Numerical Study of Kakeya Maximal Inequalities

Terry Harris†, Kaiyi Huang†, Tianle Chen†, Daniel Alexander David†, Vivek Saravanan†, Marissa Stolt†

†University of Wisconsin - Madison

Problem Introduction

The Kakeya conjecture is a fundamental problem in geometric measure theory and harmonic analysis. It concerns the minimal possible size of sets that contain a unit line segment in every direction in $\mathbb{R}^n$. One way to approach this problem is through the study of Kakeya maximal function inequalities.

The aim of our project is to find some numerical evidence for the Kakeya maximal function conjecture in 3 dimensions. This conjecture roughly states that a collection of tubes in 3-space of equal radii and separated directions is mostly disjoint. We aimed to find a way to search the parameter space for counterexamples to test whether $\frac{3}{2}$ is the correct exponent on the given bound.

We must test the bound by assigning unique starting positions to each tube carefully because we have also shown that a random sampling approach fails because it cannot achieve the sharpness of the inequality we seek. The sharpness —or worst-case scenario— usually occurs when many tubes overlap significantly within a small region, such as when all of them pass through a tiny ball. Due to the law of large numbers, randomization distributes the tubes more uniformly than we would like, and because the inequality holds with high probability, we don't see the extreme clustering of tubes required to maximize the left-hand side of the inequality. To methodically place the tubes we first implemented a greedy algorithm in our code and have since explored various programming approaches.

Inequality Formula

We want to test if the inequality

$$\int_0^1\int_0^1\int_0^1 \left|\sum_{j=1}^{N} \chi_{T_j}(x,y,z)\right|^{3/2} dx\,dy\,dz \le C_\epsilon \delta^{-\epsilon} N\delta^2, \qquad (1)$$

is true, for all $\epsilon > 0$, for any $\delta > 0$, for any $N \ge 1$, where $\{T_1, \dots, T_N\}$ is any finite set of δ-tubes in $\mathbb{R}^3$ with δ-separated directions. The inequality quantifies how the overlaps of thin tubes in the unit cube are bounded when these tubes point in δ-separated directions.

Specifically, the inequality states that the total measure of the tubes, calculated by integrating the sum of their characteristic functions raised to the $\frac{3}{2}$ power, cannot exceed a constant times $N\delta^2$ up to a small loss $\delta^{-\epsilon}$. If the tubes were completely disjoint and entirely within the unit cube, the left-hand side would be approximately $\pi\delta^2 N$, which matches the right-hand side up to a constant factor. Thus, the inequality suggests that the tubes are "approximately" disjoint in terms of their contribution to the integral, implying that sets containing lines in every direction must occupy a substantial volume.

Converted Problem for Computer Testing

It is equivalent to test (1) with $\delta = 1/n$ where n is any positive integer. Under this assumption, an equivalent version of (1) is, for any vectors $a_1, a_2, \dots, a_N$ in $\mathbb{R}^3$, and any δ-separated unit vectors $v_1, \dots, v_N$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} n^{-3}\left|\left\{\ell : \left|\left(\frac{i}{n},\frac{j}{n},\frac{k}{n}\right) - a_\ell - \left\langle\left(\frac{i}{n},\frac{j}{n},\frac{k}{n}\right) - a_\ell, v_\ell\right\rangle v_\ell\right| < \frac{1}{n}\right\}\right|^{3/2} \le C_\epsilon N n^{\epsilon-2}. \qquad (2)$$

By discretizing the inequality we represent it as a finite sum over grid points in the unit cube, making it more accessible for computation and analysis. This discrete form helps us explore the behavior as N becomes large, allowing us to investigate specific tube configurations that might challenge the inequality more than random placements. While random tube arrangements may satisfy the inequality with high probability, programming enables us to test edge cases and worst-case scenarios, providing a more rigorous validation of the inequality.

Greedy Algorithm

1. We choose some n (for example: $n = 10$).Let $N = n^2$ and $\epsilon = 0.1$, using which we calculate all $v_{\ell,j} = \left(v_{\ell,1}, v_{\ell,2}, \sqrt{1 - v_{\ell,1}^2 - v_{\ell,2}^2}\right)$, $v_{\ell,1} = \frac{i}{2n}, v_{\ell,2} = \frac{j}{2n}$ and, every a_ℓ of the form $a_\ell = (x_\ell, y_\ell, 0)$, $0 \le x_\ell, y_\ell \le 1, x_\ell, y_\ell \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$
2. Starting with $a_0 = (0,0,0)$, and $v_0 = (0,0,1)$ we iterate through every a_ℓ and compute the left hand side of equation (2) to find the a_ℓ that maximizes the equation. We achieve this by iterating through points $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ in the unit cube and computing the number of tubes that pass through that point.

The greedy algorithm with reasonable parameters always produced the 'ball example'. The ball example shows the sharpness of the power, $p = \frac{3}{2}$ in equation(2). This can be seen when we integrate the indicator functions of the tubes from the family $\mathbb{T}$ of $1 \times \delta \times \delta$-tubes with δ-separated directions over the δ ball to some power $p > 1$, $|\mathbb{T}| \sim \delta^{-2}$,

$$\delta^3(\delta^{-2})^p \lessapprox \int_{\delta-ball}\left|\sum_{T\in\mathbb{T}}\chi_T\right|^p \le \int_{\mathbb{R}^3}\left|\sum_{T\in\mathbb{T}}\chi_T\right|^p.$$

If the Kakeya maxima inequality $\int_{\mathbb{R}^3}\left|\sum_{T\in\mathbb{T}}\chi_T\right|^p \lessapprox \delta^2|\mathbb{T}|$ were true (for all small δ), this would give $\delta^{3-2p} \lessapprox \delta^2|\mathbb{T}| \approx 1$ and thus $p \le 3/2$, hence sharp at $p = 3/2$.

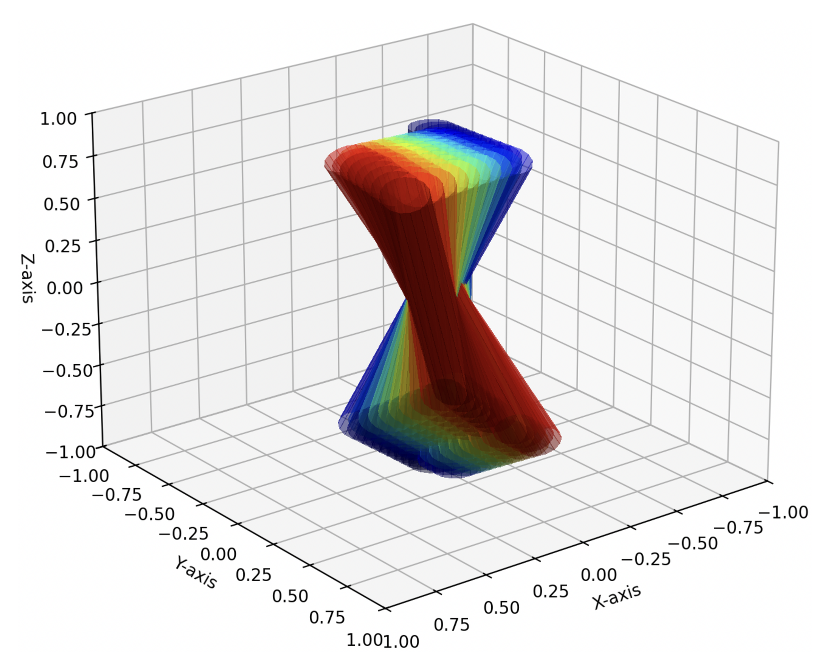

Fig. 1: Ball Example

Reinforcement Learning Approach

To improve on the greedy algorithm, we decided to adapt a reinforcement learning approach based on [2, 4], which was originally developed to find counterexamples to conjectures in graph theory. The key idea is to optimize over discrete structures using a reward function to guide a search process, which we were able to adapt for our use case.

1. The search starts with a batch of candidate tube placements, each represented as a vector of (x, y)-plane intercepts.
2. A sequence of probability distributions over possible actions (i.e. moving a tube) guides the selection of components for these vectors.
3. After evaluating the reward (LHS of eq. 2) for each solution, the top-performing solutions are used to update the probability distributions via the cross-entropy method.
4. New candidate solutions are generated using the updated distributions, iterating until the distributions converge, representing an optimal tube placement.

This approach is adaptable to any combinatorial optimization problem, and can be a very powerful algorithm for solving other math problems[1].

With this new approach, we were able to explore many more possibilities than the greedy algorithm. However, they all seemed to converge to the same Ball example (Fig. 1). To find a new example, we modified our RL algorithm to only check tubes which are in SL_2 (i.e. can be written in the form $\ell = \{(a,b,0) + t(c,d,1) : t \in \mathbb{R}\}$, for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$). This led us to the SL_2 Hairbrush example (Fig. 2).

The SL2 Example

Katz, Wu, and Zahl suggested [3] that, if $\mathbb{T}$ is a collection of SL_2 tubes, of length 1 and radius δ, with δ-separated directions, then the inequality

$$\int_{[0,1]^3}\left|\sum_{T\in\mathbb{T}}\chi_T\right|^p \lessapprox |\mathbb{T}|\,\delta^2,$$

might be true with $p = 2$. There are now a few known counterexamples which show that the p above must be at most $3/2$, one of which is the following "hairbrush" example, due to Zahl [5].

Given a single SL_2-tube T_0, the hairbrush example consists of δ^{-2} many SL_2 δ-tubes intersecting T_0. The tube T_0 is roughly a union of δ^{-1} many δ-balls, and each such δ-ball has roughly δ^{-1} many SL_2-tubes passing through it. If T_0 is partitioned into segments of length $\delta^{1/2}$, then for each segment, there is a rectangular prism of side lengths $\delta^{1/2} \times \delta^{1/2} \times \delta$ containing this segment, and each point in this prism has roughly δ^{-1} many SL_2 tubes passing through it.

This example took some time to be found by hand, but because it happens to have a reasonably simple description, it was found fairly quickly by the RL program we used. In fact, the tube T_0 does not need to be SL_2 for the example to work. Therefore, the example found by the program showed that making T_0 an SL_2 tube was a redundant assumption.

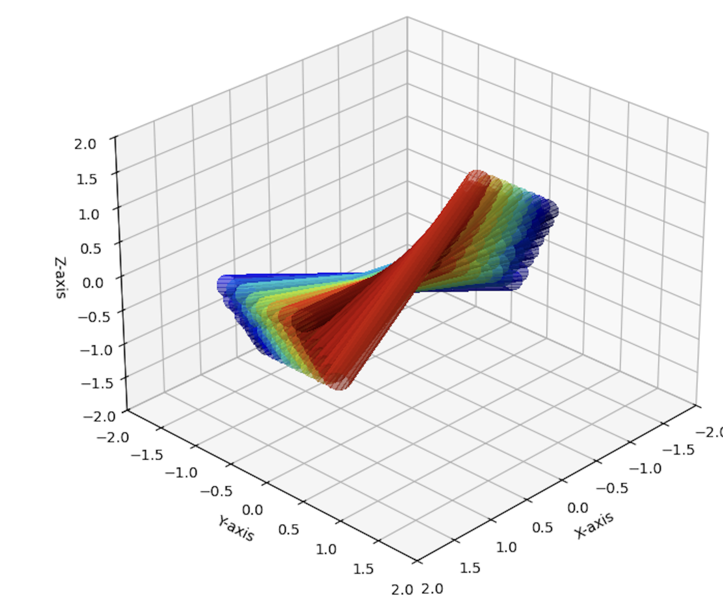

Fig. 2: SL_2 Hairbrush Example

Next Steps

One way to find other examples other than the Ball example would be to consider the "Minkowski dimension" version of the conjecture rather than the Kakeya maximal inequality. We would change the reward function to minimize the volume of the tubes, rather than maximizing the L^p norm of the sum of their indicator functions. The conjecture is that the volume of a maximal direction separated family of tubes is $\gtrapprox 1$ (where $\gtrapprox$ allows factors like $C_\epsilon\delta^{-\epsilon}$). In this setting, the ball example is not optimal, as it has volume ~ 1, but (due to the existence of area zero kakeya sets), we know that there exist collections of δ-tubes with δ-separated directions whose area can be made arbitrarily small (provided δ is sufficiently small).

References

1. Flora Angileri, Giulia Lombardi, Andrea Fois, Renato Faraone, Carlo Metta, Michele Salvi, Luigi Amedeo Bianchi, Marco Fantozzi, Silvia Giulia Galfrè, Daniele Pavesi, Maurizio Parton, Francesco Morandin: A Systematization of the Wagner Framework: Graph Theory Conjectures and Reinforcement Learning. arXiv:2406.12667v2 (2024)
2. Ghebleh, M., Al-Yakoob, S., Kanso, A., Stevanovic, D.: Reinforcement learning for graph theory, I. Reimplementation of Wagner's approach. arXiv:2403.18429v2 (2024)
3. Katz, N., Tao, T.: Recent progress on the Kakeya conjecture. arXiv:math/0010069v2 (2000)
4. Katz, N., Wu, S., Zahl, J.: Kakeya sets from lines in SL_2. Ars Inven. Anal., Paper No. 6, 23 pp. (2023)
5. Wagner, A.: Constructions in combinatorics via neural networks. arXiv:2104.14516v1 (2021)
6. Wolff, T.: Lectures on Harmonic Analysis. I. Laba and C. Shubin, eds., University Lecture Series, vol. 29, American Mathematical Society (2003)