Medical Image Analysis, handin 1

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1 The Radon transform

Computed tomography (CT) is a technique used to investigate the internal structure of an object. It uses the emission of multiple parallel beams of x-rays that are rotated around the object to project its image along a set of angles. A set of projections can then be used to reconstruct a slice image of the object, and therefore, to get an idea of its internal structure.

The linear projection of an object, for all projection angles and all positions on the detector, is an integral operation called Radon transform. Applying the Radon transform to an image f(x,y), corresponds to compute the sum of the intensity of the pixels (line integral) for each projection angle and the resulting projection is called a sinogram. Each column of the sinogram corresponds to a 1D projection of the image, obtained by placing the x-ray source and the sensor at a certain angle θ. At each angle θ, the object's contrast is detected as the sum of the intensities of the pixel in a straight line, i.e. the line integral. The original image can be reconstructed from the sinogram by performing a filtered back projection (FBP), which is a method to perform the inverse Radon transform.

For a given 2D object function $f(x, y)$, the mathematical formula of the Radon transform can be written as

$$P(t, \theta) = \mathbf{R}\{f(x, y)\} = \int_L f(x, y) dl \tag{1}$$

, where θ is the projection direction, defined as the angle between the x-ray beam $L(p, \theta)$ and the y axis (or between the detector line and the x axis), and p is the position where the x-ray intersect with the detector line (perpendicular to the x-ray beam). On the detector line, defined as the collection of points on the coordinate system where $x \cos\theta + y \sin\theta = p$, we measure the detector function $P(p, \theta)$. $P(p, \theta)$ can be defined as the linear projection of the function $f(x, y)$ under the angle θ and the detector position p (polar coordinates), or in other words, the integral of $f(x, y)$ along the path L.
Changing the coordinate system using (p, q), such that $x \cos\theta + y \sin\theta = p$ and $-x \sin\theta + y \cos\theta = q$, the Radon transform can be written as

$$P(p, \theta) = \int_{-\infty}^{\infty} f(p\cos\theta - q\sin\theta, p\sin\theta + q\cos\theta)\, dq \tag{2}$$

1.1 Simple things first

Fig 1 shows the sinograms of the Shepp-Logan phantom obtained using three different set of angles. In our experiment (Fig 1), one can see that halving the

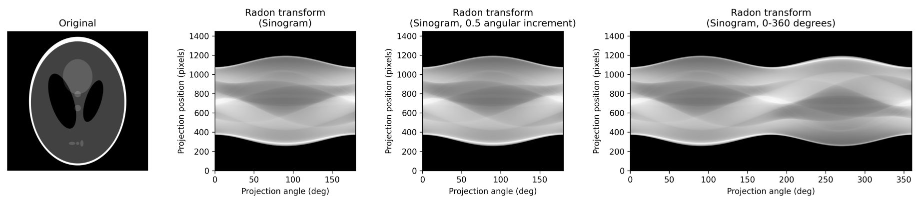


Fig. 1. Shepp-Logan phantom and sinograms. From left to right, the figure shows the original image of the Shepp-Logan phantom, the sinogram obtained performing the random transform with default setting (0-180°, increment of 1), the sinogram obtained by using a smaller angular increment (0-180°, increment of 0.5), and the sinogram obtained using a larger range of angles (0-360°)

angular increment, therefore doubling the number of projection angles, did not result in a drastic better projection. Fig 1 also shows the sinogram obtained using an angular range up to 360 degrees. This sinogram resulted to be symmetric in respect to the y axis, in fact, its left side (up to 179°) correspond to the sinogram obtained using an angular range of 0-179 degrees (second plot from left to right), while the right side corresponds to the same image rotate of 180°. This was expected and it shows that we only need an angular range of 0-179 degrees to obtain a complete projection of an object. In fact, performing the Radon transform with angles of 0° and 180° will correspond to perform the integrals of the pixels intensities along the same line, with the difference that the source and the sensor will be placed at opposite direction.

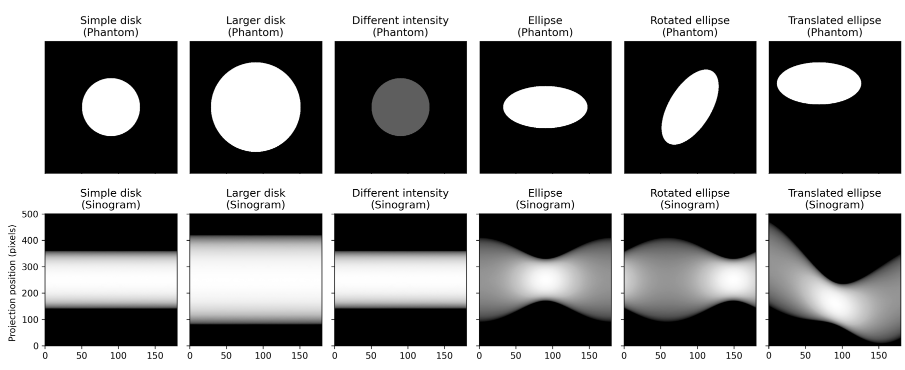
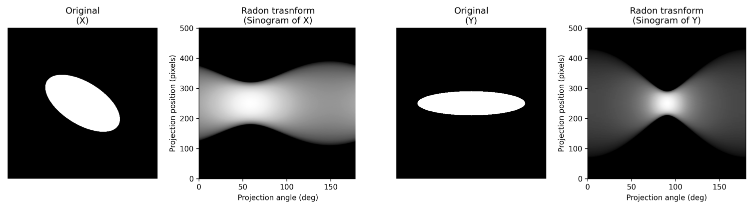
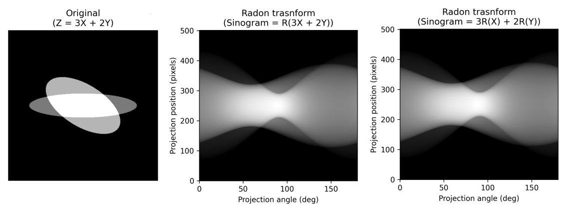


Fig. 2. User-designed phantoms and their sinograms. The first line of plots shows different custom-designed phantoms, the second line shows their associated sinograms.

Fig. 2 shows how the sinogram changes as function of the phantom position, size, orientation and pixels intensity. It is possible to see that applying a Radon



[a.]



[b.]

Fig. 3. Linearity of the Radon transform. a shows two test images x and y and their sinograms. **b** shows the image $z = 3y + 2y$, the sinogram $R(3x + 2y)$, and the sinogram $3R(x) + 2R(y)$.

transform to a phantom with different but uniform pixel intensity did not change the resulting sinogram. Also, we can see that the projection of the translated ellipse is the only sinogram which did not result to be symmetrical. In general, we can see that the thickness of the sinogram increases when the projection angles corresponds to a line in the figure that contains a larger number of white pixels.

1.2 Now on to a bit of mathematics

In order to verify analytically that the Radon transform is a linear transformation we can prove that $R(ax + by) = aR(x) + bR(y)$:

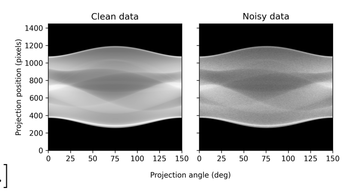
$$\begin{aligned} R(ax + by) &= \int_L ax + by \; dl = \int_L ax \; dl + \int_L by \; dl \\ &= a \int_L x \; dl + b \int_L y \; dl = aR(x) + bR(y) \end{aligned} \tag{3}$$

Furthermore, we can also show that $R[\gamma f(x, y)] = \gamma R[f(x, y)]$:

$$R[\gamma f(x, y)] = \int_L \gamma f(x, y) \; dl = \gamma \int_L f(x, y) \; dl = \gamma R[f(x, y)] \tag{4}$$

The linearity of the Radon transform can also be proved demonstrating numerically that $R(ax+by) = aR(x)+bR(y)$. Fig 3 shows that the sinogram $R(3x+2y)$ seems to be equal to the sinogram $3R(x)+2R(y)$. To provide a numerical proof, the root mean squared error (RMSE) between the two sinograms was computed, obtaining a value of 5.2102e-14. The resulting RMSE proved the linearity of the integral operation, in fact, the small difference between the two sinograms is

[a.]



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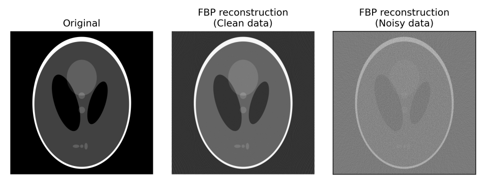


Fig. 4. Noisy and clean data. a shows the sinograms of the clean and noisy data, while **b**, from left to right, shows the original Shepp-Logan phantom, the FBP reconstruction of the clean data, and FBP reconstruction of the noisy data.

supposed to be due to rounding error and interpolation and, therefore, they can be considered equal.

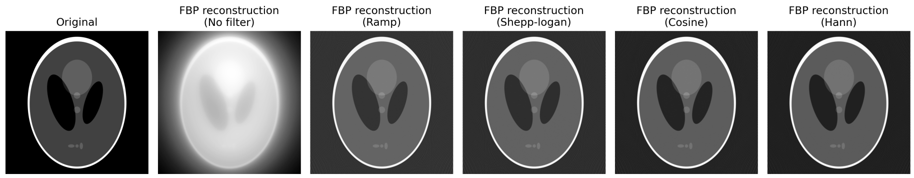
2 The Filtered Back-Projection

The filtered back projection (FBP) is a linear analytic algorithm used in image reconstruction, in particular in parallel-beam tomography, to reconstruct an object image from a set of projections (sinogram). Its mathematical foundation relies in the Fourier slice theorem. It has two phases, filtration and projection. In the filtration phase, the sinogram is Fourier transformed using the fast Fourier transform algorithm (FFT). Since a discrete number of both, angles and beams of x-rays, were used during sampling, interpolation is performed in the Fourier domain to overcome the sampling problem. Also, a high pass filter (e.g. Ram-lak filter) is applied to the Fourier transform of the projection, reducing the over representation of the low frequencies, therefore allowing the reconstruction of a sharp image. After interpolation and filtering, an inverse fast Fourier transform (IFFT) is applied to the spectrum, obtaining a filtered projection. In the second phase, the back projection algorithm is applied to the filtered sinogram, smearing it out over the object domain. The reconstructed image is obtained by projecting back the line integrals onto the plane at their respective angles, or in other words, smearing out the projection over all projection angles and summing up the values from each direction.

Fig 4 shows the clean and noisy projections of the Shepp-Logan phantom, their FBP reconstructions, and the original image. The RMSE between the images was computed resulting in a value of 5.73 for the FBP reconstructions of clean and noisy data, 0.036 for the original image and the FBP reconstruction of clean data, and 5.731 for the original image and the FBP reconstruction of noisy data.

Fig 5 shows the effect of performing FBP with different filters and interpolation methods.

Lastly, Fig 6 shows the linearity of the FBP. In fact, it is possible to observe that the FBP reconstruction of the Shepp-Logan image, obtained as the sum of the reconstruction from parts of the data, is equal to the FBP reconstruction obtained from all data. This was expected because the filtered back projection



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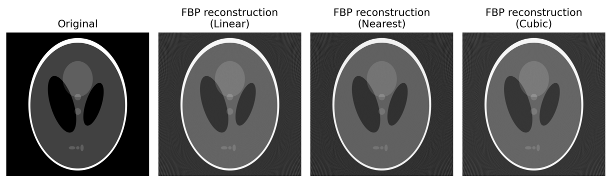
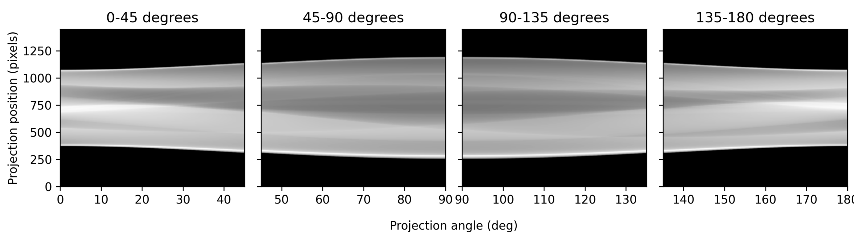


[b.]

Fig. 5. Experiment with FBP reconstruction. The fiure shows the original Shepp-logan phantom and its FBP reconstruction using different filters (**a**) and different interpolation methods (**b**).



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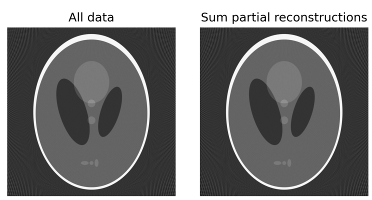

[b.]

Fig. 6. Linearity of FBP. a shows four projections (partial sinograms) obtained by performing the Radon transforms of the Shepp-Logan phantom using four different sets of angles. **b** shows the two FBP reconstructions of the image, the left one is obtained from all data, while the right one is obtained by summing the FBP reconstruction of the four partial sinograms.

is a linear reconstruction algorithm, and a function can be considered as linear only if it can be written as a linear combination of its argument's components. Therefore, the FBP of a sinogram must be equal to the sum of the FBP of the partial sinograms.