Lab05-Linear Programming

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2019.

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1. A company intends to invest 0.3 million dollars in 2018, with a proper combination of the following 3 projects:

   - **Project 1:** Invest at the beginning of a year, and can receive a 20% profit of the investment in this project at the end of this year. Both the capital and profit can be invested at the beginning of next year;
   - **Project 2:** Invest at the beginning of 2018, and can receive a 50% profit of the investment in this project at the end of 2019. The investment in this project cannot exceed 0.15 million dollars;
   - **Project 3:** Invest at the beginning of 2019, and can receive a 40% profit of the investment in this project at the end of 2019. The investment in this project cannot exceed 0.1 million dollars.

   Assume that the company will invest *all* its money at the beginning of a year. Please design a scheme of investment in 2018 and 2019 which maximizes the overall sum of capital and profit at the end of 2019.

   (a) Formulate a linear programming with necessary explanations.

   (b) Transform your LP into its standard form and slack form.

   (c) Transform your LP into its dual form.

   (d) Use the simplex method to solve your LP by step.

   Solution. (a) Let us denote x_1 as the amount of money invested into Project 1 at the beginning of 2018, x_2 as that at the beginning of 2019, and x_3, x_4 as the amount of money invested into Project 2 and 3 accordingly. Based on their starting and ending date and return on investment, we can give a linear programming as the following:

$$\begin{aligned} \max \quad & 1.2x_2 + 1.5x_3 + 1.4x_4 \\ s.t. \quad & x_1 + x_3 = 0.3, \\ & x_2 + x_4 = 1.2x_1, \\ & x_3 \leq 0.15, \\ & x_4 \leq 0.1, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

   (b) To obtain its standard form, we only have to change the equivalent constraints into inequivalent ones, then change $\geq$ into $\leq$. The detail is shown as follows.

$$\begin{aligned} \max \quad & 1.2x_2 + 1.5x_3 + 1.4x_4 \\ s.t. \quad & x_1 + x_3 \leq 0.3, \\ & x_1 + x_3 \geq 0.3, \\ & x_2 + x_4 \leq 1.2x_1, \\ & x_2 + x_4 \geq 1.2x_1, \\ & x_3 \leq 0.15, \\ & x_4 \leq 0.1, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 1.2x_2 + 1.5x_3 + 1.4x_4 \\ s.t. \quad & x_1 + x_3 \leq 0.3, \\ & -x_1 - x_3 \leq -0.3, \\ & -1.2x_1 + x_2 + x_4 \leq 0, \\ & 1.2x_1 - x_2 - x_4 \leq 0, \\ & x_3 \leq 0.15, \\ & x_4 \leq 0.1, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

And to obtain its slack form, we only need to change the inequilities in (a) into equilities. We introduce x_5, x_6 as slack variables, and it should be noticed that x_1, x_2 themselves act like slack variables to some extent, as we will see this in (d).

$$
\begin{aligned}
\max \ \ & 1.2x_2 + 1.5x_3 + 1.4x_4 \\
s.t. \ \ & x_1 + x_3 = 0.3, \\
& x_2 + x_4 = 1.2x_1, \\
& x_3 \le 0.15, \\
& x_4 \le 0.1, \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\max \ \ & 1.2x_2 + 1.5x_3 + 1.4x_4 \\
s.t. \ \ & x_1 + x_3 = 0.3, \\
& x_2 + x_4 = 1.2x_1, \\
& x_3 + x_5 = 0.15, \\
& x_4 + x_6 = 0.1, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.
\end{aligned}
$$

(c) Observing its standard form, we have

$$
\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ -1.2 & 1 & 0 & 1 \\ 1.2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.3 \\ -0.3 \\ 0 \\ 0 \\ 0.15 \\ 0.1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1.2 \\ 1.5 \\ 1.4 \end{bmatrix}
$$

With the transforming formula between primal and dual form show in the slides:

$$
\begin{aligned}
\max \ \ & \mathbf{c}^T\mathbf{x} \\
s.t. \ \ & \mathbf{A}\mathbf{x} \le \mathbf{b}, \\
& \mathbf{x} \ge \mathbf{0}.
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min \ \ & \mathbf{y}^T\mathbf{b} \\
s.t. \ \ & \mathbf{y}^T\mathbf{A} \ge \mathbf{c}^T, \\
& \mathbf{y} \ge \mathbf{0}.
\end{aligned}
$$

We can easily write down its dual form:

$$
\begin{aligned}
\min \ \ & 0.3y_1 - 0.3y_2 + 0.15y_5 + 0.1y_6 \\
s.t. \ \ & y_1 - y_2 - 1.2y_3 + 1.2y_4 \ge 0, \\
& y3 - y4 \ge 1.2, \\
& y_1 - y_2 + y_5 \ge 1.5, \\
& y_3 - y_4 + y_5 \ge 1.4, \\
& y_i \ge 0 (1 \le i \le 6).
\end{aligned}
$$

(d) We have already had its slack form, so the next step is to assign a basic solution. Observe that there are four equations in slack form yet 6 variables. Without loss of generality, let us define x_3, x_4 as nonbasic variables and x_1, x_2, x_5, x_6 as basic variables. With this we transform the original slack form so that only nonbasic variables exist in the objec function, and in constraints nonbasic and basic variables are in different hands.

$$
\begin{aligned}
\max \ \ & 1.2(1.2(0.3 - x_3) - x_4) + 1.5x_3 + 1.4x_4 = 0.432 + 0.06x_3 + 0.2x_4 \\
s.t. \ \ & 0.3 - x_3 = x_1, \\
& 1.2x_1 - x_4 = x_2, \\
& 0.15 - x_3 = x_5, \\
& 0.1 - x_4 = x_6, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.
\end{aligned}
$$

By setting all nonbasic variables to 0, we get the basic solution:

$$
\overline{x} = (\overline{x_1}, \overline{x_2}, \cdots, \overline{x_6}) = (0.3, 0.36, 0, 0, 0.15, 0.1)
$$

Then we choose the nonbasic variable x_3, and find that $0.15 - x_3 x_5$ is the tightest constraint for x_3. We exchange the state of x_3 and x_5, making x_5 the new nonbasic variable. Namely, we have

$$\begin{aligned} \max \quad & 0.432 + 0.06x_3 + 0.2x_4 \\ s.t. \quad & 0.3 - x_3 = x_1, \\ & 1.2x_1 - x_4 = x_2, \\ & 0.15 - x_3 = x_5, \\ & 0.1 - x_4 = x_6, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 0.432 + 0.06(0.15 - x_5) + 0.2x_4 \\ s.t. \quad & 0.3 - x_3 = x_1, \\ & 1.2x_1 - x_4 = x_2, \\ & 0.15 - x_5 = x_3, \\ & 0.1 - x_4 = x_6, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

$$\overline{x} = \{0.3, 0.36, 0, 0, 0.15, 0.1\} \quad \Rightarrow \quad \overline{x} = \{0.15, 0.18, 0.15, 0, 0, 0.1\}$$

We repeat this step with x_4, and finally we get a form with all coefficients in object function are negative. That is,

$$\begin{aligned} \max \quad & 0.461 - 0.06x_5 - 0.2x_6 \\ s.t. \quad & 0.3 - x_3 = x_1, \\ & 1.2x_1 - x_4 = x_2, \\ & 0.15 - x_5 = x_3, \\ & 0.1 - x_6 = x_4, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \\ & \overline{x} = \{0.15, 0.08, 0.15, 0.1, 0, 0\} \end{aligned}$$

The maximum of objective function is 0.461, when $x_5 = x_6 = 0$. Therefore, the optimal solution for the original problem is

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0.15, 0.08, 0.15, 0.1),$$

and the maximum overall sum is 0.461.

□

2. An engineering factory makes seven products (PROD 1 to PROD 7) on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and one planer. Each product yields a certain contribution to profit (in £/unit). These quantities (in £/unit) together with the unit production times (hours) required on each process are given below. A dash indicates that a product does not require a process.

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
Contribution to profit	10	6	8	4	11	9	3
Grinding	0.5	0.7	-	-	0.3	0.2	0.5
Vertical drilling	0.1	0.2	-	0.3	-	0.6	-
Horizontal drilling	0.2	-	0.8	-	-	-	0.6
Boring	0.05	0.03	-	0.07	0.1	-	0.08
Planing	-	-	0.01	-	0.05	-	0.05

There are marketing limitations on each product in each month, given in the following table:

It is possible to store up to 100 of each product at a time at a cost of £0.5 per unit per month (charged at the end of each month according to the amount held at that time). There are no stocks at present, but it is desired to have a stock of exactly 50 of each type of product at

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

the end of June. The factory works six days a week with two shifts of 8h each day. It may be assumed that each month consists of only 24 working days. Each machine must be down for maintenance in one month of the six. No sequencing problems need to be considered.

When and what should the factory make in order to maximize the total net profit?

(a) Use *CPLEX Optimization Studio* to solve this problem. Describe your model in *Optimization Programming Language* (OPL). Remember to use a separate data file (.dat) rather than embedding the data into the model file (.mod).

(b) Solve your model and give the following results.

   i. For each machine:

      A. the month for maintenance.

   ii. For each product:

      A. The amount to make in each month.

      B. The amount to sell in each month.

      C. The amount to hold at the end of each month.

   iii. The total selling profit.

   iv. The total holding cost.

   v. The total net profit (selling profit minus holding cost).

Solution. (a) The month for maintenance for each machine. The table shows the number of machine down in each month.

	grinder	vertical drill	horizontal drill	borer	planer
January	0	0	1	0	0
February	1	1	0	0	0
March	0	0	2	0	0
April	3	0	0	1	1
May	0	1	0	0	0
June	0	0	0	0	0

(b) i. The amount to make in each month

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	400	700	100	100	600	400	200
April	0	0	0	500	0	0	0
May	0	100	500	100	1000	300	0
June	550	550	150	350	1150	550	110

ii. The amount to sell in each month

iii. The amount to hold at the end of each month

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	100	100	100	100	0	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
January	0	0	0	0	0	0	0
February	0	0	0	0	0	0	0
March	100	100	100	100	100	0	100
April	0	0	0	0	0	0	0
May	0	0	0	0	0	0	0
June	50	50	50	50	50	50	50

(c) The total selling point is 103730.

(d) The total holding cost is 475.

(e) The total net profit is 103255.

□

Remark: You need to include your .mod, .dat, .pdf and .tex files in your uploaded .zip file.