Exercise 10.1.1

$x^1(t) = v_0(x_1)t + x_1$, $x^2(t) = v_0(x_2)t + x_2$

$v_0(x_1)t + x_1 = v_0(x_2)t + x_2$

$\Leftrightarrow t(v_0(x_1) - v_0(x_2)) = x_2 - x_1 \Leftrightarrow t = \dfrac{x_2 - x_1}{v_0(x_1) - v_0(x_2)}$

$$\boxed{\therefore\ x^1(t) \text{ and } x^2(t) \text{ intersect at } t = -\left(\frac{x_2 - x_1}{v_0(x_2) - v_0(x_1)}\right)}$$

Exercise 10.1.2

The break time is the minimum possible value for t at the intersection. So:

$$T_b = \min\left(-\left(\frac{x_2 - x_1}{u_0(x_2) - u_0(x_1)}\right)\right) = \frac{-1}{\min\left(\frac{u_0(x_2) - u_0(x_1)}{x_2 - x_1}\right)}$$

By the Fundamental Theorem of Calculus,

$$= \frac{-1}{\min\left(\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} u_0'(x)dx\right)}$$

By the Mean Value Theorem for integrals, $\exists x$ s.t.

$$\int_{x_1}^{x_2} u_0'(x)\,dx = u'(x)[x_2 - x_1]$$

So,

$$= \frac{-1}{\left(\min \frac{1}{(x_2 - x_1)} \cdot u_0'(x)[x_2 - x_1]\right)} = \frac{-1}{\min(u_0'(x))}$$

$$\boxed{\therefore T_b = \frac{-1}{\min(u_0'(x))}\ ;\ u_0'(x) \text{ has to be somewhere negative because otherwise } T_b < 0 \text{ or undefined.}}$$

②

Exercise 10.2.1

Characteristic Speed:

$$f'(\rho) = \frac{\partial f}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\rho u_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)\right)$$

$$= \frac{\partial}{\partial \rho}\left(\rho u_{max} - \frac{\rho^2 u_{max}}{\rho_{max}}\right)$$

$$= u_{max} - \frac{2\rho u_{max}}{\rho_{max}} = \boxed{u_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)}$$

Shock Speed:

$$S = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} = \frac{\rho_l u_{max}\left(1 - \frac{\rho_l}{\rho_{max}}\right) - \rho_r u_{max}\left(1 - \frac{\rho_r}{\rho_{max}}\right)}{\rho_l - \rho_r}$$

$$= \left[\rho_l u_{max} - \frac{\rho_l^2 u_{max}}{\rho_{max}} - \frac{\rho_r u_{max} + \rho_r^2 u_{max}}{\rho_{max}}\right] \frac{1}{\rho_l - \rho_r}$$

$$= \frac{1}{\rho_l - \rho_r}\left[(\rho_l - \rho_r) u_{max} + \frac{\rho_r^2 u_{max} - \rho_l^2 u_{max}}{\rho_{max}}\right]$$

$$= u_{max} + \frac{u_{max}\left[\rho_r^2 - \rho_l^2\right]}{(\rho_{max})(\rho_l - \rho_r)} = u_{max} + \frac{u_{max}(\rho_l - \rho_r)(\rho_l + \rho_r)}{(\rho_{max})(\rho_l - \rho_r)}$$

$$= \boxed{u_{max}\left(1 - \frac{(\rho_l + \rho_r)}{\rho_{max}}\right)}$$

Exercise 10.3.1

Choice : Lax - Wendroff

$$v_m^{n+1} = v_m^n - \frac{k}{h}\cdot\frac{1}{2}\left(f(v_{m+1}^n) - f(v_{m-1}^n)\right) + \frac{k^2}{2h}\left(a_{m+\frac{1}{2}}^n\left(f(v_{m+1}^n) - f(v_m^n)\right)\right.$$
$$\left. - a_{m-\frac{1}{2}}^n\left(f(v_m^n) - f(v_{m-1}^n)\right)\right)$$

where $a_{m+\frac{1}{2}}^n = f'\left(\frac{v_m^n + v_{m+1}^n}{2}\right)$

Check Conservative:

Introduce an intermediate term in the $\frac{k}{2h}$ expression:

$$v_m^{n+1} = v_m^n - \frac{k}{h}\cdot\frac{1}{2}\left(f(v_{m+1}^n) + f(v_m^n) - f(v_m^n) - f(v_{m-1}^n)\right)$$
$$+ \frac{k}{h}\cdot\frac{1}{2}\left(\frac{k}{h}\left(a_{m+\frac{1}{2}}^n\left(f(v_{m+1}^n) - f(v_m^n)\right) - a_{m-\frac{1}{2}}^n\left(f(v_m^n) - f(v_{m-1}^n)\right)\right)\right)$$

Let $\mathcal{F}(v_{m+1}^n, v_m^n) = \frac{1}{2}\left[f(v_{m+1}^n) + f(v_m^n) + \frac{k}{h}a_{m+\frac{1}{2}}^n\left(f(v_{m+1}^n) - f(v_m^n)\right)\right]$

$\Rightarrow \mathcal{F}(v_m^n, v_{m-1}^n) = \frac{1}{2}\left[f(v_m^n) + f(v_{m-1}^n) + \frac{k}{h}a_{m-\frac{1}{2}}^n\left(f(v_m^n) - f(v_{m-1}^n)\right)\right]$

$\Rightarrow v_m^{n+1} = v_m^n - \frac{k}{h}\left[\mathcal{F}(v_{m+1}^n, v_m^n) - \mathcal{F}(v_m^n, v_{m-1}^n)\right]$

Thus Lax - Wendroff is conservative.

Check consistent:

1) $\mathcal{F}(u,u) = \mathcal{F}(u)$

$$\mathcal{F}(u,u) = \frac{1}{2}\left(f(u) + f(u) + \frac{K}{h} f'\left(\frac{u+u}{2}\right)[f(u) - f(u)]\right)$$

$$= \frac{1}{2}(2f(u)) = f(u) \iff \mathcal{F}(u,u) = f(u)$$

2) $|\mathcal{F}(v,w) - \mathcal{F}(u,u)| \leq K|(v,w) - (u,u)|$

$$|\mathcal{F}(v,w) - \mathcal{F}(u,u)| = \left| \frac{1}{2}\left(f(v) + f(w) + \frac{K}{h} f'\left(\frac{w+v}{2}\right)(f(v) - f(w))\right) - \frac{1}{2}(f(u) + f(u) + 0)\right|$$

$$= \left|\frac{1}{2}\left((f(v) - f(u)) - (f(w) - f(u)) + \frac{K}{h} f'\left(\frac{w+v}{2}\right)(f(v) - f(w))\right)\right|$$

$$\leq \frac{1}{2}\left|(f(v) - f(u)) - (f(w) - f(u))\right| + \frac{K}{h}\left|f'\left(\frac{w+v}{2}\right)(f(v) - f(w))\right|$$

by the Triangle Inequality.

Now, assume f to be Lipschitz continuous with associated Lipschitz constant K_f

Thus the RHS of our inequality is

$$\frac{K_f}{2}\left|(v-u) - (w-u)\right| + \frac{K_f K}{h}\left|f'\left(\frac{w+v}{2}\right)(v-w)\right| \quad \ldots$$

$$\le \frac{K_f}{2} \max\{|v-u|, |w-u|\} + \frac{K_f K}{h} \left| f'\left(\frac{w+v}{2}\right)(v-w) \right|$$

Assume f is continuous from w to v. By the Mean Value Theorem for derivatives,

$$f'\left(\frac{w+v}{2}\right) = \frac{f(v)-f(w)}{(v-w)}$$

Thus, after substitution, the RHS of our inequality becomes

$$\frac{K_f}{2} \max\{|v-u|, |w-u|\} + \frac{K_f K}{h} |f(v)-f(w)|$$

$$= \frac{K_f}{2} \max\{|v-u|, |w-u|\} + \frac{K_f^2 K}{h} |v-u|$$

$$\le \left(\frac{K_f}{2} + \frac{K_f^2 K}{2}\right) \max\{|v-u|, |w-u|\} \le \left(\frac{K_f}{2} + \frac{K_f^2 K}{2}\right) |(v,w)-(u,u)|$$

$$\therefore |\mathcal{F}(v,w) - \mathcal{F}(u,u)| \le \left(\frac{K_f}{2} + \frac{K_f^2 K}{2}\right) |(v,w)-(u,u)|$$

$\Rightarrow$ $\mathcal{F}$ is Lipschitz continuous with $K_{\mathcal{F}} = \dfrac{K_f(1+K_f K)}{2}$

By 1) and 2) Lax-Wendroff is consistent.

. . .

Extra Credit: Exercise 10.3.2

$$v_m^{n+1} = v_m^n + D_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right) - C_{m-\frac{1}{2}}\left(v_m^n - v_{m-1}^n\right)$$

$$\Rightarrow v_{m+1}^{n+1} = v_{m+1}^n + D_{(m+1)+\frac{1}{2}}\left(v_{m+2}^n - v_{m+1}^n\right) - C_{(m+1)-\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right)$$

$$= v_{m+1}^n + D_{m+\frac{3}{2}}\left(v_{m+2}^n - v_{m+1}^n\right) - C_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right)$$

$\Rightarrow$

$$v_{m+1}^{n+1} - v_m^{n+1} = \left(v_{m+1}^n - v_m^n\right) + D_{m+\frac{3}{2}}\left(v_{m+2}^n - v_{m+1}^n\right) - D_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right)$$

$$- C_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right) + C_{m-\frac{1}{2}}\left(v_m^n - v_{m-1}^n\right)$$

$\Rightarrow$

$$\sum_{m=-\infty}^{\infty} \left|v_{m+1}^{n+1} - v_m^{n+1}\right| = \sum_{m=-\infty}^{\infty} \left|\left(v_{m+1}^n - v_m^n\right) + D_{m+\frac{3}{2}}\left(v_{m+2}^n - v_{m+1}^n\right)\right.$$

$$- D_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right) - C_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right)$$

$$\left. + C_{m-\frac{1}{2}}\left(v_m^n - v_{m-1}^n\right)\right|$$

$$= \|v^{n+1}\|_{TV}$$

. . .

By repeated applications of the Triangle Inequality and by distributing the summation,

$$\|v^{n+1}\|_{TV} \leq \sum_{m=-\infty}^{\infty} \left|\left(1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}}\right)\left(v_{m+1}^{n} - v_{m}^{n}\right)\right|$$

$$+ \sum_{m=-\infty}^{\infty} \left| D_{m+\frac{3}{2}}\left(v_{m+2}^{n} - v_{m+1}^{n}\right)\right| + \sum_{m=-\infty}^{\infty} \left| C_{m-\frac{1}{2}}\left(v_{m}^{n} - v_{m-1}^{n}\right)\right|$$

Since the summations iterate from $-\infty$ to ∞, let us reindex the terms which contain $D_{m+\frac{3}{2}}$ to $D_{m+\frac{1}{2}}$ and $C_{m-\frac{1}{2}}$ to $C_{m+\frac{1}{2}}$

The RHS of our inequality becomes:

$$\sum_{m=-\infty}^{\infty} \left|\left(1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}}\right)\left(v_{m+1}^{n} - v_{m}^{n}\right)\right| + \sum_{m=-\infty}^{\infty} \left| D_{m+\frac{1}{2}}\left(v_{m+1}^{n} - v_{m}^{n}\right)\right|$$

$$+ \sum_{m=-\infty}^{\infty} \left| C_{m+\frac{1}{2}}\left(v_{m+1}^{n} - v_{m}^{n}\right)\right|$$

Recombine the sum and factor

$$= \sum_{m=-\infty}^{\infty} \left|v_{m+1}^{n} - v_{m}^{n}\right| \cdot \left(\left|1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}}\right| + \left|D_{m+\frac{1}{2}}\right| + \left|C_{m+\frac{1}{2}}\right|\right)$$

Now, since $C_{m+\frac{1}{2}} \geq 0$ and $D_{m+\frac{1}{2}} \geq 0$

$$= \sum_{m=-\infty}^{\infty} \left|v_{m+1}^{n} - v_{m}^{n}\right| \cdot \left(\left|1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}}\right| + D_{m+\frac{1}{2}} + C_{m+\frac{1}{2}}\right)$$

...

Furthermore, since $C+D \le 1$, $1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}} \ge 0$; so

$$= \sum_{m=-\infty}^{\infty} \left| v_{m+1}^{n} - v_{m}^{n} \right| \cdot \left(1 - C_{m+\frac{1}{2}} - D_{m+\frac{1}{2}} + D_{m+\frac{1}{2}} + C_{m+\frac{1}{2}} \right)$$

$$= \sum_{m=-\infty}^{\infty} \left| v_{m+1}^{n} - v_{m}^{n} \right| = \|v^n\|_{TV}$$

$\Rightarrow$ $\boxed{\|v^{n+1}\|_{TV} \le \|v^n\|_{TV}}$, and thus this scheme is total variation diminishing

Extra Credit: Exercise 10.3.3

Lax-Friedrichs Conservative Form:

$$v_m^{n+1} = v_m^n - \frac{k}{h}\left(\mathcal{F}(v_{m+1}^n, v_m^n) - \mathcal{F}(v_m^n, v_{m-1}^n) \right)$$

with $\mathcal{F}(v_{m+1}^n, v_m^n = \frac{h}{2k}(v_m^n - v_{m+1}^n + \frac{1}{2}(f(v_m^n) + f(v_{m+1}^n))$

Substitute:

$$v_m^{n+1} = v_m^n - \frac{k}{h}\Big[\Big(\frac{h}{2k}(v_m^n - v_{m+1}^n) + \frac{1}{2}(f(v_m^n) + f(v_{m+1}^n)\Big)$$
$$- \Big(\frac{h}{2k}(v_{m-1}^n - v_m^n) + \frac{1}{2}(f(v_{m-1}^n) + f(v_m^n)\Big)\Big]$$

$$= v_m^n - \frac{k}{h}\Big[\frac{h}{2k}(v_m^n - v_{m+1}^n + v_m^n - v_{m-1}^n) + \frac{1}{2}(f(v_m^n) + f(v_{m+1}^n)$$
$$- f(v_m^n) - f(v_{m-1}^n)\Big]$$

$$= v_m^n - \frac{1}{2}(v_m^n - v_{m+1}^n + v_m^n - v_{m-1}^n) - \frac{k}{2h}(f(v_{m+1}^n) - f(v_{m-1}^n))$$

$$= v_m^n - \frac{1}{2}(v_m^n - v_{m+1}^n) - \frac{1}{2}(v_m^n - v_{m-1}^n) - \frac{k}{2h}(f(v_{m+1}^n) - f(v_{m-1}^n))$$

$$= v_m^n + \frac{1}{2}(v_{m+1}^n - v_m^n) - \frac{1}{2}(v_m^n - v_{m-1}^n) - \frac{k}{2h}(f(v_{m+1}^n) - f(v_{m-1}^n))$$

Let $D_{m+\frac{1}{2}} = \frac{1}{2} = C_{m-\frac{1}{2}}$

TC

$$= v_m^n + D_{m+\frac{1}{2}}\left(v_{m+1}^n - v_m^n\right) - C_{m-\frac{1}{2}}\left(v_m^n - v_{m-1}^n\right) - \frac{k}{2h}\left(f(v_{m+1}^n) - f(v_{m-1}^n)\right)$$

The restriction that we need to place is

$$\frac{k}{2h}\left(f(v_{m+1}^n) - f(v_{m-1}^n)\right) = 0$$