应用数理统计第二章作业答案*

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2.1 (3)

For $\theta > 0$, then $E(X) = \int_0^1 f(x)dx = \int_0^1 x \cdot \theta x^{\theta-1}dx = \frac{\theta}{\theta+1} \neq 1$, which equivalent to $\theta = \frac{E(X)}{1-E(X)}$, hence the moment estimator of θ is $\hat{\theta} = \frac{\bar{x}}{1-\bar{x}}$

2.2 (5)

$$L(a,\lambda) = \prod f(x_i; a, \lambda) = \begin{cases} \lambda^n e^{-\lambda \sum (x_i - a)}, & \min\{x_1, ..., x_n\} \geq a, \\ 0, & \text{else.} \end{cases}$$

For $\min\{x_i\} \geq a$,

$$\ln L(a,\lambda) = n \ln \lambda - \lambda \sum (x_i - a)$$

$$\frac{\partial \ln L(a,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum (x_i - a), \qquad \frac{\partial \ln L(a,\lambda)}{\partial a} = n\lambda > 0$$

Then the maximum likelihood estimation of a is

$$\hat{a} = \min\{x_i\}$$

and through $\frac{\partial \ln L(a,\lambda)}{\partial \lambda} = 0$ we can get

$$\hat{\lambda} = \frac{1}{\bar{x} - \min\{x_i\}}$$ [1]

[1] Consider what will happen if $\bar{x} = \min\{x_i\}$

2.3

$L(p;x_i) = \prod p(1-p)^{x_i-1} = p^n(1-p)^{\sum(x_i-1)}$, and $\ln L(p;x_i) = n\ln p + \sum(x_i-1)\ln(1-p)$ through $\frac{\partial \ln L}{\partial p} = 0$, which equivalent to $\frac{n}{p} - \frac{\sum(x_i-1)}{1-p} = 0$, we can get $\hat{p} = 1/\bar{x}$.

2.6

Since $R = x_{max} - x_{min} = 2.14 - 2.09 = 0.05$, $\hat{\sigma} = \frac{R}{d_5} = 0.4299 \times 0.05 = 0.0215$

By simply grouping in order, we can get

$$\begin{aligned} \text{first:} \quad & 2.14, 2.10, 2.15, 2.13, 2.12, 2.13 \\ \text{second:} \quad & 2.10, 2.15, 2.12, 2.14, 2.10, 2.13 \\ \text{third:} \quad & 2.11, 2.14, 2.10, 2.11, 2.15, 2.10 \end{aligned}$$

Average range is $\bar{R} = (0.05 + 0.05 + 0.05)/3 = 0.05$, then $\hat{\sigma} = \frac{\bar{R}}{d_6} = 0.3946 \times 0.05 = 0.0197$.

2.10

Due to the Poisson distribution of X, we know $E(X) = \lambda$, $Var(X) = \lambda$. And it is easy to demonstrate that $E(\bar{x}) = \lambda$ and $E(s^2) = \lambda$, thus

$$E(\alpha\bar{x} + (1-\alpha)s^2) = \alpha\lambda + (1-\alpha)\lambda = \lambda$$

for all $\alpha \in [0, 1]$.

2.14

Due to the Poisson distribution of X, we know

$$p(x;\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}, \quad x = 0, 1, \ldots$$

The existence of Fisher information can be verified, and

$$\ln p(x;\lambda) = x\ln\lambda - \lambda - \ln(x!), \qquad \frac{\partial}{\partial\lambda}\ln p(x;\lambda) = \frac{x}{\lambda} - 1,$$

hence

$$I(\lambda) = E\left(\frac{x-\lambda}{\lambda}\right)^2 = \frac{1}{\lambda}.$$

Let T is an arbitrary unbiased estimate of θ, where $\theta = g(\lambda) = \lambda^2$, then

$$Var(T) \geq [g'(\lambda)]^2/(nI(\lambda)) = \frac{4\lambda^3}{n}.$$

2.17

Due to $X \sim U(0,\theta)$, $E(X) = \frac{\theta}{2}$, $Var(X) = \frac{\theta^2}{12}$. Thus we can get

$$E(\bar{x}) = \frac{\theta}{2}, \quad Var(\bar{x}) = \frac{\theta^2}{12n} = E(\bar{x}^2) - E(\bar{x})^2,$$

where $E(\bar{x}) = \frac{\theta^2}{12n} + \frac{\theta^2}{4}$ can be deduced immediately. By using the follow expression

$$\begin{aligned} E(2\bar{x} - \theta)^2 &= 4E(\bar{x}^2) - 4\theta \cdot E(\bar{x}) + \theta^2 \\ &= \tfrac{\theta^2}{3n} + \theta^2 - 4\theta \cdot \tfrac{\theta}{2} + \theta^2 \\ &= \tfrac{\theta^2}{3n} \end{aligned}$$

and $\lim\limits_{n\to+\infty} \frac{\theta^2}{3n} = 0$, we know $2\bar{x}$ is the mean square consistent estimate of θ and so is the consistent estimate.

Next the situation of $x_{(n)}$ is similar. Through the density function $f(x)$ and distribution function $F(x)$ of X, we can acquire the density function of $x_{(n)}$, where

$$p_n(x) = nF(x)^{n-1}f(x) = \begin{cases} n\left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} = \frac{nx^{n-1}}{\theta^n}, & x \in (0,\theta); \\ 0, & else. \end{cases}$$

thus

$$E(x_{(n)}) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^{\theta} x \frac{n x^{n-1}}{\theta^n} dx = \frac{n\theta}{n+1}$$

$$E(x_{(n)}^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_0^{\theta} x^2 \frac{n x^{n-1}}{\theta^n} dx = \frac{n\theta^2}{n+2}$$

$$E(x_{(n)} - \theta)^2 = E(x^2) + \theta^2 - 2\theta E(x) = \frac{n\theta^2}{n+2} + \theta^2 - 2\frac{n\theta^2}{n+1}$$

$$\lim_{n\to+\infty} E\left(x_{(n)} - \theta\right)^2 = \lim_{n\to+\infty} \left(\frac{n\theta^2}{n+2} + \theta^2 - 2\frac{n\theta^2}{n+1}\right) = \left(\lim_{n\to+\infty} \frac{\theta^2}{1+\frac{2}{n}} + \theta^2 - 2\frac{\theta^2}{1+\frac{1}{n}}\right) = 0.$$

2.22

Due to the exponential distribution of X, consider the sufficient statistics $\bar{x}$ of parameter $\frac{1}{\lambda}$, where $n\bar{x} = \sum x_i \sim Ga(n, \lambda)$, we can use

$$\mathcal{X}^2 = 2\lambda n\bar{x} \sim \mathcal{X}^2(2n)$$

as test statistics by the nature of gamma distribution.

Let $\alpha = 0.1$, we can immediately acquire

$$P\left(\mathcal{X}^2_{\alpha/2}(2n) \le \mathcal{X}^2 \le \mathcal{X}^2_{1-\alpha/2}(2n)\right) = 1 - \alpha = 90\%,$$

thus the confidence interval, under 90% confidence, of λ is

$$\left[\frac{\mathcal{X}^2_{\alpha/2}(2n)}{2n\bar{x}}, \frac{\mathcal{X}^2_{1-\alpha/2}(2n)}{2n\bar{x}}\right] = [0.00056, 0.00147]$$

though the sufficient statistics $\bar{x}$ of parameter μ, we can similarly get the confidence interval, under 90% confidence, of μ by updating $\mathcal{X}^2 = \frac{2n\bar{x}}{\mu} \sim \mathcal{X}^2(2n)$, where

$$\left[\frac{2n\bar{x}}{\mathcal{X}^2_{1-\alpha/2}(2n)}, \frac{2n\bar{x}}{\mathcal{X}^2_{\alpha/2}(2n)}\right] = [681.6, 1792.3]$$

Completely similar, by using

$$P\left(\mathcal{X}^2_{\alpha}(2n) \leq \mathcal{X}^2\right) = P\left(\mu \leq \frac{2n\bar{x}}{\mathcal{X}^2_{\alpha}(2n)}\right) = 1-\alpha, \quad P\left(\mathcal{X}^2 \leq \mathcal{X}^2_{1-\alpha}(2n)\right) = P\left(\mu \geq \frac{2n\bar{x}}{\mathcal{X}^2_{1-\alpha}(2n)}\right) = 1-\alpha,$$

the upper and lower confidence limits can be acquire as $\dfrac{2n\bar{x}}{\mathcal{X}^2_{\alpha}(2n)}, \dfrac{2n\bar{x}}{\mathcal{X}^2_{1-\alpha}(2n)}$ respectively, which are calculated as 1585.0, 747.7 .

2.29

It is simply to demonstrate $\sum_i^n (x_i - \mu)^2/\sigma^2 \sim \mathcal{X}^2(n)$, which implies

$$P\left(\mathcal{X}^2_{\alpha/2}(n) \le \frac{\sum_i^n (x_i - \mu)^2}{\sigma^2} \le \mathcal{X}^2_{1-\alpha/2}(n)\right) = P\left(\frac{\sum_i^n (x_i - \mu)^2}{\mathcal{X}^2_{1-\alpha/2}(n)} \le \sigma^2 \le \frac{\sum_i^n (x_i - \mu)^2}{\mathcal{X}^2_{\alpha/2}(n)}\right) = 1 - \alpha$$

By chosing $\alpha = 0.05$, we can the confidence interval of λ under 95% confidence is $[0.0242, 0.2829]$.

Similarly, from $(n-1)s^2/\sigma^2 \sim \mathcal{X}^2(n-1)$, we can get

$$P\left(\mathcal{X}^2_{\alpha/2}(n-1) \le \frac{(n-1)s^2}{\sigma^2} \le \mathcal{X}^2_{1-\alpha/2}(n-1)\right) = P\left(\frac{(n-1)s^2}{\mathcal{X}^2_{1-\alpha/2}(n)} \le \sigma^2 \le \frac{(n-1)s^2}{\mathcal{X}^2_{\alpha/2}(n)}\right) = 1 - \alpha$$

and the computation is $[0.0271, 0.4192]$

2.34

It is simply to demonstrate $\frac{\sqrt{n}(\bar{x}-\mu_c)}{s} \sim t(n-1)$, where the $\bar{x}$ and s are computed by the sample of $X - Y$ and $n = 10$, which implies

$$P\left(t_{\alpha/2}(n-1) \le \frac{\sqrt{n}(\bar{x} - \mu_c)}{s} \le t_{1-\alpha/2}(n-1)\right) = P\left(\bar{x} - \frac{t_{1-\alpha/2}(n-1)s}{\sqrt{n}} \le \mu_c \le \bar{x} - \frac{t_{\alpha/2}(n-1)s}{\sqrt{n}}\right) = 1 - \alpha.$$

By chosing $\alpha = 0.05$, we can get the computation is$[-6.2956, 0.7400]$

2.35

Due to $(n-1)s_A^2/\sigma^2 \sim \mathcal{X}^2(n-1)$, $(n-1)s_B^2/\sigma^2 \sim \mathcal{X}^2(n-1)$ and the independence between s_A^2 and s_B^2, the pivot could be designed as

$$F = \frac{s_A^2/\sigma_A^2}{s_B^2/\sigma_B^2} \sim F(n-1, n-1).$$

For a given confidence level α, where $\alpha = 0.05$, there is

$$P\left(F_{\alpha/2}(n-1, n-1) \leq \frac{s_A^2}{s_B^2} \cdot \frac{\sigma_B^2}{\sigma_A^2} \leq F_{1-\alpha/2}(n-1, n-1)\right) = 1-\alpha$$

The following $1-\alpha$ confidence interval of σ_A^2/σ_B^2 can be given by inequality deformation.

$$\left[\frac{\sigma_A^2}{\sigma_B^2} \cdot \frac{1}{F_{1-\alpha/2}(n-1, n-1)}, \ \frac{\sigma_A^2}{\sigma_B^2} \cdot \frac{1}{F_{\alpha/2}(n-1, n-1)}\right] = [0.2217, 3.6008]$$

Similarly, the lower and upper confidence limits can be acquire as

$$\frac{\sigma_A^2}{\sigma_B^2} \cdot \frac{1}{F_{1-\alpha}(n-1, n-1)}, \ \frac{\sigma_A^2}{\sigma_B^2} \cdot \frac{1}{F_{\alpha}(n-1, n-1)}$$

respectively, which are calculated as 0.2810, 2.8413 .