1 Basics of Vectors and Matrices

Exercise1.1

要求 $|v_1\rangle$ 和 $|v_2\rangle$ 线性无关，就是要求方程组

$$\begin{pmatrix} x & 2 \\ y & x-y \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

仅存在零解．因此矩阵

$$\begin{pmatrix} x & 2 \\ y & x-y \\ 3 & 1 \end{pmatrix}$$

是必须是满秩的．将其做初等行变换

$$\begin{pmatrix} 3 & 1 \\ 0 & x-6 \\ 0 & 3x-4y \end{pmatrix}$$

所以得到 $|v_1\rangle$ 和 $|v_2\rangle$ 线性无关的条件

$$x \neq 6, \text{且 } 3x \neq 4y$$

Exercise1.2

证明三个向量 $\{|v_i\rangle\}, i = 1,2,3$ 构成 $\mathbb{C}^3$ 的基，只需要证明它们是线性无关的即可．对矩阵

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

做初等行变换得到

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

很明显矩阵是满秩的，说明这组向量是线性无关的．因此向量 $\{|v_i\rangle\}, i = 1,2,3$ 构成 $\mathbb{C}^3$ 的基．

Exercise1.3

$$||\,|x\rangle\,|| = \left[1 + 1 + (2^2 + 1)\right]^{1/2} = \sqrt{7}$$

$$\langle x \,|\, y \rangle = (2-i) - i + (2-i)(2+i) = 7 - 2i$$

$$\langle y \,|\, x \rangle = (2+i) + i + (2-i)(2+i) = 7 + 2i$$

Exercise1.4

$$\begin{aligned}
\langle x \,|\, y \rangle &= \sum_{i=1}^{n} x_i^* y_i \\
&= \sum_{i=1}^{n} (x_i y_i^*)^* \\
&= \left(\sum_{i=1}^{n} y_i^* x_i \right)^* \\
&= \langle y \,|\, x \rangle^*
\end{aligned}$$

Exercise1.5

$$c_1 = \langle e_1 \,|\, v \rangle = \frac{5}{\sqrt{2}}$$

$$c_2 = \langle e_2 \,|\, v \rangle = \frac{1}{\sqrt{2}}$$

Exercise1.6

(1)

$$|e_1\rangle = \frac{1}{3}(-1, 2, 2)^t$$

$$|f_2\rangle = (2, -1, 2)^t$$

$$|e_2\rangle = \frac{1}{3}(2, -1, 2)^t$$

$$|f_3\rangle = (2, 2, -1)^t$$

$$|e_3\rangle = \frac{1}{3}(2, 2, -1)^t$$

(2)

$$c_1 = \langle e_1 \,|\, u \rangle = 3$$

$$c_2 = \langle e_2 \,|\, u \rangle = 6$$

$$c_3 = \langle e_3 \,|\, u \rangle = -3$$

Exercise1.7

$$|e_1\rangle = \frac{|v_1\rangle}{||\,|v_1\rangle\,||} = \frac{\sqrt{3}}{3}(1, i, 1)^t$$

$$|f_2\rangle = (2, 1-i, i-1)^t$$

$$|e_2\rangle = 2\sqrt{2}(2, 1-i, i-1)^t$$

Exercise1.8

$$[(cA)^\dagger]_{jk} = (cA)^*_{kj} = c^* A^*_{kj} = c^*(A^\dagger)_{jk} \to (cA^\dagger) = c^* A^\dagger$$

$$[(A+B)^\dagger]_{jk} = (A+B)^*_{kj} = A^*_{kj}+B^*_{kj} = (A^\dagger)_{jk}+(B^\dagger)_{jk} \to (A+B)^\dagger = A^\dagger+B^\dagger$$

$$\begin{aligned}[(AB)^\dagger]_{jk} = (AB)^*_{kj} &= \Big(\sum_i A_{ki}B_{ij}\Big)^* \\ &= \sum_i A^*_{ki}B^*_{ij} = \sum_i (A^\dagger)_{ik}(B^\dagger)_{ji} \to (AB)^\dagger = B^\dagger A^\dagger \\ &= (B^\dagger A^\dagger)_{jk}\end{aligned}$$

Exercise1.9

$$\det\begin{vmatrix} -\lambda & \frac{1+i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} & -\lambda \end{vmatrix} = 0 \to \lambda^2 = 1$$

A 的本征值为 ± 1．对于 $\lambda_1 = 1$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

再结合 $|v_1|^2 + |v_2|^2 = 1$，得到本征矢

$$|\lambda_1\rangle = \frac{1}{2}(\sqrt{2}, 1-i)^t$$

对于 $\lambda_2 = -1$，得到本征矢

$$|\lambda_2\rangle = \frac{1}{2}(\sqrt{2}, i-1)^t$$

$$\langle\lambda_1 \,|\, \lambda_2\rangle = \frac{1}{4}(2-2) = 0$$

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2| = \frac{1}{4}\begin{pmatrix} 2 & \sqrt{2}(1+i) \\ \sqrt{2}(1-i) & 2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 & -\sqrt{2}(1+i) \\ \sqrt{2}(i-1) & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

使得 A 对角化的酉矩阵为

$$\frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 1-i & i-1 \end{pmatrix}$$

Exercise1.10

问题 (1)：首先从 A 的本征值方程出发

$$A\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle \to \left\langle\lambda_i\right| A^\dagger = \lambda_i^* \left\langle\lambda_i\right| \to -\left\langle\lambda_i\right| A = \lambda_i^* \left\langle\lambda_i\right|$$

两边同时与 $\left|\lambda_i\right\rangle$ 内积

$$-\left\langle\lambda_i \mid A \mid \lambda_i\right\rangle = \lambda_i^* \left\langle\lambda_i \mid \lambda_i\right\rangle \to -\lambda_i = \lambda_i^*$$

显然 λ_i 是纯虚数.

问题 (2)：同样是算符 U 的本征方程

$$U\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle \to \left\langle\lambda_i\right| U^\dagger = \lambda_i^* \left\langle\lambda_i\right|$$

将两个式子结合在一起得到

$$\left\langle\lambda_i \middle| U^\dagger U \middle| \lambda_i\right\rangle = \lambda_i^*\lambda_i \left\langle\lambda_i \mid \lambda_i\right\rangle \to \left\langle\lambda_i \mid \lambda_i\right\rangle = \lambda_i^*\lambda_i \left\langle\lambda_i \mid \lambda_i\right\rangle \to |\lambda|^2 = 1$$

问题 (3)：若 A 是 Hermitian 矩阵，则 A 显然是正规矩阵，本征值为实数. 反之，若 A 的本征值为实数，则

$$(A-\lambda_i)\left|\lambda_i\right\rangle = 0 \to \left\langle\lambda_i\right|(A^\dagger-\lambda_i) = 0$$

再结合对易关系 $[A^\dagger, A] = 0$

$$\left\langle\lambda_i \middle| (A^\dagger-\lambda_i)(A-\lambda_i) \middle| \lambda_i\right\rangle = \left\langle\lambda_i \middle| (A-\lambda_i)(A^\dagger-\lambda_i) \middle| \lambda_i\right\rangle = 0$$

可得到

$$A^\dagger\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle$$

所以 $A^\dagger = A$，A 是 Hermitian 矩阵.

Exercise1.11

将矩阵的基 $\{1,2,3\}$ 轮换为 $\{2,3,1\}$，得到矩阵

$$U' = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

显然这个矩阵具有块对角的形式，其中 2×2 部分是 $i\sigma_x$. 因此本征值分别为 $i, i, -i$，（未归一化的）本征矢分别为

$$(1,0,0)^t, \quad (0,1,1)^t, \quad (0,1,-1)^t$$

而 U 的本征矢需要将基重新轮换回去，并且归一化

$$(0,1,0)^t, \quad \frac{1}{\sqrt{2}}(1,0,1)^t, \quad \frac{1}{\sqrt{2}}(-1,0,1)^t$$

Exercise1.12

已知 H 是 Hermitian 矩阵

$$H\left|\lambda_j\right\rangle = \lambda_j\left|\lambda_j\right\rangle$$

容易证明

$$\begin{aligned}
(I+iH)\left|\lambda_j\right\rangle &= (1+i\lambda_j)\left|\lambda_j\right\rangle \\
(I-iH)\left|\lambda_j\right\rangle &= (1-i\lambda_j)\left|\lambda_j\right\rangle \\
(I+iH)^{-1}\left|\lambda_j\right\rangle &= (1+i\lambda_j)^{-1}\left|\lambda_j\right\rangle \\
(I-iH)^{-1}\left|\lambda_j\right\rangle &= (1-i\lambda_j)^{-1}\left|\lambda_j\right\rangle
\end{aligned}$$

我们考虑 U^{-1} 和 $U^\dagger$ 作用在任意向量上

$$\begin{aligned}
U^{-1}\left|v\right\rangle = (I-iH)(I+iH)^{-1}\left|v\right\rangle &= \sum_j (I-iH)(I+iH)^{-1}\left|\lambda_j\right\rangle\left\langle\lambda_j \mid v\right\rangle \\
&= \sum_j (1-i\lambda_j)(1+i\lambda_j)^{-1}\left|\lambda_j\right\rangle\left\langle\lambda_j \mid v\right\rangle
\end{aligned}$$

$$\begin{aligned}
U^{\dagger}\left|v\right\rangle = (I+iH)^{-1}(I-iH)\left|v\right\rangle &= \sum_j (I+iH)^{-1}(I-iH)\left|\lambda_j\right\rangle\left\langle\lambda_j \mid v\right\rangle \\
&= \sum_j (1+i\lambda_j)^{-1}(1-i\lambda_j)\left|\lambda_j\right\rangle\left\langle\lambda_j \mid v\right\rangle
\end{aligned}$$

显然 $U^{-1}\left|v\right\rangle = U^\dagger\left|v\right\rangle$，所以 U 是酉矩阵.