Time-Optimal Path Tracking for Robots
a Convex Optimization Approach

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Outline

1. Introduction
   - Time-optimal motion planning
   - Convex optimization
2. Time-optimal path tracking
   - Problem formulation
   - Existing solution methods
3. Reformulation and solution method
   - Convex reformulation
   - Direct transcription
   - SOCP formulation
   - Implementation
4. Examples
5. Conclusions

Outline

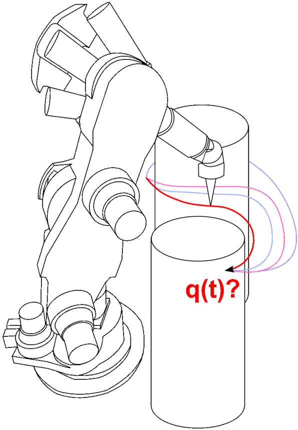
Time-optimal motion planning

Time-optimal motion planning

- important for maximizing the productivity of robot systems.
- reduces need for time-consuming manual programming.

Two approaches

- direct approach.
- decoupled approach (Choset et al. 2005).



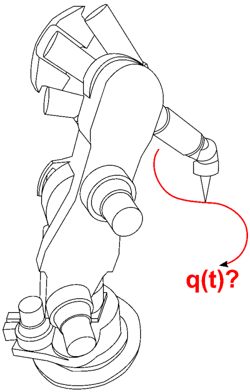
Time-optimal motion planning

The direct approach

- solve motion planning directly in system's state space ($2 \times n$ states).
- complex because of nonlinear robot dynamics.

The decoupled approach

- solve in two stages:
  - path planning: geometric.
  - path tracking: dynamic (2 states).
- easier and computationally cheaper.



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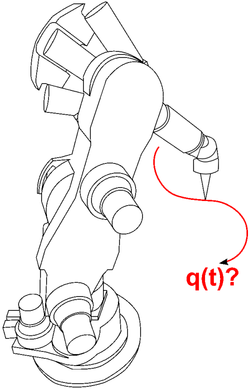

Optimization problems

Optimization problems in engineering

- nonlinear.
- large number of variables.
- optimality of the solution?

Convex optimization problems

"In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."
- Rockafellar

Convex optimization problems

Convex optimization problems

Can be formulated as (Boyd and Vandenberghe 2004):

$$\min_{\mathbf{x}} \; f_o(\mathbf{x}), \tag{1}$$

$$\text{subject to } \mathbf{F}\mathbf{x} = \mathbf{g}, \tag{2}$$

$$f_j(\mathbf{x}) \leq 0, \tag{3}$$

$$\text{for } j = 1 \ldots m.$$

- convex objective function $f_o(\mathbf{x})$.
- linear equality constraints!
- convex nonlinear inequality constraints $f_j(\mathbf{x})$.

Convex optimization problems

Why should we care?

- any local optimum is also globally optimal.
- convergence to optimal value using any nonlinear solver.
- efficient solvers for particular types of convex problems.
- let numerical experts worry about numerical issues.

There is one catch: formulation as a particular type of convex problem is not always obvious.

Types of convex optimization problems

Types of convex optimization problems

- linear program (LP).
- quadratic program (QP).
- second-order cone program (SOCP).
- semidefinite program (SDP).

For numerical efficiency, it is important to use the least general formulation.

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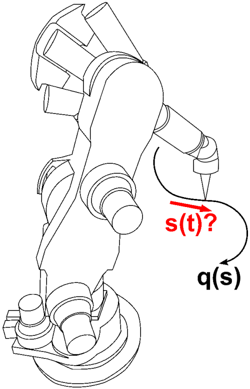
Time-optimal path tracking

Time-optimal path tracking

- equations of motion

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{F}_s(\mathbf{q})\mathrm{sgn}(\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}),$$

- path $\mathbf{q}(s)$, function of path coordinate s.
- lower and upper bounds $\underline{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\tau}}$ on torques.



Problem

Find relation $s(t)$ that minimizes trajectory duration T and satisfies torque constraints.

Time-optimal path tracking

Path $\mathbf{q}(s)$ function of the path coordinate s.

$$\dot{\mathbf{q}}(s) = \mathbf{q}'(s)\dot{s}, \tag{4}$$

$$\ddot{\mathbf{q}}(s) = \mathbf{q}'(s)\ddot{s} + \mathbf{q}''(s)\dot{s}^2, \tag{5}$$

Equations of motion can be rewritten as

$$\boldsymbol{\tau}(s) = \mathbf{m}(s)\ddot{s} + \mathbf{c}(s)\dot{s}^2 + \mathbf{g}(s), \tag{6}$$

where

$$\mathbf{m}(s) = \mathbf{M}(\mathbf{q}(s))\mathbf{q}'(s), \tag{7}$$

$$\mathbf{c}(s) = \mathbf{M}(\mathbf{q}(s))\mathbf{q}''(s) + \mathbf{C}(\mathbf{q}(s), \mathbf{q}'(s))\mathbf{q}'(s), \tag{8}$$

$$\mathbf{g}(s) = \mathbf{F}_s(\mathbf{q}(s))\mathrm{sgn}(\mathbf{q}'(s)) + \mathbf{G}(\mathbf{q}(s)), \tag{9}$$

Time-optimal path tracking

Time-optimal path tracking subject to actuator constraints

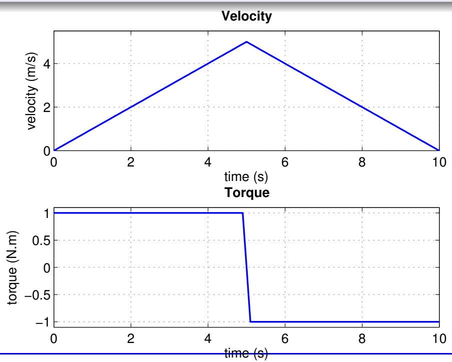
$$
\begin{aligned}
\min_{T, s(\cdot), \boldsymbol{\tau}(\cdot)} \quad & T, && (10)\\
\text{subject to } \quad & \boldsymbol{\tau}(t) = \mathbf{m}(s(t))\ddot{s}(t) + \mathbf{c}(s(t))\dot{s}(t)^2 + \mathbf{g}(s(t)), \\
& s(0) = 0 \text{ and } s(T) = 1, && (11)\\
& \dot{s}(0) = \dot{s}_0 \text{ and } \dot{s}(T) = \dot{s}_T, && (12)\\
& \dot{s}(t) \geq 0, && (13)\\
& \underline{\boldsymbol{\tau}}(s(t)) \leq \boldsymbol{\tau}(t) \leq \overline{\boldsymbol{\tau}}(s(t)), && (14)\\
& \text{for } t \in [0, T].
\end{aligned}
$$

Optimal control problem with 2 states s and $\dot{s}$ instead of $2 \times n$.

Time-optimal path tracking

Structure of the solution

- velocity along the path $\dot{s}$ as high as possible.
- at least one actuator saturated at each point on the path .
- fully determined by "switching points" along the path.



Existing solution methods

Existing solution methods:

Indirect methods

Numerical searches and forward and backward integration to determine "switching points"

- tedious implementation.
- exhaustive searches for candidate switching points.
- not very flexible.

Other methods

- Dynamic programming.
- Direct methods.

Outline

Reformulation

Nonlinear optimal control problem

$\Downarrow$

Convex optimal control problem

$\Downarrow$

Discrete large scale nonlinear optimization problem

$\Downarrow$

Discrete large scale SOCP

Convex reformulation

Convex reformulation:

- objective function can be rewritten as

$$T = \int_0^T 1 dt = \int_{s(0)}^{s(T)} \frac{1}{\dot{s}} ds = \int_0^1 \frac{1}{\dot{s}} ds. \tag{15}$$

- introduce optimization variables

$$a(s) = \ddot{s}, \tag{16}$$

$$b(s) = \dot{s}^2 \tag{17}$$

- additional linear constraint

$$b'(s) = 2a(s), \tag{18}$$

Convex reformulation

Convex reformulation

$$
\begin{aligned}
&\min_{a(\cdot),b(\cdot),\boldsymbol{\tau}(\cdot)} \int_0^1 \frac{1}{\sqrt{b(s)}} ds, && (19)\\
&\text{subject to } \boldsymbol{\tau}(s) = \mathbf{m}(s)a(s) + \mathbf{c}(s)b(s) + \mathbf{g}(s), && (20)\\
&\qquad b(0) = \dot{s}_0^2 \text{ and } b(1) = \dot{s}_T^2, && (21)\\
&\qquad b'(s) = 2a(s), && (22)\\
&\qquad b(s) \geq 0, && (23)\\
&\qquad \underline{\boldsymbol{\tau}}(s) \leq \boldsymbol{\tau}(s) \leq \overline{\boldsymbol{\tau}}(s), && (24)\\
&\qquad \text{for } s \in [0, 1].
\end{aligned}
$$

Convex objective function, linear constraints and time is eliminated.

Reformulation

Nonlinear optimal control problem

$\Downarrow$

Convex optimal control problem

$\Downarrow$

Discrete large scale nonlinear optimization problem

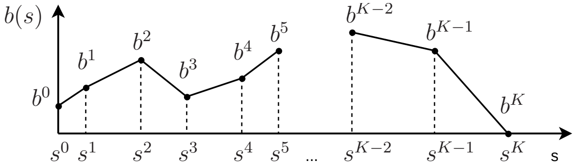
$\Downarrow$

Discrete large scale SOCP

Direct transcription

Direct transcription:

- discretize the path coordinate s on $[0, 1]$.
- discretize $b(s)$, $a(s)$ and $\tau_i(s)$.
- $b(s) = \dot{s}^2$ is assumed to be piecewise linear.



Direct transcription

- objective function can be approximated as

$$\int_0^1 \frac{1}{\sqrt{b(s)}} ds \approx \sum_{k=0}^{K-1} \frac{2\Delta s^k}{\sqrt{b^{k+1}} + \sqrt{b^k}}. \tag{25}$$

- differential constraint $b'(s) = 2a(s)$ is rewritten as

$$(b^{k+1} - b^k) = 2a^k \Delta s^k. \tag{26}$$

- ...

Direct transcription

Large-scale optimization problem

$$\min_{a^k, b^k, \boldsymbol{\tau}^k} \sum_{k=0}^{K-1} \frac{2\Delta s^k}{\sqrt{b^{k+1}} + \sqrt{b^k}},$$

$$\text{subject to } \boldsymbol{\tau}^k = \mathbf{m}(s^{k+1/2})a^k + \mathbf{c}(s^{k+1/2})b^{k+1/2} + \mathbf{g}(s^{k+1/2}),$$

$$b^0 = \dot{s}_0^2 \text{ and } b^K = \dot{s}_T^2, \tag{27}$$

$$(b^{k+1} - b^k) = 2a^k \Delta s^k, \tag{28}$$

$$b^k \geq 0 \text{ and } b^K \geq 0, \tag{29}$$

$$\underline{\boldsymbol{\tau}}(s^{k+1/2}) \leq \boldsymbol{\tau}^k \leq \overline{\boldsymbol{\tau}}(s^{k+1/2}), \tag{30}$$

$$\text{for } k = 0 \dots K-1.$$

Reformulation

Nonlinear optimal control problem

$\Downarrow$

Convex optimal control problem

$\Downarrow$

Discrete large scale nonlinear optimization problem

$\Downarrow$

Discrete large scale SOCP

SOCP formulation

Second-order cone program

Standard form (Boyd and Vandenberghe 2004):

$$\min_{\mathbf{x}} \ \mathbf{f}^T\mathbf{x}, \tag{31}$$

$$\text{subject to } \mathbf{F}\mathbf{x} = \mathbf{g}, \tag{32}$$

$$\|\mathbf{M}_j\mathbf{x} + \mathbf{n}_j\|_2 \leq \mathbf{p}_j^T\mathbf{x} + q_j, \tag{33}$$

$$\text{for } j = 1 \ldots m.$$

Linear objective and equality constraints, and second-order cone inequality constraints.

SOCP formulation

SOCP formulation:

- introduce variables d^k.
- new linear objective function

$$\sum_{k=0}^{K-1} 2\Delta s^k d^k. \tag{34}$$

- additional inequality constraints

$$\frac{1}{\sqrt{b^{k+1}} + \sqrt{b^k}} \leq d^k. \tag{35}$$

- can be replaced by two equivalent second-order cone constraints by introducing variables c^k.

SOCP formulation

SOCP formulation

$$\min_{a^k, b^k, \boldsymbol{\tau}^k, c^k, d^k} \sum_{k=0}^{K-1} 2\Delta s^k d^k, \tag{36}$$

$$\text{subject to } \boldsymbol{\tau}^k = \mathbf{m}(s^{k+1/2})a^k + \mathbf{c}(s^{k+1/2})b^{k+1/2} + \mathbf{g}(s^{k+1/2}),$$

$$b^0 = \dot{s}_0^2 \text{ and } b^K = \dot{s}_T^2, \tag{37}$$

$$(b^{k+1} - b^k) = 2a^k \Delta s^k, \tag{38}$$

$$\underline{\boldsymbol{\tau}}(s^{k+1/2}) \leq \boldsymbol{\tau}^k \leq \overline{\boldsymbol{\tau}}(s^{k+1/2}), \tag{39}$$

$$\left\| \begin{matrix} 2 \\ c^{k+1} + c^k - d^k \end{matrix} \right\|_2 \leq c^{k+1} + c^k + d^k, \tag{40}$$

$$\left\| \begin{matrix} 2c^k \\ b^k - 1 \end{matrix} \right\|_2 \leq b^k + 1. \tag{41}$$

SOCP formulation

OK, but...

- this sounds all very complicated...
- and why should I care?

Easy and efficient

Once the (complicated) formulation is done, implementation is

- very easy.
- automatically efficient.
- requires almost no optimization knowledge.

Matlab implementation

Matlab implementation using YALMIP (Löfberg 2001):

- objective function $\sum_{k=0}^{K-1} 2\Delta s^k d^k$
  t = sum(2*dvar.*ds);
- constraints

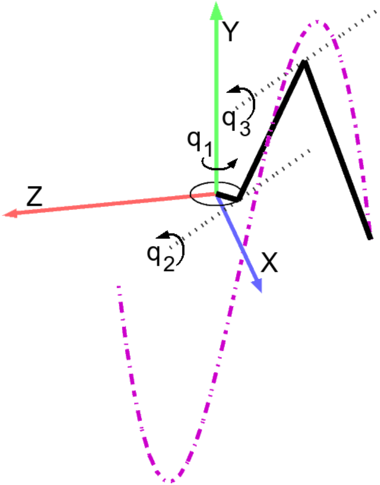
$$\left\| \begin{array}{c} 2c^k \\ b^k - 1 \end{array} \right\|_2 \leq b^k + 1 \qquad (42)$$

  F = F + set(cone([2*cvar(1,k); bvar(1,k)-1], bvar(1,k)+1));
- solve with SeDuMi (Sturm 1999)
  diagnostics = solvesdp(F,t);
- no numerical "hassle": tolerances, scaling, ...
- implementation straightforward, solution automatically efficient.

Outline

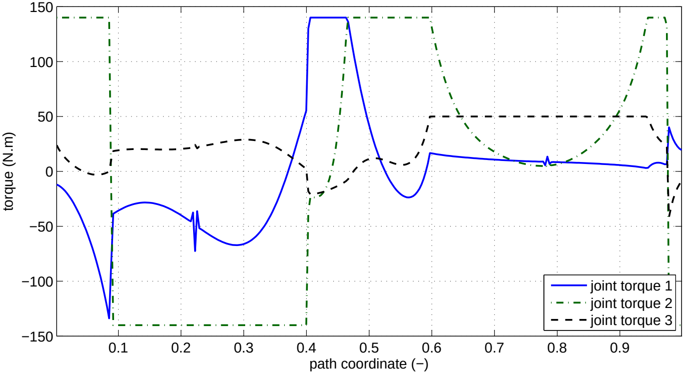
Three-DOF elbow manipulator

Example in (Pfeiffer and Johanni 1987):



Three-DOF elbow manipulator

Torques:



Three-DOF elbow manipulator

Numerical jitter

- because solution is not uniquely defined (singular point).
- major problem for indirect methods. Singular arcs can result in infinitely many "switching points".
- not a problem for the solver.

Undesirable in practice

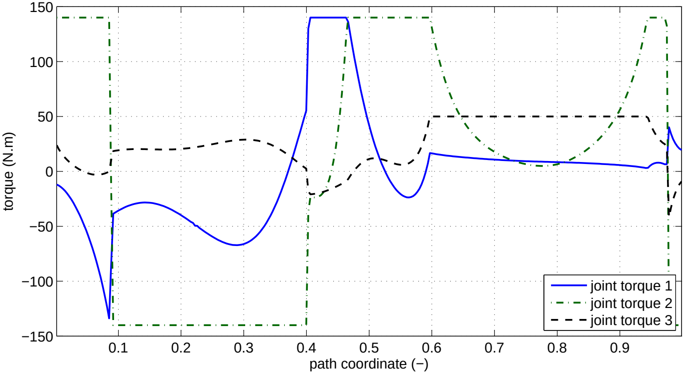
- just add a suitable penalty term to the objective function

$$\int_0^T |\dot{\tau}_i(s)|\, dt = \int_0^1 \frac{|\tau_i'(s)\dot{s}|}{\dot{s}} ds = \int_0^1 \left|\tau_i'(s)\right| ds, \qquad (43)$$

- still SOCP.

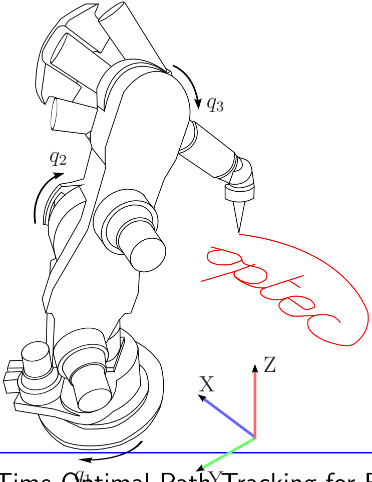
Three-DOF elbow manipulator

Torques with penalization of torque jumps:



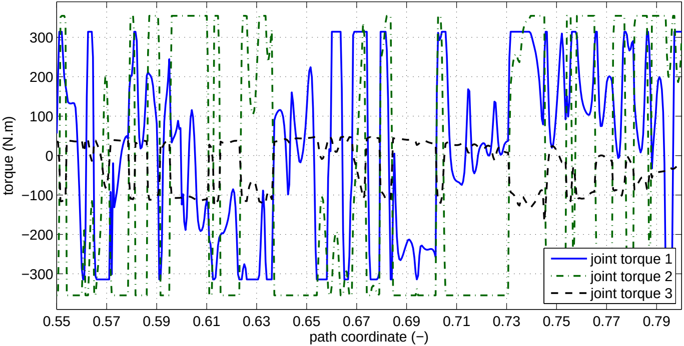
Complicated example

A more complicated example: a six-DOF manipulator carrying out an non-smooth path.



Complicated example

Torques: complicated and a lot of switching!



2000 grid points. Solver time 2.87 s on a Pentium 4 running at 3.60 GHz. (Video)

Complicated example

Time-optimal path tracking

In practice, the pure time-optimal solution is too agressive.

Smoothing

- easy to incorporate penalty terms, such as actuator energy (integral of torques squared).
- still SOCP.
- other constraints and objective functions also possible (though not necessarily SOCP or convex).

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Conclusions

Conclusions:

- convex optimization approach to time-optimal path tracking.
- easy to implement.
- very efficient.
- flexible.

Future work:

- constraints on the rate of change of the torques.
- explore various applications, such as minimization of reaction forces at the robot base.
- integrate with path planning (iterative/sequential)

References

Carried out in the context of OPTEC:

- Optimization in Engineering Center: interdisciplinary research on development of efficient optimization algorithms for engineering applications.
- free matlab implementation available: `http://homes.esat.kuleuven.be/~optec/software/timeopt/`
- D. Verscheure, B. Demeulenaere, J. Swevers, J. De Schutter, M. Diehl, "Practical Time-Optimal Trajectory Planning for Robots: a Convex Optimization Approach", IEEE Transactions on Automatic Control, submitted

Thank you for your attention

- Thank you for your attention!
- Comments/Suggestions/Questions?

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