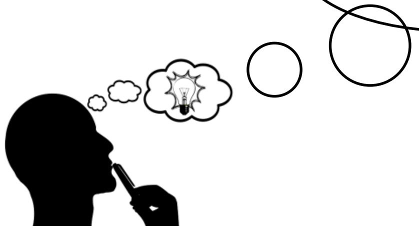
Analyse Numérique 2

Ecole Hassania des Travaux Publics

Salem Nafiri

Approximation des EDOs



Finite difference method

Finite difference method

First Derivative

FORWARD: $dy/dx = (y_{i+1} - y_i)/\Delta x + O(\Delta x)$.

CENTERED: $dy/dx = (y_{i+1} - y_{i-1})/(2 \cdot \Delta x) + O(\Delta x^2)$.

BACKWARD: $dy/dx = (y_i - y_{i-1})/\Delta x + O(\Delta x)$.

Finite difference method

Second Derivative

FORWARD: $d^2y/dx^2 = (y_{i+2}-2 \cdot y_{i+1}+y_i)/(\Delta x^2)+O(\Delta x)$.

CENTERED: $d^2y/dx^2 = (y_{i+1}-2 \cdot y_i+y_{i-1})/(\Delta x^2)+O(\Delta x^2)$.

BACKWARD: $d^2y/dx^2 = (y_i-2 \cdot y_{i-1}+y_{i-2})/(\Delta x^2)+O(\Delta x)$.

Numerical solution for first-order ODE

<u>*Example 1*</u> -- Solve the ODE

$$dy/dx = y \sin(x),$$

with initial conditions $y(0) = 1$, in the interval $0 < x < 5$. Use $\Delta x = 0.5$, or $n = (5\text{-}0)/0.5 + 1 = 11$.

1. *Calculer puis tracer la solution analytique avec la commande ode de scilab.*
2. *Faites un programme de shéma* ***d'Euler implicite*** *qui calule et trace la solution numérique.*
3. *Faites un programme de shéma* ***d'Euler explicite*** *qui calule et trace la solution numérique.*
4. *Conclure.*

Exact solution: the exact is $y(x) = exp(-cos(x))/(cosh(1)-sinh(1))$.

Numerical solution: Using a centered difference formula for dy/dx, i.e.,

$$dy/dx = (y_{i+1} - y_{i-1})/(2\cdot\Delta x),$$

into the ODE, we get $(y_{i+1} - y_{i-1})/(2\cdot\Delta x) = y_i\ sin(x_i)$, which results in the $(n\text{-}2)$ implicit equations:

$$y_{i-1} + 2\cdot\Delta x\cdot sin(x_i)\cdot y_i - y_{i+1} = 0,\ (i = 2, 3, \ldots, n\text{-}1).$$

We already know that

$$y_1 = 1$$

(initial condition), thus we have $(n\text{-}1)$ unknowns left. We still need to come up with an additional equation, which could be obtained by using a forward difference formula for $i = 1$, i.e.,

$$dy/dx|_{x=1} = (y_2 - y_1)/\Delta x = -y_1\ sin(x_1),$$

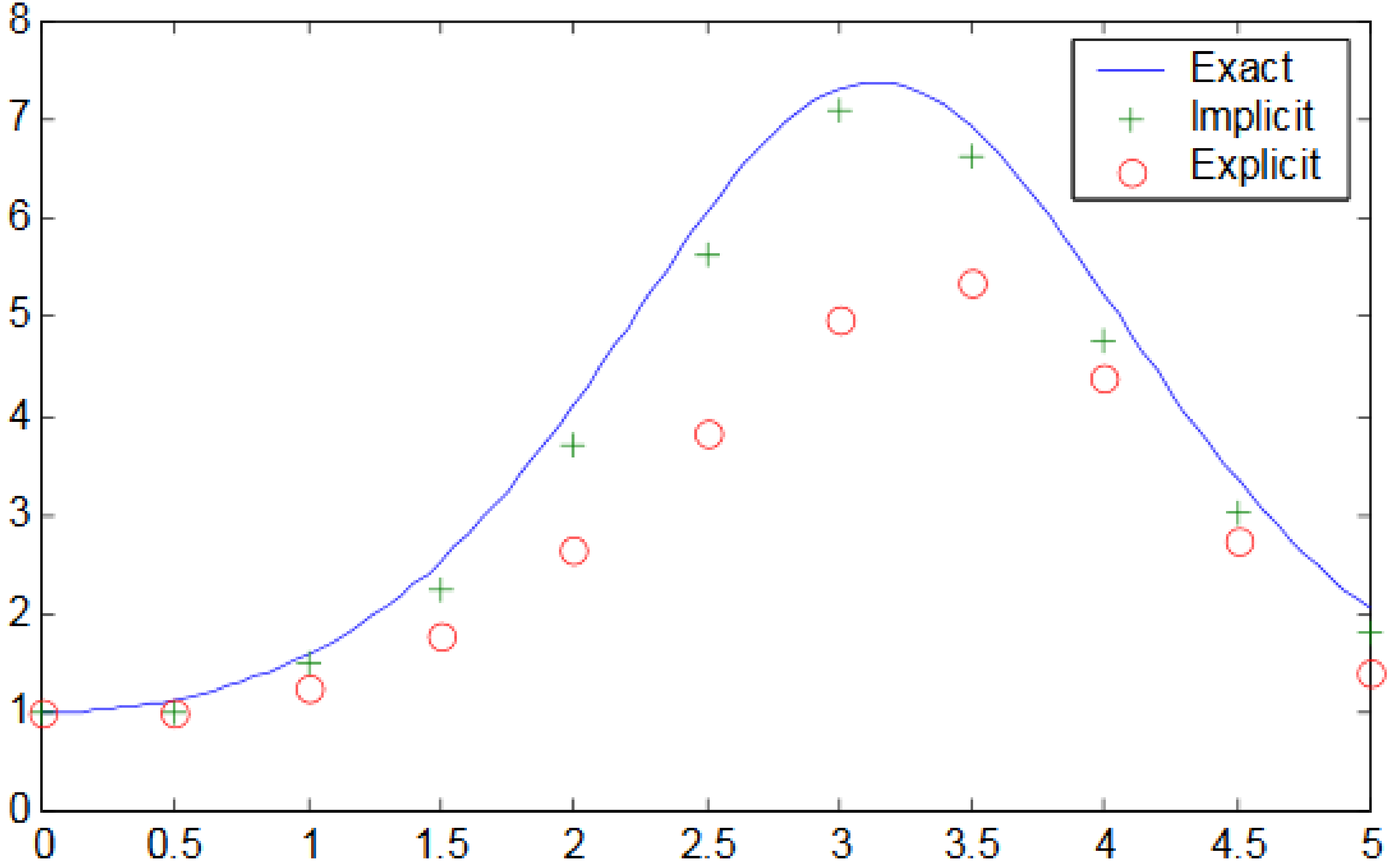
or

$$(1+\Delta x\ sin(x_1))\cdot y_1 - \cdot y_2 = 0.$$

These equations can be written in the form of a matrix equation, for example, for n = 5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1+\Delta x\cdot\sin(x_1) & -1 & 0 & 0 & 0 \\ 1 & 2\cdot\Delta x\cdot\sin(x_2) & -1 & 0 & 0 \\ 0 & 1 & 2\cdot\Delta x\cdot\sin(x_3) & -1 & 0 \\ 0 & 0 & 1 & 2\cdot\Delta x\cdot\sin(x_4) & -1 \end{bmatrix}\cdot\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}=\begin{bmatrix} y0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where *y0* represents the initial condition for y. [**Note**: The data requires $n = 11$. The example for $n = 5$ is presented above to provide a sense of the algorithm to fill out the matrix of data]. The matricial equation can be written as **A·y** = **b**. Matrix A and column vector b can be defined using MATLAB, as indicated below, and the solution found by using left-division. First, we enter the basic data for the problem:



Exact
Implicit
Explicit
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
1
2
3
4
5
6
7
8

Numerical solution for a second-order ODE

For example, to solve the ODE

$$d^2y/dx^2+y = 0,$$

in the x-interval (0,20) subject to $y(0) = 1$, $dy/dx = 1$ at $y = 0$. Use $\Delta x = 0.1$.

First, we discretize the differential equation using the finite difference approximation

$$d^2y/dx^2 = (y_{i+2}-2\cdot y_{i+1}+y_i)/(\Delta x^2)\ ,$$

which results in

$$(y_{i+2}-2\cdot y_{i+1}+y_i)/(\Delta x^2)+y_i = 0.$$

An explicit solution can be obtained from the recurrence equation:

$$y_{i+2} = 2\cdot y_{i+1}-(1+\Delta x^2)\cdot y_i,\ \ i = 1,\ 2,\ ...,\ n\text{-}2;.$$

Numerical solution for a second-order ODE

This equation is based on the two previous values of y_i, therefore, to get started we need the values $y = y_1$, and $y = y_2$. The value y_1 is provided in the initial condition, $y(0) = 1$, i.e.,

$$y_1 = 1.$$

The value of y_2 can be obtained from the second initial condition, $dy/dx = 1$, by replacing the derivative with the finite difference approximation:

$$dy/dx = (y_2 - y_1)/\Delta x,$$

which results in

$$(y_2 - y_1)/\Delta x = 1,$$

or

$$y_2 = y_1 + \Delta x.$$

The x-domain is discretized in a similar fashion as in the previous examples for first derivatives, i.e., by making $x_1 = a$, and $x_n = b$, and computing the values of x_i, $i = 2,3, \ldots n$, with

$$x_i = x_1 + (i-1)\cdot\Delta x = a + (i-1)\cdot\Delta x,$$

where,

$$n = (x_n - x_1)/\Delta x + 1 = (b-a)/\Delta x + 1.$$

The implementation of the solution for this example is left as an exercise for the reader.

We use the same problem from the previous section: solve the ODE

$$d^2y/dx^2+y = 0,$$

in the x-interval (0,20) subject to $y(0) = 1$, $dy/dx = 1$ at $x = 0$. Use $\Delta x = 0.1$.

We discretize the differential equation using the finite difference approximation

$$d^2y/dx^2 = (y_{i+2}-2\cdot y_{i+1}+y_i)/(\Delta x^2)\ ,$$

which results in

$$(y_{i+1}-2^*y_i+y_{i-1})/(\Delta x^2)+y_i = 0.$$

From this result we get the following implicit equations:

$$y_{i-1}-(2-\Delta x^2)\cdot y_i+y_{i+1} = 0,$$

for $i = 2,3, \ldots, n-1$. There are a total of $(n-2)$ equations. Since we have n unknowns, i.e., $y_1, y_2, \ldots, y_n$, we need two more equations to solve a system of linear equations. The remaining equations are provided by the two initial conditions:

From the initial condition, $y(0) = 1$, we can write $y_1 = 1$. For the second initial condition, $dy/dx = 1$, at $x = 0$, we will use a forward difference, i.e.,

$$dy/dx = (y_2 - y_1)/ \Delta x,$$

or

$$y_2 - y_1 = \Delta x.$$

The x-domain is discretized in a similar fashion as in the previous examples. The n equations resulting from discretizing the domain can be written as a matrix equation similar to that of Example 1. Solution to the matrix equation can be accomplished, for example, through the use of left-division for matrices. The implementation of the solution for this example is left as an exercise for the reader.

Systems of ordinary differential equations

To introduce the idea of systems of differential equations we will limit the coverage of the subject to first-order, linear equations with constant coefficients. A system of ordinary differential equations consists of a set of two or more equations with an equal number of unknown functions, $y_1(x)$, $y_2(x)$, etc. As an example consider the following *homogeneous* system:

$$\frac{dy_1}{dx} + 3\,y_1 - 2\,y_2 = 0\,, \quad \frac{dy_2}{dx} - y_1 + y_2 = 0\,.$$

In a homogeneous system the right-hand sides of the equations are zero. The following example represents a *non-homogeneous* system of ordinary differential equations:

$$\frac{dy_1}{dx} + 2\,y_1 - 5\,y_2 = \sin(x)\,, \quad \frac{dy_2}{dx} - 4\,y_1 + 3\,y_2 = \mathrm{e}^x\,.$$

A homogeneous system of ODEs can be written as a single matrix differential equation by using vector functions and a matrix of coefficients as illustrated in the following example. First, we re-write the homogeneous system presented above to read:

$$\frac{dy_1}{dx} = -3\,y_1 + 2\,y_2\,,$$

$$\frac{dy_2}{dx} = y_1 - y_2\,.$$

Then, we define the vector function $\mathbf{f}(x) = [y_1(x)\ y_2(x)]^T$, and the matrix $\mathbf{A} = [-3\ 2;\ 1\ -1]$, and write the differential equation:

$$\frac{d}{dx}\,\mathbf{f}(x) = \mathbf{A}\,\mathbf{f}(x).$$

This result is equivalent to writing:

$$\frac{d}{dx}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}.$$

The non-homogeneous system presented earlier can be re-written as

$$\frac{dy_1}{dx} = -2\,y_1 + 5\,y_2 - \sin(x)\,,$$

$$\frac{dy_2}{dx} = 4\,y_1 - 3\,y_2 + \mathbf{e}^x\,.$$

For this system we will use the same vector function **f**(x) defined earlier, but change the matrix **A** to **A** = [-2 5; 4 -3]. We also need to define a new vector function, **g**(x) = [-sin(x) exp(x)]^T. With these definitions, we can re-write the non-homogeneous system as:

$$\frac{d}{dx}\ \mathbf{f}(\mathrm{x}) = \mathbf{A}\ \mathbf{f}(\mathrm{x}) + \mathbf{g}(\mathrm{x}),$$

or

$$\frac{d}{dx}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} -\sin(x) \\ \exp(x) \end{bmatrix}.$$

Converting 2nd order linear eqns to a system of eqns

A second-order linear ODE of the form $\frac{d^2 y}{dx^2} + \frac{b\,dy}{dx} + c\,y = \mathrm{r}(x)$, can be transformed into a linear system of equations by introducing the relationship, $\mathrm{u}(x) = \frac{dy}{dx}$, so that $\frac{d^2 y}{dx^2} = \frac{du}{dx}$, thus, the equation reduces to $\frac{du}{dx} + b\,u + c\,y = \mathrm{r}(x)$, or $\frac{du}{dx} = -b\,u - c\,y + \mathrm{r}(x)$. The resulting system of equations is:

$$\frac{du}{dx} = -b\,u - c\,y + \mathrm{r}(x),$$

$$\frac{dy}{dx} = u.$$

Which can be written in matricial form as d**f**/dx = **A f**(x)+**g**(x), with

$$f(x) = \begin{bmatrix} u \\ y \end{bmatrix}, A = \begin{bmatrix} -b & -c \\ 1 & 0 \end{bmatrix}, g(x) = \begin{bmatrix} r(x) \\ 0 \end{bmatrix}.$$

For example, the solution to the second order differential equation

$$\frac{d^2 y}{dx^2} + \frac{5\,dy}{dx} - 3\,y = x\,,$$

can be obtained by solving the equivalent first-order linear system:

$$\frac{du}{dx} = -5\,u + 3\,y + x\,,$$

$$\frac{dy}{dx} = u\,.$$

The procedure outlined above to transform a second order linear equation can be used to convert a linear equation of order n into a system of first-order linear equations. For example, if the original ODE is written as:

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{(n-1)} y}{dx^{(n-1)}} + \ldots + a_2\frac{d^2 y}{dx^2} + \frac{a_1\, dy}{dx} + a_0 y = \mathrm{r}(x)\,,$$

we can re-write it as

$$\frac{d^n y}{dx^n} = -a_{n-1}\frac{d^{(n-1)} y}{dx^{(n-1)}} - \ldots - a_2\frac{d^2 y}{dx^2} - \frac{a_1\, dy}{dx} - a_0 y + \mathrm{r}(x)\,,$$

and transform it into a system of n first-order linear equations given by:

$$\frac{du_{n-1}}{dx} = -a_{n-1}\, u_{n-1} - a_{n-2}\, u_{n-2} - \ldots - a_2\, u_2 - a_1 u_1 - a_0 y + \mathrm{r}(x)\,,$$

$$\frac{du_{n-2}}{dx} = u_{n-1}\,, \quad \frac{du_{n-3}}{dx} = u_{n-2}\,, \ldots, \quad \frac{du_1}{dx} = u_2\,, \quad \frac{dy}{dx} = u_1\,,$$

or, in matricial form,

$$\mathbf{f}(x)=\begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_2 \\ u_1 \\ y \end{bmatrix}, \quad A=,\begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad g(x)=\begin{bmatrix} r(x) \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For example, to transform the following fourth-order (*n=4*) linear ODE

$$\frac{d^4 y}{dx^4}+\frac{3\, d^3 y}{dx^3}-\frac{2\, d^2 y}{dx^2}+\frac{5\, dy}{dx}+y=0\,,$$

subjected to $y=1$, $\frac{dy}{dx}=-1$, $\frac{d^2 y}{dx^2}=0$, $\frac{d^3 y}{dx^3}=-1$, at $x=0$, into a first-order linear system, we would write:

$du_3/dx = -3u_3(x)+2u_2(x)-5u_1(x)-y(x)+x^2/2$, $du_2/dx = u_3(x)$, $du_1/dx = u_2(x)$, and $dy/dx = u_1(x)$,

or

$$\frac{d}{dx}\begin{bmatrix} u_3(x) \\ u_2(x) \\ u_1(x) \\ y(x) \end{bmatrix} = \begin{bmatrix} -3 & 2 & -5 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_3(x) \\ u_2(x) \\ u_1(x) \\ y(x) \end{bmatrix} + \begin{bmatrix} x^2/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

with $v(x) = [u_3(x);u_2(x);u_1(x);\, y(x)]^T$, $A = [-3,2,-5,-1;1,0,0,0;0,1,0,0;0,0,1,0]$; and $g(x) = [x^2/;0;\, 0;\, 0]^T$, the system of differential equations is written as $dv/dx = Av+g(x)$. The initial conditions are $y(0) = 1$, $u_1(0) = dy/dx = -1$, $u_2(0) = du_1/dx = d^2y/dx^2 = 0$, $u_3(0) = du_2/dx = d^2u_1/dx^2 = d^3y/dx^3 = -1$, or $u_0 = [\,-1;0;-1;1]$.

Initial-value problems

Any differential equation of the form d**f**/dx = **A** **f**(x)+**g**(x), subject to initial conditions **f**(x_0) = **f**$_0$, is referred to as an *initial-value problem* (IVP). For a constant matrix **A**, the solution can be found using one of several functions available in Matlab. Some of these functions are:

- ode23: IVP solver of order 2 or 3
- ode45: IVP solver of order 4 or 5

- **ode**: Scilab command

Use the following commands to get additional information on these functions:

```
» help ode23

» help ode45
```