# Homework

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EXERCISE 1. **Affine Sets**

SOLUTION. 1.(a) ($\Rightarrow$) Since $U$ is an affine set, for all $\mathbf{x}, \mathbf{y} \in U$ and $\theta \in \mathbb{R}$, we have

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} \in U.$$

Since $\mathbf{0} \in U$, let $\mathbf{y} = \mathbf{0}$. Therefore, for all $\mathbf{x} \in U$ and $\theta \in \mathbb{R}$, we have $\theta\mathbf{x} \in U$. Let $\theta = \frac{1}{2}$, then

$$\forall\, \mathbf{x}, \mathbf{y} \in U, \mathbf{x} + \mathbf{y} = 2(\frac{\mathbf{x}+\mathbf{y}}{2}) \in U.$$

Thus, $U$ is a subspace.

($\Leftarrow$) Since $U$ is a subspace, for all $\mathbf{x}, \mathbf{y} \in U$ and $\theta \in \mathbb{R}$, we have

$$\theta\mathbf{x} \in U, (1-\theta)\mathbf{y} \in U \Rightarrow \theta\mathbf{x} + (1-\theta)\mathbf{y} \in U.$$

Thus, $U$ is an affine set.

(b) Suppose that $V_1$ and $V_2$ are two subspaces that satisfy the condition. Let $\mathbf{u}_0 \in U$. Then we have

$$V_1 = U - \mathbf{u}_0 = V_2,$$

which implies that $V$ must be unique.

Now, let $V = U - \mathbf{u}_0$, then $V$ is also an affine set. Since $\mathbf{0} = \mathbf{u}_0 - \mathbf{u}_0 \in V$, from exercise 1.1(a) we know that $V$ is a subspace. Let $\mathbf{u}_1$ be an arbitrary vector in $U$. We will prove that

$$V + \mathbf{u}_1 = U.$$

In fact, for all $\mathbf{u}_2 \in U$, since $\mathbf{u}_1$ and $\mathbf{u}_0 \in U$, let $\theta = \frac{1}{2}$. We have

$$\frac{\mathbf{u}_1 + \mathbf{u}_0}{2}, \quad \frac{\mathbf{u}_2 + \mathbf{u}_0}{2} \in U \Rightarrow \frac{\mathbf{u}_1 - \mathbf{u}_0}{2}, \quad \frac{\mathbf{u}_2 - \mathbf{u}_0}{2} \in V.$$

Since $V$ is a subspace, we have

$$\mathbf{u}_2 - \mathbf{u}_1 = 2\left(\frac{\mathbf{u}_2 - \mathbf{u}_0}{2} - \frac{\mathbf{u}_1 - \mathbf{u}_0}{2}\right) \in V \Rightarrow \mathbf{u}_2 \in V + \mathbf{u}_1.$$

Therefore, we conclude that

$$U \subset V + \mathbf{u}_1.$$

Since

$$|V + \mathbf{u}_1| = |U - \mathbf{u}_0 + \mathbf{u}_1| = |U|,$$

we have

$$V + \mathbf{u}_1 = U, \forall\, \mathbf{u}_1 \in U.$$

Thus, we conclude that

$$U = \mathbf{u} + V, \forall\, \mathbf{u} \in U.$$

2.(a) Suppose that $\mathbf{x}_1$, $\mathbf{x}_2 \in C$ and $\theta \in \mathbb{R}$, then we have

$$\mathbf{A}(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) = \theta\mathbf{A}\mathbf{x}_1 + (1-\theta)\mathbf{A}\mathbf{x}_2$$

$$= \theta\mathbf{b} + (1-\theta)\mathbf{b} = \mathbf{b},$$

which implies that $\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C$. Thus $C$ is an affine set.

(b) From exercise 1.1(b), let $\mathbf{u}_0 \in U$. Then $V = U - \mathbf{u}_0$ is a subspace. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a basis of $V$, where $k \leq n$.
Since $\mathbf{X} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_k^T)^T \in \mathbb{R}^{k \times n}$, the equation $\mathbf{X}\mathbf{a} = \mathbf{0}$ has at least one nontrivial solution, denoted by $\mathbf{a}_0$. Therefore, we have

$$\mathbf{a}_0^T \mathbf{x}_i = 0, \quad i = 1, 2, \ldots, k.$$

Let $\mathbf{A} = (\mathbf{a}_0, \mathbf{a}_0, \ldots, \mathbf{a}_0)^T \in \mathbb{R}^{m \times n}$. Since

$$\mathbf{u} - \mathbf{u}_0 = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i, \forall \mathbf{u} \in U,$$

we have

$$\mathbf{A}(\mathbf{u} - \mathbf{u}_0) = \sum_{i=1}^{k} \lambda_i \mathbf{A}\mathbf{x}_i = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{u}_0 \triangleq \mathbf{b} \in \mathbb{R}^{m \times 1}, \quad \forall \mathbf{u} \in U.$$

Thus, we have proved that

$$U \subset \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

On the other hand, suppose that $\mathbf{x}_0 \in \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$. This implies

$$\mathbf{x}_0 - \mathbf{u}_0 \in \mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{X}).$$

Since $V$ is a subspace, $\mathcal{C}(\mathbf{X}) = V$, which implies that $\mathbf{x}_0 \in \mathbf{U}$. Therefore, we conclude that

$$\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} = U.$$

Exercise 2. **Convex Sets**

Solution. 1.(a) For all $\mathbf{x} \in \bar{C}$, since $\bar{C} = C \cap C'$, we consider two cases:
(i) If $\mathbf{x} \in C$, let $\mathbf{x}_n = \mathbf{x}$ for $n = 1, 2, \ldots$. Then $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$.
(ii) If $\mathbf{x} \in C'$, since $\mathbf{x}$ is a limit point of $C$, we can find a sequence $\{\mathbf{x}_n\} \subset C$ such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$.
In both cases, we conclude that for all $\mathbf{x} \in \bar{C}$, there exists a sequence $\{\mathbf{x}_n\} \subset C$ such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$.
Now, for all $\mathbf{x}, \mathbf{y} \in \bar{C}$ and $\theta \in \mathbb{R}$, let $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$. Since $C$ is convex, we have

$$\theta \mathbf{x}_n + (1-\theta)\mathbf{y}_n \in C \subset \bar{C}, \forall n.$$

Since $\bar{C}$ is closed, we have

$$\theta \mathbf{x} + (1-\theta)\mathbf{y} = \lim_{n\to\infty} (\theta \mathbf{x}_n + (1-\theta)\mathbf{y}_n) \in \bar{C}.$$

Therefore, we conclude that $\bar{C}$ is also convex.
For all $\mathbf{x} \in C^\circ$, there exists $r \geq 0$ such that $B_r(\mathbf{x}) \subset C$. Therefore, for all $\mathbf{x}, \mathbf{y} \in C^\circ$ and $\theta \in \mathbb{R}$, there exist $r \geq 0$ such that $B_r(\mathbf{x}) \subset C$ and $B_r(\mathbf{y}) \subset C$. Now, for all $\mathbf{z} \in B_r(\theta\mathbf{x} + (1-\theta)\mathbf{y})$, we have

$$B_r(\theta\mathbf{x} + (1-\theta)\mathbf{y}) = \theta\mathbf{x} + (1-\theta)\mathbf{y} + B_r(\mathbf{0}).$$

For any $\mathbf{r} \in B_r(\mathbf{0})$, we can express

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} = \theta(\mathbf{x}+\mathbf{r}) + (1-\theta)(\mathbf{y}+\mathbf{r}).$$

Since $\mathbf{x}+\mathbf{r} \in B_r(\mathbf{x}) \subset C$ and $\mathbf{y}+\mathbf{r} \in B_r(\mathbf{y}) \subset C$, and since $C$ is convex, we have

$$\theta(\mathbf{x}+\mathbf{r}) + (1-\theta)(\mathbf{y}+\mathbf{r}) \in C$$

for all $\mathbf{r} \in B_r(\mathbf{0})$. Therefore, we conclude that

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} + B_r(\mathbf{0}) \subset C,$$

which implies that

$$\theta \mathbf{x} + (1-\theta)\mathbf{y} \in C^\circ.$$

Thus, $C^\circ$ is also convex.

(b) For all $\mathbf{x} \in \textbf{relint } C$, there exists $r \geq 0$ such that $B_r(\mathbf{x}) \cap \textbf{aff } C \subset C$. Therefore, for all $\mathbf{x}, \mathbf{y} \in \textbf{relint } C$ and $\theta \in \mathbb{R}$, there exist $r \geq 0$ such that $B_r(\mathbf{x}) \cap \textbf{aff } C \subset C$ and $B_r(\mathbf{y}) \cap \textbf{aff } C \subset C$.

Without loss of generality, assume that $\theta \neq 0$ and $\theta \neq 1$. Let $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y}$. Consider

$$B_r(\mathbf{z}) \cap \textbf{aff } C = \mathbf{z} + \{\mathbf{r} : \mathbf{z} + \mathbf{r} \in \textbf{aff } C,\ \mathbf{r} \in B_r(\mathbf{0})\} = \mathbf{z} + R.$$

For all $\mathbf{r} \in R$, since $\mathbf{z} + \mathbf{r}, \mathbf{x}, \mathbf{y} \in \textbf{aff } C$, we have

$$\mathbf{y} + \frac{\mathbf{r}}{1-\theta} = \frac{-\theta}{1-\theta}\mathbf{x} + \frac{1}{1-\theta}(\theta\mathbf{x} + (1-\theta)\mathbf{y} + \mathbf{r}) = \frac{-\theta}{1-\theta}\mathbf{x} + \frac{1}{1-\theta}(\mathbf{z} + \mathbf{r}) \in \textbf{aff } C.$$

Therefore,

$$\mathbf{y} + \mathbf{r} = (1-\theta)\left(\mathbf{y} + \frac{\mathbf{r}}{1-\theta}\right) + \theta\mathbf{y} \in \textbf{aff } C.$$

Moreover, since $\mathbf{y} + \mathbf{r} \in B_r(\mathbf{y})$, we have $\mathbf{y} + \mathbf{r} \in B_r(\mathbf{y}) \cap \textbf{aff } C \subset C$. Similarly, $\mathbf{x} + \mathbf{r} \in C$. Therefore, for all $\mathbf{r} \in R$, since $\mathbf{x} + \mathbf{r}, \mathbf{y} + \mathbf{r} \in C$ and $C$ is convex, we have

$$\mathbf{z} + \mathbf{r} = \theta(\mathbf{x} + \mathbf{r}) + (1-\theta)(\mathbf{y} + \mathbf{r}) \in C.$$

Thus, $B_r(\mathbf{z}) \cap \textbf{aff } C \subset C$, which implies that $\mathbf{z} \in \textbf{relint } C$. Therefore, $\textbf{relint } C$ is also convex.

(c) For all $\mathbf{x}, \mathbf{y} \in \bigcap_{i \in I} C_i$ and $\theta \in \mathbb{R}$, since each $C_i$ is convex, we have

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} \in C_i, \forall\, i \in I \quad \Rightarrow \quad \theta\mathbf{x} + (1-\theta)\mathbf{y} \in \bigcap_{i \in I} C_i,$$

which implies that $\bigcap_{i \in I} C_i$ is convex.

(d) Denote the set by $P$. $\forall \mathbf{y}_1 = \mathbf{A}\mathbf{x}_1 + \mathbf{a}$, $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2 + \mathbf{a} \in \mathbf{P}$ and $\theta \in \mathbb{R}$, we have

$$\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2 = \theta(\mathbf{A}\mathbf{x}_1 + \mathbf{a}) + (1-\theta)(\mathbf{A}\mathbf{x}_2 + \mathbf{a}) = \mathbf{A}(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) + \mathbf{a}.$$

Since $C$ is convex, we have $\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C$, which implies that $\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2 \in P$. Therefore, $P$ is convex.

(e) Denote the set by $Q$. $\forall \mathbf{y}_1$, $\mathbf{y}_2 \in \mathbf{P}$ and $\theta \in \mathbb{R}$, let $\mathbf{x}_1 = \mathbf{B}\mathbf{y}_1 + \mathbf{b}$, $\mathbf{x}_2 = \mathbf{B}\mathbf{y}_2 + \mathbf{b}$, we have

$$\mathbf{B}(\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2) + \mathbf{b} = \theta(\mathbf{B}\mathbf{y}_1 + \mathbf{b}) + (1-\theta)(\mathbf{B}\mathbf{y}_2 + \mathbf{b}) = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2.$$

Since $C$ is convex, we have $\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C$, which implies that $\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2 \in Q$. Therefore, $Q$ is convex.

2. Denote the set by $P$.
(a) $P^\circ = \emptyset$, $\mathbf{relint}\ P = P$.

(b) $P^\circ = \emptyset$, $\mathbf{relint}\ P = P$.

(c) $P^\circ = \mathbf{relint}\ P = P$.

Exercise 3. **Relative Interior and Interior**

Solution. 1. From exercise 2.1(b), we know that

$$\begin{aligned}\mathbf{x}_0 \in \textbf{relint}\ C \quad &\Leftrightarrow \quad \exists r_0 > 0,\ \text{s.t.}\ (\mathbf{x}_0 + B_{r_0}(\mathbf{0})) \cap \textbf{aff}\ C \subset C \\ &\Leftrightarrow \quad \exists r_0 > 0,\ \text{s.t.}\ \mathbf{x}_0 + (B_{r_0}(\mathbf{0}) \cap (\textbf{aff}\ C - \mathbf{x}_0)) \subset C.\end{aligned}$$

Noticing that

$$B_{r_0}(\mathbf{0}) \cap (\textbf{aff}\ C - \mathbf{x}_0) = \{\mathbf{r} : \mathbf{r} \in \textbf{aff}\ C - \mathbf{x}_0, \|\mathbf{r}\|_2 \le r_0\}.$$

Since $\textbf{aff}\ C$ is an affine set in $\mathbb{R}^n$ and $\mathbf{x}_0 \in \textbf{aff}\ C$, from exercise 1.1(a) we have that $\textbf{aff}\ C - \mathbf{x}_0$ is a subspace. Let $r = r_0$. Therefore, for all $\mathbf{v} \in \textbf{aff}\ C - \mathbf{x}_0$ and $\|\mathbf{v}\|_2 \le 1$, we have

$$r\mathbf{v} \in \textbf{aff}\ C - \mathbf{x}_0,\ \|r\mathbf{v}\|_2 \le r = r_0 \Rightarrow r\mathbf{v} \in \{\mathbf{r} : \mathbf{r} \in \textbf{aff}\ C - \mathbf{x}_0, \|\mathbf{r}\|_2 \le r_0\}.$$

Thus,

$$\mathbf{x}_0 + r\mathbf{v} \in \mathbf{x}_0 + B_{r_0}(\mathbf{0}) \cap (\textbf{aff}\ C - \mathbf{x}_0) \subset C,$$

which implies that

$$\exists r_0 > 0,\ \text{s.t.}\ B_{r_0}(\mathbf{0}) \cap (\textbf{aff}\ C - \mathbf{x}_0) \subset C$$

$$\Leftrightarrow \exists r_0 > 0,\ \text{s.t.}\ \mathbf{x}_0 + r\mathbf{v} \in C \text{ for any } \mathbf{v} \in \textbf{aff}\ C - \mathbf{x}_0 \text{ and } \|\mathbf{v}\|_2 \le 1.$$

2.(a) If there $\exists\, \gamma > 0$ and $\mathbf{y}_0 \in C$, s.t.

$$\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in C,$$

since $\frac{\gamma}{1+\gamma}, \frac{1}{1+\gamma} \in (0,1)$ and $C$ is convex, we have

$$\mathbf{x} = \frac{\gamma}{1+\gamma}\mathbf{y} + \frac{1}{1+\gamma}(\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y})) \in C.$$

Thus without loss of generality, we assume that $\mathbf{x} \in C$.

($\Rightarrow$) If $\mathbf{x} \in \textbf{relint } C$, from exercise 3.1, we have

$$\exists\, r > 0, \text{ s.t. } \mathbf{x} + \{\mathbf{r} : \mathbf{x} + \mathbf{r} \in \textbf{aff } C, \|\mathbf{r}\|_2 \le r\} \subset C.$$

Since **aff** $C$ is affine, we have

$$\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in \textbf{aff } C, \forall\, \gamma \in \mathbb{R}.$$

Set $\gamma_0 = \frac{r}{2\|\mathbf{x}-\mathbf{y}\|_2}$, we have

$$\|\gamma_0(\mathbf{x} - \mathbf{y})\|_2 \le \frac{r}{2\|\mathbf{x} - \mathbf{y}\|_2}\|\mathbf{x} - \mathbf{y}\|_2 = \frac{r}{2} < r,$$

which implies that

$$\gamma_0(\mathbf{x} - \mathbf{y}) \in \{\mathbf{r} : \mathbf{x} + \mathbf{r} \in \textbf{aff } C, \|\mathbf{r}\|_2 \le r\}.$$

Therefore,

$$\mathbf{x} + \gamma_0(\mathbf{x} - \mathbf{y}) \in \mathbf{x} + \{\mathbf{r} : \mathbf{x} + \mathbf{r} \in \textbf{aff } C, \|\mathbf{r}\|_2 \le r\} \subset C.$$

($\Leftarrow$) We will use the conclusion from the next question.

Firstly we will prove that **relint** $C \ne \emptyset$ if $C$ is a non-empty convex set. Since non-empty convex set $C$ contains a non-empty simplex $S$, applying an affine transformation if necessary, we can assume that the vertices of $S$ are the vectors $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, ..., $(0, 0, \dots, 1)$:

$$S = \{(\xi_1, \xi_2, \dots, \xi_n) : \xi_i \ge 0, \sum_{i=1}^{n} \xi_i \le 1\}.$$

But the simplex does have a non-empty interior:

$$S^\circ = \{(\xi_1, \xi_2, \dots, \xi_n) : \xi_i > 0, \sum_{i=1}^{n} \xi_i < 1\}.$$

Therefore $S^\circ$ is a subset of **relint** $C$. Thus **relint** $C \neq \emptyset$.

Since

$$\forall\, \mathbf{y} \in C,\, \exists\, \gamma > 0,\ \text{ s.t. }\ \mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in C,$$

Denote $\mathbf{x} + \gamma(\mathbf{x} + \mathbf{y})$ by $\mathbf{z}$. Since **relint** $C \neq \emptyset$, set $\mathbf{z} \in$ **relint** $C$. Therefore,

$$\mathbf{x} = \frac{1}{1+\gamma}\mathbf{z} + \frac{\gamma}{1+\gamma}\mathbf{y},\ \text{ where }\ \frac{1}{1+\gamma}, \frac{\gamma}{1+\gamma} \in (0,1).$$

Since $\mathbf{z} \in$ **relint** $C$, $\mathbf{y} \in C \subset \bar{C}$, from exercise 3.2(b), we have $\mathbf{x} \in$ **relint** $C$.

(b) Denote $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ by $\mathbf{z}_\lambda$. Since $\mathbf{x} \in$ **relint** $C$, from exercise 3.1, we have

$$\exists\, r > 0,\ \text{ s.t. }\ \mathbf{x} + \{\mathbf{r} : \mathbf{r} \in \mathbf{aff}\ C - \mathbf{x}, \|\mathbf{r}\|_2 \leq r\} \subset C.$$

From exercise 1.1(b), **aff** $C - \mathbf{x}$ is unrelated to $\mathbf{x}$. Therefore, we can denote $\{\mathbf{r} : \mathbf{r} \in \mathbf{aff}\ C - \mathbf{x}, \|\mathbf{r}\|_2 \leq t\}$ by $R_t$.

Since $\mathbf{y} \in \bar{C}$, $\forall\, \epsilon > 0$, we have

$$\mathbf{y} \in C + B_\epsilon(\mathbf{0}).$$

Therefore,

$$\begin{aligned} B_\delta(\mathbf{z}_\lambda) \cap \mathbf{aff}\ C &= \lambda\mathbf{x} + (1-\lambda)\mathbf{y} + R_\delta \\ &\subset \lambda\mathbf{x} + (1-\lambda)C + (2-\lambda)R_\delta \\ &= \lambda(\mathbf{x} + R_{\frac{2-\lambda}{\lambda}\delta}) + (1-\lambda)C. \end{aligned}$$

Set $\delta = \frac{\lambda}{2-\lambda} r$, we have

$$\mathbf{x} + R_\delta \subset \mathbf{x} + R_r \subset C,$$

therefore,

$$B_\delta(\mathbf{z}_\lambda) \cap \mathbf{aff}\ C \subset \lambda C + (1-\lambda)C \subset C,$$

since $C$ is convex. Thus, we conclude that

$$\mathbf{z}_\lambda = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathbf{relint}\ C,\, \forall\, \lambda \in (0,1].$$

Exercise 4. **Supporting Hyperplane**

Solution. 1.(a) Since the tangent line of $y = 1/x$ on point $(x_0, y_0)$ is

$$y = -\frac{x}{x_0^2} + \frac{2}{x_0},$$

we have

$$\{\mathbf{x} \in \mathbf{R}_+^2 | x_1 x_2 \geq 1\} = \bigcap_{x_0 \in \mathbb{R}_+} \{(x, y) : y \geq -\frac{x}{x_0^2} + \frac{2}{x_0}\}.$$

(b) Since the the boundary of $C$ is

$$\bigcup_{1 \leq i \leq n} \{(x_1, x_2, \ldots, x_n) : \|x_i\| = 1 \text{ and } \|x_j\| < 1, \forall\, j \neq i\}.$$

If the $i$th element of $\hat{x}$ is 1, the supporting hyperplanes of $C$ at $\hat{x}$ is

$$\{(x_1, x_2, \ldots, x_n) : x_i = 1\}.$$

If the $i$th element of $\hat{x}$ is $-1$, the supporting hyperplanes of $C$ at $\hat{x}$ is

$$\{(x_1, x_2, \ldots, x_n) : x_i = -1\}.$$

2. Denote the set by $P$.

$\forall (\mathbf{a}, b) \in P$, $\lambda > 0$, we have

$$\lambda \mathbf{a}^T \mathbf{x} \leq \lambda b \quad \Leftrightarrow \quad \mathbf{a}^T \mathbf{x} \leq b, \forall\, \mathbf{x} \in C.$$

$$\lambda \mathbf{a}^T \mathbf{x} \geq \lambda b \quad \Leftrightarrow \quad \mathbf{a}^T \mathbf{x} \geq b, \forall\, \mathbf{x} \in D.$$

Which implies that $P$ is a cone.

$\forall (\mathbf{a}_1, b_1)$, $(\mathbf{a}_2, b_2) \in P$, $\lambda \in (0, 1)$, we have

$$\begin{aligned}(\lambda \mathbf{a}_1 + (1-\lambda)\mathbf{a}_2)^T \mathbf{x} &= \lambda \mathbf{a}_1^T \mathbf{x} + (1-\lambda)\mathbf{a}_2^T \mathbf{x} \\ &\leq \lambda b_1 + (1-\lambda) b_2, \forall\, \mathbf{x} \in C.\end{aligned}$$

$$
\begin{aligned}
(\lambda \mathbf{a}_1 + (1-\lambda)\mathbf{a}_2)^T \mathbf{x} &= \lambda \mathbf{a}_1^T \mathbf{x} + (1-\lambda)\mathbf{a}_2^T \mathbf{x} \\
&\geq \lambda b_1 + (1-\lambda) b_2, \ \forall\, \mathbf{x} \in D.
\end{aligned}
$$

Therefore,

$$(\lambda \mathbf{a}_1 + (1-\lambda)\mathbf{a}_2, \lambda b_1 + (1-\lambda) b_2) = \lambda(\mathbf{a}_1, b_1) + (1-\lambda)(\mathbf{a}_2, b_2) \in P.$$

Thus, $P$ is convex. Therefore, we can conclude that $P$ is a convex cone.

Exercise 5. **Farkas' Lemma**

Solution. 1.(a) $\forall\, \mathbf{a} = \sum_{i=1}^n \alpha_i \mathbf{a}_i$, $\mathbf{b} = \sum_{i=1}^n \beta_i \mathbf{a}_i \in \textbf{cone}\ A$ and $\lambda \in (0,1)$, we have

$$\lambda \mathbf{a} + (1-\lambda)\mathbf{b} = \sum_{i=1}^n (\lambda \alpha_i + (1-\lambda)\beta_i)\mathbf{a}_i.$$

Since $\alpha_i \geq 0$, $\beta_i \geq 0$, we have $\lambda \alpha_i + (1-\lambda)\beta_i \geq 0$, which implies that

$$\lambda \mathbf{a} + (1-\lambda)\mathbf{b} \in \textbf{cone}\ A, \ \forall\, \mathbf{a}, \mathbf{b} \in \textbf{cone}\ A \text{ and } \lambda \in (0,1).$$

Therefore **cone** $A$ is convex.

Suppose that $\mathbf{a}_n \to \mathbf{a}$ as $n \to \infty$. Since $\|\mathbf{a}_n - \mathbf{a}\|_2 \geq |a_{n_i} - a_i|$, we have $a_i = \lim\limits_{n\to\infty} a_{n_i} \geq 0$. Therefore $\mathbf{a} \in \textbf{cone}\ A$, which implies that **cone** $A$ is closed.

2. Since $\mathbf{b} = \sum_{i=1}^n \beta_i \mathbf{a}_i$, set $\mathbf{x} = (\beta_1, \beta_2, \ldots, \beta_n)$, we have $\mathbf{A}\mathbf{x} = \mathbf{b}$.

3. Since **cone** $A$ is a nonempty closed convex set, from the seperation theorem, we have

$$\exists\, \mathbf{y}_0 \neq \mathbf{0} \text{ and } \alpha < \beta, \ \text{ s.t. } \mathbf{A}^T \mathbf{y}_0 \geq \boldsymbol{\beta} = (\beta, \beta, \ldots, \beta), \ \mathbf{b}^T \mathbf{y}_0 \leq \alpha.$$

Since $\mathbf{0} \in \textbf{cone}\ A$, we have $\mathbf{b} \neq \mathbf{0}$. Without loss of generality, we can assume that $b_1 \neq 0$.

If $\alpha = 0$ or $\beta = 0$ then the proof is trivial. Suppose that $\alpha$, $\beta \neq 0$, set $\mathbf{z} = (\frac{\beta}{b_1}, 0, 0, \ldots, 0)$, we have

$$\mathbf{A}^T(\mathbf{y}_0 - \mathbf{z}) \geq \boldsymbol{\beta} - \boldsymbol{\beta} = \mathbf{0},\ \mathbf{b}^T(\mathbf{y}_0 - \mathbf{z}) \leq \alpha - \beta < 0.$$

Therefore, $\mathbf{y}_0 - \mathbf{z} \in \mathbb{R}^m$ satisfies the condition.

4. Consider two cases:

(i) $\mathbf{b} \in$ **cone** $A$. From exercise 5.2, there exists $\mathbf{x} \in \mathbb{R}^n$, s.t. $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

(ii) $\mathbf{b} \notin$ **cone** $A$. From exercise 5.3, there exists $\mathbf{y} \in \mathbb{R}^m$, s.t. $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$.

Since we have either $\mathbf{b} \in$ **cone** $A$ or $\mathbf{b} \notin$ **cone** $A$, at least one of the two statements hold.

On the other hand, suppose that two statements hold together. Since $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$, we have

$$\mathbf{b}^T\mathbf{y} = (\mathbf{Ax})^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \geq 0,$$

contradicting to $\mathbf{b}^T\mathbf{y} < 0$. Therefore, one and only one of the two statements hold.