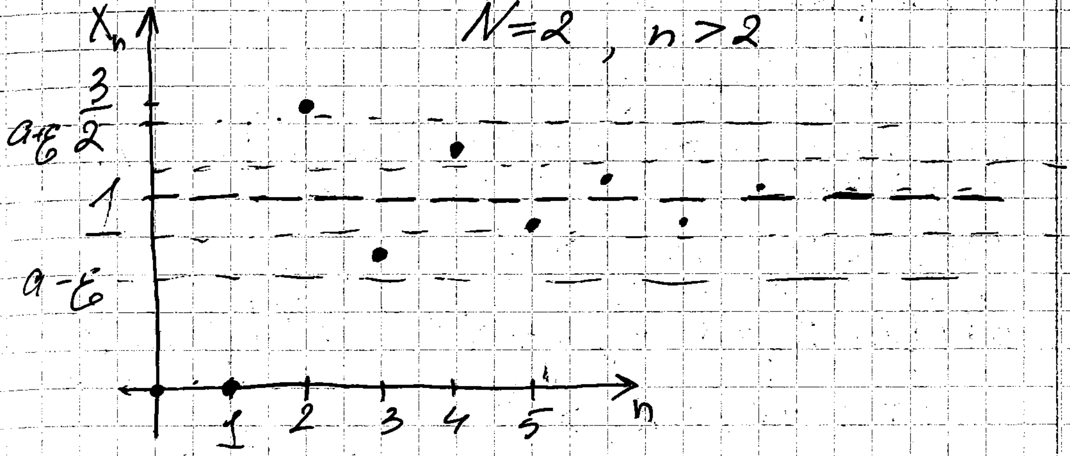
Семинар 5 (06.10.16)

~~Предел~~ Пример

$$x_n = 1 + \frac{(-1)^n}{n}$$



$$\lim_{n\to\infty} x_n = 1$$

Опр. Число $a \in \mathbb{R}$ называется **пределом** последовательности x_n, если для любого $\varepsilon > 0$ сущ-ет число $N \in \mathbb{N}$ такое, что если $n > N$, то $|x_n - a| < \varepsilon$

Обозначение: $\lim\limits_{n\to\infty} x_n = a$ или $x_n \to a$ при $n \to \infty$

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N): |x_n - a| < \varepsilon$$

$$-\varepsilon < x_n - a < \varepsilon$$
$$a - \varepsilon < x_n < a + \varepsilon$$

1) $\left(\lim\limits_{n\to\infty} x_n = a\right) \Leftrightarrow (\forall \varepsilon > 0)(\exists N \in \mathbb{N})$
$(\forall n > N): |x_n - a| < \varepsilon.$

2) $\left(\lim\limits_{n\to\infty} x_n = \infty\right) \Leftrightarrow (\forall M > 0)(\exists N \in \mathbb{N})$
$(\forall n > N): |x_n| > M$

3) $\left(\lim x_n = +\infty\right) \Leftrightarrow (\forall M > 0)(\exists N \in \mathbb{N})$
$(\forall n > N): x_n > M$

4) $\left(\lim x_n = -\infty\right) \Leftrightarrow (\forall M > 0)(\exists N \in \mathbb{N})$
$(\forall n > N): x_n < -M.$

~~5)~~ Если послед. (x_n) имеет конечный предел, то говорят, что она <u>сходится</u>. Если послед. (x_n) имеет бесконечный предел или не имеет никакого (ни конечного, ни бесконечного) предела, то говорят, что посл (x_n) <u>расходится</u>

Последовательность (x_n)

имеет конечный предел $a \in \mathbb{R}$ — сходится

имеет бескон. предел $(\infty, +\infty, -\infty)$

не имеет никакого предела

расходится

бесконечно большая послед-ть

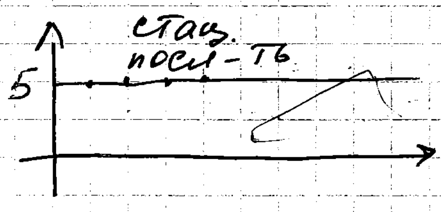
$\lim\limits_{n \to \infty} x_n = 0$ — бесконечно малая послед.

Задача

$$\lim_{n \to \infty} \frac{5n^2 - 8n}{7n^2 + 3n - 11} = \lim_{n \to \infty} \frac{n^2\left(5 - \frac{8}{n} + \frac{2}{n^2}\right)}{n^2\left(7 + \frac{3}{n} - \frac{11}{n^2}\right)} =$$

$$= \lim_{n \to \infty} \frac{5 - \frac{8}{n} + \frac{2}{n^2}}{7 + \frac{3}{n} - \frac{11}{n^2}} = \frac{\lim\limits_{n \to \infty}\left(5 - \frac{8}{n} + \frac{2}{n^2}\right)}{\lim\limits_{n \to \infty}\left(7 + \frac{3}{n} - \frac{11}{n^2}\right)} =$$

$$= \frac{\lim\limits_{n \to \infty} 5 - \lim\limits_{n \to \infty} \frac{8}{n} + \lim\limits_{n \to \infty} \frac{2}{n^2}}{\lim\limits_{n \to \infty} 7 + \lim\limits_{n \to \infty} \frac{3}{n} - \lim\limits_{n \to \infty} \frac{11}{n^2}} = \frac{5 - 0 + 0}{7 + 0 - 0} = \frac{5}{7}$$



$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} = 0 \quad (\alpha > 0)$$

$$\lim_{n\to\infty} \frac{5n^2-8n+2}{7n^3+3n-11} = \lim_{n\to\infty} \frac{n^2\left(5-\frac{8}{n}+\frac{2}{n^2}\right)}{n^3\left(7+\frac{3}{n^2}-\frac{11}{n^2}\right)} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{5-\frac{8}{n}+\frac{2}{n^2}}{7+\frac{3}{n^2}-\frac{11}{n^3}} = 0 \cdot \frac{5}{7} = 0$$

$$\lim_{n\to\infty} \frac{5n^3-8n+2}{7n^2+3n-11} = \lim_{n\to\infty} \frac{n^3\left(5-\frac{8}{n^2}+\frac{2}{n^3}\right)}{n^2\left(7+\frac{3}{n}-\frac{11}{n^2}\right)} =$$

$$= \lim_{n\to\infty} n \cdot \frac{5-\frac{8}{n^2}+\frac{2}{n^3}}{7+\frac{3}{n}-\frac{11}{n^2}} = +\infty$$

$$\lim_{n\to\infty} \frac{(7-3n^2)^2}{(1-2n^2)(2-5n^3)} = \lim_{n\to\infty} \frac{n^4\left(\frac{7}{n^2}-3\right)^2}{n^5\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^3}-5\right)} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\left(\frac{7}{n^2}-3\right)^2}{\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^3}-5\right)} = 0 \cdot \frac{9}{10} = 0$$

$$\lim_{n\to\infty} = \frac{(7-3n^2)^3}{(1-2n^2)(2-5n^3)} = \lim_{n\to\infty} \frac{n^6\left(\frac{7}{n^2}-3\right)^3}{n^5\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^3}-5\right)} =$$

$$= \lim n \cdot \frac{\left(\frac{7}{n^2}-3\right)^3}{n^5\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^3}-5\right)} = \left[+\infty \cdot \left(-\frac{27}{10}\right)\right] =$$

$$= -\infty$$

$$\lim_{n\to\infty}\frac{(7-3n^2)^3}{(1-2n^2)(2-5n^4)}=\lim_{n\to\infty}\frac{n^6\left(\frac{7}{n^2}-3\right)^3}{n^6\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^4}-5\right)}=$$

$$=\lim_{n\to\infty}\frac{\left(\frac{7}{n^2}-3\right)^3}{\left(\frac{1}{n^2}-2\right)\left(\frac{2}{n^4}-5\right)}=\frac{-27}{10}=-2,7.$$

$$\lim_{n\to\infty}\frac{\sqrt{2n^4-n^3+10}}{(3n-5)\sqrt[3]{n+2}}=\lim_{n\to\infty}\frac{n^2\sqrt{2-\frac{1}{n}+\frac{10}{n^4}}}{n^{\frac{4}{3}}\left(3-\frac{5}{n}\right)\sqrt[3]{1+\frac{2}{n}}}=$$

$$\lim_{n\to\infty} n^{\frac{2}{3}}\cdot\frac{\sqrt{2-\frac{1}{n}+\frac{10}{n^4}}}{\left(3-\frac{5}{n}\right)\sqrt[3]{1+\frac{2}{n}}}=\left[+\infty\cdot\frac{\sqrt{2}}{3}\right]=+\infty$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{2n^4-n^3+10}}{(3n-5)\sqrt{n+2}}=\frac{n^{\frac{4}{3}}\sqrt[3]{2-\frac{1}{n}+\frac{10}{n^4}}}{n^{\frac{3}{2}}\sqrt{1+\frac{2}{n}}\left(3-\frac{5}{n}\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{n^{\frac{1}{6}}}\cdot\frac{\sqrt[3]{2-\frac{1}{n}+\frac{10}{n^4}}}{\sqrt{1+\frac{2}{n}}\left(3-\frac{5}{n}\right)}=0\cdot\frac{\sqrt[3]{2}}{3}=0$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{2n^4-n^3+10}}{(3n-5)\sqrt[3]{n+2}}=\lim_{n\to\infty}\frac{n^{\frac{4}{3}}\sqrt[3]{2-\frac{1}{n}+\frac{10}{n^4}}}{n^{\frac{4}{3}}\left(3-\frac{5}{n}\right)\sqrt[3]{1+\frac{2}{n}}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[3]{2-\frac{1}{n}+\frac{10}{n^4}}}{\left(3-\frac{5}{n}\right)\sqrt[3]{1+\frac{2}{n}}}=\frac{\sqrt[3]{2}}{3}$$

$$\lim_{n\to\infty}\left(\frac{n-2}{2n-1}-\frac{5n+1}{7n-1}\right)=\lim_{n\to\infty}\frac{n-2}{2n-1}-\lim_{n\to\infty}\frac{5n+1}{7n-1}=$$

$$=\lim_{n\to\infty}\frac{n\left(1-\frac{2}{n}\right)}{n\left(2-\frac{1}{n}\right)}-\lim_{n\to\infty}\frac{5+\frac{1}{n}}{7-\frac{1}{n}}=\frac{1}{2}-\frac{5}{7}=-\frac{3}{14}$$

$$\lim_{n\to\infty}\left(\frac{n^2-n}{2n-1}-\frac{3n^2-1}{6n+1}\right)=[\infty-\infty]$$

$$\cancel{\lim_{n\to\infty}\frac{n^2\left(1-\frac{1}{n}\right)}{n\left(2-\frac{1}{n}\right)}-\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n\left(6+\frac{1}{n}\right)}=}$$

$$\cancel{=\lim_{n\to\infty} n\cdot\frac{1-\frac{1}{n}}{2-\frac{1}{n}}-\lim n\cdot\frac{3-\frac{1}{n^2}}{6+\frac{1}{n}}=}$$

$$\cancel{=\left[+\infty\cdot\frac{1}{2}-\left(+\infty\cdot\frac{1}{2}\right)\right]=0}$$

$$=\lim_{n\to\infty}\frac{\cancel{6n^3}-6n^2+n^2-n-\cancel{6n^3}+2n+3n^2-1}{(2n-1)(6n+1)}=$$

$$=\lim_{n\to\infty}\frac{-2n^2+n-1}{n^2\left(2-\frac{1}{n}\right)\left(6+\frac{1}{n}\right)}=\cancel{n^2}\lim_{n\to\infty}\frac{-2+\frac{1}{n}-\frac{1}{n^2}}{\left(2-\frac{1}{n}\right)\left(6+\frac{1}{n}\right)}=$$

$$=-\frac{1}{6}$$

$$\lim_{n\to\infty}\left(\sqrt{3n^2+5}-n\sqrt{3}\right)=[\infty-\infty]=$$

$$\lim_{n\to\infty}\frac{\left(\sqrt{3n^2+5}-n\sqrt{3}\right)\left(\sqrt{3n^2+5}+n\sqrt{3}\right)}{\sqrt{3n^2+5}+n\sqrt{3}}=$$

$$=\lim_{n\to\infty}\frac{3n^2+5-3n^2}{\sqrt{3n^2+5}+n\sqrt{3}}=\lim_{n\to\infty}\frac{5}{n\left(\sqrt{3+\frac{5}{n^2}}+\sqrt{3}\right)}=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{5}{\sqrt{3+\frac{5}{n^2}}+\sqrt{3}}$$

$$=0\cdot\frac{5}{2\sqrt{3}}=0$$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+2n^2}-n\right)=\lim_{n\to\infty}\frac{n^3+2n^2-n^3}{(n^3+2n^2)^{\frac{2}{3}}+n\sqrt[3]{n^3+2n^2}+n^2}=$$

$$=\lim_{n\to\infty}\frac{2n^2}{n^2\left(\left(1+\frac{2}{n}\right)^{\frac{2}{3}}+\sqrt[3]{1+\frac{2}{n}}+1\right)}=$$

$$=\lim_{n\to\infty}\frac{2}{\left(1+\frac{2}{n}\right)^{\frac{2}{3}}+\sqrt[3]{1+\frac{2}{n}}+1}=\frac{2}{3}$$

$$\lim_{n\to\infty}\frac{3^{2n}+2^{n+3}}{3^{n-2}-2^{3n}}=\lim_{n\to\infty}\frac{9^n+8\cdot2^n}{\frac{1}{9}\cdot3^n-8^n}=$$

$$=\lim_{n\to\infty}\frac{9^n\left(1+\frac{8\cdot2^n}{9^n}\right)}{8^n\left(\frac{1\cdot3^n}{9\cdot8^n}-1\right)}=\lim\left(\frac{9}{8}\right)^n\cdot\frac{1+8\left(\frac{2}{9}\right)^n}{\frac{1}{9}\cdot\left(\frac{3}{8}\right)^n-1}=$$

$$=\left[+\infty\cdot(-1)\right]=-\infty$$

$$\lim_{n\to\infty}\frac{3^n+2^{2n+3}}{3^{n-2}-2^{3n}}=\lim_{n\to\infty}\frac{3^n+8\cdot4^n}{\frac{1}{9}\cdot3^n-8^n}=$$

$$=\lim_{n\to\infty}\frac{4^n\left(\left(\frac{3}{4}\right)^n+8\right)}{8^n\left(\frac{1}{9}\left(\frac{3}{8}\right)^n-1\right)}=\lim_{n\to\infty}\left(\frac{1}{2}\right)^n\cdot\frac{\left(\frac{3}{4}\right)^n+8}{\frac{1}{9}\left(\frac{3}{8}\right)^n-1}=0\cdot(-8)=$$

$$=0$$

$$\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n+\sqrt{n}}}-\sqrt{n}\right)=\frac{1}{2}$$

$$\lim_{n\to\infty}\frac{(3^n+4^n)^3}{(7^n+8^n)^2}=1$$

83 $$\lim_{n\to\infty}n\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)=$$

~~lim~~ Задача $x_n = \frac{7}{6} + \frac{7}{30} + \frac{7}{150} + \ldots \frac{7}{6\cdot 5^{n-1}}$

Найти: $\lim\limits_{n\to\infty} x_n$

$$x_n = \cancel{\frac{7}{6}}\,\frac{7}{6}\cdot\left(\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}}\right.$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{7}{6}\cdot\frac{1-\left(\frac{1}{5}\right)^n}{\cancel{1}\,\frac{4}{5}} = \frac{7}{6}\cdot\frac{5}{4} = \frac{35}{24}$$

$\cancel{= \lim}$

Задача $x_n = 0{,}5\underbrace{144144\ldots144}_{n}$

$$x_n = 0{,}5 + \cancel{0{,}0144}\cdot 144\cdot 10^{\cancel{-4}\,-4} + 144^{-7} + \ldots + 144\cdot 10^{\cancel{-4x-3(n-1)}\,-1-3n}$$

$\cancel{x_n = 0{,}5 + 144\cdot}$

$\cancel{x_n = 0{,}5144*}$

$$x_n = 0{,}5 + 144\,\frac{1-10^{-3n}}{1-10^{-3}}\cdot 10^{-4}$$

$$\lim_{n\to\infty} x_n = 0{,}5 + \lim 144\cdot 10^{-4}\cdot\frac{1-\frac{1}{10^{3n}}}{0{,}999} =$$

$$= \frac{1}{2} + \frac{144}{10^4\cdot 0{,}999} = \frac{1}{2} + \frac{144}{9990} = \frac{1}{2} + \frac{16}{1110} =$$

$$= \frac{571}{1110}$$

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Задача $x_n = \frac{11}{2\cdot 7} + \frac{11}{7\cdot 12} + \frac{11}{12\cdot 17} + \ldots \frac{11}{(5n-3)(5n+2)}$

$$x_n = \frac{11}{5}\left[\left(\frac{1}{2} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{12}}\right) + \left(\cancel{\frac{1}{12}} - \cancel{\frac{1}{17}}\right) + \ldots\right.$$

$$\left. + \left(\cancel{\frac{1}{5n-3}} - \frac{1}{5n+2}\right)\right] = \frac{11}{5}\left(\frac{1}{2} - \frac{1}{5n+2}\right) =$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{11}{5}\left(\frac{1}{2} - \frac{1}{5n+2}\right) = \frac{11}{10}$$

Задача $x_n = \cancel{\log}\ \ln(1+1) + \ln\left(1+\frac{1}{2}\right) + \ln\left(1+\frac{1}{3}\right) + \ldots + \ln\left(1+\frac{1}{n}\right)$ $\quad \sum \ln\left(1-\frac{1}{n^2}\right) - Д3.$

$$x_n = \ln 2 + \ln\frac{3}{2} + \ln\frac{4}{3} + \ldots \ln\frac{n+1}{n}$$

$$x_n = \ln \cancel{2}\cdot\frac{3}{\cancel{2}}\cdot\frac{4}{\cancel{3}}\cdot\ldots\cdot\frac{n+1}{\cancel{n}} = \ln(n+1)$$

$$\lim_{n\to\infty} x_n = \lim \ln(n+1) = +\infty$$

Задача $x_n = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{n! + (n+1)!}{(n+2)! - n(n+1)!} = \lim_{n\to\infty} \frac{n! + n!(n+1)}{n!(n+1)(n+2) - n\cdot n!(n+1)} =$$

$$= \lim_{n\to\infty} \frac{2+n}{(n+1)(n+2) - n(n+1)} = \lim_{n\to\infty} \frac{\cancel{n\left(\frac{2}{n}+1\right)}\ 2+n}{(n+1)(n+2-n)} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}}\cdot\frac{\cancel{\left(\frac{2}{n}+1\right)}\ 2+n}{2n+2} = \frac{1}{2}$$

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Утв. Если последовательность (x_n) бесконечно малая (то есть $\lim\limits_{n\to\infty} x_n = 0$), а последовательность (y_n) ограниченная, то послед. $(x_n y_n)$ бесконечно малая

Задача

$$\lim_{n\to\infty} \frac{\sqrt{n+3}}{n+5}\cdot \sin(n^2+10) = 0$$

$$\lim_{n\to\infty} \frac{\sqrt{n+3}}{n+5} = \frac{n^{\frac{1}{2}}\sqrt{1+\frac{3}{n}}}{n\left(1+\frac{5}{n}\right)} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{\sqrt{1+\frac{3}{n}}}{1+\frac{5}{n}} =$$

$= 0\cdot 1 = 0$

Послед. $y_n = \sin(n^2+10)$ ограниченная

$|\sin\alpha| \le 1$

Задача $\lim\limits_{n\to\infty} \underbrace{\arcsin\frac{n}{n^2+1}}_{\to 0}\cdot \underbrace{\operatorname{arctg}\frac{n^2+1}{n}}_{\to\frac{\pi}{2}} =$

$= 0$

ограничен.

$e^{100\sin n}$

Теорема о промежуточной последовательности.

$(\forall n \in \mathbb{N})\ x_n \le z_n \le y_n$

$\lim\limits_{n\to\infty} x_n = a = \lim\limits_{n\to\infty} y_n$

Тогда $\lim\limits_{n\to\infty} z_n = a$

Задача

$\lim\limits_{n\to\infty} \cancel{x^n} \frac{5^n}{n^n} =$

Если $n \ge 10$, то

$0 < \frac{5^n}{n^n} \le \frac{5^n}{10^n} = \left(\frac{1}{2}\right)^n$

$0 \to 0$, $\left(\frac{1}{2}\right)^n \Rightarrow 0$

По теореме о промеж. послед.

$\lim \frac{5^n}{n^n} = 0$

Задача $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \dots + \frac{1}{\sqrt{n^2+n}}$

$\frac{n}{\sqrt{n^2+n}} \le x_n \le \frac{n}{\sqrt{n^2+1}}$

$\frac{n}{\sqrt{n^2+n}} \Rightarrow 1$, $\frac{n}{\sqrt{n^2+1}} \Rightarrow 1$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=$$

$$= \cancel{\alpha}\, 1$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+\frac{1}{n^2}}}=$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$$

$$\lim_{n\to\infty} x_n = 1.$$

Семинар 6 (13.10.16)

Задача: $\lim\limits_{n\to\infty}\cos(\pi\sqrt{4n^2+n})=$

$$=\lim_{n\to\infty}\cos(\pi\sqrt{4n^2+n}-2\pi n)=$$

$$=\lim_{n\to\infty}\cos\left(\pi\left(\sqrt{4n^2+n}-2n\right)\right)=$$

$$=\lim_{n\to\infty}\cos\left(\pi\frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n}\right)=$$

$$=\lim_{n\to\infty}\cos\left(\pi\frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n}\right)=$$

$$=\lim_{n\to\infty}\cos\left(\pi\frac{n}{n(\sqrt{4+\frac{1}{n}}+2)}\right)=$$

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