Lec 1 - Sp 2018

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1} \qquad \underline{X} = \begin{bmatrix} 1 & X_{11} & \cdots & X_{p1} \\ 1 & X_{12} & \cdots & X_{p2} \\ 1 & X_{13} & \cdots & X_{p3} \\ \vdots & \vdots & \ddots & \\ 1 & X_{1n} & \cdots & X_{pn} \end{bmatrix}_{n \times p+1} \qquad \underline{\beta} = [\beta_0 \ \beta_1 \cdots \beta_p]$$

$n \times p+1$

$$\underline{Y} \sim N(\underline{X}\,\underline{\beta}, \underline{\Sigma})$$

$$\underline{\Sigma} = \begin{pmatrix} \sigma^2 & & & & 0 \\ & \sigma^2 & & & \\ & & \sigma^2 & & \\ & & & \ddots & \\ 0 & & & & \sigma^2 \end{pmatrix}$$

$n \times n$

$$\mathcal{L}(\underline{\beta}, \sigma^2 \mid \underline{X}, \underline{Y}) = (2\pi)^{-n/2} \det(\underline{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\underline{Y} - \underline{X}\underline{\beta})^t \Sigma^{-1} (\underline{Y} - \underline{X}\underline{\beta})\right)$$

$$\ell(\underline{\beta}, \underline{\sigma}^2 \mid \underline{X}, \underline{Y}) \propto -\frac{n \ln \sigma^2}{2} - \frac{1}{2\sigma^2}(\underline{Y} - \underline{X}\underline{\beta})'(\underline{Y} - \underline{X}\underline{\beta})$$

$$\frac{\partial \ell}{\partial \underline{\beta}} \propto \left[(-\underline{X})'(\underline{y}-\underline{X}\underline{\beta}) + (-\underline{X})'(\underline{y}-\underline{X}\underline{\beta}) \right]$$

$$\propto -2(\underline{y}-\underline{X}\underline{\beta})'\underline{X}$$

$$\propto -(\underline{y}'-\underline{\beta}'\underline{X}')X$$

$$\propto -\underline{y}'\underline{X} - \underline{\beta}'\underline{X}'\underline{X} = 0$$

$$\hat{\underline{\beta}}'_{MLE}(X'X) = y'X'$$

$$\hat{\underline{\beta}}'_{MLE} = y'X(X'X)^{-1}$$

$$\hat{\underline{\beta}}_{MLE} = (X'X)X'y$$

$$\hat{\underline{y}} = \underline{X}\hat{\underline{\beta}} = \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{y}$$

$$\frac{\partial \ell}{\partial \sigma^2} \propto -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2}\frac{1}{(\sigma^2)^2}(Y-X\beta)'(Y-X\beta)$$

$$\propto \frac{1}{2\sigma^2}\left(-n + \frac{1}{\sigma^2}(Y-X\beta)'(Y-X\beta)\right) = 0$$

Assume $\sigma^2 > 0$

$$-n + \frac{1}{\hat{\sigma}^2}(Y-X\beta)'(Y-X\beta) = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}(Y-X\beta)'(Y-X\beta)$$

$$= \frac{1}{n}(Y-\hat{Y})'(Y-\hat{Y})$$

$$= \frac{1}{n}\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2$$

Posterior | $\underline{\underline{\sigma^2}}$

$$[\boldsymbol\beta, \sigma^2 \mid \mathbf{Y}, \mathbf{X}] \propto (\cancel{2}\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{Y}-\mathbf{X}\boldsymbol\beta)'(\mathbf{Y}-\mathbf{X}\boldsymbol\beta)\right)$$

$$\cancel{(2\pi\sigma_\beta^2)^{-p/2}\exp\left(-\frac{1}{2\sigma_\beta^2}\boldsymbol\beta'\boldsymbol\beta\right)}$$

$$\cancel{\frac{b^a}{\Gamma(a)}}(\sigma^2)^{-a-1}\exp\left(-\frac{b}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\right)$$

$$\sigma^{2(-a-1)} \exp\left(-\frac{b}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-(a+n/2)-1} \exp\left(-\frac{1}{\sigma^2}\left[b + \frac{(Y-X\beta)'(Y-X\beta)}{2}\right]\right)$$

$$\Rightarrow$$

$$\sigma^2 \mid Y, X \sim \text{Inv-Gamma}\left(a+\frac{n}{2},\ b+\frac{(Y-X\beta)'(Y-X\beta)}{2}\right)$$

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$$[\boldsymbol{\beta}, \sigma^2 \mid \mathbf{Y}, \mathbf{X}] \propto (2\pi\cancel{\sigma^2})^{\cancel{-n/2}} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})\right)$$

$$\cancel{(2\pi\sigma^2_\beta)^{-p/2}} \exp\left(-\frac{1}{2\sigma^2_\beta}\boldsymbol{\beta}'\boldsymbol{\beta}\right)$$

$$\cancel{\frac{b^a}{\Gamma(a)}(\sigma^2)^{-a-1} \exp\left(-\frac{b}{\sigma^2}\right)}$$

$$\boxed{S = \frac{\sigma^2}{\sigma^2_\beta} \qquad \frac{1}{\sigma^2_\beta} = \frac{S}{\sigma^2}}$$

$$\beta \mid Y, X, \sigma^2 \propto \exp\left[-\frac{1}{2\sigma^2}(Y' - \beta' X')(Y - X\beta) - \frac{S}{2\sigma^2}\beta'\beta\right]$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\left[\left(Y'Y - \beta' X' Y - Y' X \beta + \beta' X' X \beta\right) + S\beta'\beta\right]\right]$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\left(\beta'\left(\underline{X'X} + S\mathbb{1}\right)\beta - \beta' X' Y - Y' X \beta\right)\right]$$

$$\boxed{\text{Backwards}}$$

$$\beta \sim N(\mu_p, \Sigma_p)$$

$$\beta \mid \cdot \propto \exp\left[-\frac{1}{2}(\beta - \mu_p)' \Sigma_p^{-1} (\beta - \mu_p)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\beta' \Sigma_p^{-1} \beta - \beta' \Sigma_p^{-1} \mu_p - \mu_p' \Sigma_p^{-1} \beta\right)\right]$$

$$\Sigma_\beta^{-1} = X'X + s\,\mathbb{1}$$

$$\Sigma_\beta = \left(X'X + \frac{\sigma_\beta^2}{\sigma^2}\,\mathbb{1} \right)^{-1}$$

$$\beta' \Sigma_\beta^{-1} \mu_\beta = \beta' X' y$$

$$\mu_\beta = \Sigma_\beta X' y = \left(X'X + \frac{\sigma_\beta^2}{\sigma^2}\,\mathbb{1} \right)^{-1} X' y$$