Exos et TP avec Scilab

Exercice 1

On cherche à évaluer sur ordinateur de façon précise, pour de petites valeurs de x, les deux fonctions suivantes

$$f(x) = \frac{1}{1 - \sqrt{1 - x^2}}.$$

$$f(x) = \frac{1 + \sqrt{1 - x^2}}{x^2}$$

1. Pour x=sqrt(%eps)/2. Calculer f(x) dans les deux cas.
2. Conclure.

Exercice 2

Calcul des racines de $x^2 - 2px + 1$, quand $p \gg 1$ (ex : $p = 10^7$)

Algorithme 1

$$x^+ = p + \sqrt{p^2 - 1}$$

$$x^- = p - \sqrt{p^2 - 1}$$

Algorithme 2

$$x^+ = p + \sqrt{p^2 - 1}$$

$$x^- = 1/(p + \sqrt{p^2 - 1})$$

1. Calculer les deux racines pour les deux algorithmes.
2. Conclure.

Exercice 3

On considère la $u_{n+1} = \alpha u_n + \beta, \quad n = 0, 1, \ldots, \qquad u_0$ donné

1. Trouver l'expression de u_n en fonction de n.
2. Quand est ce que u_n est constante ?
3. Faites un programme scilab qui calcule u_n pour n=0…30.
4. Pour u_0=1/3*(1-delta), avec delta=eps_machine. Montrer que u_n converge vers –Inf.

Exercice 4: erreur de Cancellation

In this example we compare the following two equivalent expressions:

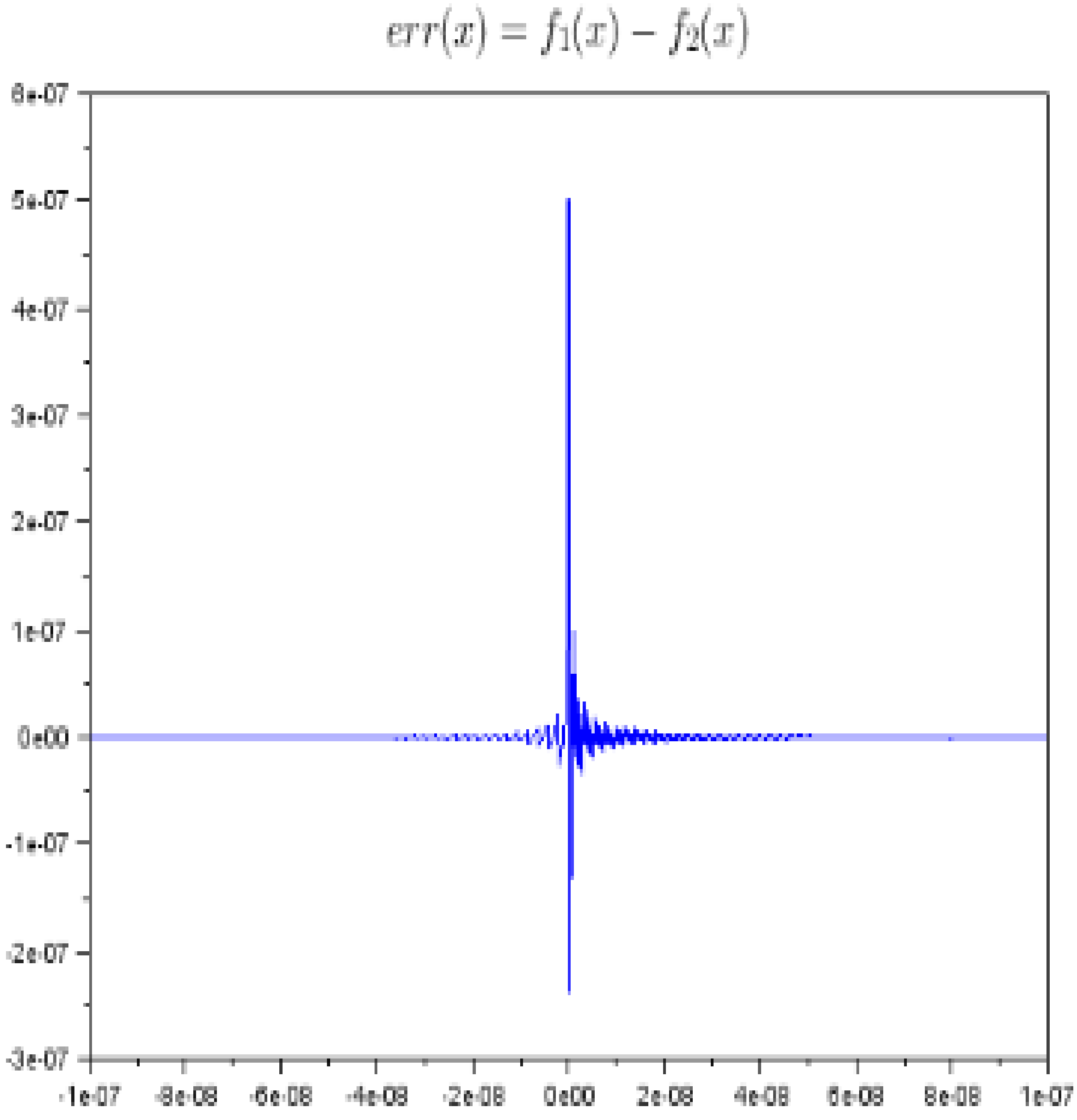
$$f_1(x) = \frac{(1+x)-1}{x}$$

$$f_2(x) = 1$$

These two expressions are equivalent from an algebraic point of view; the second expression is simply obtained from the first one through simplification.

The first function suffers of the cancellation error when x goes to zero. This phenomenon has been explained in the previous step.

In the reported figure we have plotted the error $err = f_1(x) - f_2(x)$. We may see that the cancellation error is huge when x is close to zero. This phenomenon produces a loss of significant digits.



$err(x) = f_1(x) - f_2(x)$
6e-07
5e-07
4e-07
3e-07
2e-07
1e-07
0e00
-1e-07
-2e-07
-3e-07
-1e-07
-8e-08
-6e-08
-4e-08
-2e-08
0e00
2e-08
4e-08
6e-08
8e-08
1e-07

Exercice 5: erreur de Cancellation:

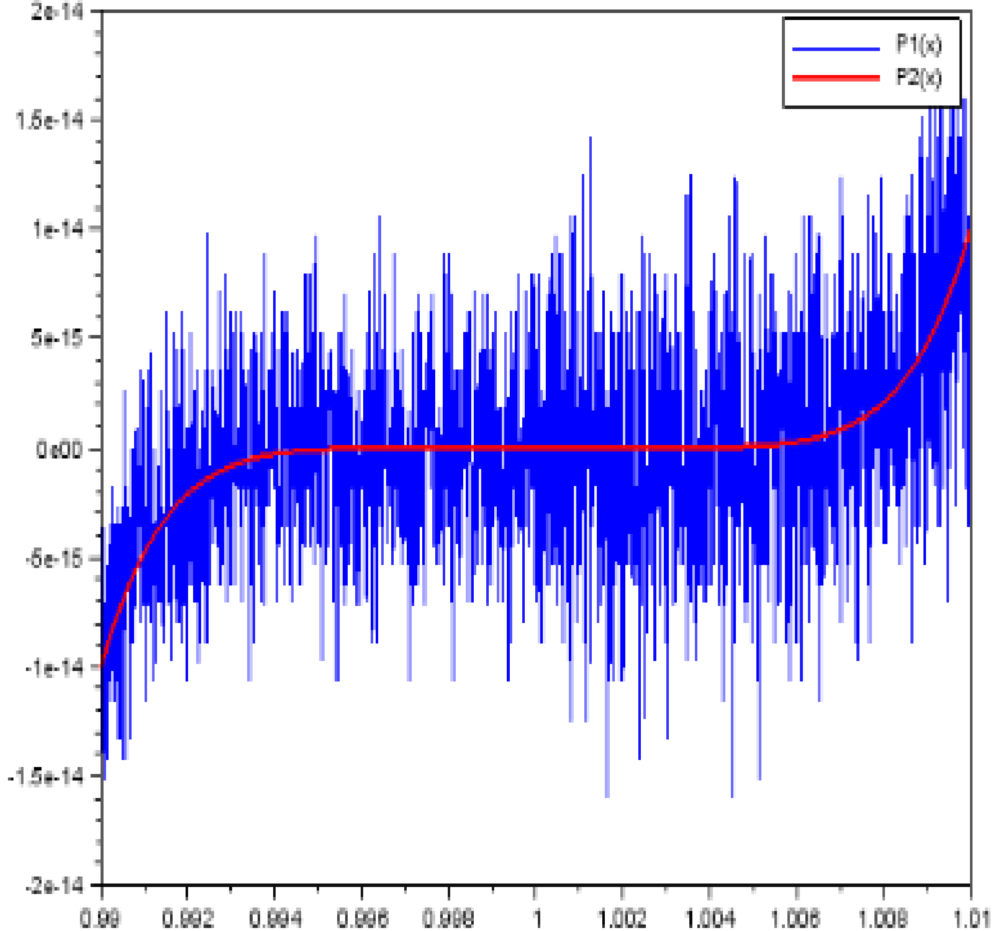
In this example, we compare the two following polynomial expressions:

$$P_1(x) = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$$

$$P_2(x) = (x-1)^7$$

where the first expression is obtained from the second by expansion.

As depicted on the right, the second expression is numerically stable while the first one is more unstable.



P1(x)
P2(x)
2e-14
1.5e-14
1e-14
5e-15
0e00
-5e-15
-1e-14
-1.5e-14
-2e-14
0.99
0.992
0.994
0.996
0.998
1
1.002
1.004
1.006
1.008
1.01

Exercice 6: erreur de Cancellation:

In this step, we study the computation of a derivative from a numerical point of view. To approximate the first derivate, we use here the difference quotient (a.k.a. Newton's quotient):

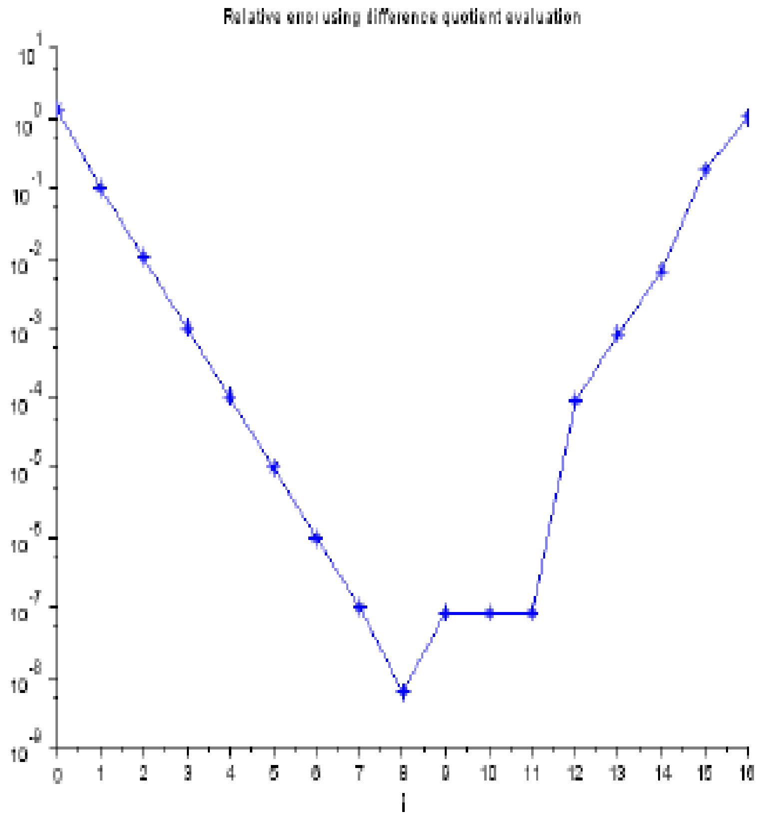
$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For example, we may analyze the formula for the function $f(x) = x^3 + 1$ at point $x = 1$ varying h in the following way:

$$h = 10^0, 10^{-1}, 10^{-2}, \dots, 10^{-16}$$

The plot on the right visualizes the results.

From a mathematical point of view, when h goes to zero, the formula should assume the value of the first derivative. Intuitively, we may expect the smaller the h, the better the approximation. Unfortunately, because of cancellation errors, it happens that the most reliable value for the derivate is reached in $h = 10^{-8}$ (for $h = 10^{-16}$, the value of the derivate is the same as $h = 10$).



i	h	Der. exact	Der. approx	Abs. error.	Rel. error.
0	1.0e+000	3.000000e+000	7.0000000000e+000	4.00000e+000	1.33333e+000
1	1.0e-001	3.000000e+000	3.3100000000e+000	3.10000e-001	1.03333e-001
2	1.0e-002	3.000000e+000	3.0301000000e+000	3.01000e-002	1.00333e-002
3	1.0e-003	3.000000e+000	3.0030010000e+000	3.00100e-003	1.00033e-003
4	1.0e-004	3.000000e+000	3.0003000100e+000	3.00010e-004	1.00003e-004
5	1.0e-005	3.000000e+000	3.0000300001e+000	3.00001e-005	1.00000e-005
6	1.0e-006	3.000000e+000	3.0000029998e+000	2.99980e-006	9.99933e-007
7	1.0e-007	3.000000e+000	3.0000003015e+000	3.01512e-007	1.00504e-007
8	1.0e-008	3.000000e+000	2.9999999818e+000	1.82324e-008	6.07747e-009
9	1.0e-009	3.000000e+000	3.0000002482e+000	2.48221e-007	8.27404e-008
10	1.0e-010	3.000000e+000	3.0000002482e+000	2.48221e-007	8.27404e-008
11	1.0e-011	3.000000e+000	3.0000002482e+000	2.48221e-007	8.27404e-008
12	1.0e-012	3.000000e+000	3.0002667017e+000	2.66702e-004	8.89006e-005
13	1.0e-013	3.000000e+000	2.9976021665e+000	2.39783e-003	7.99278e-004
14	1.0e-014	3.000000e+000	3.0198066270e+000	1.98066e-002	6.60221e-003
15	1.0e-015	3.000000e+000	3.5527136788e+000	5.52714e-001	1.84238e-001
16	1.0e-016	3.000000e+000	0.0000000000e+000	3.00000e+000	1.00000e+000

Exercice 7: erreur de Cancellation:

It is possible to compute the second derivatives of a function using the following approximation

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Apply the above formula to compute the second derivative of the function $f(x) = \cos(x)$ at a given point $x = \frac{1}{2}$.

Moreover, prove the following estimate for the total error:

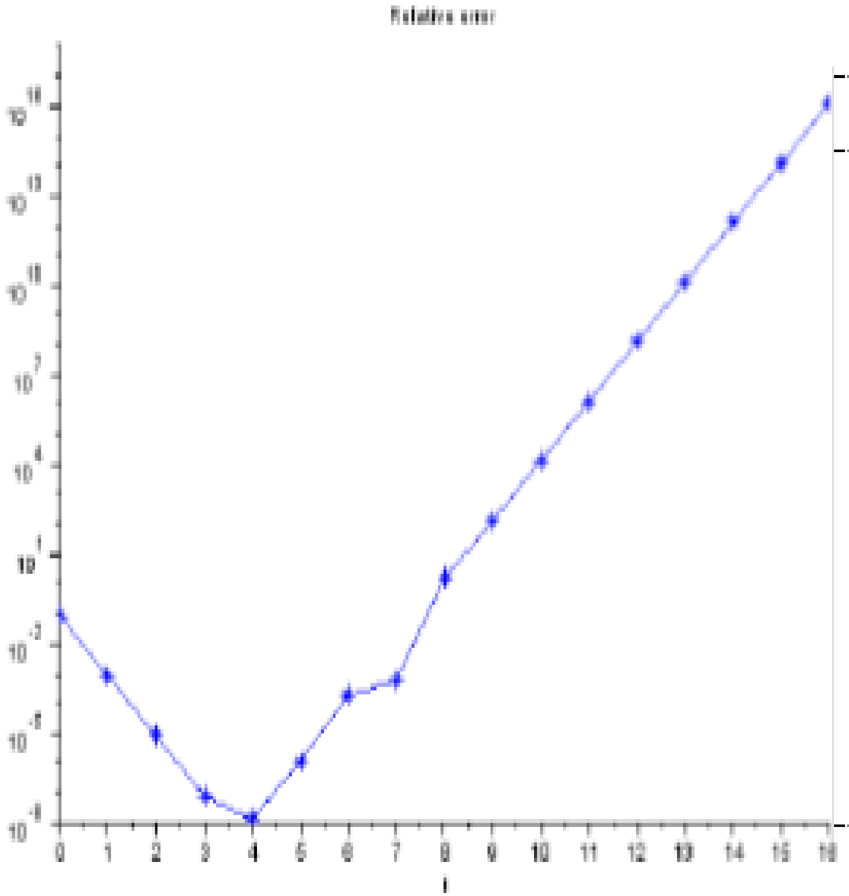
$$\left| f''(x) - \frac{fl\big(f(x+h)\big) - fl\big(2f(x)\big) + fl\big(f(x-h)\big)}{h^2} \right| \leq \frac{3\,\bar{\varepsilon}}{h^2} M_1 + \frac{h^2}{12} M_2$$

where $M_1 = \max_{x\in[x-h,x+h]} |f(x)|$ and $M_2 = \max_{x\in[x-h,x+h]} \left|f^{(iv)}(x)\right|$.

The upper bound error is minimized when h has the value

$$h^* = \sqrt[4]{\frac{36\,\bar{\varepsilon}\,M_1}{M_2}}$$

Hints: Using Taylor expansion for the function $f(x+h)$ and $f(x-h)$ to the fourth order paying attention to the signs.



i	h	Der. exact	Der. approx	Abs. error.	Rel. error.
0	1.0e+000	-8.775826e-001	-8.0684536022e-001	7.07372e-002	8.06046e-002
1	1.0e-001	-8.775826e-001	-8.7685148682e-001	7.31075e-004	8.33056e-004
2	1.0e-002	-8.775826e-001	-8.7757524873e-001	7.31316e-006	8.33330e-006
3	1.0e-003	-8.775826e-001	-8.7758248890e-001	7.29897e-008	8.31713e-008
4	1.0e-004	-8.775826e-001	-8.7758257328e-001	1.13873e-008	1.29757e-008
5	1.0e-005	-8.775826e-001	-8.7758356138e-001	9.99486e-007	1.13891e-006
6	1.0e-006	-8.775826e-001	-8.7774232327e-001	1.59761e-004	1.82047e-004
7	1.0e-007	-8.775826e-001	-8.7707618945e-001	5.06372e-004	5.77008e-004
8	1.0e-008	-8.775826e-001	-2.2204460493e+000	1.34286e+000	1.53018e+000
9	1.0e-009	-8.775826e-001	-1.1102230246e+002	1.10145e+002	1.25509e+002
10	1.0e-010	-8.775826e-001	-1.1102230246e+004	1.11014e+004	1.26499e+004
11	1.0e-011	-8.775826e-001	-1.1102230246e+006	1.11022e+006	1.26509e+006
12	1.0e-012	-8.775826e-001	-1.1102230246e+008	1.11022e+008	1.26509e+008
13	1.0e-013	-8.775826e-001	-1.1102230246e+010	1.11022e+010	1.26509e+010
14	1.0e-014	-8.775826e-001	-1.1102230246e+012	1.11022e+012	1.26509e+012
15	1.0e-015	-8.775826e-001	-1.1102230246e+014	1.11022e+014	1.26509e+014
16	1.0e-016	-8.775826e-001	-1.1102230246e+016	1.11022e+016	1.26509e+016

Exercice 8

On considère la suite récurrente suiv

$$\begin{cases} u_0 & = & 2 \\ u_1 & = & -4 \\ u_n & = & 111 - \dfrac{1130}{u_{n-1}} + \dfrac{3000}{u_{n-1}u_{n-2}}. \end{cases}$$

1. Montrer que u_n converge vers 6.
2. Montrer que

$$u_n = \frac{\alpha \cdot 100^{n+1} + \beta \cdot 6^{n+1} + \gamma \cdot 5^{n+1}}{\alpha \cdot 100^n + \beta \cdot 6^n + \gamma \cdot 5^n},$$

3. Ecrire un programme Scilab qui aff la
valeur exacte de u_n .

n	Computed value	Exact value
3	18.5	18.5
4	9.378378378378379	9.3783783783783784
5	7.801152737752179	7.8011527377521613833
6	7.154414480975334	7.1544144809752493535
11	6.2744386627644761	6.2744385982163279138
12	6.2186967691620172	6.2186957398023977883
16	6.1661267427176769	6.0947394393336811283
17	7.2356654170119432	6.0777223048472427363
18	22.069559154531031	6.0639403224998087553
19	78.58489258126825	6.0527217610161521934
20	98.350416551346285	6.0435521101892688678
21	99.898626342184102	6.0360318810818567800
22	99.993874441253126	6.0298473250239018567
23	99.999630595494608	6.0247496523668478987
30	99.999999999998948	6.0067860930312057585
31	99.999999999999943	6.0056486887714202679

Exercice:*Calcul approché de π

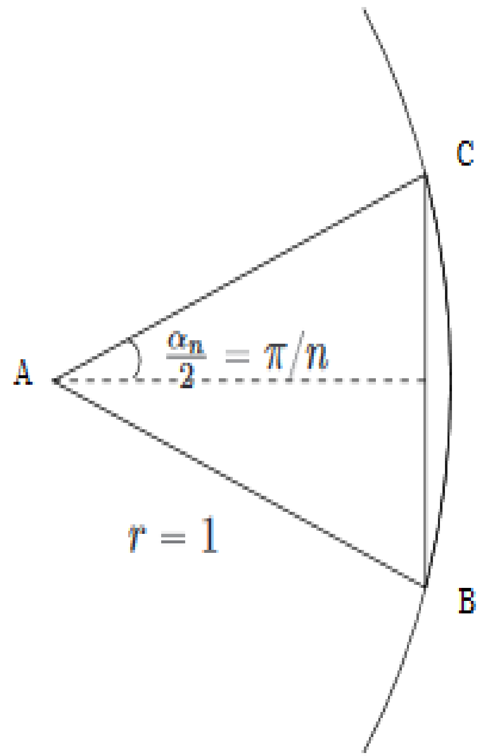
Regardons l'algorithme de calcul par les polygônes inscrits. On considère un cercle de rayon $r = 1$ et on note A_n l'aire associée au polygône inscrit à n côtés. En notant $\alpha_n = \frac{2\pi}{n}$, A_n est égale à n fois l'aire du triangle ABC représenté sur la figure 1.1, c'est-à-dire

$$A_n = n \cos \frac{\alpha_n}{2} \sin \frac{\alpha_n}{2},$$

que l'on peut réécrire

$$A_n = \frac{n}{2}(2 \cos \frac{\alpha_n}{2} \sin \frac{\alpha_n}{2}) = \frac{n}{2} \sin \alpha_n = \frac{n}{2} \sin(\frac{2\pi}{n}).$$

Exercice*: Calcul approché de π



Exercice*: Calcul approché de π

Comme on cherche à calculer π à l'aide de A_n, on ne peut pas utiliser l'expression ci-dessus pour calculer A_n, mais on peut exprimer A_{2n} en fonction de A_n en utilisant la relation

$$\sin\frac{\alpha_n}{2} = \sqrt{\frac{1-\cos\alpha_n}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\alpha_n}}{2}}.$$

Ainsi, en prenant $n = 2^k$, on définit l'approximation de π par récurrence

$$x_k = A_{2^k} = \frac{2^k}{2}s_k, \quad \text{avec } s_k = \sin(\frac{2\pi}{2^k}) = \sqrt{\frac{1-\sqrt{1-s_{k-1}}}{2}}$$

En partant de $k = 2$ (i.e. $n = 4$ et $s = 1$) on obtient l'algorithme suivant :

Algorithm 1.1 Algorithme de calcul de π, version naïve

1: $s \leftarrow 1, n \leftarrow 4$ ▷ Initialisations
2: **Tantque** $s > 1e-10$ **faire** ▷ Arrêt si $s = \sin(\alpha)$ est petit
3: $\quad s \leftarrow sqrt((1-sqrt(1-s*s))/2$ ▷ nouvelle valeur de $\sin(\alpha/2)$
4: $\quad n \leftarrow 2*n$ ▷ nouvelle valeur de n
5: $\quad A \leftarrow (n/2)*s$ ▷ nouvelle valeur de l'aire du polygône
6: **fin Tantque**

Exercice*: Calcul approché de π

1. Montrer que x_n converge vers π.
2. Ecrire un code Scilab qui approche la valeur de π.
3. Conclure.
4. Modifier votre code, en considérant la formule ci-après. Conclure.

$$\sin\frac{\alpha_n}{2} = \sqrt{\frac{1-\sqrt{1-\sin^2\alpha_n}}{2}} = \sqrt{\frac{1-(1-\sin^2\alpha_n)}{2(1+\sqrt{1-\sin^2\alpha_n})}} = \frac{\sin\alpha_n}{\sqrt{2(1+\sqrt{1+\sin^2\alpha_n})}}.$$