of the interval and the derivatives of the solution are bounded. In the similar way we can obtain bound for the case $a(x) > 0$, i.e, boundary layer occurs at the left end of the interval.

Remark 2. Let $[p_1, q_1]$ be a subinterval of $[-1, 1]$ contained in an interval (p, q) such that $[p, q]$ contains none of the points $\{-1, 1, 0\}$. Assume $f,\ a,\ b$ are in $C^m[-1, 1]$ with m a positive integer and let $S_2(m)$ denote the set $\{||a||_m,\ ||b||_m,\ ||f||_m,\ \min_{p\leq x\leq q}|a(x)|, q-p,\ q-q_1,\ p_1-p,\ b_0,$ $|\phi(-1)|,\ |\gamma|\}$. Then, there is a constant C depending only on $S_2(m)$ such that

$$|u^{(k)}(x)| \leq C \quad \text{for } i = 1, ..., m+1,\ p_1 \leq x \leq q_1. \qquad (2.8)$$

Theorem 2.3 and Remark 2 reduce the matter of a priori estimates for $u(x)$ to producing bounds on the derivatives of $u(x)$ in a neighbourhood of the turning point.

Theorem 2.4. *Assume (1.3-1.6), $\beta < 0,\ a,\ b,\ f$ be in $C^m[-1,1]$ with m a positive integer and define*

$$S_4(m) = \{||a||_m,\ ||b||_m,\ ||f||_m,\ \beta_s,\ b_0,\ |\phi(-1)|,\ |\gamma|,\ m\}.$$

Then, there exist a constant C depending only on $S_4(m)$ such that

$$|u^{(k)}(x)| \leq C, \quad \text{for } k = 1,\dots,m, \text{ and } |x| \leq 1/2.$$

Proof. Using mean value theorem, (1.5) and (1.7), we get

$$|a(x)| = |a(x) - a(0)| = |x||a_x(\zeta)| \geq |x||a_x(0)|/2 \geq .5|x|b_0/\beta_s. \tag{2.9}$$

Remark 2 implies that $|u^{(k)}(\pm 1/2)| \leq C_1$ for $k = 1,...,m$ where C_1 depends only on $S_4(m)$ and differentiating (2.2) once and putting $z = u'$, we get

$$-(\varepsilon - \delta a(x))z''(x) + (\delta a_x - a(x))z'(x) + (b(x) - a'(x))z(x) = f'(x) - b'(x)u(x).$$

This equation has same form as (2.2). As solution of (2.2) is bounded, therefore we obtain

$$|z(x)| = |u'(x)| \leq C.$$

Now, for $k = 2,...,m$, differentiating (2.2) k times, we get the equation satisfied by $z(x) \equiv u^{(k)}(x)$ as

$$-(\varepsilon - \delta a(x))z''(x) - (-k\delta a_x + a(x))z'(x) + (k\delta a''(x) + b(x) - ka'(x))z(x) = r(x)$$

where $r(x)$ depends on $u(x), u'(x), ..., u^{(k)}(x)$ and at most kth order derivative of $a,\ b$ and f. Applying Lemma 2.2 with b replaced by $(k\delta a''(x) + b(x) - ka'(x))$ and using inductive argument we obtain the desired bounds. □

For $\beta > 0$, there is an internal layer at the turning point whose nature depends upon the value of β. Theorem 2.4 provides bound on the derivatives for the case $\beta < 0$, so we are left with finding the bounds on the derivatives of the solution to the case ((1.3)-(1.6)) along with $\beta > 0$. We now state a result for the case $\beta > 0$.

Theorem 2.5. *Let $\beta = m + \lambda$ where m is a positive integer and $\beta_l < |\lambda| < \beta_s$. In addition, let ((1.3)-(1.6)) along with $\beta > 0$ and $a(x),\ b(x),\ f(x) \in C^{m+k}[-1,1]$ where $k \geq 2$. Then, there exist a constant C depending only on $S_5(m+k)$ such that the solution $u(x)$of (2.2) satisfies*

$$|u^{(i)}(x)| \leq C, \quad \text{for } -1 \leq x \leq 1, \text{ and } i = 1,\dots,m,$$

$$|u^{(i)}(x)| \leq C(|x| + C_\varepsilon^{1/2})^{\lambda-i} I(x, C_\varepsilon, \lambda),$$
$$\text{for } -1 \leq x \leq 1, \text{ and } i = m+1, \ldots, m+k+1,$$

where

$$S_5(m) = \{||a||_2,\ ||b||_1, ||f||_1,\ b_0,\ \beta_l,\ \beta_s,\ |\phi(-1)|,\ |\gamma|,\ ||a||_m,\ ||b||_m,\ ||f||_m,\ m\},$$

and $I(x, C_\varepsilon, \lambda) \equiv \int_{x^2+C_\varepsilon}^{c} s^{(-\lambda-1)/2} ds, \quad c > 2.$

Proof. The proof same as in Berger et al. [1], just replace ε by C_ε which is constant part of $C_\varepsilon(x) = \varepsilon - \delta a(x)$. ☐

Remark 3. If $\beta > 1$ then above theorem holds whereas if $\beta < 0$ then we can take $\lambda = \beta$ and still above result holds good. Therefore in corresponding sections we take $\lambda = \beta$ as proving the result for $\beta > 1$ is then straightforward.

3. Numerical Scheme

We construct a scheme based on exponential scheme of El-Mistikawy and Werle [21] to approximate the solution $u(x)$ of (2.2). Let uniform partition of the interval $[-1, 1]$ be given by $x_i = -1 + ih$ for $i = 0, 1, \ldots, n$, where $h = 2/n$. Let u^h denote discrete approximation to the solution u of (2.2). Let g_i denotes $g(x_i)$ for any function g defined on this mesh. In this scheme, we replace (2.2) by piecewise constant coefficient approximating differential equation. The solution u^h of the problem

$$\begin{aligned} L_h u^h \equiv -(\varepsilon - \delta A(x))(u^h)''(x) - A(x)(u^h)'(x) + B(x)u^h(x) = F(x), \\ u^h(-1) = u(-1) = \phi(-1), \quad u^h(1) = u(1) = \gamma, \end{aligned} \tag{3.1}$$

is used as approximation to the solution $u(x)$. Here, $A,\ B,\ F$ are constants in each subinterval $(x_{i-1}, x_i),\ 1 \leq i \leq n$, but their value can vary from interval to interval. Here, $A,\ B,\ F$ satisfy the inequality

$$|A(x) - a(x)| + |B(x) - b(x)| + |F(x) - f(x)| \leq Ch,$$
$$\text{for } x \in X' = \cup_{i=1}^{n}(x_{i-1}, x_i) \tag{3.2}$$

and

$$B(x) \geq b_0, \quad \text{for } x \in X' = \cup_{i=1}^{n}(x_{i-1}, x_i), \tag{3.3}$$

where C depends only on $\{||a||_1,\ ||b||_1,\ ||f||_1\}$, $A(x) = (a_{i-1} + a_i)/2,\ B(x) = (b_{i-1} + b_i)/2$ and $F(x) = (f_{i-1} + f_i)/2,$ on (x_{i-1}, x_i) for each i. Problem (3.1) has a unique solution for the above choice of coefficients and at each interior grid point, x has following tridiagonal relationship

$$-(\varepsilon-\delta A_i)h^{-2}(r_i^- u_{i-1}^h + r_i^c u_i^h + r_i^+ u_{i+1}^h) = s_i^- f_{i-1} + s_i^c f_i + s_i^+ f_{i+1},$$
$$1 \le i \le n-1, \quad (3.4)$$

where

$$\begin{array}{rcl} r_i^- &=& \exp(n_i)/g(n_i-k_i), \quad r_i^+ = \exp(-k_{i+1})/g(n_{i+1}-k_{i+1}), \\ r_i^c &=& r_i^1 + r_i^2, \\ r_i^1 &=& -n_i - 1/g(n_i-k_i), \quad r_i^2 = n_{i+1} - 1/g(n_{i+1}-k_{i+1}), \\ s_i^- &=& g(n_i)v_i - \exp(n_i)g(-k_i)v_i, \\ s_i^+ &=& g(-k_{i+1})v_{i+1} - \exp(-k_{i+1})g(n_{i+1})v_{i+1}, \\ s_i^c &=& s_i^- + s_i^+, \\ v_i &=& 1/2\,[1-\exp(n_i-k_i)]^{-1}, \ i = 0\ (1)\ n-1 \\ g(w) &=& (\exp(w)-1)/w, \ \text{with } g(0) \equiv 1. \end{array} \quad (3.5)$$

Here, $n_i = h\bar{n_i}$ and $k_i = h\bar{k_i}$ where $\bar{n_i}$, $\bar{k_i}$ denotes the the negative and positive roots of

$$-(\varepsilon - \delta A^{(i)})w^2 - A^{(i)}w + B^{(i)} = 0, \ \text{respectively.}$$

In the above equation $A^{(i)}$, B^i denote the value of $A(x)$, $B(x)$ on (x_{i-1}, x_i), respectively.

We will use comparison function argument to derive the error estimate. For this, we require following Lemma:

Lemma 3.1. *Consider operator $L_h w(x) = -(\varepsilon - \delta A(x))w''(x) - A(x)w'(x) + B(x)w(x)$, where $(\varepsilon - \delta A(x))$, $A(x)$ and $B(x)$ are constants on each subinterval (x_{i-1}, x_i), $i = 1, ..., n$. Suppose $w(x)$ is in $C^1[-1,1]$, $w(x)$ restricted to $[x_{i-1}, x_i]$ is in $C^2[x_{i-1}, x_i]$ for each i, $w(-1) \ge 0$, $w(1) \ge 0$, and $Lw(x) \ge 0$ for all x in X'. Then $w(x) \ge 0$ for $-1 \le x \le 1$.*

Proof. Let x_0 be a point in $(-1,1)$ at which w attains its minimum. If $w(x_0) > 0$ then nothing to prove. Hence, suppose $w(x_0) < 0$. Then $w'(x_0) = 0$ and $w''(x_0) > 0$. Now

$$L_h w(x_0) = -(\varepsilon - \delta A(x_0))w''(x_0) - A(x_0)w'(x_0) + Bw(x_0).$$

Using (3.3) and the fact that $\varepsilon - \delta A(x_0) > 0$, we get $Lw(x_0) < 0$ which is contradiction. Hence $w(x_0) > 0$. Since x_0 is chosen arbitrarily, this result holds for all $x \in [-1,1]$ and hence $w(x) > 0$ for $-1 \le x \le 1$. $\square$

Let $e(x) \equiv u^h(x) - u(x)$, then we get

$$\begin{aligned} L_h e(x) &= \delta(A(x) - a(x))e''(x) + F(x) - f(x) + (A(x) - a(x))e'(x) \\ &+(b(x) - B(x))e(x) \equiv G(x), \quad x \in X', \end{aligned} \tag{3.6}$$

$$e(-1) = 0, \quad e(+1) = 0. \tag{3.7}$$

We choose suitable comparison function ψ in $C^2[-1,1]$ such that

$$\psi(\pm 1) \geq 0 \quad \text{and} \quad L_h\psi(x) \geq |G(x)| \ \text{ for } x \in X' \tag{3.8}$$

Then applying Lemma 2.3 and barrier function $w(x) = \psi(x) \pm e(x)$ we get

$$|e(x)| \leq \psi(x) \quad \text{for } -1 \leq x \leq 1. \tag{3.9}$$

We use a priori estimates to bound $G(x)$ and choose $\psi(x)$ suitably so that it satisfies (3.8) thus yielding the error estimate. We state following Lemmas which we require in deriving error estimate:

Lemma 3.2. *There is a positive constant c_2 depending only on β_l and β_s such that for ε in $[0,1]$ and $\beta_l < |\beta| < \beta_s$*

$$(x^2 + C_\varepsilon)^{(\beta-1)/2} I(x, C_\varepsilon, \beta) \geq c_2 \quad \text{for } -1 \leq x \leq 1$$

where $I(x, C_\varepsilon, \beta) \equiv \int_{x^2+C_\varepsilon}^{c} s^{\frac{-(\beta+1)}{2}}$.

Proof.

$$\begin{aligned} \textit{Let} \quad \phi(z) &= z^{\frac{\beta-1}{2}} I(x, C_\varepsilon, \beta) \quad \text{where } z = (x^2 + C_\varepsilon) \\ &= z^{\frac{\beta-1}{2}} \int_z^4 s^{\frac{-\beta-1}{2}} ds \\ &= z^{\frac{\beta-1}{2}} \left[\frac{2s^{\frac{1-\beta}{2}}}{1-\beta} \right]_z^4 \\ \Rightarrow \phi(z) &= \frac{2}{1-\beta} 4^{\frac{1-\beta}{2}} z^{\frac{\beta-1}{2}} - \frac{2}{1-\beta} \\ \Rightarrow \phi'(z) &= -4^{\frac{1-\beta}{2}} z^{\frac{\beta-3}{2}} < 0. \end{aligned}$$

Hence $\phi(x^2 + C_\varepsilon) \geq \phi(2) \geq c_2$ for $|x| \leq 1, \ \beta_l \leq |\beta| \leq \beta_s$. □

For deriving error estimates for the case $\beta > 0$ we define following comparison function

$$\phi(x, c) = (c^2x^2 + C_\varepsilon)^{(\beta-1)/2} I(cx, C_\varepsilon, \beta). \tag{3.10}$$

where c is constant such that $c \in (0,1]$. Above Lemma shows that for $c = 1$, $\phi(x, c)$ is bounded below.

Lemma 3.3. [2] *For any c in $(0,1]$, $\phi'(x,c) < 0$ for $0 < x \leq 1$, and hence if $0 < c_1 < c_2 \leq 1$ and $|x| \leq 1$, then $\phi(x,c_1) \geq \phi(x,c_2)$. Assuming the hypothesis of theorem 2.5, there are positive constants $c < 1$ and C_3 depending only on $S_5(1)$ such that*

$$L_h\phi(x,c) \geq C_3\phi(x,c) \quad \text{for } x \in X'. \tag{3.11}$$

Theorem 3.4. *Assuming conditions* (1.3)-(1.6) *and* (3.2)-(3.3) *be satisfied and $\beta > 0$. Suppose $A(x) \geq 0$ for $x \geq 0$ and $A(x) \leq 0$ for $x \leq 0$. Then, there are positive constants c and C_1 depending only on $S_5(1)$ such that for u the solution of* (2.2) *and u^h the solution of* (3.2)

$$|u^h(x) - u(x)| \leq C_1 h\phi(x,c) \quad \text{for } -1 \leq x \leq 1 \tag{3.12}$$

Proof. Using Lemma 3.3 we get above estimate as an immediate consequence of (3.6), (3.3), Lemma 2.2, and Theorem 2.5. The above bounds on the error given by (3.12) suffers large growth when $|x| < h$, $\beta \leq 1$ and $\varepsilon - \delta A(x) = C_\varepsilon(x)$ is small. In order to prevent loss of accuracy when $(\varepsilon - \delta A(x)) \ll h$, $|x| < h$, and $\beta < 1$, we require that $|A(x)| \leq M|x|$, $\forall x \in X'$, where M is a constant independent of h and ε. This condition is satisfied by modifying the choice of $A(x)$ near the turning point. If there is a mesh point x_i coinciding with the turning point $x = 0$, we put $A(x) = 0$ on (x_{i-1}, x_{i+1}). If the turning point is in the interior of (x_i, x_{i+1}) then $A(x) = a(x_i)$ on (x_{i-1}, x_i), $A(x) = 0$ on (x_i, x_{i+1}), and $A(x) = a(x_{i+1})$ on (x_{i+1}, x_{i+2}). □

Theorem 3.5. *Let the above conditions* (1.3)-(1.6) *and* (3.2)-(3.3) *be satisfied and $\beta > 0$. Let $|A(x)| \leq C_4|x|$. Then, there is a constant C_5 independent of C_ε and h and depending only on C_4 and $S_5(1)$ such that*

$$|u^h(x) - u(x)| \leq C_5 h\phi(h,c) \quad \text{for } |x| \leq 1. \tag{3.13}$$

Proof. Using (3.2), (3.6)-(3.9), and choosing ψ a large constant, it is sufficient to prove that $G(x)$ in (3.6) is bounded by $Ch\phi(h,c)$. By Lemma 3.3, we know that $\phi(x,c)$ is a decreasing function for $x > 0$, so we only need to prove the latter for $x \in X'$ with $0 \leq x \leq h$ and the case $x < 0$ can be proved symmetrically. For proving $g(x)$ to be bounded by $Ch\phi(h,c)$ it is sufficient to prove that

$$|A(x) - a(x)|\phi(x,c) \leq Ch\phi(h,c) \quad \text{for } x \in X' \text{ with } 0 \leq x \leq h,$$

which reduces to proving

$$x\phi(x,c) \leq Ch\phi(h,c) \quad \text{for } 0 \leq x \leq h.$$

By using mean value theorem and denoting $x\phi(x,c)=t(x)$, we get

$$\begin{array}{rcl} x\phi(x,c) & = & t(x) \\ & = & t(h)-(h-x)t_x(\xi) \\ & = & h\phi(h,c)-(h-x)[\phi(\xi,c)+\xi\phi_x(\xi,c)]. \end{array}$$

Now, we combine terms in $\xi\phi_x(\xi,c)$ containing $I(c\xi,C_\varepsilon,\beta)$ with $\phi(\xi,c)$. It can be easily verified that for ξ in $(0,h)$ $\phi(\xi,c)+\xi\phi_x(\xi,c)\geq -2$ and hence, $x\phi(x,c)\leq h\phi(h,c)+ch$. This along with Lemma 3.2 gives us desired estimate.

Now from Theorem 3.4, we get $O(h)$ accuracy away from the turning point for any β in $\beta_l,\ \beta_s$, and (3.13) yields

$$||u^h(x_i)-u_i||\leq M_1(h^\beta+h), \quad \text{for } \beta\neq 1,$$

$$||u^h(x_i)-u_i||\leq M_2 h\ \ln\frac{6}{ch^2}, \quad \text{for } \beta=1,$$

where M_1 and M_2 are constants independent of h and C_ε. □

4. Test Examples and Numerical Experiments

In this section we present some numerical examples in support of the theoretical results and to show the effect of shift on the solutions. In the following examples $x=1/2$ is taken as turning point. Since exact solutions are not known for the problems given below, we use double mesh principle to calculate maximum errors(denoted by $E_{n,\varepsilon}$) [8]:

$E_{n,\varepsilon}=\max_{0\leq j\leq n}|(u^h)^n_j-(u^h)^n_{2j}|, \quad E^n=\max_{0<\varepsilon<1}E_{n,\varepsilon}$.

Example 4.1 Consider the problem

$$-\varepsilon y''+2(1-2x)y'(x-\delta)+4y=0, \quad \text{on } x\in(0,1)$$

$$\text{with} \quad y(x)=1 \text{ on } -\delta\leq x\leq 0, \quad y(1)=1$$

Example 4.2 Consider the problem

$$-\varepsilon y''+2(1-2x)y'(x-\delta)+4y=4(1-4x), \quad \text{on } x\in(0,1)$$

$$\text{with} \quad y(x)=1 \text{ on } -\delta\leq x\leq 0, \quad y(1)=1.$$

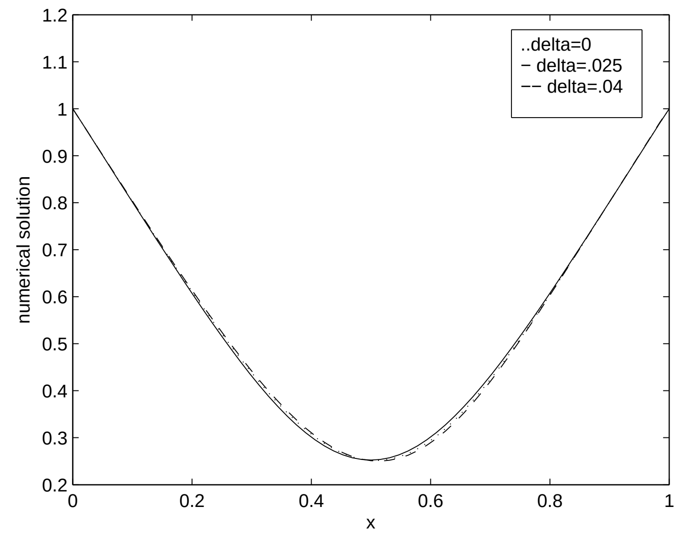

Figure 1: The numerical solution for example 1 ($\varepsilon = .1$,)

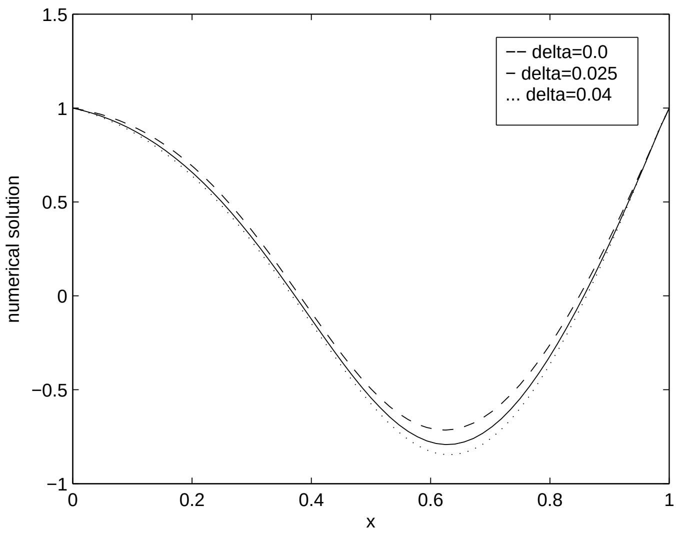

Figure 2: The numerical solution for example 2 ($\varepsilon = .1$,)

$\varepsilon \downarrow$ $n \rightarrow$	32	64	128	256	512	1024
1	$1.716e-4$	$4.436e-5$	$1.127e-5$	$2.839e-6$	$7.125e-7$	$1.785e-7$
2^{-2}	$6.284e-4$	$1.695e-4$	$4.403e-5$	$1.122e-5$	$2.833e-6$	$7.12e-7$
2^{-4}	$1.948e-3$	$5.910e-4$	$1.640e-4$	$4.33e-5$	$1.113e-5$	$2.821e-6$
2^{-6}	$3.63e-3$	$1.475e-3$	$5.047e-4$	$1.508e-4$	$4.146e-5$	$1.089e-5$
2^{-8}	$3.987e-3$	$1.952e-3$	$9.168e-4$	$3.733e-4$	$1.272e-4$	$3.789e-5$
2^{-10}	$7.147e-3$	$1.979e-3$	$9.814e-4$	$4.866e-4$	$2.297e-4$	$9.359e-5$
2^{-12}	$5.562e-3$	$3.583e-3$	$9.837e-4$	$4.898e-4$	$2.444e-4$	$1.215e-4$
2^{-14}	$4.045e-3$	$2.778e-3$	$1.794e-3$	$4.908e-4$	$2.446e-4$	$1.222e-4$
2^{-16}	$4.045e-3$	$1.986e-3$	$1.388e-3$	$8.975e-4$	$2.460e-4$	$1.222e-4$
2^{-18}	$4.046e-3$	$1.986e-3$	$9.844e-4$	$6.938e-4$	$4.489e-4$	$1.232e-4$
2^{-20}	$4.046e-3$	$1.986e-3$	$9.844e-4$	$4.902e-4$	$3.468e-4$	$2.245e-4$

Table 1: Maxmimum pointwise error ($E_{n,\varepsilon}$) for $\delta = 0$ for Example 1

$\varepsilon \downarrow$, $n \rightarrow$	32	64	128	256	512	1024
1	$3.223e-4$	$8.58e-5$	$2.215e-5$	$5.629e-6$	$1.419e-6$	$3.562e-7$
2^{-2}	$1.126e-3$	$3.194e-4$	$8.538e-5$	$2.21e-5$	$5.622e-6$	$1.418e-6$
2^{-4}	$2.857e-3$	$9.864e-4$	$2.975e-4$	$8.230e-5$	$2.169e-5$	$5.569e-6$
2^{-6}	$3.898e-3$	$1.819e-3$	$7.403e-4$	$2.53e-4$	$7.551e-5$	$2.075e-5$
2^{-8}	$4.078e-3$	$1.971e-3$	$9.73e-4$	$4.585e-4$	$1.868e-4$	$6.366e-5$
2^{-10}	$7.146e-3$	$2.007e-3$	$9.865e-4$	$4.893e-4$	$2.431e-4$	$1.148e-4$
2^{-12}	$5.561e-3$	$3.583e-3$	$9.91e-4$	$4.92e-4$	$2.446e-4$	$1.221e-4$
2^{-14}	$4.172e-3$	$2.778e-3$	$1.794e-3$	$4.92e-4$	$2.450e-4$	$1.223e-4$
2^{-16}	$4.173e-3$	$2.017e-3$	$1.388e-3$	$8.974e-4$	$2.460e-4$	$1.223e-4$
2^{-18}	$4.173e-3$	$2.017e-3$	$9.921e-4$	$6.938e-4$	$4.489e-4$	$1.232e-4$
2^{-20}	$4.174e-3$	$2.017e-3$	$9.922e-4$	$4.921e-4$	$3.468e-4$	$2.245e-4$

Table 2: Maximum pointwise error ($E_{n,\varepsilon}$) for $\delta = \varepsilon/4$ for Example 1

$\varepsilon\downarrow\ n\rightarrow$	32	64	128	256	512	1024
1	$6.779e-4$	$1.957e-4$	$5.282e-5$	$1.374e-5$	$3.504e-6$	$8.850e-7$
2^{-2}	$2.086e-3$	$6.727e-4$	$1.947e-4$	$5.266e-5$	$1.372e-5$	$3.501e-6$
2^{-4}	$3.657e-3$	$1.566e-3$	$5.688e-4$	$1.78e-4$	$5.023e-5$	$1.339e-5$
2^{-6}	$3.969e-3$	$1.949e-3$	$9.388e-4$	$4.04e-4$	$1.457e-4$	$4.513e-5$
2^{-8}	$4.097e-3$	$1.977e-3$	$9.804e-4$	$4.876e-4$	$2.363e-4$	$1.018e-4$
2^{-10}	$7.144e-3$	$2.015e-3$	$9.864e-4$	$4.897e-4$	$2.444e-4$	$1.219e-4$
2^{-12}	$5.562e-3$	$3.583e-3$	$9.933e-4$	$4.921e-4$	$2.446e-4$	$1.222e-4$
2^{-14}	$4.217e-3$	$2.778e-3$	$1.794e-3$	$4.926e-4$	$2.451e-4$	$1.223e-4$
2^{-16}	$4.218e-3$	$2.027e-3$	$1.388e-3$	$8.974e-4$	$2.460e-4$	$1.223e-4$
2^{-18}	$4.218e-3$	$2.028e-3$	$9.948e-4$	$6.938e-4$	$4.488e-4$	$1.232e-4$
2^{-20}	$4.218e-3$	$2.028e-3$	$9.948e-4$	$4.928e-4$	$3.468e-4$	$2.245e-4$

Table 3: Maximum pointwise error ($E_{n,\varepsilon}$) for $\delta = 2\varepsilon/5$ for Example 1

$\varepsilon\downarrow\ n\rightarrow$	32	64	128	256	512	1024
1	$2.203e-3$	$5.726e-4$	$1.459e-4$	$3.684e-5$	$9.254e-6$	$2.319e-6$
2^{-2}	$8.165e-3$	$2.202e-3$	$5.722e-4$	$1.459e-4$	$3.683e-5$	$9.25e-6$
2^{-4}	$2.533e-2$	$7.683e-3$	$2.132e-3$	$5.628e-4$	$1.447e-4$	$3.667e-5$
2^{-6}	$4.720e-2$	$1.918e-2$	$6.561e-3$	$1.960e-3$	$5.340e-4$	$1.415e-4$
2^{-8}	$5.173e-2$	$2.537e-2$	$1.192e-2$	$4.853e-3$	$1.654e-3$	$4.926e-4$
2^{-10}	$5.222e-2$	$2.571e-2$	$1.276e-2$	$6.325e-3$	$2.986e-3$	$1.217e-3$
2^{-12}	$5.235e-2$	$2.577e-2$	$1.278e-2$	$6.368e-3$	$3.178e-3$	$1.580e-3$
2^{-14}	$5.238e-2$	$2.578e-2$	$1.279e-2$	$6.371e-3$	$3.179e-3$	$1.588e-3$
2^{-16}	$5.239e-2$	$2.579e-2$	$1.279e-2$	$7.309e-3$	$3.180e-3$	$1.588e-3$
2^{-18}	$5.239e-2$	$2.579e-2$	$1.279e-2$	$6.372e-3$	$4.127e-3$	$1.588e-3$
2^{-20}	$5.239e-2$	$2.579e-2$	$1.279e-2$	$6.372e-3$	$3.339e-3$	$2.300e-3$

Table 4: Maximum pointwise error ($E_{n,\varepsilon}$) for $\delta = 0.0$ for Example 2

$\varepsilon \downarrow n \rightarrow$	32	64	128	256	512	1024
1	$3.104e-3$	$1.679e-3$	$9.945e-4$	$5.386e-4$	$2.80e-4$	$1.427e-4$
2^{-2}	$1.265e-2$	$3.616e-3$	$1.841e-3$	$1.05e-3$	$5.601e-4$	$2.893e-4$
2^{-4}	$3.556e-2$	$1.229e-2$	$3.715e-3$	$1.245e-3$	$7.217e-4$	$3.90e-4$
2^{-6}	$5.045e-2$	$2.346e-2$	$9.526e-3$	$3.255e-3$	$9.718e-4$	$3.759e-4$
2^{-8}	$5.198e-2$	$2.560e-2$	$1.264e-2$	$5.949e-3$	$2.422e-3$	$8.255e-4$
2^{-10}	$5.229e-2$	$2.574e-2$	$1.277e-2$	$6.361e-3$	$3.160e-3$	$1.492e-3$
2^{-12}	$5.236e-2$	$2.829e-2$	$1.279e-2$	$6.370e-3$	$3.179e-3$	$1.587e-3$
2^{-14}	$5.238e-2$	$2.579e-2$	$1.628e-2$	$6.372e-3$	$3.180e-3$	$1.588e-3$
2^{-16}	$5.239e-2$	$2.580e-2$	$1.445e-2$	$9.164e-3$	$3.486e-3$	$1.588e-3$
2^{-18}	$5.239e-2$	$2.580e-2$	$1.279e-2$	$8.093e-3$	$5.079e-3$	$1.920e-3$
2^{-20}	$5.239e-2$	$2.579e-2$	$1.279e-2$	$6.372e-3$	$4.464e-3$	$2.783e-3$

Table 5: Maximum pointwise error ($E_{n,\varepsilon}$) for $\delta = \varepsilon/4$ for Example 2

$\varepsilon \downarrow \ n \rightarrow$	32	64	128	256	512	1024
1	$7.032e-3$	$4.854e-3$	$2.828e-3$	$1.525e-3$	$7.92e-4$	$4.035e-4$
2^{-2}	$2.10e-2$	$7.302e-3$	$4.774e-3$	$2.747e-3$	$1.476e-3$	$7.656e-4$
2^{-4}	$4.516e-2$	$1.909e-2$	$6.916e-3$	$2.912e-3$	$1.734e-3$	$9.543e-4$
2^{-6}	$5.140e-2$	$2.525e-2$	$1.208e-2$	$5.170e-3$	$1.863e-3$	$8.380e-4$
2^{-8}	$5.214e-2$	$2.568e-2$	$1.274e-2$	$6.335e-3$	$3.064e-3$	$1.319e-3$
2^{-10}	$5.415e-2$	$2.576e-2$	$1.278e-2$	$6.366e-3$	$3.177e-3$	$1.585e-3$
2^{-12}	$5.237e-2$	$3.212e-2$	$1.27915e-2$	$6.371e-3$	$3.179e-3$	$1.588e-3$
2^{-14}	$5.238e-2$	$2.987e-2$	$1.83725e-2$	$7.031e-3$	$3.18e-3$	$1.588e-3$
2^{-16}	$5.239e-2$	$2.579e-2$	$1.69519e-2$	$1.026e-2$	$3.9e-3$	$1.588e-3$
2^{-18}	$5.239e-2$	$2.579e-2$	$1.27948e-2$	$9.401e-3$	$5.647e-3$	$2.133e-3$
2^{-20}	$5.239e-2$	$2.579e-2$	$1.27948e-2$	$6.793e-3$	$5.136e-3$	$3.072e-3$

Table 6: Maximum pointwise error ($E_{n,\varepsilon}$) for $\delta = 2\varepsilon/5$ for example 2

5. Conclusion and Discussion

A two point boundary value problem for a second-order singularly perturbed differential-difference equation with turning point is considered. We develop a numerical scheme based on El-Mistikawy Werle exponential finite difference scheme [21] to solve such type of differential equations. The proposed numerical method is analyzed for consistency, stability, and convergence. The numerical results are tabulated in the tables $1-6$ for the considered examples to support the predicted theory. The graphs of the solutions of the considered examples for different values of delay are plotted in fig $1-2$ to examine the questions on the effect of delay on the interior layer behavior of the solution. It can be seen that as the value of delay argument increases, graph gets more steeper.

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