Analyse Numérique 2

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Approximation des EDPs

What a PDE is ?

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g.$$

What a PDE is ?

Definition 1.5.5 *We say that the equation (1.40) is elliptic if $b^2 - 4ac < 0$, parabolic if $b^2 - 4ac = 0$, and hyperbolic if $b^2 - 4ac > 0$.*

The origin of this vocabulary is in the classification of conic sections, from which Definition 1.5.5 is copied. Indeed, it is well-known that the second degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

defines a plane curve which is (except in some degenerate cases) an ellipse if $b^2 - 4ac < 0$, a parabola if $b^2 - 4ac = 0$, and a hyperbola if $b^2 - 4ac > 0$.

Some examples of PDEs

Heat equation:

$$\begin{cases} c\dfrac{\partial \theta}{\partial t} - k\Delta\theta = f & \text{for } (x,t) \in \Omega \times \mathbb{R}_*^+ \\ \theta(t,x) = 0 & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \text{for } x \in \Omega \end{cases}$$

Black -Scholes equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - ru + 1/2rx\dfrac{\partial u}{\partial x} + 1/2\sigma^2 x^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,T) \\ u(t=T,x) = \max(x-k,0) & \text{for } x \in \mathbb{R} \end{cases}$$

Some examples of PDEs

Convection-diffusion equation:

$$\begin{cases} c\dfrac{\partial \theta}{\partial t} + cV \cdot \nabla\theta - k\Delta\theta = f & \text{in } \Omega \times \mathbb{R}_*^+ \\ \theta = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \text{in } \Omega \end{cases}$$

Advection equation

$$\begin{cases} c\dfrac{\partial \theta}{\partial t} + cV \cdot \nabla\theta = f & \text{in } \Omega \times \mathbb{R}_*^+ \\ \theta(t,x) = 0 & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_*^+ \text{ if } V(x) \cdot n(x) < 0 \\ \theta(t=0,x) = \theta_0(x) & \text{in } \Omega \end{cases}$$

Some examples of PDEs

Wave equation:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{in } \Omega \times \mathbb{R}_*^+ \\
u = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+ \\
u(t=0) = u_0 & \text{in } \Omega \\
\dfrac{\partial u}{\partial t}(t=0) = u_1 & \text{in } \Omega
\end{cases}
$$

Laplace equation

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

Some examples of PDEs

Schrödinger equation:

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u - Vu = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_*^+ \\ u(t=0) = u_0 & \text{in } \mathbb{R}^N \end{cases}$$

Lamé system

$$\begin{cases} -\mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div} u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Some examples of PDEs

Stokes system:

$$\begin{cases} \nabla p - \mu \Delta u = f & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Plate equation

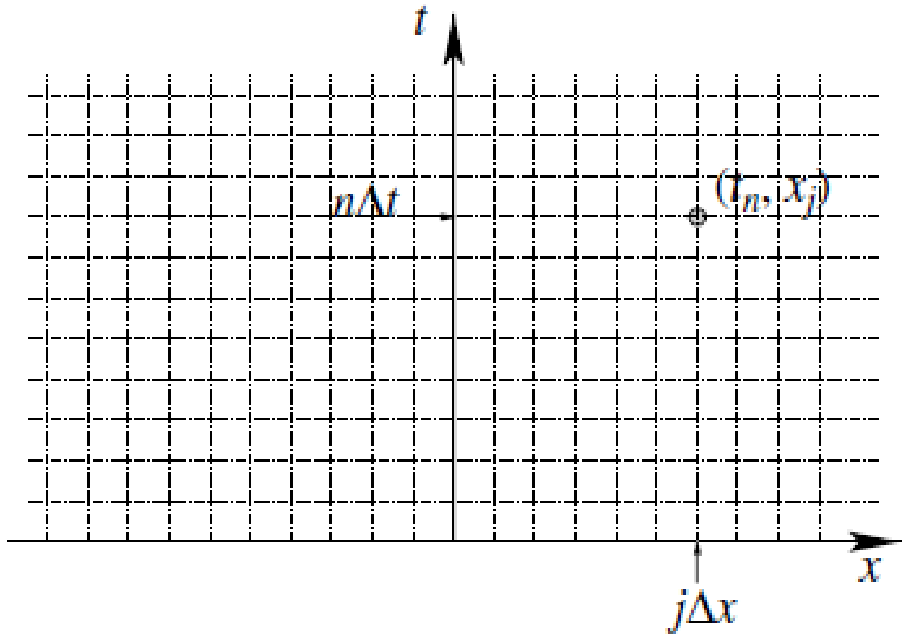
$$\begin{cases} \Delta (\Delta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Numerical scheme of heat flow equation

$$\begin{cases} c\dfrac{\partial\theta}{\partial t} - k\Delta\theta = f & \text{for } (x,t) \in \Omega \times \mathbb{R}_*^+ \\ \theta(t,x) = 0 & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \text{for } x \in \Omega \end{cases}$$

To discretise the spatio-temporal continuum, we introduce a **space step** $\Delta x > 0$ and a **time step** $\Delta t > 0$ which will be the smallest scales represented by the numerical method. We define a mesh or discrete coordinates in space and time (see Figure 1.4)

$$(t_n, x_j) = (n\Delta t, j\Delta x) \quad \text{for } n \geq 0, \;\; j \in \mathbb{Z}.$$



Finite difference mesh

We denote by u_j^n the value of the discrete solution at (t_n, x_j), and $u(t, x)$ the (unknown) exact solution. The principle of the finite difference method is to replace the derivatives by finite differences by using Taylor series in which we neglect the remainders. For example, we approximate the second space derivative (the Laplacian in one dimension) by

$$-\frac{\partial^2 u}{\partial x^2}(t_n, x_j) \approx \frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{(\Delta x)^2}$$

It only remains to do the same thing for the time derivative. Again we have a choice between finite difference schemes: centred or one sided. Let us look at three 'natural' formulas.

As a first choice, the centred finite difference

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$$

A second choice is the one-sided upwind scheme (we go back in time) which gives the **backward Euler scheme**

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^n - u_j^{n-1}}{\Delta t}$$

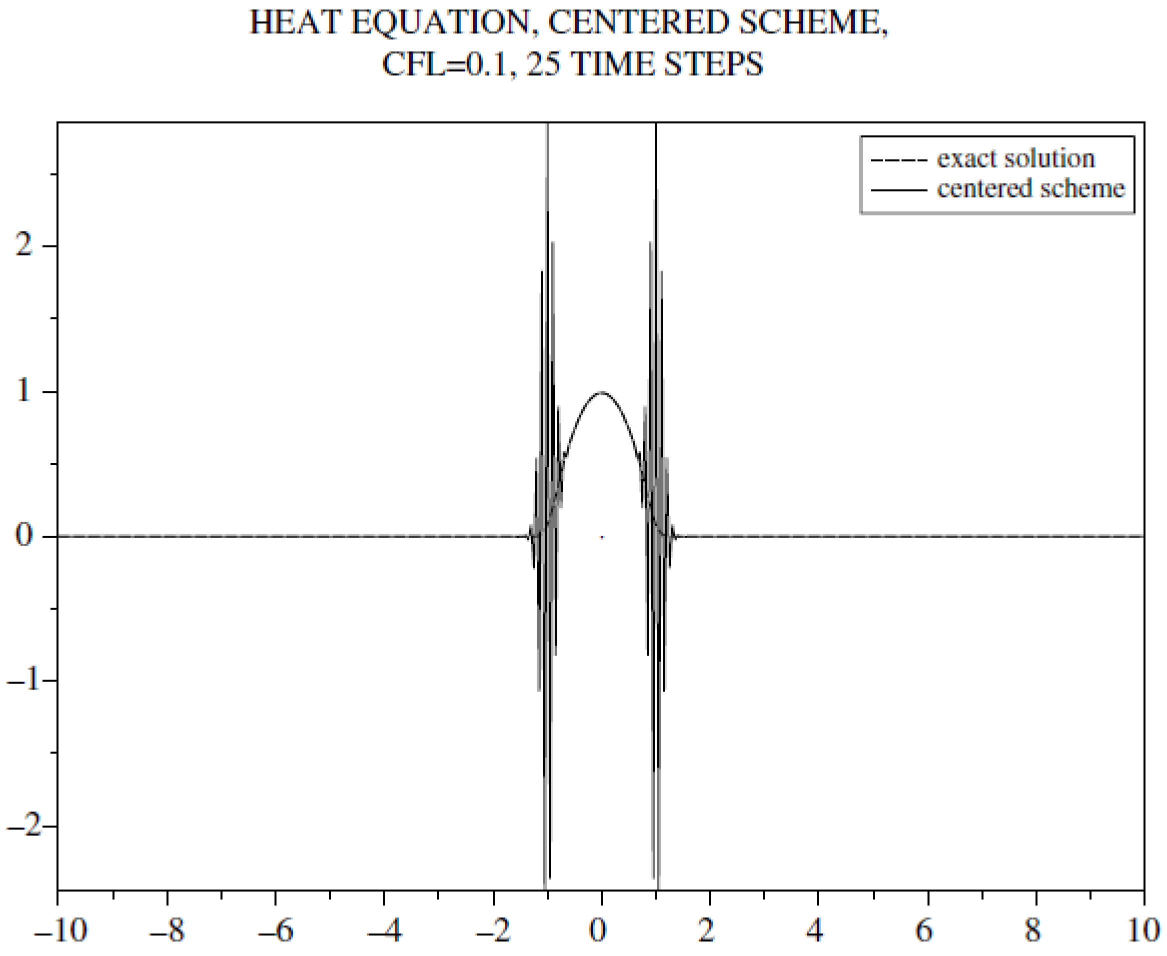
The third choice is the opposite of the preceding: the downwind one-sided finite difference (we go forward in time; we also talk of the **forward Euler scheme**)

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

We start by making some simple numerical tests in the case where $V = 0$ and $\nu = 1$, that is, **we solve the heat flow equation numerically.** As initial condition, we choose the function

$$u_0(x) = \max(1 - x^2, 0).$$

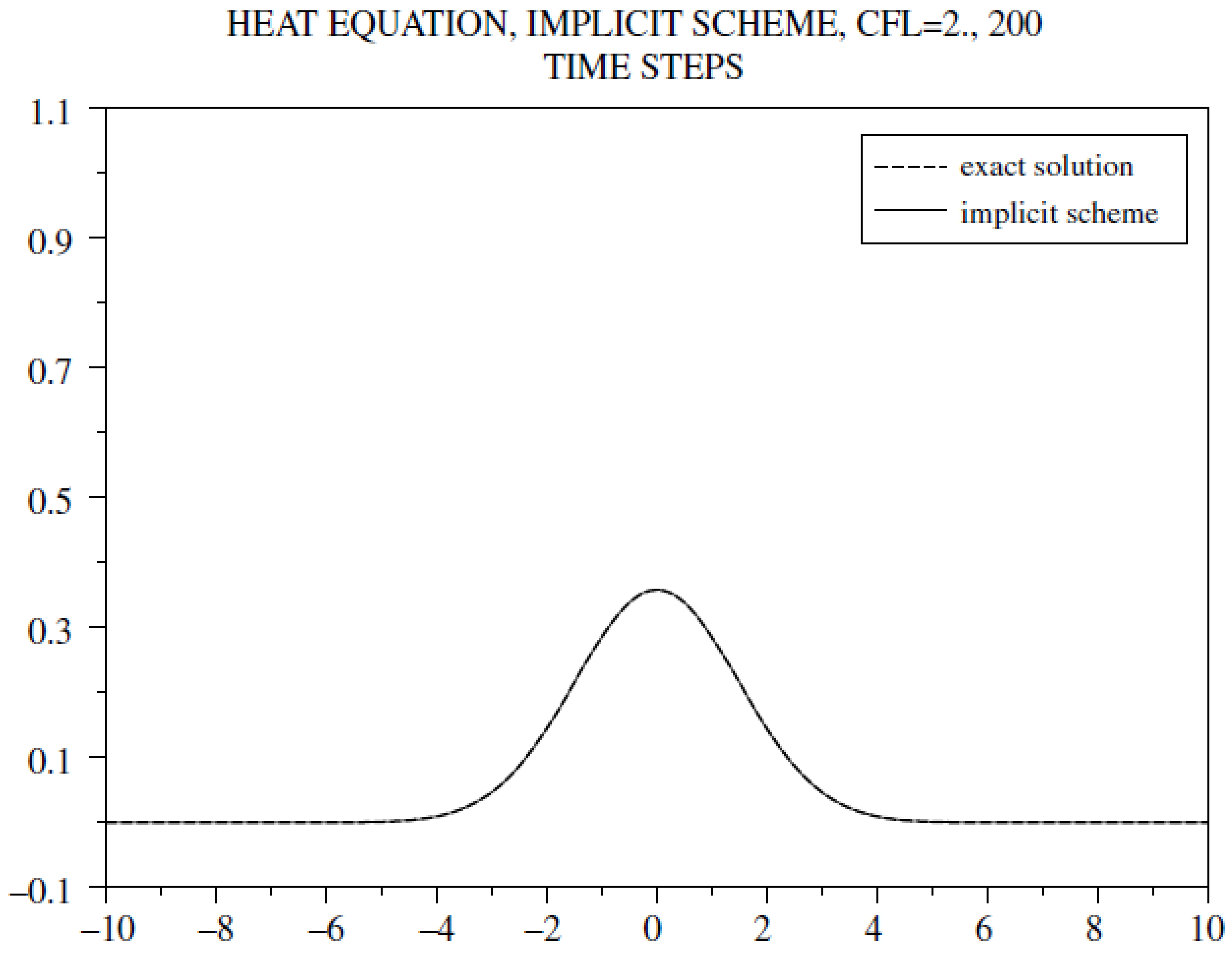
Plot1

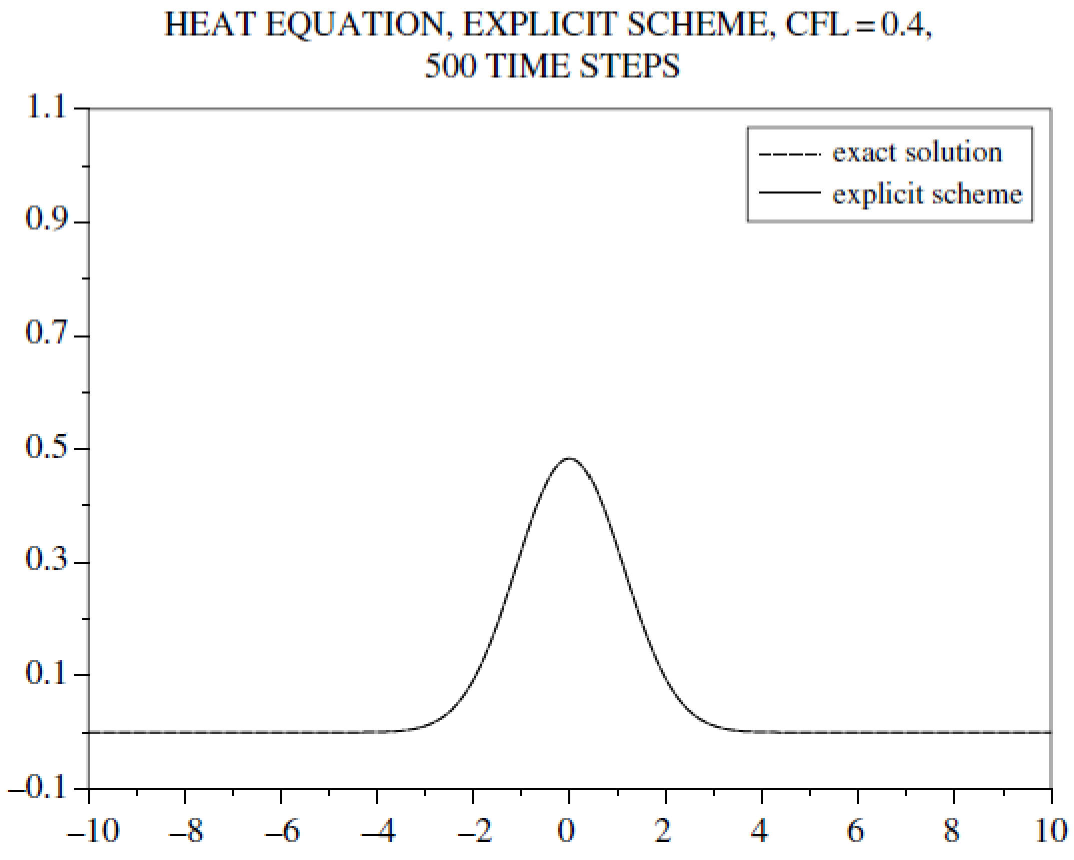


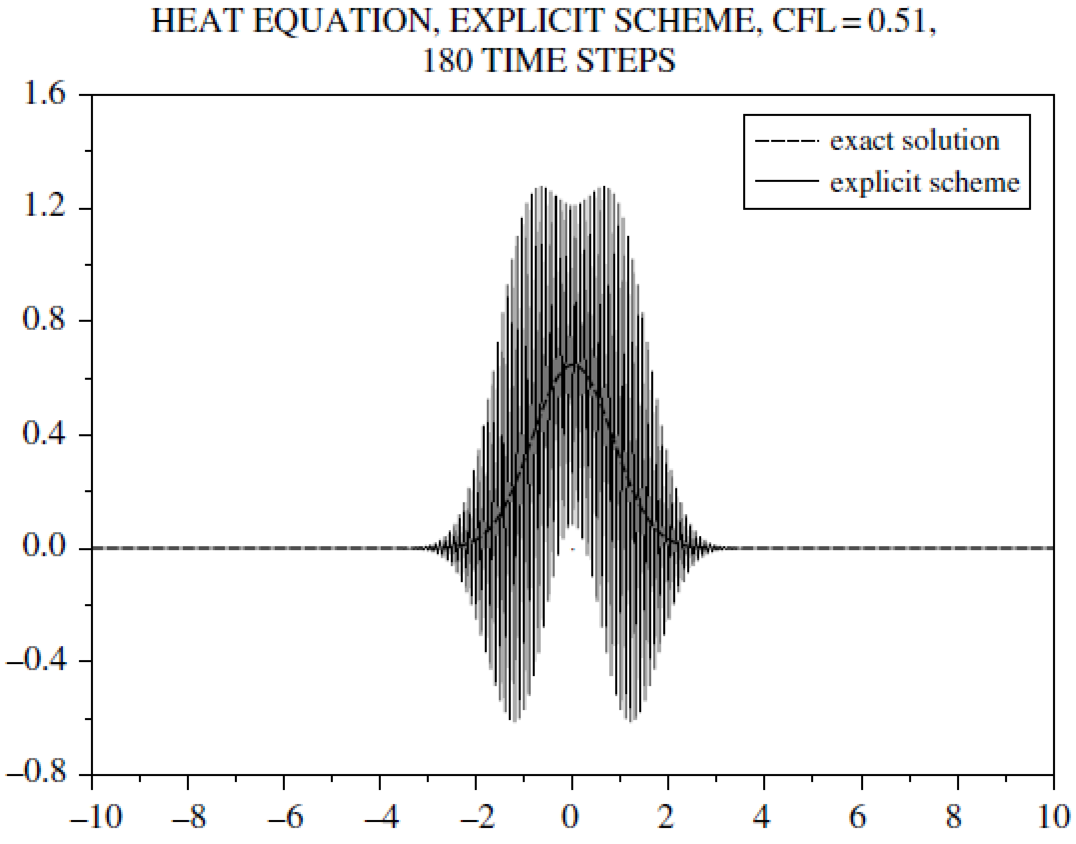
HEAT EQUATION, CENTERED SCHEME,
CFL=0.1, 25 TIME STEPS
exact solution
centered scheme
2
1
0
–1
–2
–10
–8
–6
–4
–2
0
2
4
6
8
10

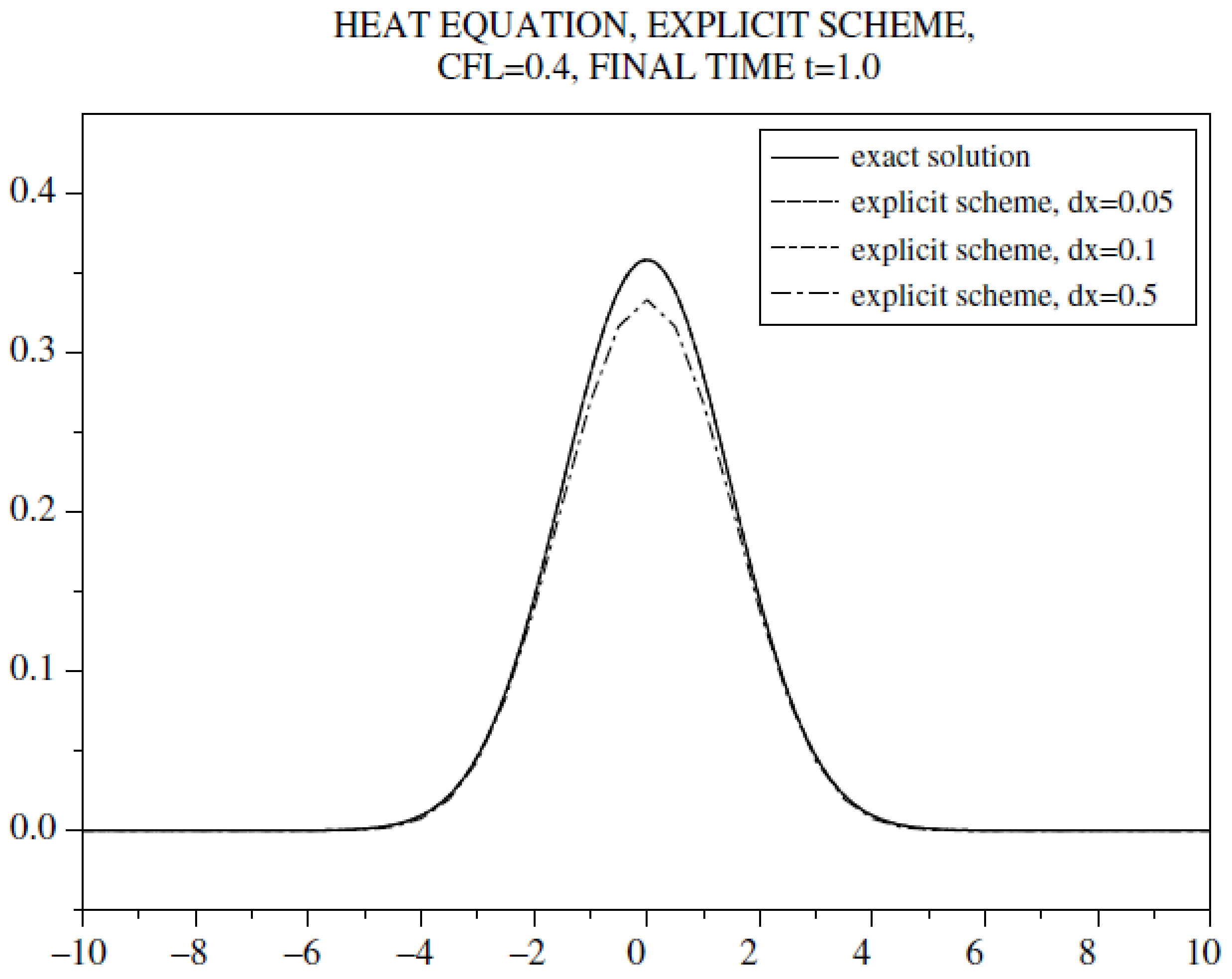
Data

To be able to compare the numerical solutions with the exact (1.14), we want to work on the infinite domain $\Omega = \mathbb{R}$, that is, calculate, for each $n \geq 0$, an infinite number of values $(u_j^n)_{j\in\mathbb{Z}}$, but the computer will not allow this as the memory is finite! To a first approximation, we therefore replace $\mathbb{R}$ by the 'large' domain $\Omega = (-10, +10)$ equipped with Dirichlet boundary conditions. The validity of this approximation is confirmed by the numerical calculations below. We fix the space step at $\Delta x = 0.05$: there are therefore 401 values $(u_j^n)_{-200\leq j\leq +200}$ to calculate. We should remember that the values u_j^n calculated by the computer are subject to rounding errors and are therefore not the exact values of the difference scheme: nevertheless, in the calculations presented here, these rounding errors are completely negligible and are in no way


HEAT EQUATION, IMPLICIT SCHEME, CFL=2., 200
TIME STEPS
exact solution
implicit scheme
1.1
0.9
0.7
0.5
0.3
0.1
–0.1
–10
–8
–6
–4
–2
0
2
4
6
8
10


HEAT EQUATION, EXPLICIT SCHEME, CFL = 0.4,
500 TIME STEPS
exact solution
explicit scheme
1.1
0.9
0.7
0.5
0.3
0.1
–0.1
–10
–8
–6
–4
–2
0
2
4
6
8
10


HEAT EQUATION, EXPLICIT SCHEME, CFL = 0.51,
180 TIME STEPS
exact solution
explicit scheme
1.6
1.2
0.8
0.4
0.0
–0.4
–0.8
–10
–8
–6
–4
–2
0
2
4
6
8
10


HEAT EQUATION, EXPLICIT SCHEME,
CFL=0.4, FINAL TIME t=1.0
exact solution
explicit scheme, dx=0.05
explicit scheme, dx=0.1
explicit scheme, dx=0.5
0.4
0.3
0.2
0.1
0.0
−10
−8
−6
−4
−2
0
2
4
6
8
10

TP Scilab

Heat equation: explicit Euler scheme

- **Heat conduction equation**
- **Diffusion equation**

$$\frac{\partial u(t,x)}{\partial t} = c\frac{\partial^2 u(t,x)}{\partial x^2}$$

- **Initial condition:**

$$u(0,x) = g(x), \quad 0 \le x \le 1$$

- **Boundary conditions:**

$$u(t,0) = \alpha, \quad u(t,1) = \beta, \quad t \ge 0$$

- **Let u_j^m be approximate solution at $x_j = j\Delta x$, $t_m = m\Delta t$**

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = \frac{c}{(\Delta x)^2}(u_{j-1}^m - 2u_j^m + u_{j+1}^m)$$

Explicit Euler Scheme

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = \frac{c}{(\Delta x)^2}(u_{j-1}^m - 2u_j^m + u_{j+1}^m), \quad \mu = \frac{c\Delta t}{(\Delta x)^2}$$

$$\begin{aligned} u_j^{m+1} &= u_j^m + \mu(u_{j-1}^m - 2u_j^m + u_{j+1}^m) \\ &= \mu u_{j-1}^m + (1-2\mu)u_j^m + \mu u_{j+1}^m \end{aligned}$$

Write this equation at every spatial grid:

$$\begin{aligned} u_1^{m+1} &= \mu u_0^m + (1-2\mu)u_1^m + \mu u_2^m \\ u_2^{m+1} &= \mu u_1^m + (1-2\mu)u_2^m + \mu u_3^m \\ &\vdots \\ u_N^{m+1} &= \mu u_{N-1}^m + (1-2\mu)u_N^m + \mu u_{N+1}^m \end{aligned}$$

Explicit Euler Scheme

$$
\begin{aligned}
u_1^{m+1} &= \mu u_0^m + (1-2\mu)u_1^m + \mu u_2^m \\
u_2^{m+1} &= \mu u_1^m + (1-2\mu)u_2^m + \mu u_3^m \\
&\vdots \\
u_N^{m+1} &= \mu u_{N-1}^m + (1-2\mu)u_N^m + \mu u_{N+1}^m
\end{aligned}
$$

In matrix form,

$$
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}^{m+1} = \begin{bmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & & \ddots & & \\ & & & \mu & 1-2\mu \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}^{m} + \begin{bmatrix} \mu u_0^m \\ 0 \\ \vdots \\ \mu u_{N+1}^m \end{bmatrix}
$$