算法说明

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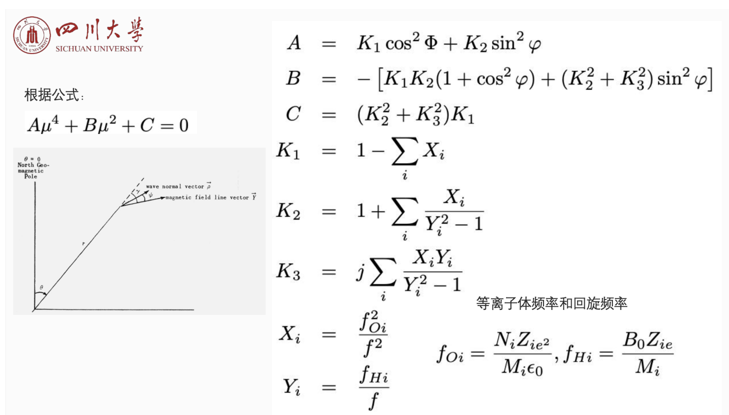

图 1: 图例

1 归一化

$$\vec{K} = \frac{\vec{K}_0 \cdot \mu}{\|\boldsymbol{K}\|}, \quad \vec{B} = \frac{\vec{B}_0}{\|\boldsymbol{K}\|}$$

这是代码最开始做的归一化，方便后续计算。

2　式 1 的计算细节

2.1　主方程

$$\frac{dr}{dt} = \frac{1}{\mu^2}\left(\rho_r - \mu\frac{\partial\mu}{\partial\rho_r}\right)$$

2.2　偏导数计算

我们只需要表示 $\frac{\partial\mu}{\partial\rho_r}$，改写为波矢 $\boldsymbol{K}$ 形式：

$$\frac{\partial\mu}{\partial\mu_r} = \frac{\partial\mu}{\partial\cos\psi}\cdot\frac{\partial\cos\psi}{\partial K_r}$$

2.2.1　分量计算

$$\begin{aligned}
\frac{\partial\mu}{\partial\cos\psi} &= -\frac{\frac{\partial D}{\partial\cos\psi}}{\frac{\partial D}{\partial\mu}} \\
\frac{\partial D}{\partial\mu} &= 4A\mu^3 + 2B\mu \\
\frac{\partial D}{\partial\cos\psi} &= \frac{\partial A}{\partial\cos\psi}\mu^4 + \frac{\partial B}{\partial\cos\psi}\mu^2
\end{aligned}$$

2.2.2　中间导数

$$\begin{aligned}
\frac{\partial A}{\partial\cos\psi} &= 2(K_1 - K_2)\cos\psi \\
\frac{\partial B}{\partial\cos\psi} &= 2\cos\psi\left[K_1K_2 - (K_2^2 + K_3^2)\right]
\end{aligned}$$

$$\frac{\partial\cos\psi}{\partial K_r} = \frac{B_r\|\boldsymbol{K}\|^2 - \|\boldsymbol{B}\|\|\boldsymbol{K}\|K_r\cos\psi}{\|\boldsymbol{B}\|\|\boldsymbol{K}\|^3} \tag{1}$$

$$\frac{\partial\cos\psi}{\partial K_r} = B_r - \frac{K_r\cos\psi}{\mu} \tag{2}$$

（代码中是 2 式，因归一化）（注：代码中的 $-2\cos\psi$ 因子是为方便重复书写，式 (1) 中的 μ 已包含在归一化中）

3 式 2 的计算细节

3.1 主方程

$$\frac{d\rho_r}{dt} = \frac{1}{\mu}\frac{\partial\mu}{\partial r} + \rho_\theta\frac{d\theta}{dt} + \rho_\phi\frac{d\phi}{dt}\sin\theta$$

3.2 梯度展开

$$\frac{\partial\mu}{\partial r} = \sum_i\left(\frac{\partial\mu}{\partial X_i}\frac{\partial X_i}{\partial r}\right) + \sum_i\left(\frac{\partial\mu}{\partial Y_i}\frac{\partial Y_i}{\partial r}\right) + \frac{\partial\mu}{\partial\cos\psi}\frac{\partial\cos\psi}{\partial r}$$

3.2.1 分量计算 1

$$\begin{aligned}
\frac{\partial\mu}{\partial X_i} &= -\frac{\frac{\partial D}{\partial X_i}}{\frac{\partial D}{\partial\mu}}\\
\frac{\partial D}{\partial X_i} &= \frac{\partial A}{\partial X_i}\mu^4 + \frac{\partial B}{\partial X_i}\mu^2 + \frac{\partial C}{\partial X_i}\\
\frac{\partial D}{\partial\mu} &= \text{(已在前页求得)}\\
\frac{\partial A}{\partial X_i} &= -\cos^2\psi(\psi\text{ 为 }\vec{B}\text{ 与 }\vec{K}\text{ 的夹角)}\\
\frac{\partial B}{\partial X_i} &= \left(\frac{K_1}{Y_i^2-1} - K_1\right)(1+\cos^2\psi) - \frac{2\left(K_i - Y_i\cdot\sum_j\frac{X_j\cdot Y_j}{Y_j^2-1}\right)\cdot\sin^2\psi}{Y_i^2-1}\\
\frac{\partial C}{\partial X_i} &= -\left[K_2^2 - \left(\sum_k\frac{X_k\cdot Y_k}{Y_k^2-1}\right)^2\right] + \frac{K_1\cdot 2\left(\sum_k\frac{X_kY_k}{Y_k^2-1}\right)\cdot Y_i^2}{Y_i^2-1}\\
\frac{X_i}{r} &= X_i\frac{\partial n_i}{\partial r}
\end{aligned}$$

3.2.2 分量计算 2

$$
\begin{aligned}
\frac{\partial \mu}{\partial Y_i} &= -\frac{\frac{\partial D}{\partial Y_i}}{\frac{\partial D}{\partial \mu}} \\
\frac{\partial D}{\partial Y_i} &= \frac{\partial A}{\partial Y_i}\mu^4 + \frac{\partial B}{\partial Y_i}\mu^2 + \frac{\partial C}{\partial Y_i} \\
\frac{\partial D}{\partial \mu} &= \text{(已在前页求得)} \\
\frac{\partial A}{\partial Y_i} &= -\frac{2X_iY_i}{(Y_i^2-1)^2}\cdot \sin^2\psi \\
\frac{\partial B}{\partial Y_i} &= \frac{2K_1X_iY_i}{(Y_i^2-1)^2(1+\cos^2\psi)} - dk5dy\cdot \sin^2\psi \\
dk5dy &= 2\frac{\sum_i \frac{X_iY_i}{Y_i^2-1}X_i(Y_i^2+1)}{(Y_i^2-1)^2} + 2\cdot dk2dy\cdot K_2 \\
dk2dy &= -2\frac{X_iY_i}{(Y_i^2-1)^2} \\
\frac{\partial C}{\partial Y_i} &= K_1 dk5dy \\
\frac{Y_i}{r} &= Y_i\frac{\partial B_i}{\partial r}
\end{aligned}
$$

3.2.3 分量计算 3

$$
\begin{aligned}
&\frac{\partial \mu}{\partial \cos\psi} = \text{(已在前页求得)} \\
&\frac{\partial \cos\psi}{\partial r} = \frac{\partial B_r}{\partial r}K_r + \frac{\partial B_\theta}{\partial r}K_\theta + \frac{\partial B_\phi}{\partial r}K_\phi \Big/ \mu \quad \text{（代码中除以 } \mu\text{，因归一化）}
\end{aligned}
$$