Homework

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2024 年 12 月 31 日

Exercise 1. **Singular Value Decomposition**

Solution. 1. (a)(d) Denote $\mathbf{A}$ by $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_n)^T$. Consider

$$\mathbf{y}_0 = \mathbf{argmin}_{\mathbf{y}\in\mathbb{R}^n} \|\mathbf{x} - \mathbf{A}\mathbf{y}\|^2,$$

we can see that $\mathbf{A}\mathbf{y}_0 = P_{\mathcal{C}(\mathbf{A})}(\mathbf{x})$. Since

$$\begin{aligned}
\|\mathbf{x} - \mathbf{A}\mathbf{y}\|^2 &= \|\mathbf{x} - \mathbf{U\Sigma V}^T\mathbf{y}\|^2 \\
&= \|\mathbf{U}^T\mathbf{x} - \mathbf{\Sigma V}^T\mathbf{y}\|^2 \\
&= \|\mathbf{U}^T\mathbf{x} - \mathbf{\Sigma z}\|^2 \\
&= \sum_{i=1}^{r}(\mathbf{u}_i^T\mathbf{x} - \sigma_i z_i)^2 + \sum_{i=r+1}^{m}(\mathbf{u}_i^T\mathbf{x})^2
\end{aligned}$$

where $\mathbf{z} = \mathbf{V}^T\mathbf{y}$. Therefore, we can set $z_i = \frac{\mathbf{u}_i^T\mathbf{x}}{\sigma_i}$ to fit the equation above, and set $z_{r+1} = \cdots = z_m = 0$ to minimize $\|\mathbf{z}\|_2$, leading to

$$\mathbf{y}_0 = \mathbf{Vz} = \mathbf{VSU}^T\mathbf{x},$$

where $\mathbf{S} = diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0)$. Thus, we have

$$P_{\mathcal{C}(\mathbf{A})}(\mathbf{x}) = \mathbf{U\Sigma V}^T\mathbf{VSU}^T\mathbf{x} = \mathbf{U\Sigma SU}^T\mathbf{x} = \mathbf{U}diag(\mathbf{I}_r, 0)\mathbf{U}^T = \mathbf{U}_1\mathbf{U}_1^T\mathbf{x}.$$

Next, we will prove (d) in three steps. Since

$$\mathbf{a}_i^T\mathbf{y} = 0,\ \forall\, \mathbf{y} \in \mathcal{N}(\mathbf{A}^T),$$

and

$$\mathbf{z} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i,\ \forall\, \mathbf{z} \in \mathcal{C}(\mathbf{A}),$$

we have

$$\mathbf{z}^T\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i^T\mathbf{y} = 0,\ \forall\, \mathbf{y} \in \mathcal{N}(\mathbf{A}^T), \mathbf{z} \in \mathcal{C}(\mathbf{A}),$$

that is, $\mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^T)$.

Since $\mathcal{C}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^T)$ are closed linear subspaces in $\mathbb{R}^m$, by the orthogonal decomposition we have learned in functional analysis class, we have

$$\forall \mathbf{x} \in \mathbb{R}^m,\ \exists\, \mathbf{y} \in \mathcal{C}(\mathbf{A}), \mathbf{z} \in \mathcal{N}(\mathbf{A}^T),\ \text{ s.t. }\ \mathbf{x} = \mathbf{y} + \mathbf{z}.$$

Moreover, $\mathbf{y}$ is the element of best approximation of $\mathbf{x}$ on the subspace $\mathcal{C}(\mathbf{A})$, which is exactly $P_{\mathcal{C}(\mathbf{A})}(\mathbf{x})$. Therefore, $\mathbf{z} = \mathbf{x} - \mathbf{y}$ satisfies

$$\mathbf{x} - \mathbf{z} = \mathbf{x} - \mathbf{x} + \mathbf{y} = \mathbf{y} = P_{\mathcal{C}(\mathbf{A})}(\mathbf{x}) \in \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \mathbf{x} - \mathbf{z} \perp \mathcal{N}(\mathbf{A}^T)$$

which is the best approximation condition. Thus, $\mathbf{z}$ is the element of best approximation of $\mathbf{x}$ on the subspace $\mathcal{N}(\mathbf{A}^T)$, which implies that

$$\mathbf{z} = P_{\mathcal{N}(\mathbf{A}^T)}(\mathbf{x}) = \mathbf{x} - \mathbf{y} = \mathbf{x} - \mathbf{U}_1\mathbf{U}_1^T\mathbf{x} = (\mathbf{I} - \mathbf{U}_1\mathbf{U}_1^T)\mathbf{x} = \mathbf{U}_2\mathbf{U}_2^T\mathbf{x},$$

thus complete the proof.

(b)(c) Since $\mathbf{A}^T = \mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^T$, we can obtain these conclusions by following the same steps in (a)(d).

2.(a) Since

$$(\mathbf{A}^T\mathbf{A})_{ii} = \sum_{j=1}^{m} a_{ji} \cdot a_{ji} = \sum_{j=1}^{m} a_{ji}^2,$$

we have

$$tr(\mathbf{A}^T\mathbf{A}) = \sum_{i=1}^{n} (\mathbf{A}^T\mathbf{A})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ji}^2 = \|\mathbf{A}\|_F^2$$

(b) Let $\mathbf{a} = \mathbf{A}\mathbf{e}_i$, $\mathbf{b} = \mathbf{B}\mathbf{e}_i$, $i \in [n]$, we have

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{e}_i^T\mathbf{A}^T\mathbf{B}\mathbf{e}_i = (\mathbf{A}^T\mathbf{B})_{ii} = \sum_{j=1}^{m} a_{ji}b_{ji} = 0.$$

which implies that

$$tr(\mathbf{A}^T\mathbf{B}) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ji}b_{ji} = 0.$$

Therefore, we have

$$\begin{aligned}
\|\mathbf{A}+\mathbf{B}\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n} (a_{ij}+b_{ij})^2 \\
&= \|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2 + 2\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} \\
&= \|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2 + 2tr(\mathbf{A}^T\mathbf{B}) \\
&= \|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2
\end{aligned}$$

Exercise 2. **Principle Component Analysis**

Solution. 1. We have

$$\begin{aligned}
f(\mathbf{GQ}) = tr((\mathbf{GQ})^T\mathbf{SCQ}) &= tr(\mathbf{Q}^T\mathbf{G}^T\mathbf{SGQ}) \\
&= tr(\mathbf{QQ}^T\mathbf{G}^T\mathbf{SG}) \\
&= tr(\mathbf{G}^T\mathbf{SG}) = f(\mathbf{G})
\end{aligned}$$

2. Since $\mathbf{g}_1^T\mathbf{S}\mathbf{g}_1 \in \mathbb{R}$, we have $f(\mathbf{g}) = \mathbf{g}_1^T\mathbf{S}\mathbf{g}_1$. Therefore, the Lagrange function

is

$$L(\mathbf{g}, \lambda) = f(\mathbf{g}) + \lambda(1 - \|\mathbf{g}\|_2^2).$$

Since $\mathbf{S}$ is symmetric, taking derivative and we get

$$\mathbf{Sg} + \mathbf{Sg} - 2\lambda\mathbf{g} = 0 \quad \Rightarrow \quad \mathbf{Sg} = \lambda\mathbf{g},$$

which implies that $\mathbf{g}$ is an eigenvector of λ. Suppose that $\mathbf{\Lambda} = \{\lambda_1, \lambda_2, ..., \lambda_r\}$ are all if the eigenvalues of $\mathbf{S}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Let $\mathbf{G} = \{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_r\}$ are corresponding eigenvectors with length 1, we have

$$\mathbf{argmax}_{\mathbf{g}_i \in \mathbf{G}}\{\mathbf{g}_i^T\mathbf{S}\mathbf{g}_i\} = \mathbf{argmax}_{\mathbf{g}_i \in \mathbf{G}}\{\lambda_i\mathbf{g}_i^T\mathbf{g}_i\} = \mathbf{argmax}_{\mathbf{g}_i \in \mathbf{G}}\{\lambda_i\} = \mathbf{g}_1.$$

which is the first principal component vector of the data.

3. The Lagrange function is

$$L(\mathbf{g}, \lambda, \mu) = f(\mathbf{g}) + \lambda(1 - \|\mathbf{g}\|_2^2) - \mu\langle\mathbf{g}, \mathbf{g}_1\rangle.$$

Taking derivative and we get

$$\begin{aligned}
&2\mathbf{Sg} - 2\lambda\mathbf{g} - \mu\mathbf{g}_1 = 0 \\
&\Rightarrow 2\mathbf{g}_1^T\mathbf{Sg} - 2\lambda\mathbf{g}_1^T\mathbf{g} - \mu\mathbf{g}_1^T\mathbf{g}_1 = 0 \\
&\Rightarrow \mu = 2\mathbf{g}_1^T\mathbf{Sg} - 2\lambda\mathbf{g}_1^T\mathbf{g} = 2\lambda_1\mathbf{g}_1^T\mathbf{g} = 0
\end{aligned}$$

Therefore, we have

$$\mathbf{g}_2 = \mathbf{argmax}_{\mathbf{g}_i \in \mathbf{G}, \mathbf{g}_i \neq \mathbf{g}_1}\{\lambda_i\} = \lambda_2.$$

is the second principal component vector of the data.

4. From the same steps, we have

$$\mathbf{g}_K = \mathbf{argmax}\{f(\mathbf{g}) : \|\mathbf{g}\| = 1, \langle\mathbf{g}_i, \mathbf{g}\rangle = 0, i = 1, 2, \ldots, K-1\}$$

5. From the content in exercise 10.1 in HW1, we know that $f(\mathbf{g}_K) = \lambda_K$, which is the K-th largest eigenvalue of $\mathbf{S}$.

Exercise 3. **Properties of Transition Matrix**

Solution. 1. Set $\mathbf{a} = (1, 1, \dots, 1)^T$, we have

$$\mathbf{Ta} = (\sum_{i=1}^{n} t_1 i, \sum_{i=1}^{n} t_2 i, \dots, \sum_{i=1}^{n} t_n i)^T = (1, 1, \dots, 1)^T,$$

which implies that 1 is a eigenvalue of $\mathbf{T}$.

2. From exercise 10.1 in HW1, we can estimate $\sup_{\|\mathbf{x}\|=1} \mathbf{x}^T\mathbf{Tx} = \lambda_{max}$ directly.

$$\begin{aligned}
\sup_{\|\mathbf{x}\|=1} \mathbf{x}^T\mathbf{Tx} &= \sup_{\sum_{i=1}^n x_i^2 = 1} \sum_{i,j=1}^{n} x_i x_j t_{ij} \\
&= \sup_{\sum_{i=1}^n x_i^2 = 1} \sum_{i,j=1}^{n} x_i \sqrt{t_{ij}} x_j \sqrt{t_{ij}} \\
&\leq \frac{1}{2} \sup_{\sum_{i=1}^n x_i^2 = 1} \sum_{i,j=1}^{n} x_i^2 t_{ij} + x_j^2 t_{ij} \\
&= \frac{1}{2} \sup_{\sum_{i=1}^n x_i^2 = 1} (\sum_{i=1}^{n} x_i^2 (\sum_{j=1}^{n} t_{ij})) + (\sum_{j=1}^{n} x_j^2 (\sum_{i=1}^{n} t_{ij})) \\
&= \frac{1}{2} \sup_{\sum_{i=1}^n x_i^2 = 1} \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{n} x_j^2 \\
&= \sup_{\sum_{i=1}^n x_i^2 = 1} \sum_{i=1}^{n} x_i^2 = 1
\end{aligned}$$

Therefore, $\lambda_{max} \le 1$. On the other hand,

$$\begin{aligned}
\lambda_{min} &= \inf_{\|\mathbf{x}\|=1} \mathbf{x}^T \mathbf{T} \mathbf{x} \\
&= \inf_{\sum_{i=1}^n x_i^2 = 1} \sum_{i,j=1}^n x_i \sqrt{t_{ij}} x_j \sqrt{t_{ij}} \\
&\ge \frac{1}{2} \inf_{\sum_{i=1}^n x_i^2 = 1} \sum_{i,j=1}^n -x_i^2 t_{ij} - x_j^2 t_{ij} \\
&= -1
\end{aligned}$$

Combining together, we have $|\lambda| \le 1$.

3. From the discussion below, we have

$$\inf_{\|\mathbf{x}\|=1} \mathbf{x}^T (\mathbf{I} - \gamma \mathbf{T}) \mathbf{x} = 1 - \gamma \sup_{\|\mathbf{x}\|=1} \mathbf{x}^T \mathbf{T} \mathbf{x} \ge 1 - \gamma > 0.$$

Since $\mathbf{x}^T (\mathbf{I} - \gamma \mathbf{T}) \mathbf{x} = \frac{\mathbf{x}^T}{\|\mathbf{x}\|} (\mathbf{I} - \gamma \mathbf{T}) \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \|\mathbf{x}\|^2$, we have

$$\mathbf{x}^T (\mathbf{I} - \gamma \mathbf{T}) \mathbf{x} \ge \|\mathbf{x}\|^2 \inf_{\|\mathbf{x}\|=1} \mathbf{x}^T (\mathbf{I} - \gamma \mathbf{T}) \mathbf{x} \ge \|\mathbf{x}\|^2 (1 - \gamma) > 0,$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\|\mathbf{x}\| \ne 0$, which implies that $\mathbf{I} - \gamma \mathbf{T} \succ 0$, thus $\mathbf{I} - \gamma \mathbf{T}$ is invertible.

Exercise 4. **Planning with a Two-Armed Bandit**

Solution. 1. We have

$$\mathcal{S} = \{state\_1, state\_2\} = \{s_1, s_2\},$$

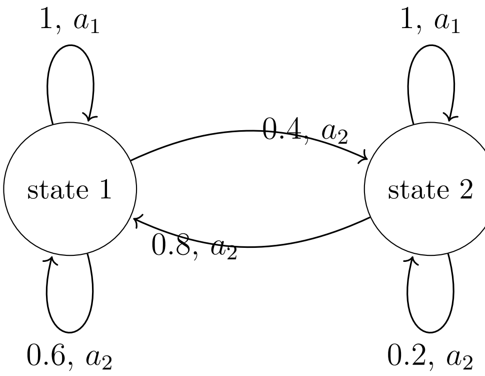
and

$$\mathcal{A} = \{pull\_bandit\_1, pull\_bandit\_2\} = \{a_1, a_2\}.$$

Therefore,

$$P(s_1|s_1,a_1)=1, \qquad P(s_2|s_1,a_1)=0.$$
$$P(s_1|s_1,a_2)=0.6, \qquad P(s_2|s_1,a_2)=0.4.$$
$$P(s_1|s_2,a_1)=0, \qquad P(s_2|s_2,a_1)=1.$$
$$P(s_1|s_2,a_2)=0.8, \qquad P(s_2|s_2,a_2)=0.2.$$

Markov process diagram:



2.(a) We have

$$\begin{aligned}V^{\pi_1}(s_1)&=\mathbb{E}[r(s_1,a_2)]+\gamma\mathbb{E}[G_{t+1}]\\&=0+\gamma(P(s'=s_1)V^{\pi_1}(s_1)+P(s'=s_2)V^{\pi_1}(s_2))\\&=0.54V^{\pi_1}(s_1)+0.36V^{\pi_1}(s_2)\end{aligned}$$

and

$$\begin{aligned}V^{\pi_1}(s_2)&=\mathbb{E}[r(s_2,a_2)]+\gamma\mathbb{E}[G_{t+1}]\\&=3+\gamma(P(s'=s_1)V^{\pi_1}(s_1)+P(s'=s_2)V^{\pi_1}(s_2))\\&=3+0.72V^{\pi_1}(s_1)+0.18V^{\pi_1}(s_2)\end{aligned}$$

Combining together, we have

$$V^{\pi_1}(s_1)=9.1525, \quad V^{\pi_1}(s_2)=11.6949.$$

(b) Similarly,

$$\begin{aligned} V^{\pi_2}(s_1) &= \mathbb{E}[r(s_1, a_2)] + \gamma \mathbb{E}[G_{t+1}] \\ &= 0 + \gamma(P(s' = s_1)V^{\pi_2}(s_1) + P(s' = s_2)V^{\pi_2}(s_2)) \\ &= 0.54V^{\pi_2}(s_1) + 0.36V^{\pi_2}(s_2) \end{aligned}$$

$$\begin{aligned} V^{\pi_2}(s_2) &= \mathbb{E}[r(s_2, a_1)] + \gamma \mathbb{E}[G_{t+1}] \\ &= 2 + \gamma(P(s' = s_1)V^{\pi_2}(s_1) + P(s' = s_2)V^{\pi_2}(s_2)) \\ &= 2 + 0.9V^{\pi_2}(s_2) \end{aligned}$$

We have

$$V^{\pi_1}(s_1) = 15.6522, \qquad V^{\pi_1}(s_2) = 20.$$

3. We have these policies:

$$\begin{aligned} &\pi_1 : \pi_1(s_1) = a_1, \pi_1(s_2) = a_1. \\ &\pi_2 : \pi_1(s_1) = a_2, \pi_1(s_2) = a_2. \\ &\pi_3 : \pi_1(s_1) = a_1, \pi_1(s_2) = a_2. \\ &\pi_4 : \pi_1(s_1) = a_2, \pi_1(s_2) = a_1. \end{aligned}$$

Since $V = (\mathbf{I} - \gamma\mathbf{T})^{-1}\mathbf{R}$, we have

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{T}_2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{pmatrix}, \mathbf{T}_3 = \begin{pmatrix} 1 & 0 \\ 0.8 & 0.2 \end{pmatrix}, \mathbf{T}_4 = \begin{pmatrix} 0.6 & 0.4 \\ 0 & 1 \end{pmatrix}.$$

and

$$\mathbf{R}_1 = (1, 2)^T, \mathbf{R}_2 = (0, 3)^T, \mathbf{R}_3 = (1, 3)^T, \mathbf{R}_4 = (0, 2)^T.$$

For $\gamma = 0.1$,

$$V_1 = (1.11, 2.22)^T, V_2 = (0.13, 3.07)^T, V_3 = (1.11, 3.15)^T, V_4 = (0.10, 2.22)^T,$$

and we can see that the best policy here is π_3.
For $\gamma = 0.99$, similarly we get

$$V_1 = (100, 200)^T, V_2 = (99.17, 101.67)^T, V_3 = (100, 102.49)^T, V_4 = (195.07, 200)^T,$$

which means we have the best policy π_4 here.
γ can be understood as the discount rate in finance, which represents the ratio of expected returns in a certain period of time in the future to present value based on the time value of money. High γ value mean a greater focus on future earnings, while low γ value aims to get more profits in short term.