Graphical Markov Models

3 Lectures

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1 Conditional independence
Graphs (DAG, UG, BG, RCG)

Markov Properties

Conditional Indep. tests
Examples

2 DAGs and Regression chain graphs
MLE for Gaussian distributions

Examples

3 MLE for categorical (binary) data
Structure learning

Examples

LECTURE 1

Summary

- Conditional independence
- Properties
- Graphical Markov Models

  Directed acyclic graphs

  Undirected graphs

  Regression graphs

References

- Højsgaard, Edwards, Lauritzen (2012) Graphical Models in R, Springer V.
- Scutari, Denis (2022) Bayesian Networks CRC Press.
- Cox, Wermuth (1996) Multivariate dependencies, Chapman & Hall.
- Roverato (2017) Graphical models for categorical variables.
- Whittaker (1990) Graphical models in applied multivariate statistics Wiley.

Useful R packages

bnlearn	(Scutari et al)
gRbase gRim	(Højsgaard et al)
ggm	(Marchetti et al)
ct2	(J. Lang)

Look at the CRAN Task View

↳ https://CRAN.R-project.org/view=GraphicalModels

Other References.

J. Pearl et al (2016) Causal Inference in Statistics. Wiley.

Hernán, Robins (2020). What if. CRC Press.

Look at the CRAN Task View

↳ https://cran.r-project.org/web/views/CausalInference.html

Data

We will consider a system of d variables

$$\underset{\sim}{X} = (X_1, X_2, X_3, \ldots, X_d)$$

Can be
- continuous (Gaussian)
- categorical or binary.
- mixed

Example

- Gaussian $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \Sigma)$
- Multinomial $\underset{\sim}{X} \sim \text{Mult}(n, \underset{\sim}{\pi})$

Exercise 1

Simulate 1000 observations from 3 binary variables $X_1 \, X_2 \, X_3$ with a joint prob.

$$\underset{\sim}{\pi} = (0.38, 0.01, 0.07, 0.01, 0.42, 0.02, 0.08, 0.01)$$

Exercise 2

Simulate 100 observations from $N_3(\underline{\mu}, \underline{\Sigma})$ with

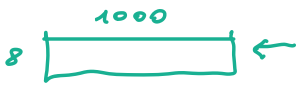
$$\Sigma^{-1} = \begin{bmatrix} 1.2 & 0.9 & 0 \\ & 2.7 & 1.4 \\ & & 1.6 \end{bmatrix}$$

Solutions

Exercise 1

$\underset{\sim}{\pi} > 0$

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$



```
n = 1000
p <- c(0.38, 0.01, 0.07, 0.01,
       0.42, 0.02, 0.08, 0.01)
X<- expand.grid(X1 = factor(0:1),
                X2 = factor(0:1),
                X3 = factor(0:1))
Z <- rmultinom(n, size = 1, prob = p)
cell <- apply(Z, 2, function(x) which(x==1))
data <- X[cell,]
rownames(data) <- 1:n

head(data)
tail(data)

table(data)
as.data.frame(table(data))
```

Exercise 2

```
library(mnormt)
K <- matrix(c(1.2, 0.9,   0,
              0.9, 2.7, 1.4,
              0,   1.4, 1.6), 3, 3)
dimnames(K) <- list(c("X1","X2", "X3"),
                    c("X1","X2", "X3"))
Sigma <- solve(K)
dimnames(Sigma) <- list(c("X1","X2", "X3"),
                        c("X1","X2", "X3"))
round(Sigma, 2)
round(cov2cor(Sigma),2)
X <- rmnorm(n = 100, varcov = Sigma)
```

Conditional independence

Random vector $(X_1, \dots, X_d)$ ↗ all contin.
↘ all discrete

Denote the joint distribution by

$$p_V(x_1, \dots, x_d) \begin{array}{l} \nearrow \text{pmf} \\ \searrow \text{pdf} \end{array}$$

where

$$V = \{1, 2, \dots, d\}$$

Example If (X_1, X_2) $p_{12}(x_1, x_2)$

Conditional Distributions

$X_1 \mid X_2$	$P_{1/2}(x_1 \mid x_2)$
$X_1 \mid X_2 \; X_3$	$P_{1/23}(x_1 \mid x_2, x_3)$

Conditional independence

Given 3 variables X_1, X_2, X_3

X_1 conditionally independent of X_2 given X_3

if

$$P_{123}(x_1, x_2 \mid x_3) = P_{1/3}(x_1 \mid x_3) P_{2/3}(x_2, x_3)$$

for all (x_1, x_2, x_3) such that $p_3(x_3) > 0$

Properties of CI

Notation $X_1 \perp\!\!\!\perp X_2 \mid X_3$

abbreviated $1 \perp\!\!\!\perp 2 \mid 3$

Equivalent (omit the suffix)

$$p(x_1 \mid x_2\, x_3) = p(x_1 \mid x_3) \quad p(x_2, x_3) > 0$$

Factorizations

$$p(x_1, x_2, x_3) = p(x_1 \mid x_3)\, p(x_2 \mid x_3)\, p(x_3)$$

$$= p(x_1 \mid x_3)\, p(x_3 \mid x_2)\, p(x_2)$$

Marginal independence

$X_1 \perp\!\!\!\perp X_2$ if $p_{12}(x_1, x_2) = p_1(x_1)\, p_2(x_2)$

for all x_1, x_2.

It is a quite different constraint.

Exercise 3

Two binary variables have marginal probabilities

$p_1(x_1) = \{0.2, 0.8\}$ $p_2(x_2) = \{0.6, 0.4\}$

and odds-ratio $= 1$ $odr = \frac{\pi_{11}\,\pi_{22}}{\pi_{12}\,\pi_{21}}$

What is the joint distribution ?

Exercise 4

Does $X_1 \perp\!\!\!\perp X_2 \Rightarrow X_1 \perp\!\!\!\perp X_2 \mid X_3$?

Does $X_1 \perp\!\!\!\perp X_2 \mid X_3 \Rightarrow X_1 \perp\!\!\!\perp X_2$?

Solutions

Exercise 3

$$\text{odr} = 1 \iff X_1 \perp\!\!\!\perp X_2$$

Thus, $$p_{12}(x_1, x_2) = \begin{array}{cc|c} 0.12 & & 0.2 \\ & & 0.8 \\ \hline 0.6 & 0.4 & \end{array}$$

Exercise 4

Counterexample 1

$X_3 = 1$

X_1 X_2	0	1
0	37	8
1	3	2

odr = 3.08

$X_3 = 2$

	0	1
0	9	38
1	1	2

odr = 0.47

Marginal X_1, X_2

	0	1
0	46	46
1	4	4

odr = 1

$X_1 \perp\!\!\!\perp X_2$

Counterexample 2

$X_3 = 0$

X_1 X_2	0	1
0	384	96
1	16	4

odr = 1

$X_3 = 1$

	0	1
0	40	360
1	10	90

odr = 1

424	456
26	94

odr = 3.36

$X_1 \perp\!\!\!\perp X_2 \mid X_3$

A fundamental property of CI

Let $d = 4$ (X_1, X_2, X_3, X_4) and consider the joint (conditional) indep.

$$X_1 \perp\!\!\!\perp X_2 X_3 \mid X_4 \qquad 1 \perp\!\!\!\perp 23 \mid 4$$

Proposition

$$1 \perp\!\!\!\perp 23 \mid 4 \iff \begin{array}{c} 1 \perp\!\!\!\perp 2 \mid 4 \\ \text{and} \\ 1 \perp\!\!\!\perp 3 \mid 24 \end{array}$$

This property makes it possible to decompose a CI involving random vectors into a set of pairwise CIs

Semigraphoids

↳ look at Pearl, Judea. 1988. Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. San Mateo, CA: Morgan Kaufmann.

Further properties

- Intersection
- Composition

... lateron

Concentration matrix

Let $\underline{X} = (X_1, \dots, X_d)$ be a Gaussian r.v.

Its covariance matrix is

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \cdot & \sigma_{22} & \cdots & \sigma_{2d} \\ & & \ddots & \\ & & & \sigma_{dd} \end{bmatrix} > 0$$

Its concentration matrix is

$$K = \Sigma^{-1} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1d} \\ & K_{22} & \cdots & K_{2d} \\ & & \ddots & \\ & & & K_{dd} \end{bmatrix} > 0$$

Block covariance matrix

Sometimes we consider a partition of the variables. Example

$V = \{1, 2, 3, 4, 5\}$ $a = \{1, 2\}$ $b = \{3, 4, 5\}$

$\underline{X} = (X_a, X_b)$

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \cdot & \Sigma_{bb} \end{pmatrix} \qquad K = \Sigma^{-1} = \begin{pmatrix} K_{aa} & K_{ab} \\ \cdot & K_{bb} \end{pmatrix}$$

$$K_{aa} \neq \Sigma_{aa}^{-1} \qquad K_{aa} = \left(\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}\right)^{-1}$$

Conditional independence in Gaussian distributions.

Let $\underset{\sim}{X} = (X_1, \dots, X_d) \sim N_d(\overset{\to 0}{\underline{\mu}}, \Sigma)$

Well-Known $\quad X_1 \perp\!\!\!\perp X_2 \iff \sigma_{12} = 0$

$$\boxed{\text{When } \quad X_1 \perp\!\!\!\perp X_2 \mid X_3 \quad ?}$$

① Conditional covariance $\operatorname{cov}(X_1, X_2 \mid X_3) = 0$

$$\sigma_{12} - \sigma_{13}\,\sigma_{32}/\sigma_{33} = 0$$

② Conditional regression coefficient $\beta_{12} = 0$

$$X_1 \mid X_2, X_3 \sim N\big(\underbrace{E(X_1 \mid X_2\, X_3)}_{\text{linear}},\ \underbrace{var(X_1 \mid X_2, X_3)}_{\text{constant}}\big)$$

$$\begin{tikzcd} E(X_1 \mid X_2\, X_3) \arrow[d] & var(X_1 \mid X_2, X_3) \arrow[d] \\ \underset{-\frac{K_{12}}{K_{11}}}{\underset{\downarrow}{\beta_{12}}} X_2 + \underset{-\frac{K_{13}}{K_{11}}}{\underset{\downarrow}{\beta_{13}}} X_3 & \frac{1}{K_{11}} \end{tikzcd}$$

$$X_1 \perp\!\!\!\perp X_2 \mid X_3 \iff E(X_1 \mid X_2\, X_3) = \beta_{13} X_3$$

$$\iff \beta_{12} = 0$$

$$\iff K_{12} = 0$$

Graphical models

It is useful to regard conditional indep. as expressing the notion of irrelevance

$X \perp\!\!\!\perp Y \mid Z$

if we know Z, information about Y is irrelevant for knowledge of X

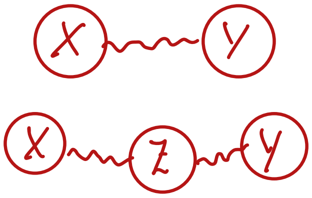
$X \perp\!\!\!\perp Y$

if we ignore all other variables Y looks irrelevant for X

Irrelevance

separation

Relevance



connection

Idea: use graphs to represent a set of conditional or marginal ind. expressed by separation

Graphs

$G = (V, E)$

nodes $= \{1, 2, \ldots, d\}$

edges connecting two nodes (i, j)

Edges : arrows $i \to j$

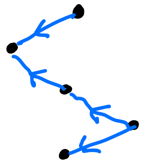
full lines $i - j$

arcs $i \leftrightarrow j$

Types of graph

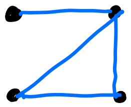
Directed acyclic DAG

only arrows – no cyles



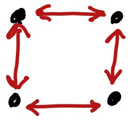
Undirected UG

only full lines



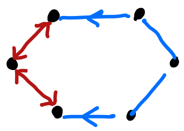
Bidirected BG

only bi-directed edges



Mixed MG

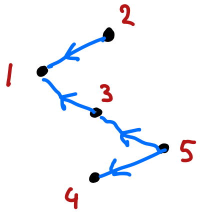
All three types of edge



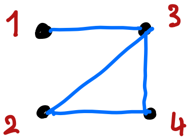
require(ggm)

Define graphs with ggm

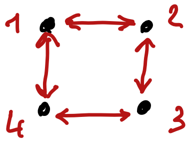
```
DAG(X1 ~ X2 + X3, X3 ~ X5, X4 ~ X5)

   X1 X2 X3 X5 X4
X1  0  0  0  0  0
X2  1  0  0  0  0
X3  1  0  0  0  0
X5  0  0  1  0  1
X4  0  0  0  0  0
```



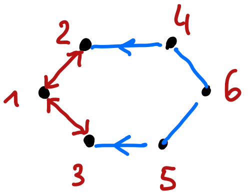
```
UG(~ X1*X3 + X2*X3*X4 )

   X1 X3 X2 X4
X1  0  1  0  0
X3  1  0  1  1
X2  0  1  0  1
X4  0  1  1  0
```



```
100 * UG(~ X1*X2 + X2*X3 + X3*X4 + X1*X4)

     X1  X2  X3  X4
X1    0 100   0 100
X2  100   0 100   0
X3    0 100   0 100
X4  100   0 100   0
```



```
makeMG(bg = UG(~ X1*X2 + X1*X3),
       dg = DAG(X2 ~ X4, X3 ~ X5),
       ug = UG(~ X4*X6 + X5*X6))

    X2 X4  X3 X5 X6  X1
X2   0  0   0  0  0 100
X4   1  0   0  0 10   0
X3   0  0   0  0  0 100
X5   0  0   1  0 10   0
X6   0 10   0 10  0   0
X1 100  0 100  0  0   0
```



Directed acyclic graph models.

Example from German labor market

A, successful job placem.	B, field of qualification			
	home economics		mechan. engineering	
	C, gender		C, gender	
	female	male	female	male
yes	15	2	4	95
	(3.61%)	(3.64%)	(20.0%)	(21.1%)
no	400	53	16	355
sum	415	55	20	450
odds-ratio	0.99		0.93	

A : successful job placement — response

B : Field of qualification — intermediate

C : Gender — Context variable

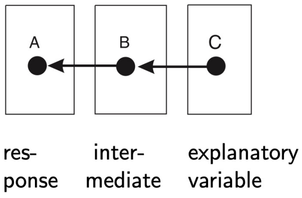
A depends on B and C

B depends on C (both explanatory and response)

C is an intrinsic v.

Data show a CI : $A \perp\!\!\!\perp C \mid B$

Graph :



The missing edge $A \leftarrow C$ implies a CI

Exercise

What happens if we ignore B = Field of qualification ?

Find the marginal distribution of A and C and the odds-ratio

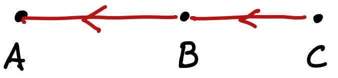
What's the interpretation ?

Solution

Discrimination against women?

A, successful job placement	C, gender female	male
yes	19 (4.4%)	97 (19.2%)
no	416	408
sum	435	505
odds-ratio	0.19	

Factorization according to a DAG



The parents of a node i are the nodes that are directly connected to i

$$pa(A) = \{B\}$$
$$pa(B) = \{C\}$$
$$pa(C) = \emptyset$$

The joint distribution of (A, B, C) can be factorized recursively

$$p(a, b, c) = p(a\,|\,bc)\; p(b\,|\,c)\; p(c)$$

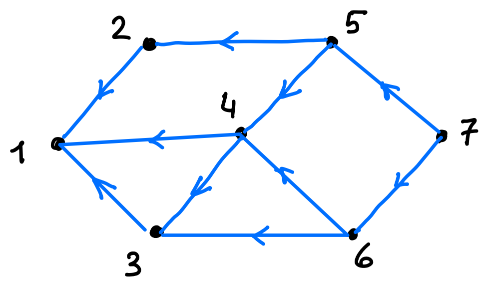
and it simplifies if there are independencies

$$p(a,b,c) = p(a\,|\,\underset{\substack{\downarrow \\ pa(A)}}{b})\; p(b\,|\,\underset{\substack{\downarrow \\ pa(B)}}{c})\; p(c\underset{\substack{\searrow \\ pa(C)}}{)}$$

General formula

$$\begin{aligned} p(x_1, x_2 \cdots x_d) &= p(x_1\,|\,pa(x_1)) \times \\ &\; p(x_2\,|\,pa(x_2)) \times \\ &\;\;\vdots \\ &\; p(x_d) \\ &= \prod_{i=1}^{d} p(x_i\,|\,pa(x_i)). \end{aligned}$$

Exercise



well ordered DAG

Find the recursive factorization.

Solution

$$p(x_1, \dots, x_7) = p(x_1 \mid x_2, x_3, x_4) \times$$
$$p(x_2 \mid x_5) \times$$
$$p(x_3 \mid x_4, x_6) \times$$
$$p(x_4 \mid x_5, x_6) \times$$
$$p(x_5 \mid x_7) \times$$
$$p(x_6 \mid x_7)\, p(x_7).$$

Exercise

Which indepencies are present ?

Solution

$X_1 \perp\!\!\!\perp X_5 X_6 X_7 \mid X_2 X_3 X_4$
$X_2 \perp\!\!\!\perp X_3 X_4 X_6 X_7 \mid X_5$
$X_3 \perp\!\!\!\perp X_5 X_7 \mid X_4 X_6$
$X_4 \perp\!\!\!\perp X_7 \mid X_5 X_6$
$X_5 \perp\!\!\!\perp X_6 \mid X_7$

Markov property.

$G \longleftrightarrow \underline{X}_V$

graph — variables

V, E

properties (adjacencies, missing edges) → Markov property rules → conditional independencies

Specifies the rules that translate the properties of the graph into conditional independencies

Ordered Markov property for DAGs

Every DAG can be well-ordered so that given two nodes

$i =$ head — tail $= j > i$

lower number — higher number

The predecessors of a node : pre(i) are the set of nodes j that are > i

Then the rule is:

$$X_i \perp\!\!\!\perp X_{pre(i) \setminus pa(i)} \mid X_{pa(i)}$$

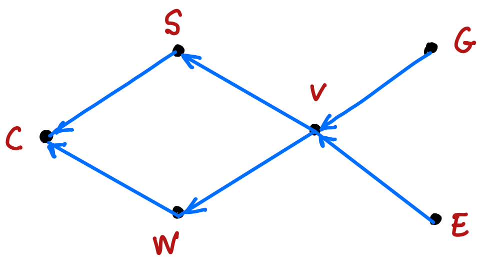
Simulated
(Scutari et al)

Example: crop data

In the analysis of a specific plant
a simple model is

C: crop (grain mass)	S: number of seeds W: seeds weight	V: Vegetative organs	G: genetic potential E: environm. potential

A DAG model shows a generating process.



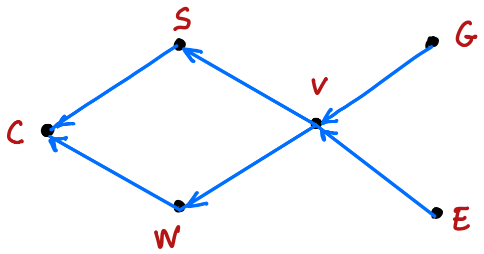
Assume that the joint p_V is Gaussian

Factorization:

$$p_V = p_{C|SW} \times p_{S|V} \times p_{W|V} \times p_{V|GE} \times p_G \times p_E$$

All the factors are univariate
conditional normal distributions

Recursive univariate regression models



Order:
C
S
W
V
G
E

Simulation

$n = 200$

Exercise!

$$C \mid SW \sim N(0.3S + 0.3W;\ 6{,}25^2)$$

$$S \mid V \sim N(45 + 0.1V;\ 9.94^2)$$

$$W \mid V \sim N(15 + 0.7V;\ 7.14^2)$$

$$V \mid GE \sim N(-10.3 + 0.5G + 0.77E;\ 5^2)$$

$$G \mid E \sim N(50;\ 10^2)$$

$$E \sim N(50;\ 10^2)$$

Parameters, variation independence

Each density depends on 2 parameters

- $\underset{\sim}{\beta}$ regression coefficients
- σ^2 conditional variance

The parameter space of the model is the Cartesian product of the separate ranges of its components

Model fitting.

- Maximum likelihood estimation
- Done by separate fit of each regression model.

```
m_full<- lm(C ~ S+W+V+G+E, data = crop)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.001      4.349   0.000
S              0.276      0.047   5.838    0.000
W              0.706      0.067  10.606    0.000
V             -0.098      0.098  -0.997    0.320
G              0.078      0.062   1.275    0.204
E              0.043      0.079   0.552    0.581
```

FIRST EQUATION

```
m_red <- lm(C ~ S + W, data = crop)
anova(m_red, m_full, test = "F")

Model 1: C ~ S + W
Model 2: C ~ S + W + V + G + E
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1    197 7851.9
2    194 7770.3  3    81.565 0.6788  0.566


        w         df         p
2.0884556 3.0000000 0.5542518
```

LRT

$C \perp\!\!\!\perp VGE \mid SW$

```
m_full <- lm(S ~ W + V + G + E, data = crop)

           Estimate Std. Error t value Pr(>|t|)
(Intercept)   54.374      5.313  10.235    0.000
W              0.020      0.101   0.199    0.842
V              0.014      0.148   0.097    0.923
G             -0.081      0.093  -0.874    0.383
E             -0.048      0.119  -0.402    0.688

m_red <- lm(S ~ 1, data = crop)
        w         df         p
0.8098683 3.0000000 0.8471052
        w         df         p
1.2909126 4.0000000 0.8629154
```

SECOND EQUATION

LRT

$S \perp\!\!\!\perp WGE \mid V$

$S \perp\!\!\!\perp WVGE$

(continued)

THIRD EQUATION

```
m_full <- lm(W ~ V + G + E, data = crop)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)   18.702      3.518   5.317    0.000
V              0.593      0.096   6.168    0.000
G              0.062      0.066   0.947    0.345
E             -0.035      0.084  -0.420    0.675

        w        df         p
2.2415432 2.0000000 0.3260281
```

LRT

$W \perp\!\!\!\perp G\,E \mid V$

4th EQUATION

```
m_full <- lm(V ~ G + E, data = crop)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -10.455      2.500  -4.182        0
G              0.455      0.036  12.501        0
E              0.743      0.033  22.357        0
```

no independence

5th EQUATION

```
m_full <- lm(G ~ E, data = crop)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)   49.692      3.365  14.766    0.000
E              0.009      0.065   0.137    0.891

m_red <- lm(G ~ 1, data = crop)

         w         df          p
0.01905948 1.00000000 0.89019611
```

LRT

$G \perp\!\!\!\perp E$

Overall fit using ggm

```
G <- DAG(C ~ S + W, S ~ V, W ~ V, V ~ G + E)
G
  C S W V G E
C 0 0 0 0 0 0
S 1 0 0 0 0 0
W 1 0 0 0 0 0
V 0 1 1 0 0 0
G 0 0 0 1 0 0
E 0 0 0 1 0 0
```

Adjacency matrix

```
ord <- colnames(G)
S <- cov(crop[,ord])
fitDag(G, S, n = 200)
```

covariance matrix in the same order of data

Fit the DAG

```
$Shat
       C      S      W       V      G       E
C 84.539 22.814 56.638  42.640 17.156  33.639
S 22.814 90.605 -2.860  -4.848 -1.950  -3.824
W 56.638 -2.860 83.725  64.107 25.792  50.574
V 42.640 -4.848 64.107 108.656 43.716  85.719
G 17.156 -1.950 25.792  43.716 96.019   0.000
E 33.639 -3.824 50.574  85.719  0.000 115.416
```

fitted covariance matrix

```
$Ahat
  C      S      W      V      G      E
C 1 -0.273 -0.686  0.000  0.000  0.000
S 0  1.000  0.000  0.045  0.000  0.000
W 0  0.000  1.000 -0.590  0.000  0.000
V 0  0.000  0.000  1.000 -0.455 -0.743
G 0  0.000  0.000  0.000  1.000  0.000
E 0  0.000  0.000  0.000  0.000  1.000
```

fitted reg. coeff with sign changed

```
$Dhat
      C      S      W      V      G       E
 39.457 90.388 45.902 25.090 96.019 115.416
```

partial variances

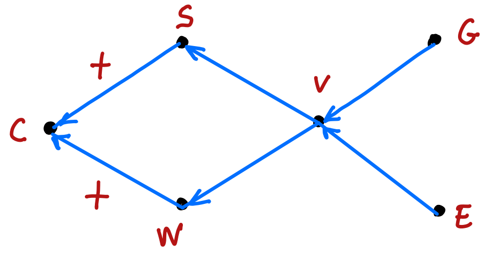
```
$dev
[1] 5.159
```

deviance

```
$df
[1] 9
```

d.f.

Separation in DAGs
and its connection
with Conditional independence



We derived some CI from this graph

however

what can you say about $S \perp\!\!\!\perp W \mid C$?

It depends on $\sigma_{SW|C} \overset{?}{=} 0$ or $\rho_{SW|C} \overset{?}{=} 0$

We can estimate it from data:

$$\hat{\rho}_{SW|C} = -0.28$$

Or test it from data

```
ci.test("S", "W", "C", data = crop)

cor = -0.2876, df = 197, p-value = 3.808e-05
```

used.
← bnlearn
Reject
$H_0: S \perp\!\!\!\perp W \mid C$

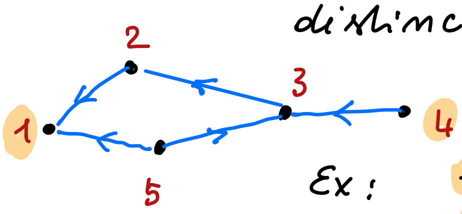
Can we say something without data?

d-separation in DAGs

It is possible to use d-separation to answer to any CI statement

$A \perp\!\!\!\perp B \mid C$ for any triple of subsets of V

Paths in DAG: a sequence of consecutive distinct edges



Ex: $1 \leftarrow 2 \rightarrow 3 \leftarrow 4$

$1 \leftarrow 5 \rightarrow 3 \leftarrow 4$

Blocked paths: by a set of nodes C iff

- the path contains a chain $a \rightarrow m \rightarrow c$ or a fork $a \leftarrow m \rightarrow c$ such that m is in C
- or the path contains a collider $a \rightarrow m \leftarrow c$ such that m is not in C and no descendants of m are in C

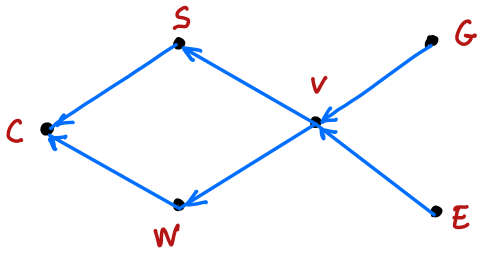
Two subsets of nodes A an B are said d-separated given C if every path between A and B is blocked by C

Global Markov Property in DAGs

Pearl 1988

Given a DAG, if $A, B, C \subset V$ are disjoint, and A and B are d-separated given C then $A \perp\!\!\!\perp B \mid C$

Example



$S \perp\!\!\!\perp W \mid C$? NO because the path $S \to C \leftarrow W$ contains a collider node C that is inside the conditioning set.

$S \perp\!\!\!\perp W \mid VG$? YES Paths

$S \to C \leftarrow W$ C not in VG

$S \leftarrow V \to W$ V is in VG

```
> dSep(G, "S", "W", "C")
[1] FALSE


> dSep(G, "S", "W", c("V", "G"))
[1] TRUE
```

check using R package ggm

$G \perp\!\!\!\perp E \mid S$? Exercise

Fit DAG models with categorical data

Canadian Women Labour-Force participation

Data on 263 married women ages 21–30 (1977)

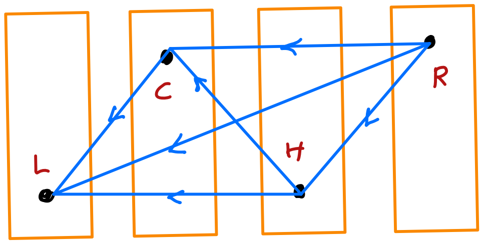
data from R canData

```
      R Atlantic BC Ontario Prairie Quebec
H C L
0 0 0        1  4       6       0      5
    1        1  4      11       3     10
  1 0       11  6      25      15     18
    1        2  0       4       4      4
1 0 0        1  4       6       1      5
    1        1  2      10       1      3
  1 0       11  8      44       7     19
    1        2  1       2       0      1
```

L	Full time work
C	Presence of children
H	Husband's income (1 = > median \$14000)
R	Region

Ordering of the variables

(L, C, H, R)



Multinomial model (cross-classified)

2 × 2 × 2 × 5 Table

$$\pi_{\ell chr} = \pi_{\ell|chr}\,\pi_{c|hr}\,\pi_{h|r}\,\pi_r$$

Logistic regression

$L \mid CHR$ $\qquad \pi_{l|chr} = P(L=1 \mid c, h, r)$

$$\text{logit}(\pi) = \log \frac{\pi}{1-\pi} \qquad (0,1) \mapsto (-\infty, +\infty)$$

generalized linear model

$L \sim C + H + R \quad (\text{link} = \text{logit})$

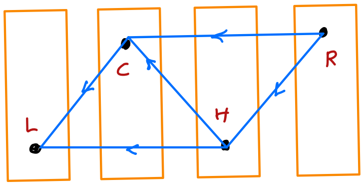
$L \sim C * H * R \quad (\text{link} = \text{logit})$

choose selected interactions!

```
glm(L ~ C + H + R, family = binomial, data =
wlfdata)

            Estimate Std. Error z value Pr(>|z|)
(Intercept)    1.041      0.607   1.715    0.086
C1            -2.609      0.361  -7.234    0.000
H1            -0.768      0.348  -2.210    0.027
RBC           -0.944      0.745  -1.266    0.206
ROntario      -0.254      0.590  -0.430    0.667
RPrairie       0.168      0.695   0.241    0.809
RQuebec       -0.342      0.627  -0.545    0.586

glm(L ~ C + H, family = binomial, data = wlfdata)
        w        df         p
2.8117211 4.0000000 0.5898111
```

LRT test of $L \perp\!\!\!\perp R \mid CH$

+ non-independence constraints.



the constraints are not visible in the graph

Recursive logistic model

$C \mid HR$ generalized linear model

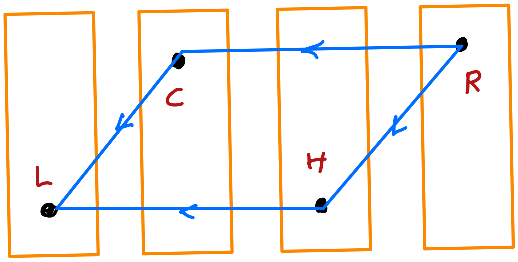
$C \sim H + R$ (link = logit)

(interaction not significant)

```
glm(C~ H+R, family = binomial, data = wlfdata)

             Estimate Std. Error z value Pr(>|z|)
(Intercept)     1.671      0.551   3.030    0.002
H1              0.437      0.282   1.546    0.122
RBC            -1.827      0.656  -2.785    0.005
ROntario       -1.092      0.579  -1.886    0.059
RPrairie       -0.136      0.730  -0.186    0.852
RQuebec        -1.250      0.598  -2.089    0.037


glm(C~ R, family = binomial, data = wlfdata)

        w        df         p
2.4134751 1.0000000 0.1202951
```

LRT of CI

$C \perp\!\!\!\perp H \mid R$



```
glm(H ~ R, family = binomial, data = wlfdata)

             Estimate Std. Error z value Pr(>|z|)
(Intercept)     0.000      0.365   0.000    1.000
RBC             0.069      0.521   0.132    0.895
ROntario        0.298      0.414   0.721    0.471
RPrairie       -0.894      0.538  -1.660    0.097
RQuebec        -0.279      0.443  -0.629    0.529

glm(H ~ 1, family = binomial, data = wlfdata)

         w         df          p
9.19254035 4.00000000 0.05646299
```

$H \not\perp\!\!\!\perp R$