Lab09-Approximation Algorithm

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2019.

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1. **Metric k-center:** Let $G = (V, E)$ be an complete undirected graph with nonnegative edge costs satisfying the triangle inequality, and k be a positive integer. For any set $S \subseteq V$ and vertex $v \in V$, define $cost(v, S)$ to be the cost of the cheapest edge from v to a vertex in S ($cost(v, S) = 0$ if $v \in S$). The problem is to find a set $S \subseteq V$, with $|S| = k$, so as to minimize $\max_v\{cost(v, S)\}$.

   (a) Design a greedy approximation algorithm (in the form of pseudo code) with approximation ratio 2 for this problem.

   (Basic idea: start with an arbitrary center, and in each round, add the 'farthest' vertex to the center set until there are totaly k centers)

   (b) Prove that your greedy algorithm achieves an approximation ratio of 2 for the metric k-center problem. (Hint: prove by contradiction and use the triangle inequality.)

Solution. (a) See Algorithm 1.

Algorithm 1: Greedy Approximation Algorithm for Metric k-center

Input: Graph $G = (V, E)$ with non-negative edge weights$c(e), e \in E$, an integer k;

Output: Subset S of V with $|S| = k$;

$S \leftarrow \emptyset$;
foreach $v \in V$ **do**
  $cost(v, S) = \infty$;
Choose a vertex $v_1 \in V$ randomly and add it to S;
for $i \leftarrow 1$ *to* $k - 1$ **do**
  $farthestdis \leftarrow 0$;
  foreach $v \in V - S$ **do**
    foreach $u \in S$ **do**
      $cost(v, S) \leftarrow \min\{cost(v, S), c(u, v)\}$;
    if $cost(v, S) > farthestdis$ **then**
      $farthestdis \leftarrow cost(v, S)$;
      $farthestver \leftarrow v$;
  $S \leftarrow S \cup \{farthestver\}$;
return S;

(b)

□

2. Let $G = (V, E)$ be a complete undirected graph with nonnegative edge costs satisfying the triangle inequality, and its vertices are partitioned into two sets, R and S. The goal is to find a minimum cost tree in G that contains R and any subset of S. Obviously, a minimum spanning tree (MST) on R is a feasible solution. Prove that finding an MST on R achieves an approximation ratio of 2 for this problem.

3. **Minimum Weighted Vertex Cover:** Consider the weighted version of the Minimum Vertex Cover problem in which a non-negative weight c_i is associated with each vertex v_i and we look for a vertex cover having minimum total weight.

   (a) Given a weighted graph $G = (V, E)$ with a non-negative weight c_i associated with each vertex v_i, please formulate the Minimum Weighted Vertex Cover problem as an integer linear program.

   (b) Prove that the following algorithm finds a feasible solution of the Minimum Weighted Vertex Cover problem with value $m_{LP}(G)$ such that $m_{LP}(G)/m^*(G) \leq 2$.

Algorithm 2: Rounding Weighted Vertex Cover

Input: Graph $G = (V, E)$ with non-negative vertex weights;
Output: Vertex cover V' of G;

Let ILP_{VC} be the integer linear programming formulation of the problem;
Let LP_{VC} be the problem obtained from ILP_{VC} by LP-relaxation;
Let $x^*(G)$ be the optimal solution for LP_{VC};
$V' \leftarrow \{v_i \mid x_i^*(G) \geq 0.5\}$;
return V';

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4. Give the corresponding $(I, sol, m, goal)$ for **Metric k-center** and **Minimum Weighted Vertex Cover** respectively.

Solution. (a) **Metric k-center**

- *Instance*: A complete undirected graph $G = (V, E)$, a cost function $c : E \to \mathbb{R}^+ \cup \{0\}$ and a positive integer k.
- *Solution*: A subset $S \subseteq V$ with $|S| = k$.
- *Measure*: A cost function for S and any vertex $v \in V$:

$$cost(v, S) = \begin{cases} \min\{c(v, u) \mid u \in S\}, & v \notin S \\ 0, & v \in S \end{cases}$$

  And we measure a solution by $\max\{cost(v, S)\}$ for all $v \in V$.
- *Goal*: Min.

(b) **Minimum Weighted Vertex Cover**

- *Instance*: A graph $G = (V, E)$, and a cost function $c : V \to \mathbb{N}$.
- *Solution*: A subset $V' \subseteq V$ that for each $(v_i, v_j) \in E$, $v_i \in V'$ or $v_j \in V'$.
- *Measure*: Total cost $\sum\limits_{v \in V'} c(v)$.
- *Goal*: Min.

□

Remark: You need to include your .pdf and .tex files in your uploaded .zip file.