CHAPTER 5

Confidence Regions

We quantify the difference between an estimator T and θ giving an interval estimate

$$[L(x) \quad R(x)]$$

Confidence interval

$X \sim p_\theta(x) \qquad \theta \in \mathbb{R}$

a confidence interval for θ is a map

$$X \longmapsto [L(X), R(X)]$$

such that

$$P_\theta\left(L(X) \leq \theta \leq R(X) \right) \geq C$$

where C is said the confidence level (usually $0.90, 0.95, 0.99$)

- Thus a CI is a stochastic interval that has a high probability of containing θ.

- The important point is that, given ANY θ, we have a fixed coverage probability of θ.

- After the data are observed the CI is just a non-stochastic interval

Example Let $X \sim N(\theta, 1)$

then

$$[X - 1.96, \quad X + 1.96]$$

is a CI for θ at level 0.95

Proof: $P_\theta(X - 1.96 \leq \theta \leq X + 1.96) =$

$$= P_\theta(-1.96 \leq \theta - X \leq 1.96) =$$

$$= P_\theta(1.96 \geq X - \theta \geq -1.96)$$

$$= P_\theta(-1.96 \leq X - \theta \leq 1.96)$$

$$= P_\theta(-1.96 \le Z \le 1.96)$$

with $Z \sim N(0,1)$

$$= \Phi(1.96) - \Phi(-1.96) = 0.95$$

If we sample $X \sim N(\theta, 1)$ we know the the interval $[X - 1.96, X + 1.96]$ contains θ 95% of the time in the long run.

Suppose that we get $X = 10$.
Then the realized CI is

$$10 \pm 1.96 = \begin{cases} 11.96 \\ 8.04 \end{cases}$$

But we can't interpret as:
we have the 95% chance of having $8.04 \le \theta \le 11.96$.

As θ is FIXED writing $P_\theta(8.04 \le \theta \le 11.96)$.
does not have any sense

Confidence region at level c for θ

is a stochastic subset G_X of Θ such that

$$P_\theta (G_X \ni \theta) \geq c$$

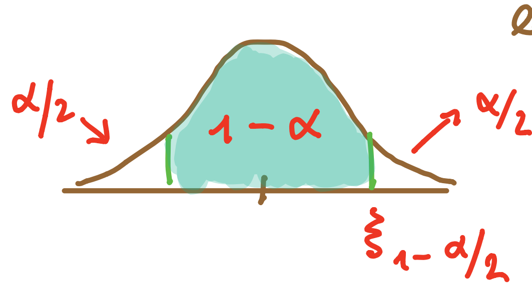
for all $\theta \in \Theta$.

EXAMPLE 5.2 $\quad (X_1, \dots, X_n) \overset{iid}{\sim} N(\mu, \sigma^2)$

with σ^2 known

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

let $c = 1 - \alpha$



$$P\left(- \xi_{1-\alpha/2} \leq Z \leq \xi_{1-\alpha/2} \right) = 1 - \alpha$$

$$\Updownarrow$$

$$P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} \xi_{1-\alpha/2} \leq \mu \leq \bar{X} + \frac{\sigma}{\sqrt{n}} \xi_{1-\alpha/2} \right) = 1 - \alpha$$

5.3 Pivots

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$ is said to be a pivot.

because $$P_\mu\left(a \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq b \right)$$

is known and does not depend on μ.

Note that the pivot is a function of the data $\underset{\sim}{X}$ and of the unknown parameter, $\theta = \mu$.

The confidence interval is obtained by inverting the pivot

Example 5.4 $(X_1 \ldots X_n) \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 unknown.

Here $$T = \frac{\bar{X} - \mu}{S_X/\sqrt{n}}$$

is a pivot because

$$P_{\mu,\sigma}\left(a \le \frac{\bar{X}-\mu}{S_X/\sqrt{n}} \le b\right) = P\left(a \le t_{n-1} \le b\right)$$

is known and does not depend on the parameter $\theta=(\mu,\sigma^2)$.

Using this we find a confidence interval

$$P_{\mu,\sigma}\left(-t_{n-1,1-\alpha/2} \le \frac{\bar{X}-\mu}{S_X/\sqrt{n}} \le t_{n-1,1-\alpha/2}\right) = 1-\alpha$$

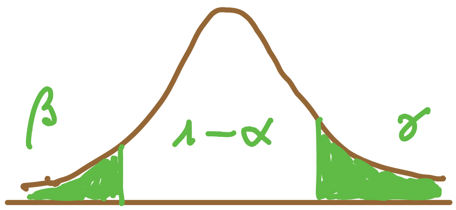
So we get the CI:

$$\boxed{\left[\bar{X} - \frac{S_X}{\sqrt{n}} t_{n-1,1-\alpha/2} \,;\, \bar{X} + \frac{S_X}{\sqrt{n}} t_{n-1,1-\alpha/2}\right]}$$

- the interval is wider than the interval
$$\bar{X} \pm \frac{\sigma}{\sqrt{n}} \xi_{1-\alpha/2}$$
- the lenght of the CI is random.
- the difference between the intervals disappears for $n \to \infty$.

- Non symmetric CI can be constructed from

$$\left[\bar{X} - \frac{S_X}{\sqrt{n}}\, t_{n-1,1-\gamma} \,;\; \bar{X} - \frac{S_X}{\sqrt{n}}\, t_{n-1,\beta}\right]$$



with $\beta + \gamma = \alpha$.

The shortest CI is obtained for $\beta = \gamma = \alpha/2$.