University of Twente

Answers to Tutorial Exercises

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Chapter 1

Introduction

Exercise 1.1

$$v(t) = \begin{cases} 0 & ,t < 0 \\ \frac{V_{max}}{T}t & ,0 \leq t < T \\ 0 & ,t \geq T \end{cases}$$

$$i(t) = \begin{cases} 0 & ,t < 0 \\ -\frac{I_{max}}{T}t + I_{max} & ,0 \geq t < T \\ 0 & ,t \geq T \end{cases}$$

a)

$$\begin{aligned} q &= \int_0^T i(t)\ dt \\ &= \int_0^T -\frac{I_{max}}{T}t + I_{max}\ dt \\ &= \Big[-\frac{I_{max}}{2T}t^2 + I_{max}\ t \Big]_0^T \\ &= -\frac{I_{max}}{2}T + I_{max}T \\ &= \frac{I_{max}T}{2}[C] \end{aligned}$$

b)

$$\begin{aligned} w_{tot} &= \int_{-\infty}^{\infty} p(t)\ dt \\ &= \int_{-\infty}^{\infty} v(t) \cdot i(t)\ dt \\ &= \int_0^T v(t) \cdot i(t)\ dt \\ &= \int_0^T \frac{V_{max}}{T}t \cdot (-\frac{I_{max}}{T}t + I_{max})\ dt \\ &= \Big[-\frac{V_{max}I_{max}}{3T^2}t^3 + \frac{V_{max}I_{max}}{2T}t^2 \Big]_0^T \\ &= -\frac{V_{max}I_{max}}{3}T + \frac{V_{max}I_{max}}{2}T \\ &= \frac{V_{max}I_{max}T}{6}[J] \end{aligned}$$

Exercise 1.2

a) Relation between V_1 and V_2, use KVL

- Configuration a) $V_1 = V_2$
- Configuration b) $V_1 = -V_2$
- Configuration c) $V_1 = -V_2$
- Configuration d) $V_1 = V_2$

b) Relation between I_1 and I_2, use KCL

- Configuration a) $I_1 = -I_2$
- Configuration b) $I_1 = I_2$
- Configuration c) $I_1 = I_2$
- Configuration d) $I_1 = -I_2$

c) Relation between V_2 and R_2, use Ohm's law

- Configuration a) $V_2 = I_2 R_2$
- Configuration b) $V_2 = I_2 R_2$
- Configuration c) $V_2 = I_2 R_2$
- Configuration d) $V_2 = I_2 R_2$

d) Relation between V_1 and I_2, use results from a) (KVL) and c) (Ohm's law)

- Configuration a) $V_1 = I_2 R_2$
- Configuration b) $V_1 = -I_2 R_2$
- Configuration c) $V_1 = -I_2 R_2$
- Configuration d) $V_1 = I_2 R_2$

Exercise 1.3

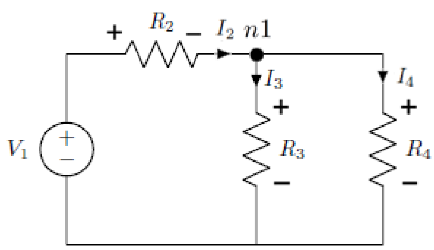


Figure 1.1

KCL at node n_1 is given by

$$-I_2 + I_3 + I_4 = 0 \quad \Rightarrow \quad I_3 = I_2 - I_4$$

- I_2 is the total current going through the equivalent resistor of R_2 and $R_3//R_4$.

$$I_2 = \frac{V_1}{R_2 + R_3//R_4}$$

- I_4 is the current through R_4

$$I_4 = \frac{V_4}{R_4}$$

Using KVL for the loop at the left side of the circuit

$$-V_1 + V_2 + V_4 = 0 \quad \Rightarrow \quad V_4 = V_1 - V_2$$

Filling this back in the expression for I_4 gives

$$\begin{aligned} &= \frac{V_1 - V_2}{R_4} \\ &= \frac{V_1 - I_2 R_2}{R_4} \\ &= \frac{V_1}{R_4} - \frac{R_2}{R_4(R_2 + R_3//R_4)} V_1 \end{aligned}$$

Now, I_3 can be evaluated

$$\begin{aligned} I_3 &= \frac{V_1}{R_2 + R_3//R_4} - \frac{V_1}{R_4} + \frac{R_2}{R_4(R_2 + R_3//R_4)} V_1 \\ &= \Big[\frac{1}{R_2 + R_3//R_4}(1 + \frac{R_2}{R_4}) - \frac{1}{R_4}\Big] V_1 \end{aligned}$$

Exercise 1.4

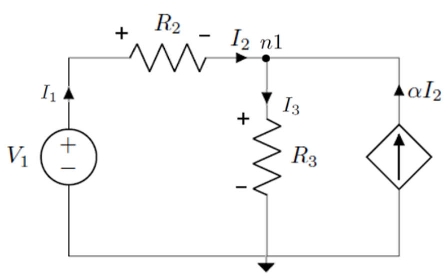


Figure 1.2

a) KCL at node n_1 is given by

$$-I_2 + I_3 - I_2 \cdot \alpha = 0 \quad \Rightarrow \quad I_3 = I_2(1 + \alpha) \tag{1.1}$$

KVL for the loop between the V1, R2 and R3

$$-V_1 + V_2 + V_3 = 0 \quad \Rightarrow \quad V_3 = V_1 - V_2 \tag{1.2}$$

Starting from Ohm's law for element R_3 and applying relation 1.2 from KVL gives:

$$\left.\begin{aligned} I_3 &= \tfrac{V_3}{R_3} \\ V_3 &= V_1 - V_2 \end{aligned}\right\} \quad \Rightarrow \quad I_3 = \frac{V_1 - V_2}{R_3}$$

$$I_3 = \frac{V_1 - I_2 \cdot R_2}{R_3} \tag{1.3}$$

Now using KCL relation 1.1 and substituting 1.3 gives:

$$\left.\begin{array}{r} I_2(1+\alpha) = I_3 \\ I_3 = \frac{V_1 - I_2 \cdot R_2}{R_3} \end{array}\right\} \Rightarrow \quad \begin{aligned} I_2(1+\alpha) &= \frac{V_1 - I_2 \cdot R_2}{R_3} \\ I_2 &= \frac{V_1}{(1+\alpha)\cdot R_3 + R_2} \end{aligned}$$

b)

$$\left.\begin{array}{r} P_2 = V_2 \cdot I_2 \\ I_2 = \frac{V_1}{(1+\alpha)\cdot R_3 + R_2} \\ V_2 = V_1 - V_3 \end{array}\right\} \Rightarrow \quad \begin{aligned} P_2 &= (V_1 - V_3) \cdot I_2 \\ &= (V_1 - I_3 \cdot R_3) \cdot I_2 \\ &= (V_1 - I_2 \cdot (1+\alpha) \cdot R_3) \cdot I_2 \\ &= V_1 \cdot I_2 - I_2^2 \cdot R_3 \cdot (1+\alpha) \\ &= \frac{V_1^2}{(1+\alpha)\cdot R_3 + R_2} - \frac{V_1^2}{((1+\alpha)\cdot R_3 + R_2)^2} \cdot (1+\alpha) \cdot R_3 \end{aligned}$$

Exercise 1.5

a)

$$\begin{aligned} R_{eq} &= 10\Omega + 6\Omega + 4\Omega//(3\Omega + 1\Omega) \\ &= 16\Omega + \frac{4\Omega \cdot 4\Omega}{4\Omega + 4\Omega} \\ &= 16\Omega + \frac{16\Omega}{8\Omega} \\ &= 16\Omega + 2\Omega \\ &= 18\Omega \end{aligned}$$

b)

$$R_{eq} = R_2 + \Big(\big((R_3 + R_8)//R_4\big) + R_6\Big)//R_7 + R_5$$

c)

$$R_{eq} = \Bigg(\bigg(\Big(\big((R_5 + R_6 + R_7)//R_{15} + R_8\big)//R_{14} + R_4\Big)//R_{13} + R_3 + R_9\bigg)//R_{12} + R_2\Bigg)//R_{11} + R_{10}$$

Exercise 1.6

a) First step is to determine the voltage over R_3.

$$V_3 = \frac{R_3//(R_4 + R_5)}{R_3//(R_4 + R_5) + R_2} \cdot V1 \tag{1.4}$$

In order to eventually find V_5 there is again a voltage divider between R_4 and R_5 dividing V_3. So:

$$\left.\begin{array}{r} V_5 = \frac{R_5}{R_5 + R_4} \cdot V_3 \\ V_3 = \frac{R_3//(R_4 + R_5)}{R_3//(R_4+R_5)+R_2} \cdot V1 \end{array}\right\} \Rightarrow \quad V_5 = \frac{R_5}{R_5 + R_4} \cdot \frac{R_3//(R_4 + R_5)}{R_3//(R_4 + R_5) + R_2} \cdot V1 \tag{1.5}$$

b) First step is to determine the voltage over R_6.

$$V_6 = \frac{(R_2 + R_3 + R_4//R_5)//R_6}{(R_2 + R_3 + R_4//R_5)//R_6 + R_7 + R_8} \cdot V1 \tag{1.6}$$

In order to eventually find V_3 there is again a voltage divider between R_2, R_3 and $R_4//R_5$ dividing V_6. So:

$$\left.\begin{array}{r} V_3 = \frac{R_3}{R_2+R_3+R_4//R_5} \cdot V_6 \\ V_6 = \frac{(R_2+R_3+R_4//R_5)//R_6}{(R_2+R_3+R_4//R_5)//R_6+R_7+R_8} \cdot V1 \end{array}\right\} \Rightarrow V_3 = \frac{R_3}{R_2 + R_3 + R_4//R_5} \cdot \frac{(R_2 + R_3 + R_4//R_5)//R_6}{(R_2 + R_3 + R_4//R_5)//R_6 + R_7 + R_8} \cdot V1 \tag{1.7}$$

c)

- $R//R = \frac{R}{2}$
- $\frac{R}{2} + R = \frac{3R}{2}$
- $\frac{3R}{2}//R = \frac{3R}{5}$
- $\frac{3R}{5} + R = \frac{8R}{5}$
- $\frac{8R}{5}//R = \frac{8R}{13}$
- $\frac{8R}{13} + R = \frac{21R}{13}$
- $\frac{21R}{13}//R = \frac{21R}{34}$
- $\frac{21R}{34} + R = \frac{55R}{34}$
- $\frac{55R}{34}//R = \frac{55R}{89}$
- $V_{out} = \frac{R}{R+\frac{55R}{89}} = \frac{89}{144}[V]$

Exercise 1.7

a) First combine R_2 and $R_3 + R_4$ in one resistor R_{eq}:

$$R_{eq} = R_2//(R_3 + R_4)$$

Now current division can be used to find I_5:

$$I_5 = \frac{R_2//(R_3 + R_4)}{R_2//(R_3 + R_4) + R_5} I_1$$

b) The first step is finding I_3, because I_3 has been divided between I_4 and I_6. I_3 can be found be using current division between R_2 and the equivalent of the rest of the resistors $R_{eq} = (R_5 + R_6 + R_7)//R_4 + R_3$:

$$I_3 = \frac{R_2}{R_2 + R_{eq}} I_1 = \frac{R_2}{R_2 + (R_5 + R_6 + R_7)//R_4 + R_3} I_1$$

$$\left.\begin{array}{rl} I_3 &= \frac{R_2}{R_2+(R_5+R_6+R_7)//R_4+R_3} I_1 \\ I_6 &= \frac{R_4}{R_4+R_5+R_6+R_7} I_3 \end{array}\right\} = I_6 = \frac{R_2 R_4}{(R_2 + R_3)(R_4 + R_5 + R_6 + R_7) + R_4(R_5 + R_6 + R_7)} I_1$$

Chapter 2

Node voltage method

Exercise 2.1

Steps 1, 2 and 3: see circuit diagram

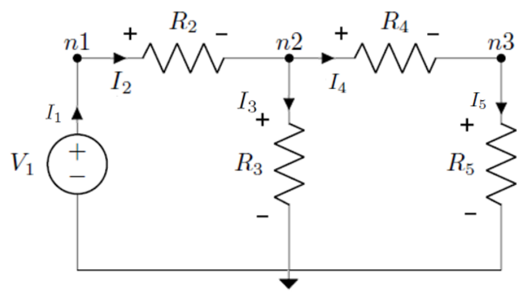


Figure 2.1

Step 4: Write down KCL for nodes n_1, n_2 and n_3

$$\begin{aligned} n_1 &: -I_1 + I_2 = 0 \\ n_2 &: -I_2 + I_3 + I_4 = 0 \\ n_3 &: -I_4 + I_5 = 0 \end{aligned}$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$\begin{aligned} n_1 &: -I_1 + \frac{V_2}{R_2} = 0 \\ n_2 &: -\frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4} = 0 \\ n_3 &: -\frac{V_4}{R_4} + \frac{V_5}{R_5} = 0 \end{aligned}$$

Step 6: Express the branch voltages in node voltages

$$\begin{aligned} b_1 &: V_{n1} = V_1 \\ n_2 &: -\frac{V_{n1} - V_{n2}}{R_2} + \frac{V_{n2}}{R_3} + \frac{V_{n2} - V_{n3}}{R_4} = 0 \\ n_3 &: -\frac{V_{n2} - V_{n3}}{R_4} + \frac{V_{n3}}{R_5} = 0 \end{aligned}$$

Step 7: Write the equations as a set of equations in node voltages

$$
\begin{aligned}
&b_1 : V_{n1} = V_1 \\
&n_2 : -\frac{1}{R_2}V_{n1} + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)V_{n2} - \frac{1}{R_4}V_{n3} = 0 \\
&n_3 : -\frac{1}{R_4}V_{n2} + \left(\frac{1}{R_4} + \frac{1}{R_5}\right)V_{n3} = 0
\end{aligned}
$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$
\begin{aligned}
&b_1 : V_{n1} = 1[V] \\
&n_2 : -V_{n1} + 3V_{n2} - V_{n3} = 0 \\
&n_3 : -V_{n2} + 2V_{n3} = 0
\end{aligned}
$$

Substituting V_{n1} into the equation from n2 gives:

$$
\begin{aligned}
&n_2 : 3V_{n2} - V_{n3} = 1 \\
&n_3 : -V_{n2} + 2V_{n3} = 0
\end{aligned}
$$

Solving a system with 2 equation and 2 unknowns results in:

$$
\begin{aligned}
V_{n1} &= 1[V] \\
V_{n2} &= \frac{2}{5}[V] \\
V_{n3} &= \frac{1}{5}[V]
\end{aligned}
$$

Exercise 2.2

Steps 1, 2 and 3: see circuit diagram

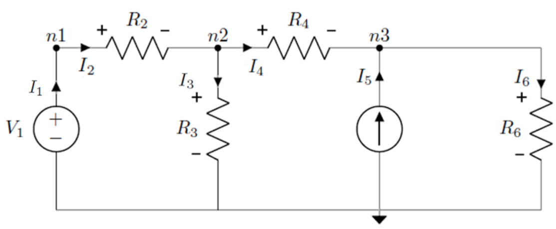


Figure 2.2

Step 4: Write down KCL for nodes n_1, n_2 and n_3

$$
\begin{aligned}
&n_1 : -I_1 + I_2 = 0 \\
&n_2 : -I_2 + I_3 + I_4 = 0 \\
&n_3 : -I_4 - I_5 + I_6 = 0
\end{aligned}
$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$\begin{aligned}
n_1 &: -I_1 + \frac{V_2}{R_2} = 0 \\
n_2 &: -\frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4} = 0 \\
n_3 &: -\frac{V_4}{R_4} - I_5 + \frac{V_6}{R_6} = 0
\end{aligned}$$

Step 6: Express the branch voltages in node voltages

$$\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\frac{V_{n1} - V_{n2}}{R_2} + \frac{V_{n2}}{R_3} + \frac{V_{n2} - V_{n3}}{R_4} = 0 \\
n_3 &: -\frac{V_{n2} - V_{n3}}{R_4} - I_5 + \frac{V_{n3}}{R_6} = 0
\end{aligned}$$

Step 7: Write the equations as a set of equations in node voltages

$$\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\frac{1}{R_2}V_{n1} + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)V_{n2} - \frac{1}{R_4}V_{n3} = 0 \\
n_3 &: -\frac{1}{R_4}V_{n2} + \left(\frac{1}{R_4} + \frac{1}{R_6}\right)V_{n3} = I_5
\end{aligned}$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$\begin{aligned}
b_1 &: V_{n1} = 2[V] \\
n_2 &: -V_{n1} + 3V_{n2} - V_{n3} = 0 \\
n_3 &: -V_{n2} + 2V_{n3} = 1
\end{aligned}$$

Substituting V_{n1} into the equation from n2 gives:

$$\begin{aligned}
n_2 &: 3V_{n2} - V_{n3} = 2 \\
n_3 &: -V_{n2} + 2V_{n3} = 1
\end{aligned}$$

Solving a system with 2 equation and 2 unknowns results in:

$$\begin{aligned}
V_{n1} &= 2[V] \\
V_{n2} &= 1[V] \\
V_{n3} &= 1[V]
\end{aligned}$$

Exercise 2.3

Steps 1, 2 and 3: see circuit diagram

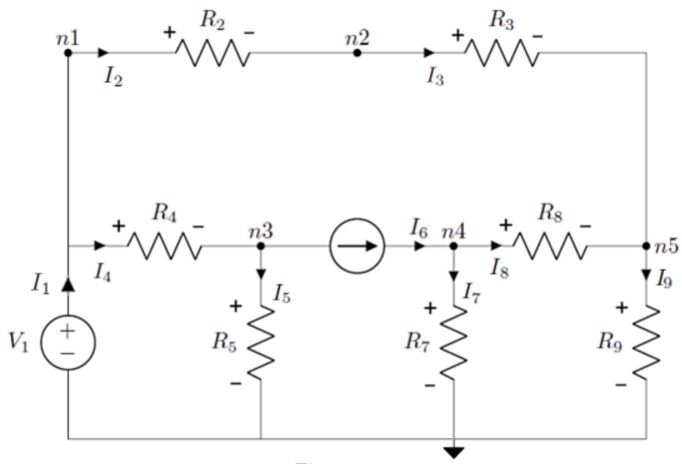


Figure 2.3

Step 4: Write down KCL for nodes n_1, n_2, n_3, n_4 and n_5

$$
\begin{aligned}
n1 &: -I_1 + I_2 + I_4 = 0 \\
n2 &: -I_2 + I_3 = 0 \\
n3 &: -I_4 + I_5 + I_6 = 0 \\
n4 &: -I_6 + I_7 + I_8 = 0 \\
n5 &: -I_3 - I_8 + I_9 = 0
\end{aligned}
$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$
\begin{aligned}
n1 &: -I_1 + \frac{V_2}{R_2} + \frac{V_4}{R_4} = 0 \\
n2 &: -\frac{V_2}{R_2} + \frac{V_3}{R_3} = 0 \\
n3 &: -\frac{V_4}{R_4} + \frac{V_5}{R_5} + I_6 = 0 \\
n4 &: -I_6 + \frac{V_7}{R_7} + \frac{V_8}{R_8} = 0 \\
n5 &: -\frac{V_3}{R_3} - \frac{V_8}{R_8} + \frac{V_9}{R_9} = 0
\end{aligned}
$$

Step 6: Express the branch voltages in node voltages

$$
\begin{aligned}
b1 &: V_{n1} = V_1 \\
n2 &: -\frac{V_{n1} - V_{n2}}{R_2} + \frac{V_{n2} - V_{n5}}{R_3} = 0 \\
n3 &: -\frac{V_{n1} - V_{n3}}{R_4} + \frac{V_{n3}}{R_5} + I_6 = 0 \\
n4 &: -I_6 + \frac{V_{n4}}{R_7} + \frac{V_{n4} - V_{n5}}{R_8} = 0 \\
n5 &: -\frac{V_{n2} - V_{n5}}{R_3} - \frac{V_{n4} - V_{n5}}{R_8} + \frac{V_{n5}}{R_9} = 0
\end{aligned}
$$

Step 7: Write the equations as a set of equations in node voltages

$$
\begin{aligned}
b1 &: V_{n1} = V_1 \\
n2 &: -\frac{1}{R_2}V_{n1} + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)V_{n2} - \frac{1}{R_3}V_{n5} = 0 \\
n3 &: -\frac{1}{R_4}V_{n1} + \left(\frac{1}{R_4} + \frac{1}{R_5}\right)V_{n3} = -I_6 \\
n4 &: \left(\frac{1}{R_7} + \frac{1}{R_8}\right)V_{n4} - \frac{1}{R_8}V_{n5} = I_6 \\
n5 &: -\frac{1}{R_3}V_{n2} - \frac{1}{R_8}V_{n4} + \left(\frac{1}{R_3} + \frac{1}{R_8} + \frac{1}{R_9}\right)V_{n5} = 0
\end{aligned}
$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$
\begin{aligned}
b1 &: V_{n1} = 1[V] \\
n2 &: -V_{n1} + 2V_{n2} - V_{n5} = 0 \\
n3 &: -V_{n1} + 2V_{n3} = -2 \\
n4 &: 2V_{n4} - V_{n5} = 2 \\
n5 &: -V_{n2} - V_{n4} + 3V_{n5} = 0
\end{aligned}
$$

Substituting V_{n1} into the equations from n2 and n3 gives:

$$
\begin{aligned}
n2 &: 2V_{n2} - V_{n5} = 1 \\
n3 &: V_{n3} = -\frac{1}{2}[V] \\
n4 &: 2V_{n4} - V_{n5} = 2 \\
n5 &: -V_{n2} - V_{n4} + 3V_{n5} = 0
\end{aligned}
$$

Rewriting the equations from n2 and n4 and substituting them into n5 gives V_{n5}. Next, V_{n2} and V_{n4} can be found.

$$
\begin{aligned}
n2 &: V_{n2} = \frac{V_{n5} + 1}{2} \\
n4 &: V_{n4} = \frac{V_{n5} + 2}{2} \\
n5 &: -V_{n2} - V_{n4} + 3V_{n5} = 0
\end{aligned}
$$

So:

$$
\begin{aligned}
V_{n1} &= 1[V] \\
V_{n2} &= \frac{7}{8}[V] \\
V_{n3} &= -\frac{1}{2}[V] \\
V_{n4} &= \frac{11}{8}[V] \\
V_{n5} &= \frac{3}{4}[V]
\end{aligned}
$$

Exercise 2.4

Steps 1, 2 and 3: see circuit diagram

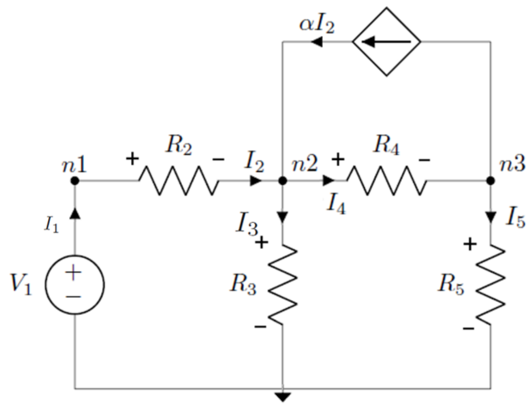


Figure 2.4

Step 4: Write down KCL for nodes n_1, n_2 and n_3

$$
\begin{aligned}
n_1 &: -I_1 + I_2 = 0 \\
n_2 &: -I_2 + I_3 + I_4 - \alpha I_2 = 0 \\
n_3 &: -I_4 + I_5 + \alpha I_2 = 0
\end{aligned}
$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$
\begin{aligned}
n_1 &: -I_1 + \frac{V_2}{R_2} = 0 \\
n_2 &: -\frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4} - \alpha\frac{V_2}{R_2} = 0 \\
n_3 &: -\frac{V_4}{R_4} + \frac{V_5}{R_5} + \alpha\frac{V_2}{R_2} = 0
\end{aligned}
$$

Step 6: Express the branch voltages in node voltages

$$
\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\frac{V_{n1} - V_{n2}}{R_2} + \frac{V_{n2}}{R_3} + \frac{V_{n2} - V_{n3}}{R_4} - \alpha\frac{V_{n1} - V_{n2}}{R_2} = 0 \\
n_3 &: -\frac{V_{n2} - V_{n3}}{R_4} + \frac{V_{n3}}{R_5} + \alpha\frac{V_{n1} - V_{n2}}{R_2} = 0
\end{aligned}
$$

Step 7: Write the equations as a set of equations in node voltages

$$
\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\Big(\frac{1}{R_2} + \frac{\alpha}{R_2}\Big)V_{n1} + \Big(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} + \frac{\alpha}{R_2}\Big)V_{n2} - \frac{1}{R_4}V_{n3} = 0 \\
n_3 &: \frac{\alpha}{R_2}V_{n1} - \Big(\frac{1}{R_4} + \frac{\alpha}{R_2}\Big)V_{n2} + \Big(\frac{1}{R_4} + \frac{1}{R_5}\Big)V_{n3} = 0
\end{aligned}
$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$\begin{aligned}
b1 &: V_{n1} = 2[V] \\
n2 &: -3V_{n1} + 5V_{n2} - V_{n3} = 0 \\
n3 &: 2V_{n1} - 3V_{n2} + 2V_{n3} = 0
\end{aligned}$$

Substituting V_{n1} into the equations from n2 and n3 gives:

$$\begin{aligned}
n2 &: 5V_{n2} - V_{n3} = 6 \\
n3 &: -3V_{n2} + 2V_{n3} = -4
\end{aligned}$$

Solving a system with 2 equation and 2 unknowns results in:

$$\begin{aligned}
V_{n1} &= 2[V] \\
V_{n2} &= \frac{8}{7}[V] \\
V_{n3} &= -\frac{2}{7}[V]
\end{aligned}$$

Exercise 2.5

Steps 1, 2 and 3: see circuit diagram

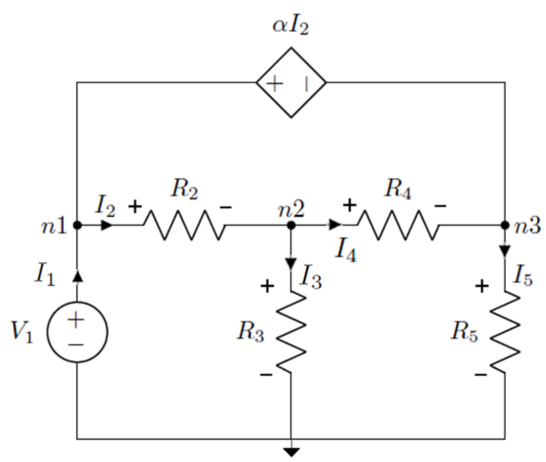


Figure 2.5

Step 4: Write down KCL for nodes n_1, n_2 and n_3

$$\begin{aligned}
&n1 : -I_1 + I_2 + I_6 = 0, \quad I_6 \text{ is the current through the current controlled voltage source} \\
&n2 : -I_2 + I_3 + I_4 = 0 \\
&n3 : -I_4 - I_6 + I_5 = 0
\end{aligned}$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$\begin{aligned}
n1 &: -I_1 + \frac{V_2}{R_2} + I_6 = 0 \\
n2 &: -\frac{V_2}{R_2} + \frac{V_3}{R_3} + \frac{V_4}{R_4} = 0 \\
n3 &: -\frac{V_4}{R_4} - I_6 + \frac{V_5}{R_5} = 0
\end{aligned}$$

Step 6: Express the branch voltages in node voltages

$$\begin{aligned}
b1 &: V_{n1} = V_1 \\
n2 &: -\frac{V_{n1} - V_{n2}}{R_2} + \frac{V_{n2}}{R_3} + \frac{V_{n2} - V_{n3}}{R_4} = 0 \\
n3 &: -\frac{V_{n2} - V_{n3}}{R_4} - I_6 + \frac{V_{n3}}{R_5} = 0 \\
b6 &: V_{n1} - V_{n3} = \alpha I_2 \\
&: V_{n1} - V_{n3} = \alpha \frac{V_2}{R_2} \\
&: V_{n1} - V_{n3} = \alpha \frac{V_{n1} - V_{n2}}{R_2}
\end{aligned}$$

Step 7: Write the equations as a set of equations in node voltages

$$\begin{aligned}
b1 &: V_{n1} = V_1 \\
n2 &: -\frac{1}{R_2} V_{n1} + \Big(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\Big) V_{n2} - \frac{1}{R_4} V_{n3} = 0 \\
n3 &: -\frac{1}{R_4} V_{n2} + \Big(\frac{1}{R_4} + \frac{1}{R_5}\Big) V_{n3} = I_6 \\
b6 &: \frac{R_2 - \alpha}{R_2} V_{n1} + \frac{\alpha}{R_2} V_{n2} - V_{n3} = 0
\end{aligned}$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$\begin{aligned}
b1 &: V_{n1} = 1[V] \\
n2 &: -V_{n1} + 3V_{n2} - V_{n3} = 0 \\
b6 &: V_{n2} - V_{n3} = 0
\end{aligned}$$

Substituting V_{n1} into the equation from n2 gives:

$$\begin{aligned}
n2 &: 3V_{n2} - V_{n3} = 1 \\
b6 &: V_{n2} - V_{n3} = 0
\end{aligned}$$

Solving a system with 2 equation and 2 unknowns results in:

$$\begin{aligned}
V_{n1} &= 1[V] \\
V_{n2} &= \frac{1}{2}[V] \\
V_{n3} &= \frac{1}{2}[V]
\end{aligned}$$

Exercise 2.6

Steps 1, 2 and 3: see circuit diagram

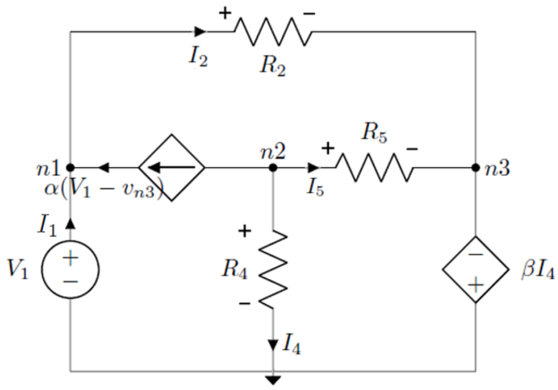


Figure 2.6

Step 4: Write down KCL for nodes n_1, n_2 and n_3

$$n1: -I_1 + I_2 - \alpha(V_1 - V_{n3}) = 0$$
$$n2: \alpha(V_1 - V_{n3}) + I_4 + I_5 = 0$$
$$n3: -I_2 - I_5 - I_6 = 0, \quad I_6 \text{ is the current through the current controlled voltage source}$$

Step 5: Express the branch currents in branch voltages using the EEQs

$$n1: -I_1 + \frac{V_2}{R_2} - \alpha(V_1 - V_{n3}) = 0$$
$$n2: \alpha(V_1 - V_{n3}) + \frac{V_4}{R_4} + \frac{V_5}{R_5} = 0$$
$$n3: -\frac{V_2}{R_2} - \frac{V_5}{R_5} - I_6 = 0$$

Step 6: Express the branch voltages in node voltages

$$b1: V_{n1} = V_1$$
$$n2: \alpha(V_1 - V_{n3}) + \frac{V_{n2}}{R_4} + \frac{V_{n2} - V_{n3}}{R_5} = 0$$
$$n3: -\frac{V_{n1} - V_{n3}}{R_2} - \frac{V_{n2} - V_{n3}}{R_5} - I_6 = 0$$
$$b6: V_{n3} = -\beta I_4$$

Step 7: Write the equations as a set of equations in node voltages

$$b1: V_{n1} = V_1$$
$$n2: \alpha V_1 + \Big(\frac{1}{R_4} + \frac{1}{R_5}\Big)V_{n2} - \Big(\alpha + \frac{1}{R_5}\Big)V_{n3} = 0$$
$$n3: -\frac{1}{R_2}V_{n1} - \frac{1}{R_5}V_{n2} + \Big(\frac{1}{R_2} + \frac{1}{R_5}\Big)V_{n3} = I_6$$
$$b6: V_{n3} = -\frac{\beta}{R_4}V_{n2}$$

Step 8: Obtaining expressions for each node voltage by substituting the components and input source values

$$\begin{aligned}
b1 &: V_{n1} = 1[V] \\
n2 &: 1 + 2V_{n2} - 2V_{n3} = 0 \\
b6 &: V_{n3} = -V_{n2}
\end{aligned}$$

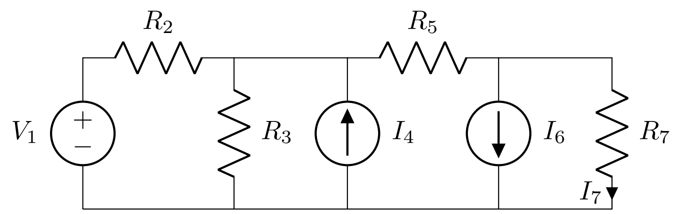
Solving a system with 2 equation and 2 unknowns results in:

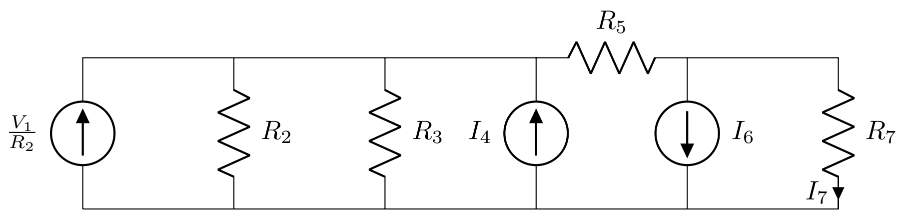
$$\begin{aligned}
V_{n1} &= 1[V] \\
V_{n2} &= -\frac{1}{4}[V] \\
V_{n3} &= \frac{1}{4}[V]
\end{aligned}$$

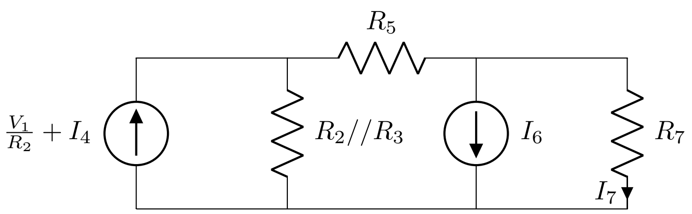
Chapter 3

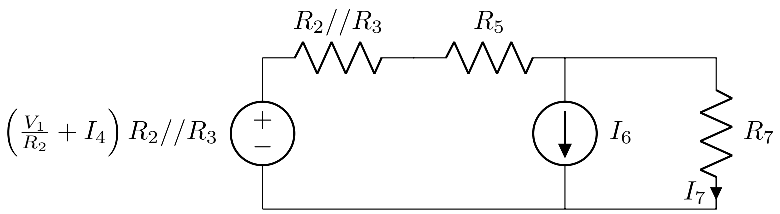
Superposition, source transformation and maximum power

Exercise 3.1









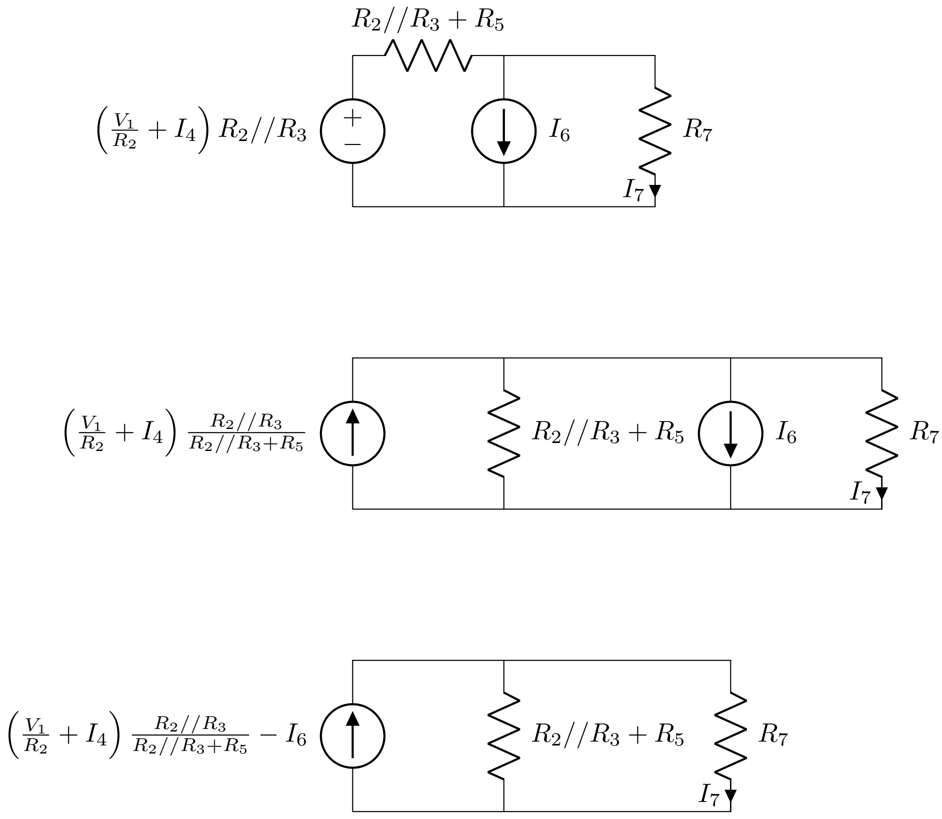


Figure 3.1: Exercise 3.1

Note that the current of the current source is divided between the two parallel resistors. Therefore:

$$I_7 = \frac{R_2//R_3 + R_5}{R_2//R_3 + R_5 + R_7}\left(\left(\frac{V_1}{R_2} + I_4\right)\frac{R_2//R_3}{R_2//R_3 + R_5} - I_6\right)$$

Exercise 3.2

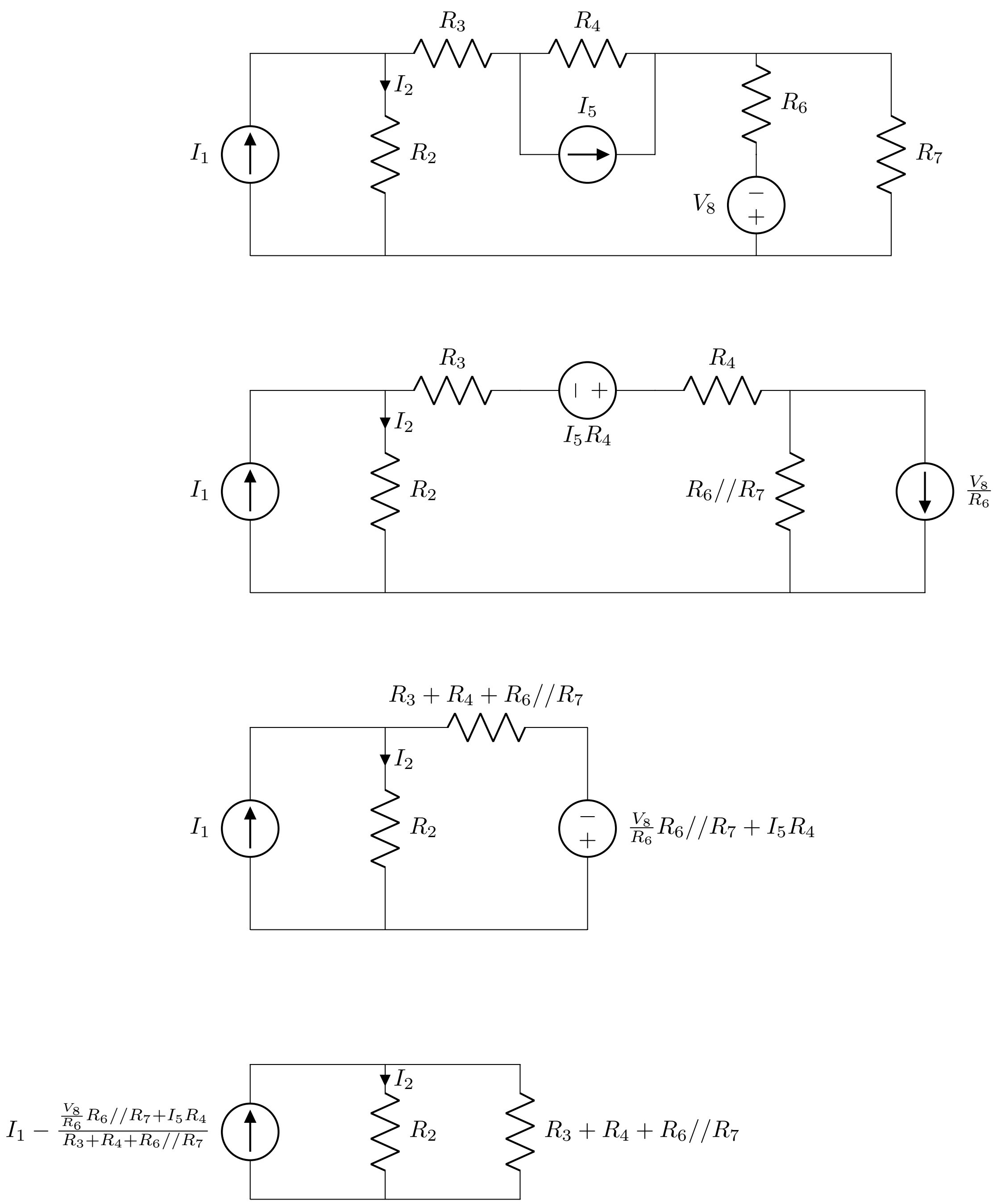


Figure 3.2: Exercise 3.2

The current of the current source is divided between the two parallel resistors:

$$I_2 = \frac{R_3 + R_4 + R_6//R_7}{R_2 + R_3 + R_4 + R_6//R_7} \left(I_1 - \frac{\frac{V_8}{R_6} R_6//R_7 + I_5 R_4}{R_3 + R_4 + R_6//R_7} \right)$$

Exercise 3.3

a) See Figure 3.3, the Thèvenin voltage is equal to the open-circuit voltage, V_{AB}. This is the voltage across R_4:

$$\begin{aligned} V_{Th} &= R_4 I_4 \\ &= \frac{R_2 R_4}{R_2 + R_3 + R_4} I_1 \end{aligned}$$

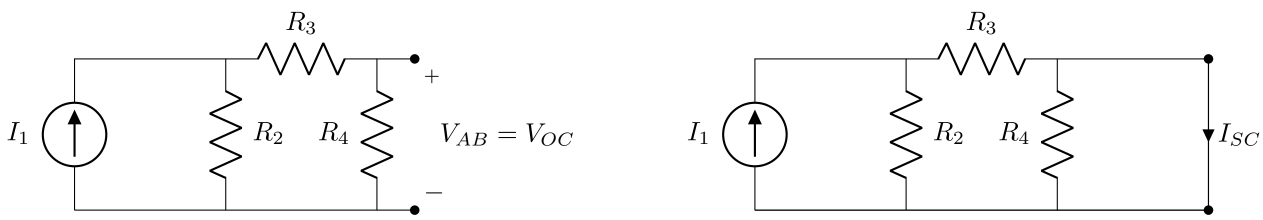


Figure 3.3: Exercise 3.3a

The short-circuit current is equal to (note that R_4 can be considered as an OC comapared with the short-circuit):

$$I_{SC} = \frac{R_2}{R_2 + R_3} I_1$$

b) The Thèvenin resistance can be determined directly by substituting the current source by an OC. R_{Th} is then equal to the total resistance between nodes A and B:

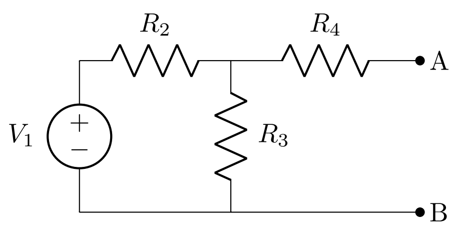
$$R_{TH} = (R_2 + R_3)//R_4$$

From a) we have:

$$\begin{aligned} R_{TH} &= \frac{V_{Th}}{I_{SC}} \\ &= \frac{\frac{R_2 R_4}{R_2+R_3+R_4} I_1}{\frac{R_2}{R_2+R_3} I_1} \\ &= \frac{(R_2 + R_3) R_4}{R_2 + R_3 + R_4} \\ &= (R_2 + R_3)//R_4 \end{aligned}$$

Exercise 3.4

a)



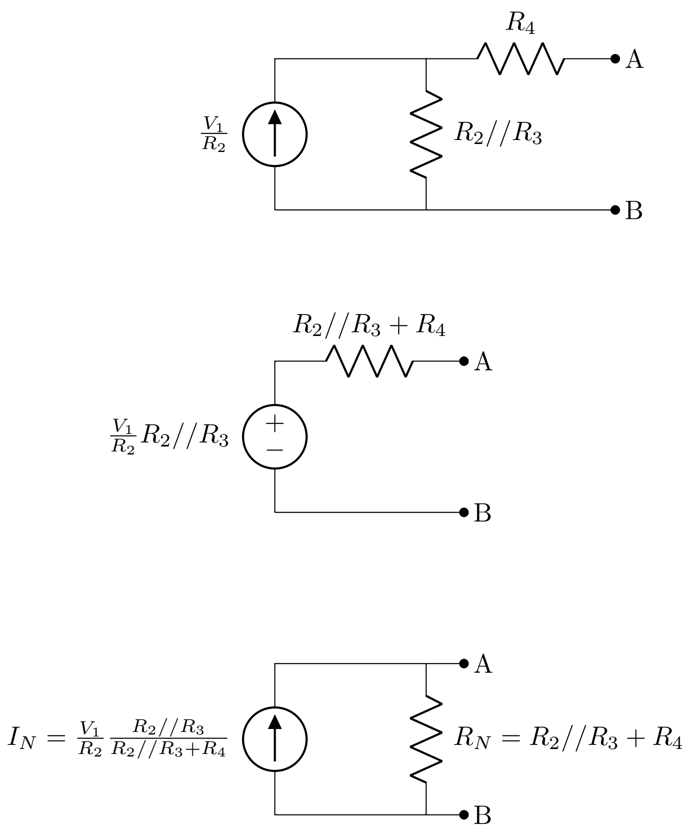


Figure 3.4: Exercise 3.4a

b) The Thèvenin voltage is equal to (note that no current flows through R_4, so $V_{AB} = V_3$):

$$V_{Th} = \frac{R_3}{R_2 + R_3} V_1$$

The short-circuit current is equal to:

$$\begin{aligned} I_{SC} &= \frac{V_4}{R_4} \\ &= \frac{R_3//R_4}{R_2 + R_3//R_4} \frac{V_1}{R_4} \end{aligned}$$

c) The Thèvenin resistance can be determined directly from the circuit by substituting the voltage by a SC. R_{Th} is then equal to the total resistance between nodes A and B:

$$R_{Th} = R_2//R_3 + R_4$$

From a) we have that:

$$R_{Th} = R_N = R_2//R_3 + R_4$$

From b) we have:

$$\begin{aligned}
R_{Th} &= \frac{V_{Th}}{I_{SC}} \\
&= \frac{\frac{R_3}{R_2+R_3}V_1}{\frac{R_3//R_4}{R_2+R_3//R_4}\frac{V_1}{R_4}} \\
&= \frac{R_3R_4(R_2+R_3//R_4)}{(R_2+R_3)R_3//R_4} \\
&= \frac{R_2R_3R_4+R_3R_4\ R_3//R_4}{(R_2+R_3)R_3//R_4} \\
&= \frac{R_2}{(R_2+R_3)}\ \frac{R_3R_4}{R_3//R_4}+\frac{R_3R_4}{(R_2+R_3)} \\
&= \frac{R_2(R_3+R_4)}{(R_2+R_3)}+\frac{R_3R_4}{(R_2+R_3)} \\
&= \frac{R_2R_3+(R_2+R_3)R_4}{R_2+R_3} \\
&= R_2//R_3+R_4
\end{aligned}$$

Exercise 3.5

a) In order to find V_{Th} and I_{SC}, we will use superposition to find the contribution of each method, see Figure 3.5

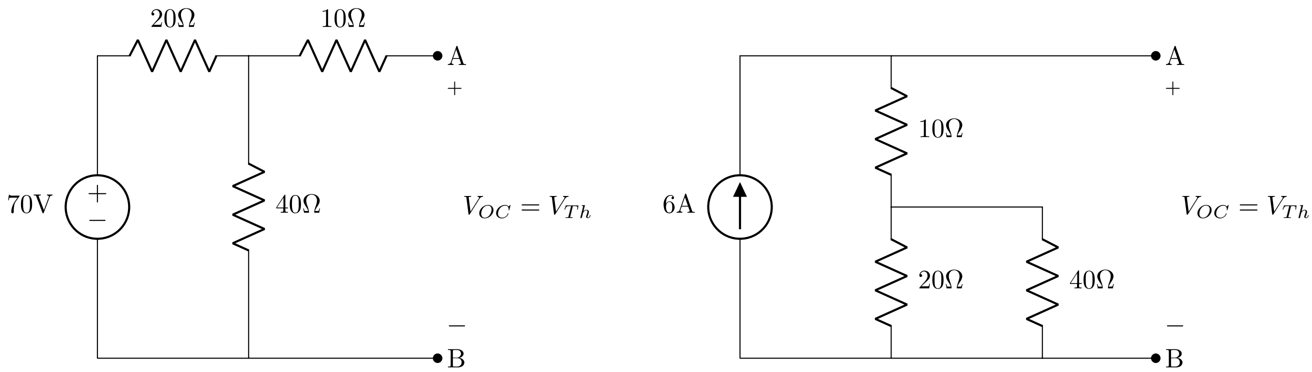


Figure 3.5: Determination of V_{Th}

Note that in the right-hand side of Figure 3.5, no current will flow through the 10Ω resistor.

$$\begin{aligned}
V_{Th1} &= \frac{40}{20+40}70 = \frac{140}{3}V \\
V_{Th2} &= (10+\frac{20\cdot 40}{20+40})\cdot 6 = 140V \\
\Rightarrow V_{Th} &= \frac{140}{3}+140 = \frac{560}{3}V
\end{aligned}$$

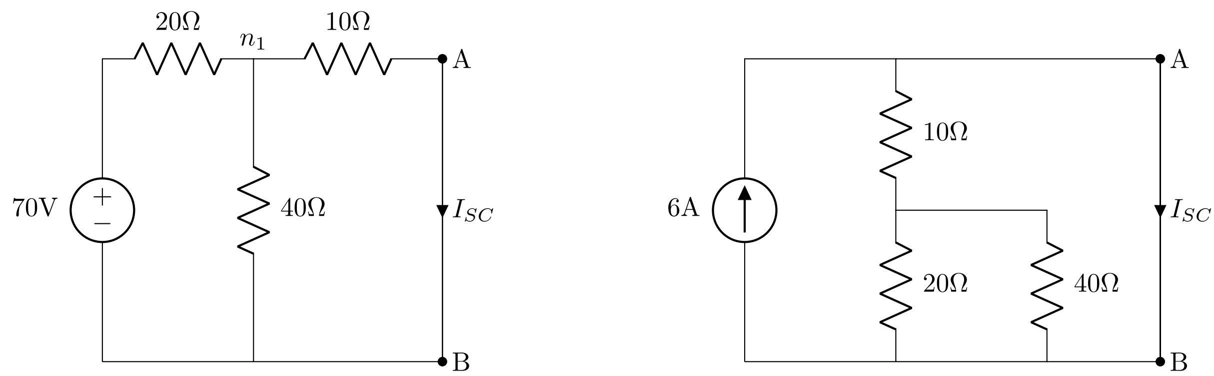


Figure 3.6: Determination of I_{SC}

Note that in the right-hand side of Figure 3.6, the current will take the 'easiest' (i.e. less-ohmic) path. Therefore, no current will flow through the resistors.

$$
\begin{aligned}
I_{SC1} &= \frac{V_{n_1}}{10\Omega} = \frac{10//40}{10 \cdot (10//40 + 20)} \cdot 70 = 2A \\
I_{SC2} &= 6A \\
\Rightarrow I_{SC} &= 2 + 6 = 8A
\end{aligned}
$$

The Thèvenin resistance is then equal to:

$$
\begin{aligned}
R_{TH} &= \frac{V_{Th}}{I_{SC}} \\
&= \frac{560/3}{8} \\
&= \frac{70}{3}\Omega
\end{aligned}
$$

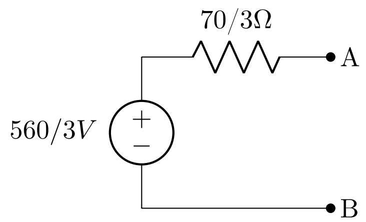


Figure 3.7: Thèvenin equivalent network

b) R_{Th} can be determined directly from the circuit by substituting the current source by an OC and the voltage source by a SC. R_{Th} then becomes:

$$
\begin{aligned}
R_{Th} &= 10 + 20//40 \\
&= 10 + \frac{20 \cdot 40}{20 + 40} \\
&= \frac{70}{3}\Omega
\end{aligned}
$$

Exercise 3.6

a) The power transferred to the load resistance R_L is given by:

$$\begin{aligned} P_L &= \frac{V_L^2}{R_L} \\ &= \frac{R_L}{(R_2+R_L)^2}V_1^2 \end{aligned}$$

In order to find the value of R_L for which P_L is maximum, we have to differentiate P_L with respect to R_L and find the value R_L for which the derivative is zero (i.e. maximum value)

$$\begin{aligned} \frac{dP_L}{dR_L} &= \frac{(R_2+R_L)^2 - R_L(2R_L+2R_2)}{(R_2+R_L)^4}V_1^2 \\ &= \frac{R_2^2 - R_L^2}{(R_2+R_L)^4}V_1^2 \end{aligned}$$

The derivative is zero if the numerator of the above result is zero, i.e.:

$$\begin{aligned} \frac{dP_L}{dR_L} = 0 \quad \Rightarrow \quad & R_2^2 - R_L^2 = 0 \\ & R_2^2 = R_L^2 \Rightarrow R_L = R_2 \end{aligned}$$

b) In case R_L is fixed, P_L is maximum when the denominator of P_L is minimum, i.e. when $R_2 = 0$. P_L is then equal to:

$$P_L = \frac{R_L}{R_L^2}V_1^2 = \frac{V_1^2}{R_L}$$

Exercise 3.7

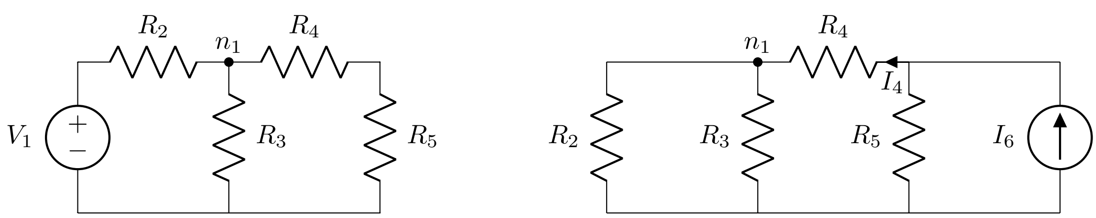


Figure 3.8: Exercise 3.7, left: contribution of V_1, right: contribution of I_6

The voltage at node n_1 is equal to:

$$
\begin{aligned}
V_{n_1|V_1} &= \frac{(R_4+R_5)//R_3}{(R_4+R_5)//R_3+R_2}V_1 \\
V_{n_1|I_6} &= I_4\ (R_2//R_3) = \frac{R_5}{R_2//R_3+R_4+R_5}\ I_6\ (R_2//R_3) \\
\Rightarrow V_{n_1} &= V_{n_1|V_1} + V_{n_1|I_6} \\
&= \frac{(R_4+R_5)//R_3}{(R_4+R_5)//R_3+R_2}V_1 + \frac{R_5}{R_2//R_3+R_4+R_5}\ (R_2//R_3)\ I_6 \\
&= \frac{R_3(R_4+R_5)}{R_3(R_4+R_5)+R_2(R_3+R_4+R_5)}V_1 + \frac{R_2R_3R_5}{R_2R_3+(R_2+R_3)(R_4+R_5)}I_6 \\
&= \frac{R_3(R_4+R_5)}{R_3(R_4+R_5)+R_2(R_3+R_4+R_5)}V_1 + \frac{R_2R_3R_5}{R_3(R_4+R_5)+R_2(R_3+R_4+R_5)}I_6 \\
&= \frac{R_3(R_4+R_5)V_1+R_2R_3R_5I_6}{R_3(R_4+R_5)+R_2(R_3+R_4+R_5)}
\end{aligned}
$$

Exercise 3.8

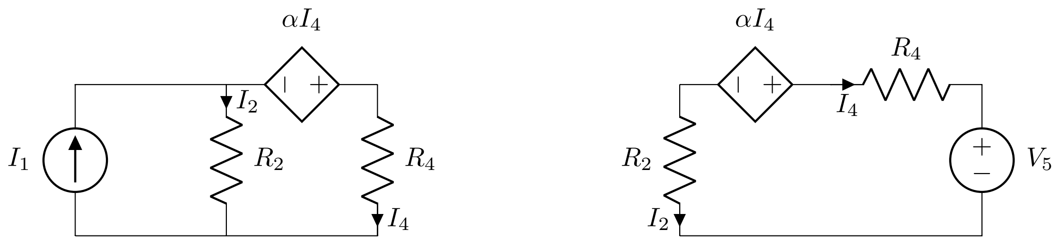


Figure 3.9: Exercise 3.8, left: contribution of I_1, right: contribution of V_5

Contribution of I_1:

$$
\begin{aligned}
\text{KCL} &: I_4 = I_1 - I_2 \\
\text{KVL} &: -I_2R_2 - \alpha I_4 + I_4R_4 = 0 \\
&: -I_2R_2 - \alpha I_1 + \alpha I_2 + I_1R_4 - I_2R_4 = 0 \\
&: (R_4-\alpha)I_1 = (R_2+R_4+\alpha)I_2 \\
&: I_2 = \frac{(R_4-\alpha)}{(R_2+R_4+\alpha)}I_1
\end{aligned}
$$

Contribution of V_5:

$$
\begin{aligned}
\text{KCL} &: I_2 = -I_4 \\
\text{KVL} &: -I_2R_2 - \alpha I_4 + I_4R_4 + V_5 = 0 \\
&: -I_2R_2 + \alpha I_2 - I_2R_4 + V_5 = 0 \\
&: V_5 = (R_2+R_4-\alpha)I_2 \\
&: I_2 = \frac{V_5}{(R_2+R_4-\alpha)}
\end{aligned}
$$

Therefore:

$$
I_2 = \frac{(R_4-\alpha)I_1 + V_5}{(R_2+R_4+\alpha)}
$$

Chapter 4

Capacitors and inductors

Exercise 4.1

a)

$$
\begin{aligned}
i_2(t) &= \frac{1}{L_2}\int_{-\infty}^{t} v_2(\tau)\ d\tau \\
&= \frac{1}{L_2}\int_{-\infty}^{t_0} v_2(\tau)\ d\tau + \frac{1}{L_2}\int_{t_0}^{t} v_2(\tau)\ d\tau \\
&= i_2(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v_2(\tau)\ d\tau
\end{aligned}
$$

For $t_0 = 0$, we have $i_2(t_0) = 0$. So,

$$
i_2(t) = \frac{1}{L_2}\int_{0}^{t} v_2(\tau)\ d\tau \quad t \geq 0 \tag{4.1}
$$

For $t \in [0, T]$ we have $v(t) = V_{max}$, so

$$
\begin{aligned}
i_2(t) &= \frac{1}{L_2}\int_{0}^{t} V_{max}\ d\tau \\
&= \frac{V_{max}}{L_2} t
\end{aligned}
$$

For $t \geq T$, we have $v(t) = 0$, so

$$
\begin{aligned}
i_2(t) &= \frac{1}{L_2}\int_{0}^{T} V_{max}\ d\tau + \frac{1}{L_2}\int_{T}^{t} 0\ d\tau \\
&= \frac{V_{max}T}{L_2}
\end{aligned}
$$

c)

$$
\begin{aligned}
p &= v(t) \cdot i_2(t) \\
&= V_{max} \cdot \frac{V_{max}}{L_2} t \\
&= \frac{V_{max}^2}{L_2}\frac{T}{2}
\end{aligned}
$$

d) The energy in an inductor is given by

$$\begin{aligned}
w &= \frac{1}{2} L_2 i_2(t)^2 \\
&= \frac{1}{2} L_2 \Big(\frac{1}{L_2} \int_0^T V_{max}\, d\tau \Big)^2 \\
&= \frac{1}{2L_2} V_{max}^2 T^2
\end{aligned}$$

Exercise 4.2

- For $0 \leq t < 10us$:

$$\begin{aligned}
V_{out} &= V_0 + \frac{1}{C} \int_0^t 20 \cdot 10^{-3} dt \\
&= 12 + 2 \cdot 10^5 \cdot t
\end{aligned}$$

- For $10us \leq t < 30us$

$$\begin{aligned}
V_{out} &= V_0 + \frac{1}{C} \int_0^{10 \cdot 10^{-6}} 20 \cdot 10^{-3} dt + \frac{1}{C} \int_{10 \cdot 10^{-6}}^t 30 \cdot 10^{-3} dt \\
&= 3 \cdot 10^5 \cdot t + 11
\end{aligned}$$

- For $30us \leq t < 40us$

$$\begin{aligned}
V_{out} &= V_0 + \frac{1}{C} \int_0^{10 \cdot 10^{-6}} 20 \cdot 10^{-3} dt + \frac{1}{C} \int_{10 \cdot 10^{-6}}^{30 \cdot 10^{-6}} 30 \cdot 10^{-3} dt + \frac{1}{C} \int_{30 \cdot 10^{-6}}^t 10 \cdot 10^{-3} dt \\
&= 10^5 \cdot t + 17
\end{aligned}$$

- For $t \geq 40us$

$$\begin{aligned}
V_{out} &= V_0 + \frac{1}{C} \int_0^{10 \cdot 10^{-6}} 20 \cdot 10^{-3} dt + \frac{1}{C} \int_{10 \cdot 10^{-6}}^{30 \cdot 10^{-6}} 30 \cdot 10^{-3} dt + \frac{1}{C} \int_{30 \cdot 10^{-6}}^{40 \cdot 10^{-6}} 10 \cdot 10^{-3} dt \\
&= 21
\end{aligned}$$

$$v(t) = \begin{cases} 2 \cdot 10^5 \cdot t + 12, & 0 \leq t < 10us \\ 3 \cdot 10^5 \cdot t + 11, & 10us \leq t < 30us \\ 10^5 \cdot t + 17, & 30us \leq t < 40us \\ 21, & t \geq 40us \end{cases}$$

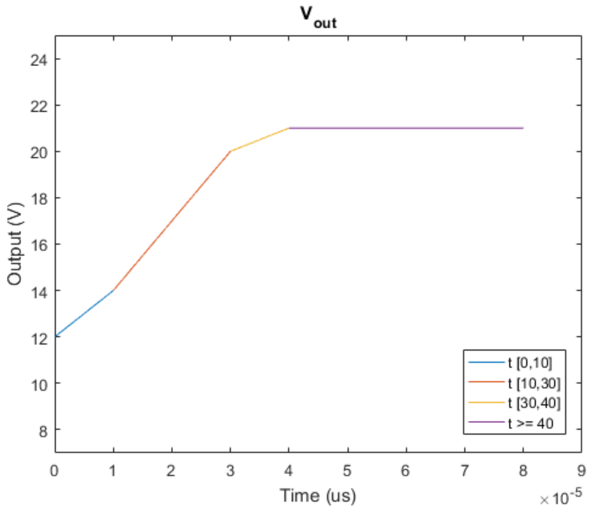


Figure 4.1

Exercise 4.3

a)

$$E_0 = \frac{1}{2}Cu^2(0) = 24.5[mJ] \tag{4.2}$$

b)

$$\left.\begin{aligned} V(0) = A_1e^0 + A_2 0e^0 \\ V(0) = 70 \end{aligned}\right\} \quad \Rightarrow \quad A_1 = 70V$$

$$\frac{dV(t)}{dt} = -1400A_1e^{-1400t} + A_2e^{-1400t} - 1400A_2te^{-1400t}$$
$$i(t) = C\frac{dV(t)}{dt} = 10^{-5}\big(- 1400A_1e^{-1400t} + A_2e^{-1400t} - 1400A_2te^{-1400t}\big)$$

$$\left.\begin{aligned} i(0) = 10^{-5}\big(- 1400A_1e^0 + A_2e^0\big) \\ i(0) = 15 \cdot 10^{-2} \\ A_1 = 70V \end{aligned}\right\} \quad \Rightarrow \quad A_2 = 113[\frac{kV}{s}]$$

So:

$$v(t) = \begin{cases} 70V, & t \le 0 \\ e^{-1400t}\left(70 + 113 \cdot 10^3 \cdot t\right), & t > 0 \end{cases}$$

c)

$$\begin{aligned} i(t) &= C\frac{dV(t)}{dt} \\ &= 10^{-5}\left(-98 \cdot 10^3 \cdot e^{-1400t} + 113 \cdot 10^3 \cdot e^{-1400t} - 113 \cdot 14 \cdot 10^5\, t \cdot e^{-1400t}\right) \\ &= 10^{-5}\Big(e^{-1400t}(113 - 98) \cdot 10^3 - 1582 \cdot 10^5 \cdot t \cdot e^{-1400t}\Big) \\ &= e^{-1400t}\left(0.15 - 1582 \cdot t\right) \end{aligned}$$

d)

$$E = \int_0^1 v(t)i(t)\,dt = \int_0^1 e^{-2800t} \cdot \left(70 + 113 \cdot 10^3 \cdot t\right)(0.15 - 1582 \cdot t)\,dt$$

Exercise 4.4

a)

$$L_{eq} = \Big(\big((L_3 + L_6//L_7)//L_5\big) + (L1//L_2)\Big)//L_4 + L_8$$

b) Note that capacitors C_3, C_4 and C_5 are short-circuited, and will, therefore, have no influence on the equivalent capacitance

$$\begin{aligned} C_{eq} &= C_1//C_2 + C_6 && \text{Note that a "+" here means a series connection, which is not equivalent} \\ & && \text{to simply adding the capacitor values for capacitors} \\ &= \frac{(C_1 + C_2)C_6}{C_1 + C_2 + C_6} \end{aligned}$$

Chapter 5

First order circuits

Exercise 5.1

a) Combine capacitors C1 and C2 in one capacitor C:

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{5.1}$$

Using KVL you can find the voltage across the new capacitor C:

$$-v_{2(t)} + v_1(t) + v_3(t) = 0 \Rightarrow v_3(t) = v_2(t) - v_1(t) = v_C(t) \tag{5.2}$$

Now using the KCL for the node at the top of the new capacitor:

$$\begin{aligned} i_C(t) + i_3(t) &= 0 \\ C\frac{dv_C(t)}{dt} + \frac{v_3(t)}{R_3} &= 0 \\ \frac{dv_3(t)}{dt} + \frac{1}{R_3 C}v_3(t) &= 0 \end{aligned} \tag{5.3}$$

b) The first thing to keep in mind is that unlike a capacitor, the voltage across a resistor *can* change instantaneously. This means that:

$$v_3(0^-) \neq v_3(0^+)$$

This can be also be seen by observing the circuit for $t < 0$ and for $t > 0$. Before the switch is closed, there is no current flowing. As a result, by Ohm's law the voltage across the resistor is zero: $\Rightarrow V_{3,t<0} = V_3(0^-) = 0V$. Once the switch closes, the voltages across the capacitors *cannot* change instantaneously and, therefore, the voltage across the resistors is 'forced' and is equal to:

$$\begin{aligned} v_3(0^+) &= v_C(0^-) = v_C(0^+) \\ &= v_2(0^-) - v_1(0^-) \\ &= v_2(0^+) - v_1(0^+) \\ &= V_2 - V_1 \end{aligned}$$

c) The general solution for a first order DE:

$$v_3(t) = Ke^{-\frac{t}{R_3 C}} \tag{5.4}$$

For t > 0, we have that $u_3(0^+) = V_2 - V_1 = K$, so the expression for $V_3(t)$

$$v_3(t) = (V_2 - V_1)e^{-\frac{t}{R_3 C}}, \quad t \geq 0 \tag{5.5}$$

For t $<$ 0, we have that $v_3(0^-) = 0 = K$, so the expression for $V_3(t)$

$$v_3(t) = 0V, \quad t < 0 \tag{5.6}$$

d) The expression for $i_3(t)$ can be easily found using Ohm's law for the resistor

$$i_3(t) = \frac{v_3(t)}{R_3} = \begin{cases} \frac{V_2 - V_1}{R_3} e^{-\frac{t}{R_3 C}} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

The expression for $v_1(t)$ and $v_2(t)$ can be obtained from the KCL and the result for $i_3(t)$ as follows. *However, note that the expression for $i_3(t)$ only holds for $t \geq 0$, and, therefore, we must add the initial voltages on each cap to the resulting expressions.*

$$\begin{aligned} i_{c1}(t) &= i_3(t) \\ C_1 \frac{dv_1(t)}{dt} &= \frac{V_2 - V_1}{R_3} e^{-\frac{t}{R_3 C}} \quad , \quad \text{integrate both sides as: } \int_0^t \\ C_1 v_1(t) &= \left[-\frac{R_3 C}{R_3} (V_2 - V_1) e^{-\frac{t}{R_3 C}} \right]_0^t \\ C_1 v_1(t) &= C (V_2 - V_1) \left[1 - e^{-\frac{t}{R_3 C}} \right] \\ v_1(t) &= \frac{C_2}{C_1 + C_2} (V_2 - V_1) \left[1 - e^{-\frac{t}{R_3 C}} \right] \end{aligned}$$

Thus, the complete solution to $v_1(t)$ is then:

$$v_1(t) = \frac{C_2}{C_1 + C_2} (V_2 - V_1) \left[1 - e^{-\frac{t}{R_3 C}} \right] + V_1$$

Following the same approach and noting that $i_{c2}(t) = -i_3(t)$, we arrive at the following expression for $v_2(t)$:

$$v_2(t) = \frac{C_1}{C_1 + C_2} (V_1 - V_2) \left[1 - e^{-\frac{t}{R_3 C}} \right] + V_2$$

e) The initial energy in the capacitors C_1 and C_2 (i.e. at $t = 0$):

$$w_1 = \frac{1}{2} C_1 V_1^2 \tag{5.7}$$

$$w_2 = \frac{1}{2} C_2 V_2^2 \tag{5.8}$$

f) The energy in stored in the capacitors as $t \to \infty$ is found using the expressions of $v_1(t)$ and $v_2(t)$:

$$w_1|_{t \to \infty} = \frac{1}{2} C_1 v_1(\infty) = \frac{1}{2} C_1 \left(\frac{C_2}{C_1 + C_2} (V_2 - V_1) + V_1 \right)^2 \tag{5.9}$$

$$w_2|_{t \to \infty} = \frac{1}{2} C_2 v_2(\infty) = \frac{1}{2} C_2 \left(\frac{C_1}{C_1 + C_2} (V_1 - V_2) + V_2 \right)^2 \tag{5.10}$$

g) You can also find the total energy delivered to the resistor by integrating the power delivered to the resistor

$$
\begin{aligned}
w &= \int_{-\infty}^{\infty} p(\tau)d\tau \\
&= \int_{-\infty}^{0} p(\tau)d\tau + \int_{0}^{\infty} p(\tau)d\tau \\
&= 0 + \int_{0}^{\infty} p(\tau)d\tau \\
&= \int_{0}^{\infty} V_3(t) \cdot i_3(t)d\tau \\
&= \int_{0}^{\infty} \frac{V_3^2(t)}{R_3}d\tau \\
&= \int_{0}^{\infty} \frac{(V_2-V_1)^2}{R_3}e^{-2\frac{\tau}{R_3C}}d\tau \\
&= \big[-\frac{R_3C}{2}\frac{(V_2-V_1)^2}{R_3}e^{-2\frac{\tau}{RC}}\big]_0^{\infty} \\
&= 0 + \frac{C}{2}(V_2-V_1)^2 \\
&= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}(V_2-V_1)^2
\end{aligned} \tag{5.11}
$$

As a sanity check: the sum of the initial energies minus the energy dissipated in the resistor must be equal to the sum of the energies stored in the capacitors, i.e.:

$$\text{Eq.5.7} + \text{Eq.5.8} - \text{Eq.5.11} = \text{Eq.5.9} + \text{Eq.5.10}$$

Check this for yourself.

Exercise 5.2

For $t \geq 0$ we consider only the right part of the circuit(where we have three elements in parallel $3H//1.5H//40\Omega$).

First step would be to make the two parallel inductors one. So:

$$L = \frac{L_1 \cdot L_2}{L_1 + L_2} = 1H$$

Now we have an inductor L in parallel with the resistor. So we can use KCL at node B to find the relation with the currents. Both currents are going away from the node so:

$$\text{KCL} \quad B \quad : \quad i_L(t) + i_R(t) = 0 \tag{5.12}$$

$$\text{KVL} \quad : \quad v_L(t) = v_R(t) \tag{5.13}$$

We will now substitute the EEQ for $i_R(t)$ and $i_L(t)$ into 5.12 which results in:

$$
\begin{aligned}
i_L(t) + \frac{v_R(t)}{R} &= 0 \\
i_L(t) + \frac{v_L(t)}{R} &= 0 \\
i_L(t) + \frac{L}{R} \cdot \frac{di_L(t)}{dt} &= 0 \\
\frac{di_L(t)}{dt} + \frac{R}{L} \cdot i_L(t) &= 0
\end{aligned}
$$

General solution:

$$i_L(t) = Ke^{-\alpha t}, \quad \text{where} \quad \alpha = \frac{R}{L} \tag{5.14}$$

Initial condition: To find the initial condition we will consider the circuit on the left now and substitute L_{3H} with short circuit. So:

$$i_L(0^-) = i_L(0^+) = \frac{V}{R} = \frac{50}{10} = 5A$$

Find $i_L(t)$: Starting from the general solution 5.14 and substituting $t = 0$ we get:

$$
\begin{aligned}
i_L(0) &= Ke^{-\alpha \cdot 0} \\
i_L(0) &= K = 5A
\end{aligned}
$$

So:

$$i_L(t) = 5e^{-\alpha t}, \quad \text{where} \quad \alpha = \frac{R}{L} \tag{5.15}$$

The question is to find the moment for which the voltage across the resistor will be 100V. So:

$$
\begin{aligned}
i_L(t) &= \frac{100}{40} \\
5e^{-\frac{R}{L}t} &= \frac{100}{40} \\
e^{-\frac{R}{L}t} &= \frac{100}{200} \\
\ln(e^{-\frac{R}{L}t}) &= \ln\frac{1}{2} \\
-\frac{R}{L}t &= \ln\frac{1}{2} \\
t &= \ln\frac{1}{2} \cdot \frac{-L}{R} \\
t &= \ln\frac{1}{2} \cdot \frac{-1}{40} \\
t &= 17.3ms
\end{aligned}
$$

Exercise 5.3

a) Take the right part of the circuit consisting of capacitor C_3, resistor R_4 and capacitor C_4 in series with each other. Start by combining the capacitors into one capacitor C:

$$C = \frac{C_3 C_5}{C_3 + C_5}$$

Next write both the KVL and KCL at node B (for the resulting circuit, assuming both currents are going out of the node):

$$\begin{aligned} &\text{KVL:} \quad -v_3(t) + v_4(t) + v_5(t) = 0 \quad \Rightarrow \quad v_4(t) = v_3(t) - v_5(t) = v_c(t) \\ &\text{KCL:} \quad i_c(t) + i_4(t) = 0 \end{aligned}$$

Now using the KCL derive the DE:

$$\begin{aligned} i_c(t) + i_4(t) &= 0 \\ C\frac{v_c(t)}{dt} + \frac{v_4(t)}{R_4} &= 0 \\ C\frac{dv_4(t)}{dt} + \frac{v_4(t)}{R_4} &= 0 \\ \frac{dv_4(t)}{dt} + \frac{1}{R_4C}v_4(t) &= 0 \end{aligned} \tag{5.16}$$

b) The initial condition can be found by finding the initial condition for $v_3(t)$ and using KVL to find the initial condition for $v_4(t)$. The initial condition for $v_3(0^-)$ can be found by taking the left part of the circuit and substituting capacitor C_3 by an OC and observing that $v_3(0^-) = v_{R_2}(0^-) = I_1R_2$

$$\begin{aligned} v_4(t) &= v_3(t) - v_5(t) \\ v_4(0^-) &= v_3(0^-) - v_5(0^-) \\ v_4(0^-) &= I_1R_2 - 0 = I_1R_2 \end{aligned} \tag{5.17}$$

c) This is again the general solution to a first order DE:

$$\begin{aligned} v_4(t) &= Ke^{-\frac{t}{R_4C}} \\ &= I_1R_2e^{-\frac{t}{R_4C}} \end{aligned} \tag{5.18}$$

The solutions to part d), e) and f) can be found using the exact same approach that has been used in exercise 5.1 part d), e) and f).

Exercise 5.4

First combine L_4 and L_6 in one inductor L. The DE in $i_5(t)$ can then be found by writing the KVL for the right part of the circuit. Please note that the polarity for the new inductor is + at node B and - at the other side. **DE**

$$\begin{aligned} -V_L + V_5 &= 0 \\ -L\frac{di_L(t)}{dt} + i_5(t)R_5 &= 0 \\ L\frac{di_5(t)}{dt} + i_5(t)R_5 &= 0 \\ \frac{di_5(t)}{dt} + \frac{R_5}{L}i_5(t) &= 0 \end{aligned} \tag{5.19}$$

Initial condition The initial condition for $i_5(t)$ can be found from the initial condition of the current of the inductor. For the initial condition of the current of L_6, it can be assumed that $i_6(0^-) = 0A$ because before the switch has been switched from position A to position B, it has been in position A for a long time, which means that all energy (that could have been in L_6) is dissipated over R_5.

$$i_5(0^-) = -i_6(0^-) = 0$$

$i_4(0^-)$ can found by considering the left part of the circuit before the switch moved to position B. By substituting L_4 by a SC, the current $i_4(0^-) = i_3(0^-) = \frac{R_2}{R_2+R_3} I_1$. So,

$$i_5(0^+) = -i_4(0^-) - i_6(0^-) = -i_4(0^+) + 0 = -\frac{R_2}{R_2 + R_3} I_1 \tag{5.20}$$

Solution to the DE
The expression for $i_5(t)$ is the solution to the first order DE in Equation 5.19.

$$\begin{aligned} i_5(t) &= Ke^{\frac{-R_5 t}{L}} \\ &= -\frac{R_2}{R_2 + R_3} I_1 e^{\frac{-R_5 t}{L}} \end{aligned} \tag{5.21}$$

b) Total energy delivered to $\mathbf{R_5}$: the total energy can be found by integrating the power delivered to R_5. This must be equal to the energy stored in the inductor L at t = 0 (check this for yourself by finding the energy stored in the inductors at t = 0).

$$\begin{aligned} w_5 &= \int_{-\infty}^{\infty} v_5(\tau) i_5(\tau) d\tau \\ &= \int_{-\infty}^{\infty} v_5(\tau) i_5(\tau) d\tau \\ &= \int_{-\infty}^{0} i_5^2(\tau) R_5 d\tau + \int_{0}^{\infty} i_5^2(\tau) R_5 d\tau \\ &= 0 + \int_{0}^{\infty} i_5^2(\tau) R_5 d\tau \\ &= \int_{0}^{\infty} (\frac{R_2}{R_2 + R_3})^2 I_1^2 R_5 e^{\frac{-2R_5 \tau}{L}} d\tau \\ &= [(\frac{R_2}{R_2 + R_3})^2 I_1^2 \frac{-R_5 L}{2R_5} e^{\frac{-2R_5 \tau}{L}}]_0^{\infty} \\ &= (\frac{R_2}{R_2 + R_3} I_1)^2 \frac{L}{2} \end{aligned} \tag{5.22}$$

Exercise 5.5

a) For $t \geq 0$ we consider only the right part of the circuit(where we have three elements an inductor, a dependent source and a resistor). We can start from KVL in the closed loop and the KCL:

$$\begin{aligned} \text{KCL} &: \; i_4(t) = i_6(t) \\ \text{KVL} &: \; - v_{L_4} + \alpha i_6 - v_{R_6} = 0 \end{aligned}$$

$$
\begin{aligned}
&-v_{L_4}+\alpha i_6-v_{R_6}=0\\
&-L_4\frac{di_{L_4}(t)}{dt}+\alpha i_6-i_6R_6=0\\
&-L_4\frac{di_{L_4}(t)}{dt}+\alpha i_4-i_4R_6=0\\
&-L_4\frac{di_{L_4}(t)}{dt}+i_4(a-R_6)=0\\
&\frac{di_{L_4}(t)}{dt}+i_4\frac{(R_6-a)}{L_4}=0
\end{aligned}
$$

b) **Find $i_L(t)$:**
The general solution is:

$$
i_L(t)=Ke^{-\beta t},\quad \text{where}\quad \beta=\frac{R_6-\alpha}{L_4} \tag{5.23}
$$

The initial condition is given to be $i_4(0^-)=\frac{V_1}{R_2}$. So plugging in $t=0$ into the general solution 5.23 gives:

$$
\begin{aligned}
&i_L(t)=Ke^{-\beta t}\\
&i_L(0^-)=i_L(0^+)=Ke^{-\beta\cdot 0}\\
&i_L(0^+)=K=\frac{V_1}{R_2}
\end{aligned}
$$

So:

$$
i_L(t)=\frac{V_1}{R_2}e^{\frac{-(R6-a)}{L4}\cdot t}
$$

Exercise 5.6

a),b)

For $0\leq t<a$:

For $0\leq t<a$ we consider only the left part of the circuit (where we have three elements a voltage source, a resistor and a capacitor. We can start from KVL in the closed loop and the KCL:

$$
\begin{aligned}
\text{KCL}\quad &:\quad i_2(t)=i_3(t)\\
\text{KVL}\quad &:\quad V_1+v_2(t)+v_3(t)=0
\end{aligned}
$$

$$
\begin{aligned}
V_1 + v_2(t) + v_3(t) &= 0 \\
V_1 + i_2(t)R_2 + v_3(t) &= 0 \\
V_1 + i_3(t)R_2 + v_3(t) &= 0 \\
V_1 + R_2C_3\frac{dv_3(t)}{dt} + v_3(t) &= 0 \\
R_2C_3\frac{dv_3(t)}{dt} + v_3 &= -V_1 \\
\frac{dv_3(t)}{dt} + \frac{1}{R_2C_3}v_3 &= -\frac{V_1}{R_2C_3}
\end{aligned}
$$

Homogeneous solution:

$$v_3(t)_h = Ke^{-\alpha_1 t}, \quad \text{where} \quad \alpha_1 = \frac{1}{R_2C_3} \tag{5.24}$$

Particular solution:

$$v_3(t)_p = -V_1 \tag{5.25}$$

General solution:

$$v_3(t) = Ke^{-\alpha_1 t} - V_1 \tag{5.26}$$

Initial condition: to find the initial condition we notice that it is given that there is no energy stored in capacitor C_3 at $t = 0$:

$$v_3(0^-) = v_3(0^+) = 0$$

Find $v_3(t)$: starting from the general solution 5.26 and plugging in $t = 0$ we get:

$$
\begin{aligned}
v_3(t) &= Ke^{-\alpha_1 t} - V_1 \\
v_3(0) &= Ke^{-\alpha_1 0} - V_1 \\
v_3(0) &= K - V_1 = 0 \\
K &= V_1
\end{aligned}
$$

So:

$$v_3(t) = V_1(e^{\frac{-t}{R_2C_3}} - 1) \tag{5.27}$$

For $t \geq a$:

For $t \geq a$ both switches are closed so we can consider the entire circuit connected. In this case it can be observed that we have two sources. So, we will perform a source transformation. The goal is to transform the circuit in such a way that it looks like the circuit we had when switch B was closed (voltage source in series with a resistor and a capacitor). We will now go through the steps to reach that goal.

- Transform the voltage source to a current source $I_1 = -\frac{V_1}{R_2}$ and make $R = R_2//R_4$
- Add the current sources $I = I_5 - \frac{V_1}{R_2}$
- Transform current source to voltage source $V = \left(I_5 - \frac{V_1}{R_2}\right) R$

Now, the circuit will be a voltage source in series with a resistor and a capacitor just like when switch B was open. So:

$$\frac{dv_3(t)}{dt} + \frac{1}{RC_3}v_3 = \frac{V}{RC_3} \tag{5.28}$$

Homogeneous solution:

$$v_3(t)_h = Ke^{\frac{-t}{RC_3}} \tag{5.29}$$

Particular solution:

$$v_3(t)_p = \left(I_5 - \frac{V_1}{R_2}\right)R \tag{5.30}$$

General solution:

$$v_3(t) = Ke^{\frac{-t}{RC_3}} + \left(I_5 - \frac{V_1}{R_2}\right)R \tag{5.31}$$

Initial condition: to find the initial condition we will consider the circuit with the capacitor being a voltage source with a voltage given by 5.27 with $t = a$. So:

$$v_3(a^-) = v_3(a^+) = V_1\left(e^{\frac{-a}{R_2C_3}} - 1\right)$$

Find $v_3(t)$: starting from the general solution 5.31 and plugging in $t = a$ we get:

$$\begin{aligned}
v_3(t) &= Ke^{\frac{-t}{RC_3}} + \left(I_5 - \frac{V_1}{R_2}\right)R \\
v_3(a) &= Ke^{\frac{-a}{RC_3}} + \left(I_5 - \frac{V_1}{R_2}\right)R \\
V_1\left(e^{\frac{-a}{R_2C_3}} - 1\right) &= Ke^{\frac{-a}{RC_3}} + \left(I_5 - \frac{V_1}{R_2}\right)R \\
K &= V_1\left(e^{\frac{-a}{R_2C_3}} - 1\right)e^{\frac{a}{RC_3}} - \left(I_5 - \frac{V_1}{R_2}\right)R\,e^{\frac{a}{RC_3}}
\end{aligned}$$

So:

$$v_3(t) = e^{\frac{-(t-a)}{RC_3}}\left[V_1\left(e^{\frac{-a}{RC_3}} - 1\right) - \left(I_5 - \frac{V_1}{R_2}\right)R\right] + \left(I_5 - \frac{V_1}{R_2}\right)R \tag{5.32}$$

Chapter 6

Second order circuits

Exercise 6.1

a) Start by writing by the KCL for the node at the top of the three elements: R_1, L_2 and C_3. Since there is only one resistor, one inductor and one capacitor, we will simply refer to these as R, L and C for the simplicity of the expressions.

$$\begin{aligned}
i_R(t) + i_L(t) + i_C(t) &= 0 \\
\frac{v_R(t)}{R} + i_L(t) + C\frac{dv_3(t)}{dt} &= 0 \\
\frac{v_3(t)}{R} + i_L(t) + C\frac{dv_3(t)}{dt} &= 0\ , \qquad \text{now differentiate both sides of the equation w.r.t. t} \\
\frac{1}{R}\frac{dv_3(t)}{dt} + \frac{di_L(t)}{dt} + C\frac{d^2v_3(t)}{dt^2} &= 0 \\
\frac{1}{R}\frac{dv_3(t)}{dt} + \frac{1}{L}v_3(t) + C\frac{d^2v_3(t)}{dt^2} &= 0 \\
\frac{d^2v_3(t)}{dt^2} + \frac{1}{RC}\frac{dv_3(t)}{dt} + \frac{1}{LC}v_3(t) &= 0 \qquad (6.1)
\end{aligned}$$

b) We start by writing the characteristic equation from the DE of Equation 6.1.

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad (6.2)$$

It is helpful to compare the resulting differential equation with the *standard form* of second-order differential equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad (6.3)$$

The solution for the underdamped, overdamped and critically-damped situations are as follows:

$$\begin{aligned}
&s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} && \text{Overdamped situation} \\
&s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d && \text{Underdamped situation} \\
&s_{1,2} = -\alpha && \text{Critically damped situation} \\
&\text{where} \quad \alpha = \frac{1}{2RC} \quad , \quad \omega_0^2 = \frac{1}{LC} \quad \text{and} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2} && (6.4)
\end{aligned}$$

The general solution to a (homogeneous) second order differential equation then follows as:

$$\begin{aligned} v_3(t) &= B_1e^{-\alpha t}\cos(\omega_d t) + B_2e^{-\alpha t}\sin(\omega_d t) && \text{Underdamped situation}\\ v_3(t) &= K_1e^{s_1 t} + K_2e^{s_2 t} && \text{Overdamped situation}\\ v_3(t) &= (K_1 t + K_2)e^{s_{1,2} t} && \text{Critically-damped situation} \end{aligned} \tag{6.5}$$

Now that the general solutions are known, using the initial conditions the solutions for $v_3(t)$ can be found.

Underdamped situation

$$\begin{aligned} v_3(0) &= B_1 = A && \Rightarrow B_1 = A\\ \frac{dv_3(0)}{dt} &= -\alpha B_1 + \omega_d B_2 = B && \Rightarrow -\alpha A + \omega_d B_2 = B\\ & && \Rightarrow B_2 = \frac{B+\alpha A}{\omega_d} \end{aligned}$$

Thus, the solution $v_3(t)$ is then given by:

$$v_3(t) = Ae^{-\alpha t}\cos(\omega_d t) + \frac{B+\alpha A}{\omega_d}e^{-\alpha t}\sin(\omega_d t)$$

Overdamped situation

$$\begin{aligned} v_3(0) &= K_1 + K_2 = A && \Rightarrow K_1 = A - K_2\\ \frac{dv_3(0)}{dt} &= s_1K_1 + s_2K_2 = B && \Rightarrow s_1(A-K_2) + s_2K_2 = B\\ & && \Rightarrow K_2 = \frac{B - s_1A}{s_2 - s_1}\\ & && \Rightarrow K_1 = \frac{s_2A - B}{s_2 - s_1} \end{aligned}$$

Thus, the solution $v_3(t)$ is then given by:

$$v_3(t) = \frac{s_2A - B}{s_2 - s_1}e^{s_1 t} + \frac{B - s_1A}{s_2 - s_1}e^{s_2 t} \tag{6.6}$$

Critically damped situation

$$\begin{aligned} v_3(0) &= K_2 = A && \Rightarrow \quad K_2 = A\\ \frac{dv_3(0)}{dt} &= K_1 + s_{1,2}K_2 = B && \Rightarrow \quad K_1 = B - s_{1,2}A\\ & && \Rightarrow \quad K_1 = B + \alpha A \end{aligned} \tag{6.7}$$

Now follows the solution as:

$$v_3(t) = ((B+\alpha A)t + A)e^{-\alpha t} \tag{6.8}$$

c) For $v_3(t)$ to be critically damped the two solutions to the characteristic equation have to be equal, or $s_1 = s_2$. This means that the determinant $(\alpha^2 - \omega_0^2)$ has to be equal to 0.

$$\begin{aligned}
\alpha^2 - \omega_0^2 &= 0 \\
\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC} &= 0 \\
\frac{1}{4(RC)^2} &= \frac{1}{LC} \\
LC = 4(RC)^2 \qquad &\Rightarrow \qquad C = \frac{L}{4R^2} = \frac{0.02}{4 \cdot 10000} = 0.5\mu F
\end{aligned}$$

Exercise 6.2

a) First replace the two capacitors C_2 and C_3, which are in parallel by a single capacitor $C = C_2 + C_3$. To derive the DE, start by writing KCL for the node at the top of the elements (and the current source). The polarities for the voltages across the capacitor and the inductor are the same as for $v_5(t)$.

$$\begin{aligned}
-I_1 + i_C(t) + i_4(t) + i_5(t) &= 0 \\
-I_1 + C\frac{dv_C(t)}{dt} + i_4(t) + \frac{v_5(t)}{R_5} &= 0 \\
C\frac{dv_4(t)}{dt} + \frac{v_4(t)}{R_5} + i_4(t) &= I_1 \\
L_4 C\frac{d^2 i_4(t)}{dt^2} + \frac{L_4}{R_5}\frac{di_4(t)}{dt} + i_4(t) &= I_1 \\
\frac{d^2 i_4(t)}{dt^2} + \frac{1}{R_5 C}\frac{di_4(t)}{dt} + \frac{1}{L_4 C} i_4(t) &= \frac{I_1}{L_4 C} \qquad (6.9)
\end{aligned}$$

b) Since it is given that at $t = 0$ L_4 contains no energy, this means that

$$\begin{aligned}
i_4(0^-) &= i_4(0^+) = 0\ A \\
\frac{di_4(0^-)}{dt} &= \frac{1}{L_4}v_4(0^-) \\
\frac{di_4(0^-)}{dt} &= \frac{1}{L_4}v_C(0^-) \\
\frac{di_4(0^-)}{dt} &= \frac{1}{L_4}v_C(0^+) \quad , \quad v_C(0^-) = v_C(0^+) = 0 \text{ (as there is no energy in the capacitor)} \\
\frac{di_4(0^-)}{dt} &= \frac{1}{L_4}v_4(0^+) \\
\frac{di_4(0^-)}{dt} &= \frac{di_4(0^+)}{dt} = 0\ A/s \qquad (6.10)
\end{aligned}$$

c) First we solve for the homogeneous solution. The characteristic equation for the differential equation is given by

$$s^2 + \frac{1}{R_5 C}s + \frac{1}{L_4 C} = 0 \qquad (6.11)$$

Since it is also given that the solution for $i_4(t)$ can be assumed to be overdamped, we have that the solutions to the characteristic equation are given:

$$\begin{aligned}
s_{1,2} &= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \\
\text{where} \quad \alpha &= \frac{1}{2R_5 C} \quad \text{and} \quad \omega_0^2 = \frac{1}{L_4 C} \qquad (6.12)
\end{aligned}$$

The homogeneous solution is then given by

$$i_{4,h}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{6.13}$$

Note that since there is an excitation source involved in the circuit (I_1), therefore there is a particular solution. Solving for the constants K_1 and K_2 must not be done at this moment, but once the complete solution for $i_4(t)$ has been found.

The particular solution can be found as follows. The term at the right hand side of the DE in Equation 6.9 is constant. Therefore the particular solution is equal to

$$\begin{aligned} \frac{1}{L_4 C} i_{4,p}(t) &= \frac{I_1}{L_4 C} \\ i_{4,p}(t) &= I_1 \end{aligned} \tag{6.14}$$

Then the complete solution for $i_4(t)$ is given by:

$$\begin{aligned} i_4(t) &= i_{4,h}(t) + i_{4,p}(t) \\ i_4(t) &= K_1 e^{s_1 t} + K_2 e^{s_2 t} + I_1 \end{aligned} \tag{6.15}$$

Now we can solve for the constants K_1 and K_2 using the initial conditions found in part b)

$$\begin{aligned} i_4(0) = K_1 + K_2 + I_1 = 0 \qquad & \Rightarrow \qquad K_1 = -(K_2 + I_1) \\ \frac{di_4(0)}{dt} = s_1 K_1 + s_2 K_2 = 0 \qquad & \Rightarrow \qquad K_2 = \frac{s_1 I_1}{s_2 - s_1} \\ & \Rightarrow \qquad K_1 = \frac{s_2 I_1}{s_1 - s_2} \end{aligned} \tag{6.16}$$

This can be substituted in Equation 6.15 giving the final solution:

$$i_4(t) = \frac{s_2 I_1}{s_1 - s_2} e^{s_1 t} + \frac{s_1 I_1}{s_2 - s_1} e^{s_2 t} + I_1 \tag{6.17}$$

d) Since the solution to $i_4(t)$ is known, the voltage across the inductor, $v_4(t)$, can be found using the EEQ of the L_4. This voltage is then equal to $v_5(t)$ since L_4 and R_5 are in parallel:

$$v_5(t) = \frac{s_1 s_2 L_4 I_1}{s_1 - s_2} e^{s_1 t} + \frac{s_1 s_2 L_4 I_1}{s_2 - s_1} e^{s_2 t} = \frac{s_1 s_2 L_4 I_1}{s_1 - s_2} \left(e^{s_1 t} - e^{s_2 t} \right) \tag{6.18}$$

Exercise 6.3

a) First step would be to write down KVL and also to notice that the current is the same for all three elements (R,L and C). So:

$$\begin{aligned} \text{KVL} \quad : \quad & V_1 = v_R(t) + v_L(t) + v_C(t) \\ & i_R(t) = i_L(t) = i_C(t) \end{aligned}$$

$$\begin{aligned} V_1 &= v_R(t) + v_L(t) + v_C(t) \\ V_1 &= i_R(t) R + L \frac{di_L(t)}{dt} + v_C(t) \\ V_1 &= i_C(t) R + L \frac{di_C(t)}{dt} + v_C(t) \\ V_1 &= RC \frac{dv_C(t)}{dt} + LC \frac{d^2 v_C(t)}{dt^2} + v_C(t) \\ \frac{V_1}{LC} &= \frac{d^2 v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) \end{aligned}$$

b) When the switch is open there is no current flowing and consequently there is no voltage across the capacitor. When we observe the circuit for $t \geq 0$ that the switch is closed (replace L with an OC because it can be seen as a current source with a zero current and C with a SC because it is a voltage source with a zero voltage), it can be noticed that again there will be no current flowing and consequently that the derivative of the voltage over the capacitor is zero. So:

$$v_C(0^-) = v_C(0^+) = 0\ V$$
$$\frac{dv_C(0^-)}{dt} = \frac{dv_C(0^+)}{dt} = \frac{i_c(t)}{C} = 0\ V/s$$

c) It is given from the exercise that $v_C(t)$ is underdamped. Starting from the homogeneous solution will be:

$$V_C(t)_h = K_1\cos(\omega_d t)e^{-\alpha t} + K_2\sin(\omega_d t)e^{-\alpha t}$$

For the particular solution:

$$\frac{1}{LC}v_C(t)_p = \frac{V_1}{LC}$$
$$v_C(t)_p = V_1$$

So $V_C(t)$ is:

$$v_C(t) = v_C(t)_h + v_C(t)_p = K_1\cos(\omega_d t)e^{-\alpha t} + K_2\sin(\omega_d t)e^{-\alpha t} + V_1$$
$$\frac{dv_C(t)}{dt} = -K_1\omega_d\sin(\omega_d t)e^{-\alpha t} - K_1\alpha\cos(\omega_d t)e^{-\alpha t} + K_2\omega_d\cos(\omega_d t)e^{-\alpha t} - K_2\alpha\sin(\omega_d t)e^{-\alpha t}$$

Now applying the initial conditions will give:

$$v_C(0) = K_1 + V_1 = 0 \qquad \Rightarrow K_1 = -V_1$$
$$\frac{dV_C(0)}{dt} = -K_1\alpha + K_2\omega_d = 0 \qquad \Rightarrow K_2 = -\frac{\alpha V_1}{\omega_d}$$

Now that the constants have also been found the complete solution for $v_C(t)$ is:

$$v_C(t) = -V_1\cos(\omega_d t)e^{-\alpha t} - \frac{\alpha V_1}{\omega_d}\sin(\omega_d t)e^{-\alpha t} + V_1$$

d) Using the EEQ of the capacitor we can obtain $i_c(t)$:

$$\begin{aligned} i_c(t) &= C\frac{dv_c(t)}{dt} \\ &= C\left[V_1\omega_d\sin(\omega_d t)e^{-\alpha t} + \alpha V_1\cos(\omega_d t)e^{-\alpha t} - \alpha V_1\cos(\omega_d t)e^{-\alpha t} + \frac{\alpha^2 V_1}{\omega_d}\sin(\omega_d t)e^{-\alpha t}\right] \\ &= CV_1\frac{\omega_d^2 + \alpha^2}{\omega_d}\sin(\omega_d t)e^{-\alpha t} \end{aligned}$$

Exercise 6.4

a) In order to find the DE for $t \geq 0$, we will consider the right part of the circuit (when the switch is connected to B). Comparing this circuit with that of exercise 6.3 a), we can see they are exactly the same, therefore:

$$\frac{V_7}{L_6C_4} = \frac{d^2v_4(t)}{dt^2} + \frac{R_5}{L_6}\frac{dv_4(t)}{dt} + \frac{1}{L_6C_4}v_4(t)$$

b) Starting at $t = 0^-$, the switch was in position A for a very long time. That means that the capacitor has been fully charged and the voltage across it is constant and is equal to the voltage across R_3. At $t = 0^-$, the capacitor has been fully charged, therefore there will be no current flowing through it: $i_4(0^-) = 0A$. At $t = 0^-$ there is also no current through the right part of the circuit due to the open circuit, thus $i_6(0^-) = 0A$. Since the current in the inductor cannot change instantaneously we also have that $i_6(0^+) = 0A$. In summary we have:

$$\begin{aligned}
v_4(0^-) = v_4(0^+) &= \frac{R_3}{R_2 + R_3} V_1 \\
\frac{dv_4(0^-)}{dt} &= \frac{1}{C_4} i_4(0^-) = 0A \\
\frac{dv_4(0^+)}{dt} &= \frac{1}{C_4} i_4(0^+) = \frac{1}{C_4} i_6(0^+) = 0A
\end{aligned}$$

c)It is given from the exercise that $v_4(t)$ is critically damped. Starting from the homogeneous solution will be:

$$v_4(t)_h = (K_1 + K_2 t)\, e^{-\alpha t}$$

For the particular solution:

$$\begin{aligned}
&\frac{1}{L_6 C_4} v_4(t)_p = \frac{V_7}{L_6 C_4} \\
&v_4(t)_p = V_7
\end{aligned}$$

So $V_C(t)$ is:

$$\begin{aligned}
v_4(t) &= v_4(t)_h + v_4(t)_p = (K_1 + K_2 t)\, e^{-\alpha t} + V_7 \\
\frac{dv_4(t)}{dt} &= (-\alpha K_1 + K_2 - \alpha K_2 t)\, e^{-at}
\end{aligned}$$

Now applying the initial conditions will give:

$$\begin{aligned}
v_4(0) = K_1 + V_7 &= \frac{R_3}{R_2 + R_3} V_1 & \Rightarrow K_1 &= \frac{R_3}{R_2 + R_3} V_1 - V_7 \\
\frac{dv_4(0)}{dt} &= -\alpha K_1 + K_2 = 0 & \Rightarrow K_2 &= \alpha \left(\frac{R_3}{R_2 + R_3} V_1 - V_7 \right)
\end{aligned}$$

Now that the constants have also been found the complete solution for $v_4(t)$ is:

$$v_4(t) = \left(\frac{R_3}{R_2 + R_3} V_1 - V_7) + \alpha \left(\frac{R_3}{R_2 + R_3} V_1 - V_7 \right) t \right) e^{-at} + V_7$$

Exercise 6.5

a)First step would be to write down KVL and KCL.

$$\begin{aligned}
\text{KVL(1)}: &\quad V_s = v_R(t) + v_C(t) && (6.19) \\
\text{KVL(2)}: &\quad v_L(t) = v_C(t) && (6.20) \\
\text{KCL}: &\quad i_R(t) = i_L(t) + i_C(t) && (6.21)
\end{aligned}$$

The starting point will be 6.19 and we will use 6.20, 6.21 on the way :

$$
\begin{aligned}
V_s &= v_R(t) + v_C(t) \\
V_s &= i_R(t)R + v_L(t) && (6.20) \\
V_s &= (i_L(t) + i_C(t))R + v_L(t) && (6.21) \\
V_s &= i_L(t)R + i_C(t)R + v_L(t) \\
V_s &= i_L(t)R + RC\frac{dv_C(t)}{dt} + L\frac{di_L(t)}{dt} \\
V_s &= i_L(t)R + RC\frac{dv_L(t)}{dt} + L\frac{di_L(t)}{dt} && (6.20) \\
V_s &= i_L(t)R + RLC\frac{d^2i_L(t)}{dt^2} + L\frac{di_L(t)}{dt} \\
\frac{V_s}{RLC} &= \frac{1}{LC}i_L(t) + \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} \\
\frac{V_s}{RLC} &= \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t)
\end{aligned}
$$

b) The exercise states that there is no energy stored in the capacitor and inductor, thus:

$$
\begin{aligned}
i_L(0^-) &= i_L(0^+) = 0A \\
\frac{di_L(0^-)}{dt} &= \frac{1}{L}v_L(0^-) = \frac{1}{L}v_C(0^-) = \frac{1}{L}v_C(0^+) = \frac{1}{L}v_L(0^+) = \frac{di_L(0^+)}{dt} = 0A/s
\end{aligned}
$$

c) Starting from the characteristic equation:

$$
\begin{aligned}
s^2 + \frac{1}{RC}s + \frac{1}{LC} &= 0 \\
s^2 + 1280s + 640000 &= 0 \\
s^2 + 2\alpha s + \omega_0^2 &= 0 \qquad (6.22)
\end{aligned}
$$

Solving equation 6.22 with the abc-formula gives a complex solution given by:

$$s_{1,2} = -\alpha \pm j\omega_d = -640 \pm j480$$

The solution to $i_L(t)$ will therefore be underdamped and is given by:

$$i_L(t)_h = B_1\cos(\omega_d t)e^{-\alpha t} + B_2\sin(\omega_d t)e^{-\alpha t}$$

For the particular solution:

$$
\begin{aligned}
\frac{1}{LC}i_L(t)_p &= \frac{V_s}{RLC} \\
i_L(t)_p &= \frac{V_s}{R} = \frac{1}{5}A
\end{aligned}
$$

Thus then we have:

$$
\begin{aligned}
i_L(t) &= i_L(t)_h + i_L(t)_p = B_1\cos(\omega_d t)e^{-at} + B_2\sin(\omega_d t)e^{-at} + \frac{1}{5} \\
\frac{di_L(t)}{dt} &= -B_1\omega_d\sin(\omega_d t)e^{-\alpha t} - B_1\alpha\cos(\omega_d t)e^{-\alpha t} + B_2\omega_d\cos(\omega_d t)e^{-\alpha t} - B_2\alpha\sin(\omega_d t)e^{-\alpha t}
\end{aligned}
$$

Now applying the initial conditions will give:

$$
\begin{aligned}
i_L(0) &= B_1 + \frac{1}{5} = 0 && \Rightarrow B_1 = -\frac{1}{5} \\
\frac{di_L(0)}{dt} &= -B_1\alpha + B_2\omega_d = 0 && \Rightarrow B_2 = -\frac{4}{15}
\end{aligned}
$$

Now that the constants have also been found the complete solution for $i_L(t)$ is:

$$i_L(t) = -\frac{1}{5}\cos(480t)e^{-640t} - \frac{4}{15}\sin(480t)e^{-640t} + \frac{1}{5}$$

Exercise 6.6

a) We start by combining the parallel resistors into one resistor of value $R_p = 320\Omega$. Then writing the KVL for the circuit on the right (switch is in position B) we get:

$$\begin{aligned}
V_{20V} &= v_L(t) + v_{R_p}(t) + v(t) \\
V_{20V} &= 0.5mH\frac{di_L(t)}{dt} + i_{R_p}(t) + v(t) \\
V_{20V} &= 0.5mH\frac{di_C(t)}{dt} + 320i_c(t) + v(t) \\
V_{20V} &= 0.5mH \cdot 12.5nF\frac{d^2v(t)}{dt^2} + 320 \cdot 12.5nF\frac{dv(t)}{dt} + v(t) \\
V_{20V} &= 6.25 \cdot 10^{-12}\frac{d^2v(t)}{dt^2} + 4 \cdot 10^{-6}\frac{dv(t)}{dt} + v(t) \\
3.2 \cdot 10^{12} &= \frac{d^2v(t)}{dt^2} + 640000\frac{dv(t)}{dt} + 1.6 \cdot 10^{11}v(t) \\
3.2 \cdot 10^{12} &= \frac{d^2v(t)}{dt^2} + 2\alpha\frac{dv(t)}{dt} + \omega_0^2\, v(t)
\end{aligned}$$

b) To obtain the initial conditions we have to observe the circuit when the switch is in position A and note that the 960Ω resistor can be neglected as it is connected to an open-circuit, that the capacitor has been fully charged and can be substituted by an open-circuit and that the inductor has been fully charged and can be substituted by a short-circuit. Thus we have a circuit consisting of two voltage sources and two resistors in series.

The voltage at node A can be obtained by using superposition and is equal to: $v(0^-) = v(0^+) = -16V$. Note that you cannot simply add the two voltage sources and calculate the voltage at node A, because the voltage at node A is in fact a combination of the voltage across the 160Ω resistor *minus* the $28V$ voltage source or the the voltage across the 480Ω resistor *plus* the $20V$ voltage source. (Or you can combine the two voltage sources in one voltage source and calculate the voltage at node A using a voltage divider configuration and adjust by the adding/subtracting the corresponding voltage source).
Since the capacitor is fully charged for $t = 0^-$ there is no current flowing through it: $i_c(0^-) = 0A$. For $t = 0^-$ the inductor is fully charged and there is a current of $75mA$ (from right to left) flowing through it. When the switch moves to position B, the current in the inductor cannot change instantaneously and thus we have that $i_L(0^-) = i_L(0^+) = i_c(0^+) = 75mA$. In summary we have:

$$\begin{aligned}
v(0^-) &= v(0^+) = -16V \\
\frac{dv(0^-)}{dt} &= \frac{1}{12.5nF}i_c(0^-) = 0 \\
\frac{dv(0^+)}{dt} &= \frac{1}{12.5nF}i_c(0^+) = \frac{75mA}{12.5nF} = 6 \cdot 10^6 A/s
\end{aligned}$$

c) We start with the homogeneous part and we write the characteristic equation as:

$$s^2 + 640000s + 1.6 \cdot 10^{11} = 0$$

We note that solving this equation results in a complex solution:

$$s_{1,2} = \alpha \pm j\omega_d = - - 320000 \pm j240000$$

The solution to $v(t)$ will therefore be underdamped and is given by:

$$v(t)_h = B_1 \cos(\omega_d t)e^{-\alpha t} + B_2 \sin(\omega_d t)e^{-\alpha t}$$

For the particular solution:

$$\begin{aligned}
1.6 \cdot 10^{11}v(t)_p &= 3.2 \cdot 10^{12} \\
v(t)_p &= 20V
\end{aligned}$$

Then we have:

$$
\begin{aligned}
v(t) &= v(t)_h + v(t)_p = B_1 \cos(\omega_d t)e^{-\alpha t} + B_2 \sin(\omega_d t)e^{-\alpha t} + 20 \\
\frac{dv(t)}{dt} &= -B_1\omega_d \sin(\omega_d t)e^{-\alpha t} - B_1\alpha \cos(\omega_d t)e^{-\alpha t} + B_2\omega_d \cos(\omega_d t)e^{-\alpha t} - B_2\alpha \sin(\omega_d t)e^{-\alpha t}
\end{aligned}
$$

Now applying the initial conditions (note that $\frac{dv(0)}{dt} = \frac{dv(0^+)}{dt}$ since for $t = 0$ the switch is in position B):

$$
\begin{aligned}
v(0) &= B_1 + 20 = -16 && \Rightarrow B_1 = -36 \\
\frac{dv(0)}{dt} &= -B_1\alpha + B_2\omega_d = 6 \cdot 10^6 && \Rightarrow B_2 = -23
\end{aligned}
$$

Now that the constants have also been found the complete solution for $v(t)$ is:

$$
v(t) = -36 \cos(240000t)e^{-320000t} - 23 \sin(240000t)e^{-320000t} + 20
$$

Chapter 7

Sinusoidal signals and phasor domain

Exercise 7.1

a)

$$f = \frac{1}{T} = \frac{1}{10 \cdot 10^{-3}} = 100\ [Hz]$$

b)

$$\omega = 2\pi f = 200\pi\ [rad/s]$$

c)

$$\left.\begin{array}{c} i(t) = A\cos(\omega t + \phi) \\ A = 10 \\ i(0) = 2 \end{array}\right\} \quad \Rightarrow \quad 2 = 10\cos(\phi)$$

$$\phi = \pm 1.369\ [\text{rad}]$$

The $\pm$ sign comes from the fact that the cosine is an even function. In the assignment it is given that the magnitude of the current is rising at $t = 0$, so 1.369 is rejected. For -1.369 the condition is fulfilled. So, $\phi = -1.369$. This is not the only phase that fulfills the condition. Adding/subtracting $2\kappa\pi$ gives all the possible solutions: $\phi = -1.369 \pm 2\kappa\pi$.

$$i(t) = 10\cos(200\pi t - 1.369)$$

d)

$$i_{rms} = \frac{i_{max}}{\sqrt{2}} = \frac{10}{\sqrt{2}}\ [\text{A}]$$

Exercise 7.2

a)

$$\omega = 140\pi\ [\text{rad/s}]$$

b)

$$f = \frac{\omega}{2\pi} = 70\ [\text{Hz}]$$

c)

$$T = \frac{1}{f} = \frac{1}{70}\ [s] = \frac{1000}{70}\ [ms]$$

d)

$$\begin{aligned} u(77 \cdot 10^{-3}) &= 250 \cos\left(140 \cdot \pi \cdot 77 \cdot 10^{-3} + \frac{\pi}{9}\right) \\ &= -235.51\ [\mathrm{V}] \end{aligned}$$

e)

$$v_{rms} = \frac{v_{max}}{\sqrt{2}} = \frac{250}{\sqrt{2}}$$

Exercise 7.3

We can extract from the graph the following information.

- $A = 2$
- $f(0) = 1.75$
- $T = 1 \Rightarrow f = 1 \Rightarrow \omega = 2\pi$

Combining all of them gives:

$$\left.\begin{array}{c} f(t) = A\cos(\omega t + \phi) \\ A = 2 \\ f(0) = 1.75 \end{array}\right\} \quad \Rightarrow \quad \begin{aligned} 1.75 &= 2\cos(\phi) \\ \\ \phi &= 0.51\ [\mathrm{rad}] \end{aligned}$$

So:

$$f(t) = 2\cos(2\pi t + 0.51)$$

Exercise 7.4

a)

$$V_{rms} = I_{rms} R = \frac{I_{max} R}{\sqrt{2}}$$

b)
In this case the I_{RMS} will be computed from the given graph using the definition.

$$\left.\begin{array}{ll} I_{rms} & = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \\ V_{rms} & = RI_{rms} \end{array}\right\} \quad \Rightarrow \quad \begin{aligned} V_{rms} &= R\sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \\ &= R\sqrt{\frac{1}{T}\int_0^{T/2} I_{max}^2 dt + \frac{1}{T}\int_{T/2}^{T} \frac{I_{max}^2}{4} dt} \\ &= R\sqrt{\frac{I_{max}^2}{2} + \frac{I_{max}^2}{8}} \\ &= R\sqrt{\frac{5I_{max}^2}{8}} \\ &= R\,\frac{\sqrt{5}}{2\sqrt{2}} I_{max} \end{aligned}$$

Exercise 7.7

It is given that the (complex) response to the complex input signal $v_s(t) = |V|e^{j\phi}e^{j\omega t}u(t)$ is $v_c(t) = \frac{V}{1+j\omega RC}e^{j\omega t}$. The question is then to calculate the response $v_c(t)$ if the input signal is given by $v_s(t) = |V|cos(\omega t + \phi)$.

Comparing the two input signals, we find that

$$|V|cos(\omega t + \phi) = Re\Big(|V|e^{j\phi}e^{j\omega t}u(t)\Big)$$

This means that the response to the real input signal is equal to the real-part of the complex response.

$$v_c(t) = Re\Big(\frac{V}{1+j\omega RC}e^{j\omega t}\Big)$$

The easiest approach to obtain the real-part of the above expression is by converting the complex fraction $\frac{1}{1+j\omega RC}$ into the polar form. This is done by finding the magnitude and the argument of the complex fraction.

- **Magnitude**:

$$\begin{aligned}\Big|\frac{1}{1+j\omega RC}\Big| &= \frac{|1|}{|1+j\omega RC|} \\ &= \frac{1}{\sqrt{1+(\omega RC)^2}}\end{aligned}$$

- **Argument:**

$$\begin{aligned}\arg\left(\frac{1}{1+j\omega RC}\right) &= \arg(1) - \arg(1+j\omega RC) \\ &= 0 - \arctan\Big(\frac{\omega RC}{1}\Big) \\ &= -\arctan(\omega RC)\end{aligned}$$

This means that the polar form of the complex fraction is given by

$$\frac{1}{1+j\omega RC} = \frac{1}{\sqrt{1+(\omega RC)^2}}e^{-j\ \arctan(\omega RC)}$$

And the complex response can be written as

$$\begin{aligned}\frac{V}{1+j\omega RC}e^{j\omega t} &= \frac{|V|}{\sqrt{1+(\omega RC)^2}}e^{j\phi}e^{-j\ \arctan(\omega RC)}e^{j\omega t} \\ &= \frac{|V|}{\sqrt{1+(\omega RC)^2}}e^{j(\omega t+\phi-\arctan(\omega RC))}\end{aligned}$$

Now the real-part is equal to

$$\begin{aligned}v_c(t) &= Re\Big(\frac{|V|}{\sqrt{1+(\omega RC)^2}}e^{j(\omega t+\phi-\arctan(\omega RC))}\Big) \\ &= \frac{|V|}{\sqrt{1+(\omega RC)^2}}cos\left(\omega t+\phi-\arctan(\omega RC)\right)\end{aligned}$$

Exercise 7.8

a) The DE is given by:

$$\frac{di_L(t)}{dt} + \frac{R}{L} i_L(t) = \frac{1}{L} v_s(t)$$

b) The current $i_L(t)$ is simply given by:

$$i_L(t) = \frac{v_s(t)}{R + j\omega L}$$

Note that both the input source and the steady state response are *complex*.

c) With an input source voltage $v_s(t) = |V|\, cos\,(\omega t + \phi) = \text{Re}\left[|V|\, e^{j\phi} e^{j\omega t}\right]$, the steady state response $i_L(t)$ is the real part of the response obtained in part b):

$$\begin{aligned}
i_L(t) &= \text{Re}\left[\frac{v_s(t)}{R + j\omega L}\right] \\
&= \text{Re}\left[\frac{|V|\, e^{j\phi} e^{j\omega t}}{R + j\omega L}\right] \\
&= \text{Re}\left[\frac{|V|\, e^{j\phi} e^{j\omega t}}{\sqrt{R^2 + (\omega L)^2} e^{j \arctan(\omega L/R)}}\right] \\
&= \frac{|V|}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \phi - \arctan\left(\omega L/R\right)\right)
\end{aligned}$$

Chapter 8

Circuit analysis in phasor domain and transfer functions

Exercise 8.1

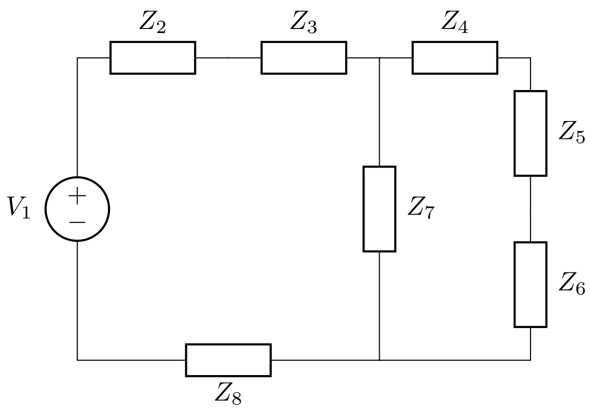


Figure 8.1: Exercise 8.1

The impedances are:

$$
\begin{aligned}
Z_2 &= \frac{1}{j\omega C_2} \\
Z_3 &= j\omega L_3 \\
Z_4 &= R_4 \\
Z_5 &= R_5 \\
Z_6 &= j\omega L_6 \\
Z_7 &= \frac{1}{j\omega C_7} \\
Z_8 &= j\omega L_8
\end{aligned}
$$

Exercise 8.2

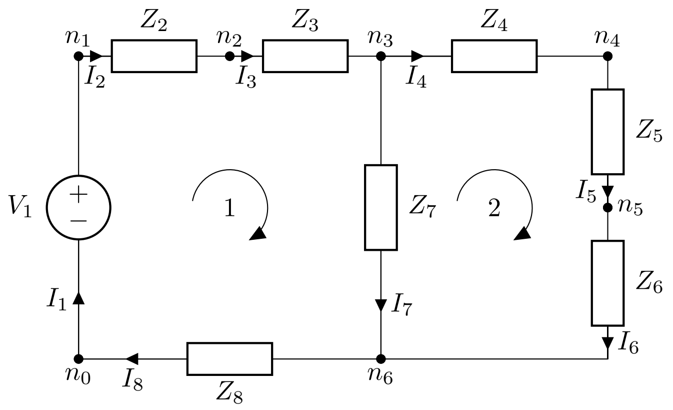


Figure 8.2: Exercise 8.2

Write down KCL for nodes n_1, n_2 and n_3:

$$\begin{aligned}
n_1 &: -I_1 + I_2 = 0 \\
n_2 &: -I_2 + I_3 = 0 \\
n_3 &: -I_3 + I_4 + I_7 = 0 \\
n_4 &: -I_4 + I_5 = 0 \\
n_5 &: -I_5 + I_6 = 0 \\
n_6 &: -I_6 - I_7 + I_8 = 0
\end{aligned}$$

Write down KVL for the loops 1 and 2 (the voltages for the impedances follow the passive sign convention):

$$\begin{aligned}
\mathbf{1} &: -V_1 + V_2 + V_3 + V_7 + V_8 = 0 \\
\mathbf{2} &: -V_7 + V_4 + V_5 + V_6 = 0
\end{aligned}$$

Exercise 8.3

$$Z_{eq} = R_2 + \frac{1}{j\omega C_3} + R_6//(R_4 + j\omega L_5) + R_7 + j\omega L_8$$

Exercise 8.4

step 1, 2 and 3:

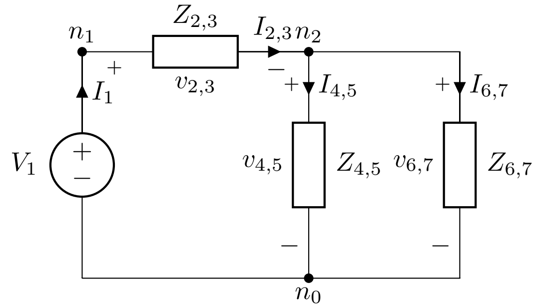


Figure 8.3: Exercise 8.4

step 4:

$$
\begin{aligned}
n_1 &: -I_1 + I_{2,3} = 0 \\
n_2 &: -I_{2,3} + I_{4,5} + I_{6,7} = 0
\end{aligned}
$$

step 5:

$$
\begin{aligned}
n_1 &: -I_1 + \frac{V_{2,3}}{Z_{2,3}} = 0 \\
n_2 &: -\frac{V_{2,3}}{Z_{2,3}} + \frac{V_{4,5}}{Z_{4,5}} + \frac{V_{6,7}}{Z_{6,7}} = 0
\end{aligned}
$$

step 6:

$$
\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\frac{V_{n1} - V_{n2}}{Z_{2,3}} + \frac{V_{n2}}{Z_{4,5}} + \frac{V_{n2}}{Z_{6,7}} = 0
\end{aligned}
$$

step 7:

$$
\begin{aligned}
b_1 &: V_{n1} = V_1 \\
n_2 &: -\frac{1}{Z_{2,3}} V_{n1} + \left(\frac{1}{Z_{2,3}} + \frac{1}{Z_{4,5}} + \frac{1}{Z_{6,7}} \right) V_{n2} = 0
\end{aligned}
$$

step 8: Applying equation b_1 into n_2 gives

$$
\begin{aligned}
-\frac{1}{Z_{2,3}} V_1 + \left(\frac{1}{Z_{2,3}} + \frac{1}{Z_{4,5}} + \frac{1}{Z_{6,7}} \right) V_{n2} = 0 \\
\left(\frac{1}{Z_{2,3}} + \frac{1}{Z_{4,5}} + \frac{1}{Z_{6,7}} \right) V_{n2} = \frac{1}{Z_{2,3}} V_1 \\
V_{n2} = \frac{Z_{4,5} \cdot Z_{6,7}}{Z_{4,5} \cdot Z_{6,7} + Z_{2,3} \cdot Z_{6,7} + Z_{2,3} \cdot Z_{4,5}} V_1
\end{aligned}
$$

Exercise 8.5

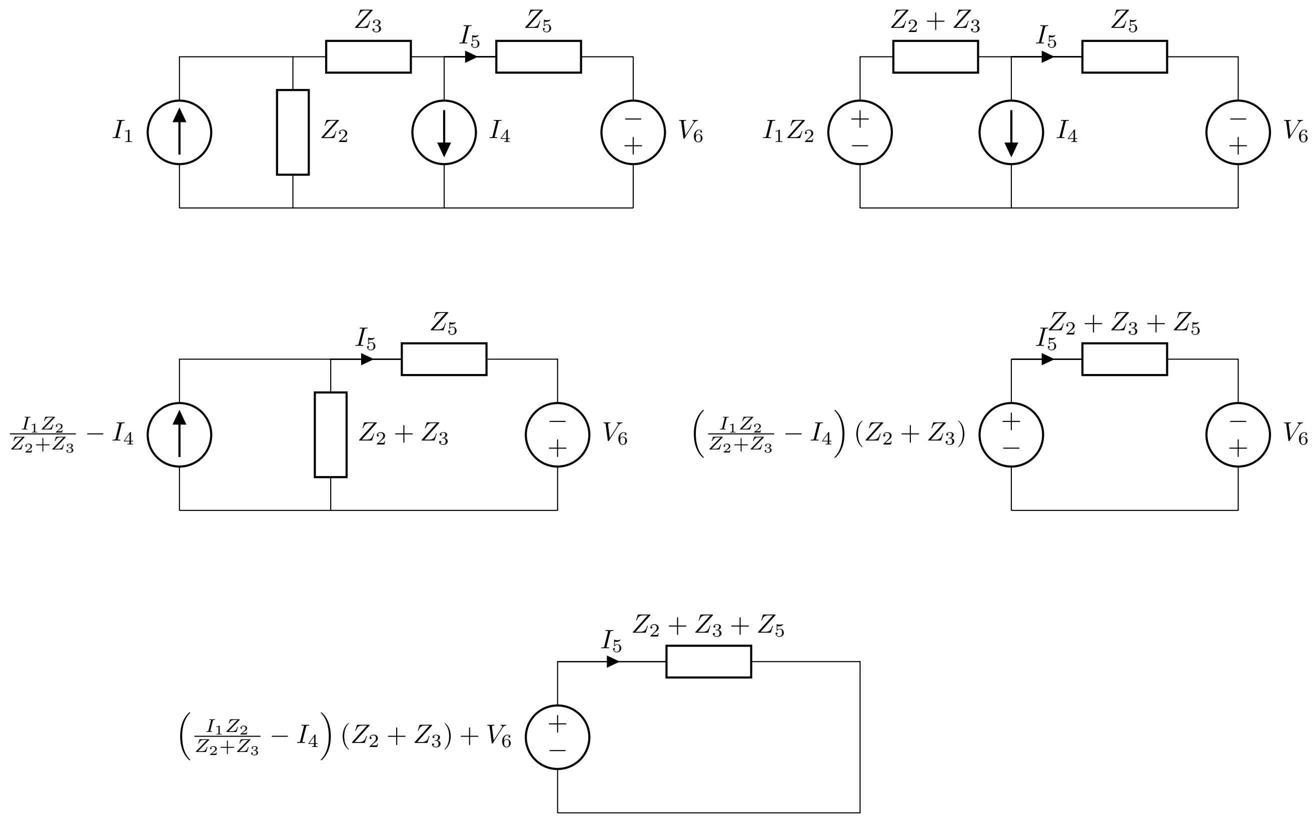


Figure 8.4: Exercise 8.5

The current is then equal to:

$$I_5 = \frac{\left(\frac{I_1 Z_2}{Z_2+Z_3} - I_4\right)(Z_2 + Z_3) + V_6}{Z_2 + Z_3 + Z_5}$$

Exercise 8.6

The Thévenin impedance can be calculated directly from the given circuit by substituting the voltage source V_1 by a short-circuit. The resulting Thévenin impedance is then given by:

$$\begin{aligned} Z_{Th} &= (Z_2 + Z_3)//(Z_4 + Z_5) \\ &= \left(j\omega L_2 + \frac{1}{j\omega C_3}\right) // \left(R_4 + j\omega L_5\right) \\ &= \frac{\left(j\omega L_2 - \frac{j}{\omega C_3}\right)(R_4 + j\omega L_5)}{R_4 + j\left(\omega(L_2 + L_5) - \frac{1}{\omega C_3}\right)} \end{aligned}$$

The Thévenin voltage is equal to the open-circuit voltage V_{AB}:

$$\begin{aligned} V_{Th} &= \frac{Z_4 + Z_5}{Z_2 + Z_3 + Z_4 + Z_5} V_1 \\ &= \frac{R_4 + j\omega L_5}{R_4 + j\left(\omega(L_2 + L_5) - \frac{1}{\omega C_3}\right)} V_1 \end{aligned}$$

The Norton current is given by:

$$\begin{aligned} I_N &= \frac{V_{Th}}{R_{Th}} \\ &= \frac{V_1}{\left(j\omega L_2 - \frac{j}{\omega C_3}\right)} \end{aligned}$$

The Thévenin and Norton equivalent circuits are shown in Figure 8.5

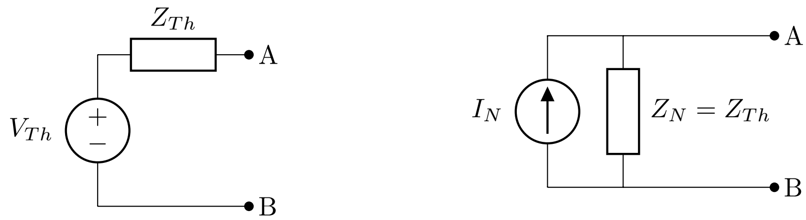


Figure 8.5: Exercise 8.6

Exercise 8.7

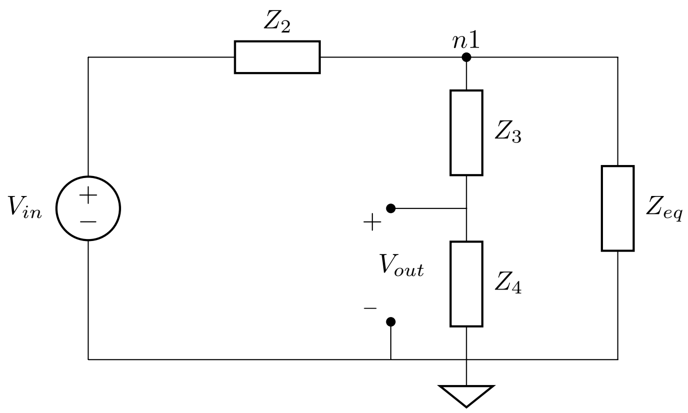


Figure 8.6: Exercise 8.7

First step is to find the voltage at node n_1:

$$V_{n1} = \frac{Z_{eq}//(Z_3 + Z_4)}{Z_{eq}//(Z_3 + Z_4) + Z_2} V_{in}$$

Now we can calculate V_{out}:

$$V_{out} = \frac{Z_4}{Z_4 + Z_3} V_{n1}$$

So,

$$\begin{aligned} H(j\omega) &= \frac{V_{out}}{V_{in}} \\ &= \frac{Z_4}{Z_4 + Z_3} \frac{Z_{eq}//(Z_3 + Z_4)}{Z_{eq}//(Z_3 + Z_4) + Z_2} \end{aligned}$$

Exercise 8.8

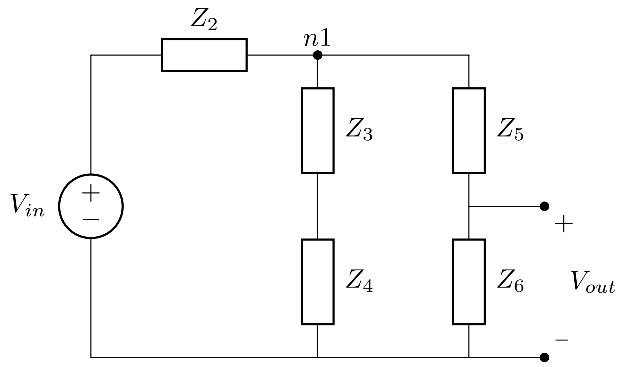


Figure 8.7: Exercise 8.8

a) First step is to find the voltage at node n_1:

$$V_{n1} = \frac{(Z_3 + Z_4)//(Z_5 + Z_6)}{(Z_3 + Z_4)//(Z_5 + Z_6) + Z_2} V_{in}$$

Now we can calculate V_{out} since it is a voltage divider between Z_5 and Z_6:

$$V_{out} = \frac{Z_6}{Z_5 + Z_6} V_{n1}$$

So, the transfer function is:

$$\begin{aligned} V_{out} &= \frac{Z_6}{Z_5 + Z_6} \frac{(Z_3 + Z_4)//(Z_5 + Z_6)}{(Z_3 + Z_4)//(Z_5 + Z_6) + Z_2} V_{in} \\ H(j\omega) = \frac{V_{out}}{V_{in}} &= \frac{Z_6}{Z_5 + Z_6} \frac{(Z_3 + Z_4)//(Z_5 + Z_6)}{(Z_3 + Z_4)//(Z_5 + Z_6) + Z_2} \end{aligned}$$

b) Substituting the impedances and with some algebraic manipulations the expression for $H(j\omega)$ results in:

$$H(j\omega) = \frac{j\omega}{3 + j\omega}$$

In order to find for which ω is the phase of V_{out} shifted by $\frac{\pi}{6}$ with respect to V_{in}:

$$\begin{aligned} \arg(H(j\omega)) &= \arg(j\omega) - \arg(3 + j\omega) \\ \frac{\pi}{6} &= \arg(j\omega) - \arg(3 + j\omega) \\ \frac{\pi}{6} &= \arctan\left(\frac{\omega}{0}\right) - \arctan\left(\frac{\omega}{3}\right) \\ \frac{\pi}{6} &= \frac{\pi}{2} - \arctan\left(\frac{\omega}{3}\right) \\ \arctan\left(\frac{\omega}{3}\right) &= \frac{\pi}{3} \\ \frac{\omega}{3} &= \tan\left(\frac{\pi}{3}\right) \\ \omega &= 3\sqrt{3}[rad/s] \end{aligned}$$

c) Starting from the magintude of the TF:

$$\begin{aligned} |H(j\omega)| &= \left|\frac{j\omega}{3 + j\omega}\right| \\ &= \frac{\omega}{\sqrt{9 + \omega^2}} \end{aligned}$$

As $\omega \to \infty$ then $9 + \omega^2 \approx \omega^2$. Using this we have:

$$\begin{aligned} |H(j\omega)| &= \frac{\omega}{\sqrt{\omega^2}} \\ &= \frac{\omega}{\omega} \\ &= 1 \end{aligned}$$

For the phase we have:

$$\begin{aligned} \arg(H(j\omega)) &= \arg(j\omega) - \arg(3 + j\omega) \\ &= \arctan\left(\frac{\omega}{0}\right) - \arctan\left(\frac{\omega}{3}\right) \end{aligned}$$

As $\omega \to \infty$ then $\frac{\omega}{3} = \infty$. Using this we have:

$$\begin{aligned} \arg(H(j\omega)) &= \arctan\left(\frac{\omega}{0}\right) - \arctan\left(\infty\right) \\ &= \frac{\pi}{2} - \frac{\pi}{2} \\ &= 0 \end{aligned}$$

Chapter 9

Power of sinusoidal signals

Exercise 9.1

$$p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t)$$

The first term, P, represents the average power. We can represent the instantaneous power using the complex exponential notation as follows:

$$p(t) = P + Re\{S \cdot e^{j2\omega t}\}$$

where:

$$S = P + jQ$$
$$|S| = \sqrt{P^2 + Q^2}$$

As $p(t)$ is a sinusoidal function, $|S|$ represents the amplitude of the sinusoidal part, i.e. $P\cos(2\omega t) + Q\sin(2\omega t)$. Therefore, the maximum and minimum of $p(t)$ are:

$$\text{Maximum} : P + \sqrt{P^2 + Q^2}$$
$$\text{Minimum} : P - \sqrt{P^2 + Q^2}$$

For example, if the average power is equal zero, i.e. $P = 0$, the power is between $[-Q, Q]$, which is as expected for a pure sine-function (with a zero DC shift). Another, more complicated, approach to obtain the minimum and maximum values of the instantaneous power is by differentiating $p(t)$ with respect to t and setting the derivative to zero and solving for t. Substituting the resulting t-values in $p(t)$ results in the above minimum and maximum values.

Exercise 9.2

a)

$$\left.\begin{array}{c} S_3 = \frac{1}{2}V_3 \cdot I_3^* \\ V_3 = V_1 \\ I_3 = \frac{V_1}{Z_{C3}} = j\omega C_3 \cdot V_1 \\ I_3^* = \left(\frac{V_1}{Z_{C3}}\right)^* = -j\omega C_3 \cdot V_1^* \end{array}\right\} \Rightarrow \quad S_3 = \frac{1}{2}V_1 \cdot (-j\omega C_3 \cdot V_1^*)$$

$$= -\frac{1}{2}j\omega C_3 \cdot |V_1|^2 \text{ [VA]}$$

b) S_3 can be writen as $S_3 = P + jQ$, where $P = 0$:

$$
\begin{aligned}
S_3 &= P + jQ \\
-\frac{1}{2} j\omega C_3 \cdot |V_1|^2 &= 0 + jQ \\
Q &= -\frac{1}{2}\omega C_3 \cdot |V_1|^2
\end{aligned}
$$

Now the power factor $\cos(\theta)$ can be calculated:

$$
\begin{aligned}
&\theta = \arctan\left(\frac{Q}{P}\right) = \arctan\left(\frac{-\frac{1}{2}\omega C_3 \cdot |V_1|^2}{0}\right) = -\frac{\pi}{2} \\
&\Rightarrow \cos(-\frac{\pi}{2}) = 0
\end{aligned}
$$

So, the power factor is equal to zero. Since θ is negative, the power graph is shifted to the right then it starts at $\frac{\pi}{2}$, so it is leading by $\frac{\pi}{2}$.

c) Combine R_2 and C_3 into one impedance Z:

$$
\begin{aligned}
Z &= \frac{R_2/\left(j\omega C_3\right)}{R_2 + 1/\left(j\omega C_3\right)} \\
&= \frac{R_2}{1 + j\omega R_2 C_3}
\end{aligned}
$$

$$
\left.\begin{aligned}
S &= \tfrac{1}{2} V \cdot I^* \\
V &= V_1 \\
I = \tfrac{V_1}{Z} &= \tfrac{V_1 \cdot (1 + j\omega R_2 C_3)}{R_2} \\
I^* &= \tfrac{V_1^* \cdot (1 - j\omega R_2 \cdot C_3)}{R_2}
\end{aligned}\right\} \Rightarrow \quad S = \frac{1}{2}|V_1|^2 \frac{1 - j\omega R_2 C_3}{R_2}
$$

$$
= \frac{|V_1|^2}{2R_2} - \frac{j\omega C_3 \cdot |V_1|^2}{2} \text{ [VA]}
$$

Exercise 9.3

$$
\begin{aligned}
Z_{tot} &= R_2 + \frac{Z_L \cdot Z_C}{Z_L + Z_C} \\
Z_{tot} &= R_2 + \frac{\frac{j\omega L}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} \\
&= R_2 + \frac{j\omega L}{1 - \omega^2 LC}
\end{aligned}
$$

$$
\begin{aligned}
S &= V_{1_{RMS}} \cdot I^*_{1_{RMS}} \\
&= Z_{tot} \cdot I_{1_{RMS}} \cdot I^*_{1_{RMS}} \\
&= Z_{tot} \cdot |I_{1_{RMS}}|^2 \\
&= \left(R_2 + \frac{j\omega L}{1 - \omega^2 LC}\right) \cdot |I_{1_{RMS}}|^2 \\
&= R_2 |I_{1_{RMS}}|^2 + \left(\frac{j\omega L}{1 - \omega^2 LC}\right) |I_{1_{RMS}}|^2
\end{aligned}
$$

Active or average power $P = Re\{S\}$:

$$\left.\begin{array}{r} P = R_2|I_{1_{RMS}}|^2 \\ |I_{1_{RMS}}|^2 = \frac{|I_1|^2}{2} \end{array}\right\} \Rightarrow \quad P = \frac{|I_1|^2}{2} \cdot R_2 \text{ [W]}, \quad \text{where } I_1 \text{ is the amplitude}$$

Reactive or blind power $Q = Im\{S\}$

$$\left.\begin{array}{r} Q = \left|\frac{\omega L}{1-\omega^2 LC}\right| \cdot |I_{1_{RMS}}|^2 \\ |I_{1_{RMS}}|^2 = \frac{|I_1|^2}{2} \end{array}\right\} \Rightarrow \quad Q = \frac{\omega L}{1-\omega^2 LC} \cdot \frac{|I_1|^2}{2} \text{ [VAR]}, \quad \text{where } I_1 \text{ is the amplitude}$$

Exercise 9.4

Please note that the solution for this exercise is not the only way to approch the problem. It is also possible for example to start working from expression **(2)** and it will still result into the same expression for I_R.
KVL:$-V_L + 30 \cdot I_\Delta + V_c + V_R = 0$ **(1)**
KCL:$-I_s + I_R + I_\Delta = 0$ **(2)**

$$I_\Delta = \frac{V_L}{j\omega L}$$

$$\left.\begin{array}{r} I_\Delta = \frac{V_L}{j\omega L} \\ V_L = 30 \cdot I_\Delta + V_c + V_R \quad \textbf{(1)} \end{array}\right\} \Rightarrow \quad j\omega L\, I_\Delta = 30 \cdot I_\Delta + V_c + V_R$$

$$\Rightarrow \quad I_\Delta = \frac{V_C + V_R}{j\omega L - 30}$$

$$\left.\begin{array}{r} I_\Delta = \frac{V_C + V_R}{j\omega L - 30} \\ V_C + V_R = I_R \cdot (\frac{1}{j\omega C} + R) \end{array}\right\} \Rightarrow \quad I_\Delta = \frac{I_R \cdot (\frac{1}{j\omega C} + R)}{j\omega L - 30}$$

$$\left.\begin{array}{r} I_R = I_s - I_\Delta, \quad \textbf{(2)} \\ I_\Delta = \frac{I_R \cdot (\frac{1}{j\omega C} + R)}{j\omega L - 30} \end{array}\right\} \Rightarrow \quad I_R = I_s \frac{j\omega L - 30}{\frac{1}{j\omega C} + R + j\omega L - 30}$$

$$= I_s \frac{j\omega L - 30}{\frac{1}{j\omega C} + j\omega L}$$

$$= I_s \frac{\omega^2 LC + j30\omega C}{\omega^2 LC - 1}$$

$$\left.\begin{array}{r} S_R = \frac{1}{2}R|I_R|^2 \\ I_R = I_s \frac{\omega^2 LC + j30\omega C}{\omega^2 LC - 1} \end{array}\right\} \Rightarrow \quad S_R = \frac{1}{2} R\, |I_s|^2 \, \frac{\left(\omega^2 LC\right)^2 + (30\omega C)^2}{\left(\omega^2 LC - 1\right)^2}$$

$$= 600 \text{ [W]}$$

Exercise 9.5

$$
\begin{aligned}
S_L &= \frac{1}{2} V_L I_L^* \\
V_L &= \frac{R_3 + j\omega L_4}{R_2 + R_3 + j\omega L_4} V_1 \\
I_L &= \frac{V_1}{R_2 + R_3 + j\omega L_4} = \frac{R_2 + R_3 - j\omega L_4}{(R_2 + R_3)^2 + (\omega L_4)^2} V_1 \\
I_L^* &= \frac{R_2 + R_3 + j\omega L_4}{(R_2 + R_3)^2 + (\omega L_4)^2} V_1^* \\
S_L &= \frac{1}{2} \frac{R_3 + j\omega L_4}{R_2 + R_3 + j\omega L_4} V_1 \frac{R_2 + R_3 + j\omega L_4}{(R_2 + R_3)^2 + (\omega L_4)^2} V_1^* \\
S_L &= \frac{1}{2} |V_1|^2 \frac{R_3 + j\omega L_4}{(R_2 + R_3)^2 + (\omega L_4)^2} \\
|S_L| &= \frac{1}{2} |V_1|^2 \frac{\sqrt{R_3^2 + (\omega L_4)^2}}{(R_2 + R_3)^2 + (\omega L_4)^2}
\end{aligned}
$$

Exercise 9.6

a)

$$
\begin{aligned}
S &= \frac{1}{2} V_{Line} I_{Line}^* \\
V_{Line} &= \frac{R_2 + j\omega L_3}{R_2 + R_4 + j\omega L_3 + j\omega L_5} V_1 \\
I_{Line} &= \frac{V_1}{R_2 + R_4 + j\omega L_3 + j\omega L_5} \\
I_{Line}^* &= V_1^* \frac{R_2 + R_4 + j\omega L_3 + j\omega L_5}{(R_2 + R_4)^2 + (\omega L_3 + \omega L_5)^2} \\
S &= \frac{1}{2} \frac{R_2 + j\omega L_3}{R_2 + R_4 + j\omega L_3 + j\omega L_5} V_1 V_1^* \frac{R_2 + R_4 + j\omega L_3 + j\omega L_5}{(R_2 + R_4)^2 + (\omega L_3 + \omega L_5)^2} \\
S &= \frac{1}{2} |V_1|^2 \frac{R_2 + j\omega L_3}{(R_2 + R_4)^2 + (\omega L_3 + \omega L_5)^2}
\end{aligned}
$$

Average power:

$$
P = Re\{S\} = \frac{1}{2} |V_1|^2 \frac{R_2}{(R_2 + R_4)^2 + (\omega L_3 + \omega L_5)^2} = 60.75 \text{ [W]}
$$

b)

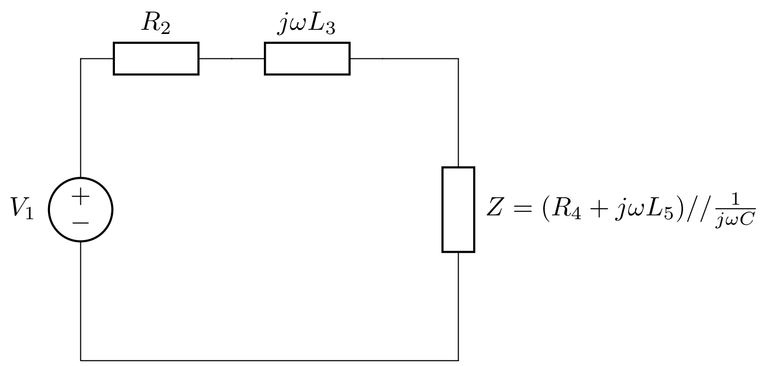


Figure 9.1: Exercise 9.6

$$Z = \frac{(R_4 + j\omega L_5)(\frac{1}{j\omega C})}{R_4 + j\omega L_5 + \frac{1}{j\omega C}}$$

We start by bringing Z into the form $R + jX$, which is done with some algebraic manipulations:

$$Z = \frac{\omega^2 L_5 C R_4 + R_4(1 - \omega^2 L_5 C)}{(1 - \omega^2 L_5 C)^2 + (\omega C R_4)^2} + j\frac{\omega L_5(1 - \omega^2 L_5 C) - \omega C R_4^2}{(1 - \omega^2 L_5 C)^2 + (\omega C R_4)^2}$$

Since we want the load to be resistive then $X = 0$

$$X = \frac{\omega L_5(1 - \omega^2 L_5 C) - \omega C R_4^2}{(1 - \omega^2 L_5 C)^2 + (\omega C R_4)^2}$$
$$0 = \frac{\omega L_5(1 - \omega^2 L_5 C) - \omega C R_4^2}{(1 - \omega^2 L_5 C)^2 + (\omega C R_4)^2}$$
$$0 = \omega L_5(1 - \omega^2 L_5 C) - \omega C R_4^2$$
$$C = \frac{L_5}{(\omega L_5)^2 + R_4^2}$$
$$C = 80[uF]$$

c)
The load Z now is equal with $Z = R + j0 = R = \frac{250}{3}\Omega$.

$$
\begin{aligned}
S &= \frac{1}{2} V_{Line} I^*_{Line} \\
V_{Line} &= \frac{R_2 + j\omega L_3}{R_2 + R + j\omega L_3} V_1 \\
I_{Line} &= \frac{V_1}{R_2 + R + j\omega L_3} \\
I^*_{Line} &= V_1^* \frac{R_2 + R + j\omega L_3}{(R_2 + R)^2 + (\omega L_3)^2} \\
S &= \frac{1}{2} \frac{R_2 + j\omega L_3}{R_2 + R + j\omega L_3} V_1 V_1^* \frac{R_2 + R + j\omega L_3}{(R_2 + R)^2 + (\omega L_3)^2} \\
S &= \frac{1}{2} |V_1|^2 \frac{R_2 + j\omega L_3}{(R_2 + R)^2 + (\omega L_3)^2} \\
|V_{1_{RMS}}|^2 &= \frac{|V_1|^2}{2} \\
S &= |V_{1_{RMS}}|^2 \frac{R_2 + j\omega L_3}{(R_2 + R)^2 + (\omega L_3)^2} \\
\Rightarrow \quad P &= Re\{S\} \\
&= |V_{1_{RMS}}|^2 \frac{R_2}{(R_2 + R)^2 + (\omega L_3)^2} \\
&\approx 27.2[W]
\end{aligned}
$$

Chapter 10

Bode diagrams

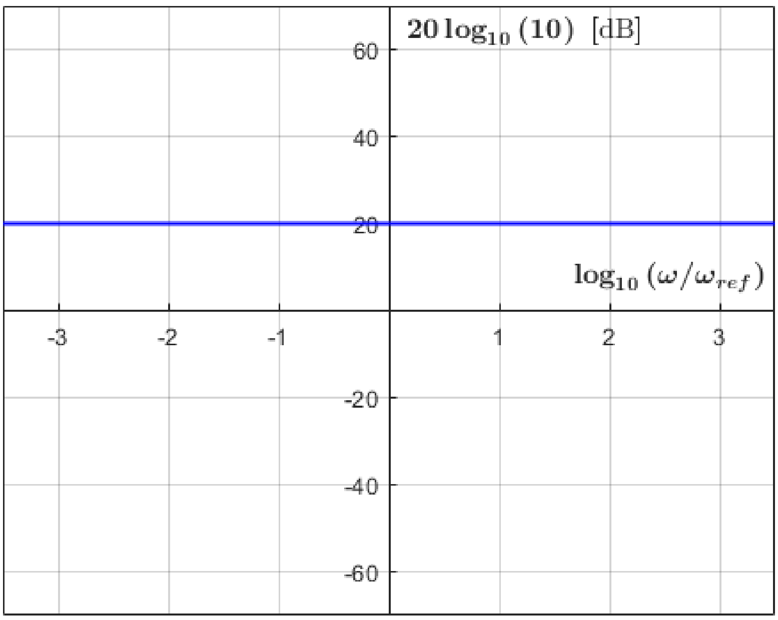
Exercise 10.3

<u>**Modulus plot:**</u>

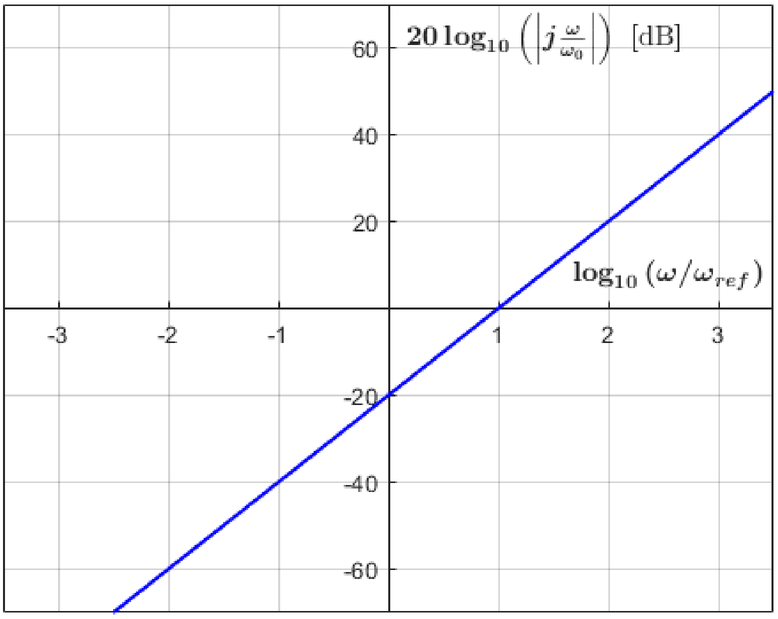
$$20log_{10}|H(j\omega)| = 20log_{10}(10) + 20log_{10}\left|j\frac{\omega}{\omega_0}\right| - 20log_{10}\left|1 + j\frac{\omega}{\omega_1}\right|$$

Solving each one separately:

- $$20log_{10}(10) = 20[dB]$$



- $$\begin{aligned} 20log_{10}\left|j\frac{\omega}{\omega_0}\right| &= 20log_{10}\left(\frac{\omega}{\omega_0}\right) \\ &= 20log_{10}\left(\frac{\omega}{\omega_0}\frac{\omega_r}{\omega_r}\right) \\ &= 20log_{10}\left(\frac{\omega}{\omega_r}\right) + 20log_{10}\left(\frac{\omega_r}{\omega_0}\right) \\ &= 20log_{10}\left(\frac{\omega}{\omega_r}\right) - 20 \end{aligned}$$



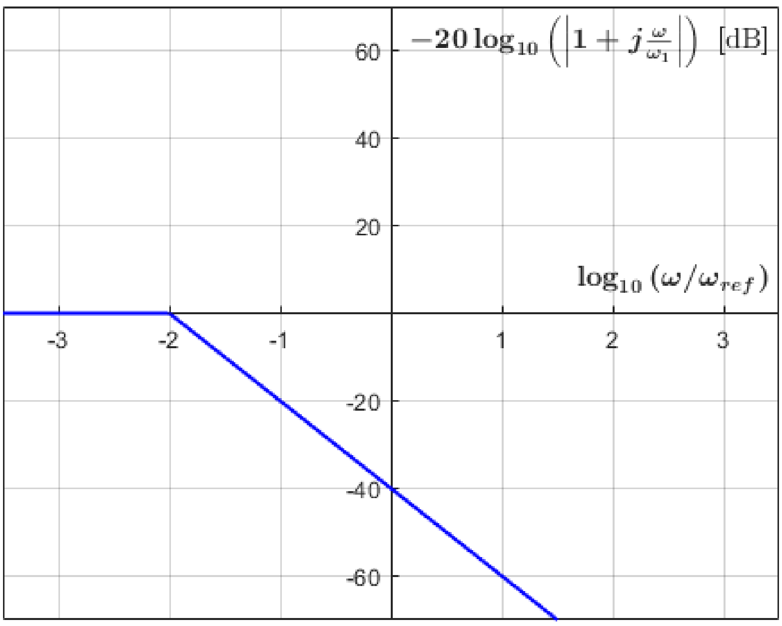
- $$-20log_{10}\left|1+j\frac{\omega}{\omega_1}\right|$$

  In order to simplify it we consider:

1. $\frac{\omega}{\omega_1} \ll 1 \quad \Rightarrow \quad 1+j\frac{\omega}{\omega_1} \approx 1 \quad \Rightarrow \quad -20log_{10}|1| \quad = 0$

2. $\frac{\omega}{\omega_1} \gg 1 \quad \Rightarrow \quad 1+j\frac{\omega}{\omega_1} \approx j\frac{\omega}{\omega_1} \quad \Rightarrow \quad -20log_{10}\left|j\frac{\omega}{\omega_1}\right|$

$$
\begin{aligned}
&= -20log_{10}\left(\frac{\omega}{\omega_1}\right)\\
&= -20log_{10}\left(\frac{\omega}{\omega_1}\frac{\omega_r}{\omega_r}\right)\\
&= -20log_{10}\left(\frac{\omega}{\omega_r}\right) - 20log_{10}\left(\frac{\omega_r}{\omega_1}\right)\\
&= -20log_{10}\left(\frac{\omega}{\omega_r}\right) - 40
\end{aligned}
$$

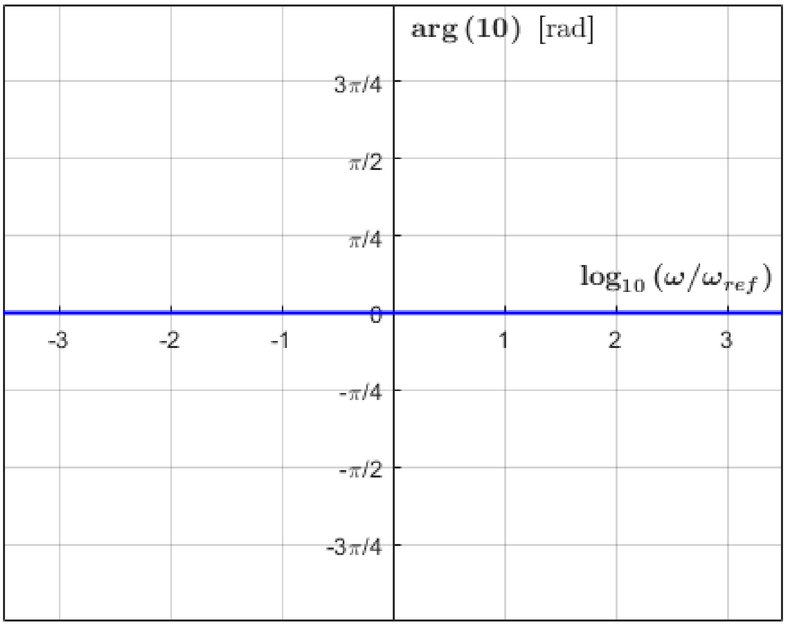


Phase plot:

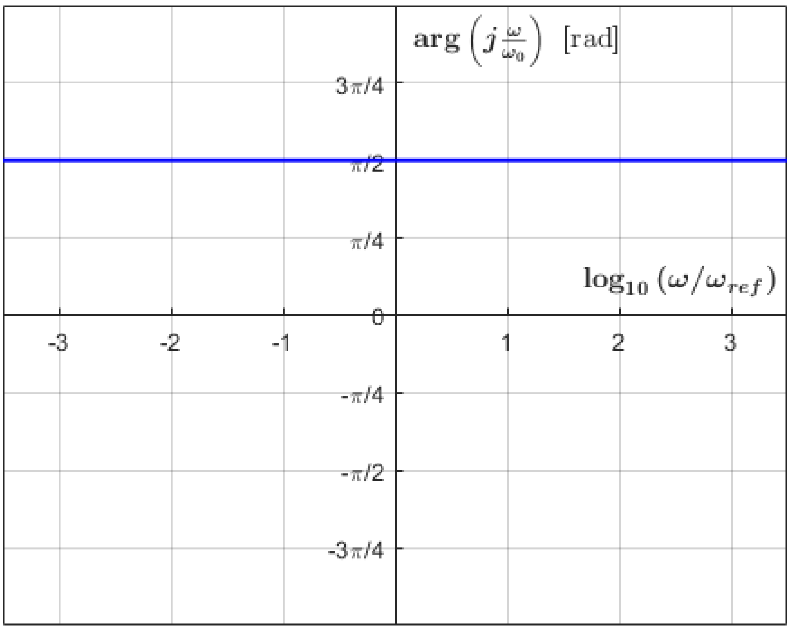
$$\arg\left(H(j\omega)\right) = \arg(10) + \arg\left(j\frac{\omega}{\omega_0}\right) - \arg\left(1 + j\frac{\omega}{\omega_1}\right)$$

Solving each one separately:

- $$\arg(10) = \arctan\left(\frac{0}{10}\right) = 0$$



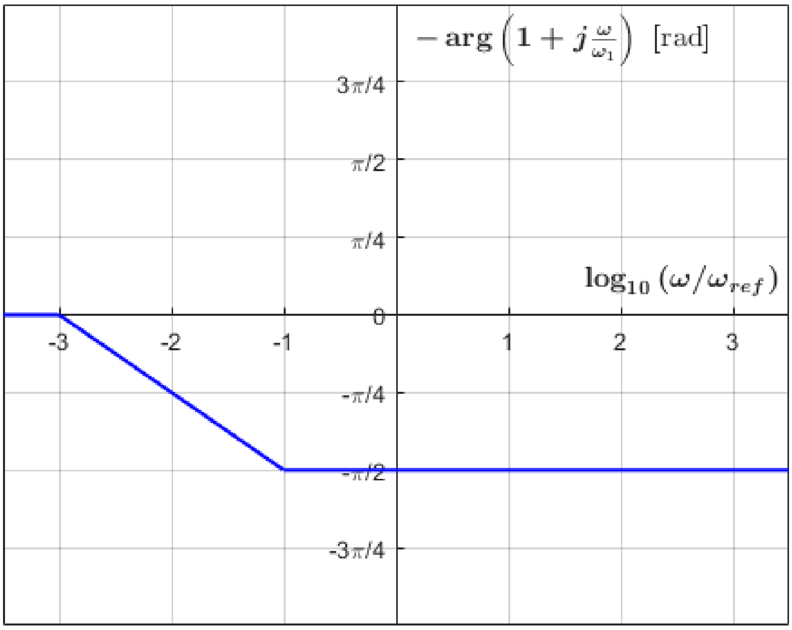
- $$\begin{aligned}\arg\left(j\frac{\omega}{\omega_0}\right) &= \arctan\left(\frac{\frac{\omega}{\omega_0}}{0}\right)\\ &= \arctan(\infty)\\ &= \frac{\pi}{2}\end{aligned}$$



- $$-\arg\left(1+j\frac{\omega}{\omega_1}\right) = -\arctan\left(\frac{\omega}{\omega_1}\right)$$

Here we consider three cases:

$$\begin{aligned}
&\textbf{1.}\quad \omega \ll \omega_1 &&\Rightarrow\quad \frac{\omega}{\omega_1}\approx 0 &&\Rightarrow\quad -\arctan(0) &&= 0\\
&\textbf{2.}\quad \omega \gg \omega_1 &&\Rightarrow\quad \frac{\omega}{\omega_1}\approx \infty &&\Rightarrow\quad -\arctan(\infty) &&= -\frac{\pi}{2}\\
&\textbf{3.}\quad \omega = \omega_1 &&\Rightarrow\quad \frac{\omega}{\omega_1}= 1 &&\Rightarrow\quad -\arctan(1) &&= -\frac{\pi}{4}
\end{aligned}$$



<table>
<tr><td rowspan="2">x-axis</td><td colspan="2">10</td><td colspan="2">$j\frac{\omega}{\omega_1}$</td><td colspan="2">$1+j\frac{\omega}{\omega_1}$</td><td colspan="2">SUM</td></tr>
<tr><td>A_{dB}</td><td>Φ</td><td>A_{dB}</td><td>Φ</td><td>A_{dB}</td><td>Φ</td><td>A_{dB}</td><td>Φ</td></tr>
<tr><td>-3</td><td>20</td><td>0</td><td>-80</td><td>$\pi/2$</td><td>0</td><td>0</td><td>-60</td><td>$\pi/2$</td></tr>
<tr><td>-2</td><td>20</td><td>0</td><td>-60</td><td>$\pi/2$</td><td>0</td><td>$-\pi/4$</td><td>-40</td><td>$\pi/4$</td></tr>
<tr><td>-1</td><td>20</td><td>0</td><td>-40</td><td>$\pi/2$</td><td>-20</td><td>$-\pi/2$</td><td>-40</td><td>0</td></tr>
<tr><td>0</td><td>20</td><td>0</td><td>-20</td><td>$\pi/2$</td><td>-40</td><td>$-\pi/2$</td><td>-40</td><td>0</td></tr>
<tr><td>1</td><td>20</td><td>0</td><td>0</td><td>$\pi/2$</td><td>-60</td><td>$-\pi/2$</td><td>-40</td><td>0</td></tr>
<tr><td>2</td><td>20</td><td>0</td><td>20</td><td>$\pi/2$</td><td>-80</td><td>$-\pi/2$</td><td>-40</td><td>0</td></tr>
<tr><td>3</td><td>20</td><td>0</td><td>40</td><td>$\pi/2$</td><td>-100</td><td>$-\pi/2$</td><td>-40</td><td>0</td></tr>
</table>

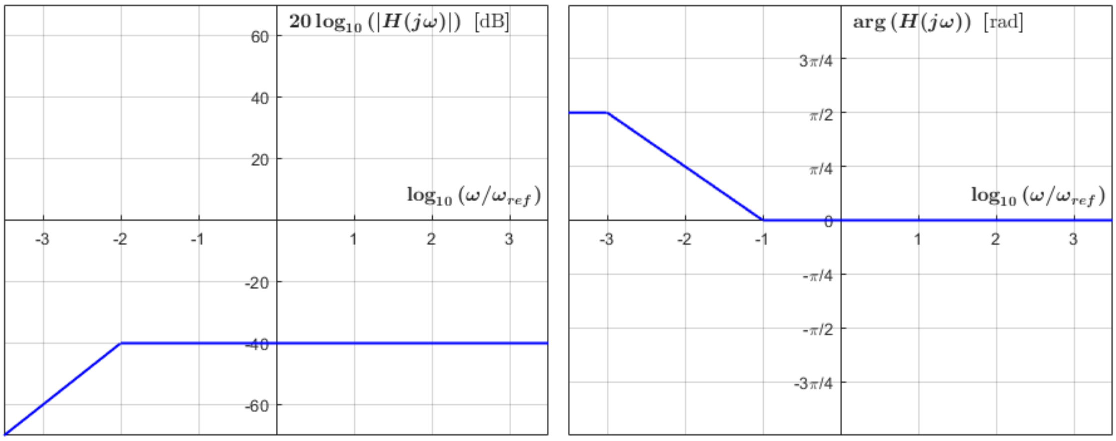


Figure 10.1: Modulus and phase plots of $H(j\omega)$

Exercise 10.4

<u>Modulus plot:</u>

$$20log_{10}|H(j\omega)| = 20log_{10}\left|1 + j\frac{\omega}{\omega_0}\right| - 20log_{10}\left|1 + j\frac{\omega}{\omega_1} + \left(j\frac{\omega}{\omega_1}\right)^2\right|$$

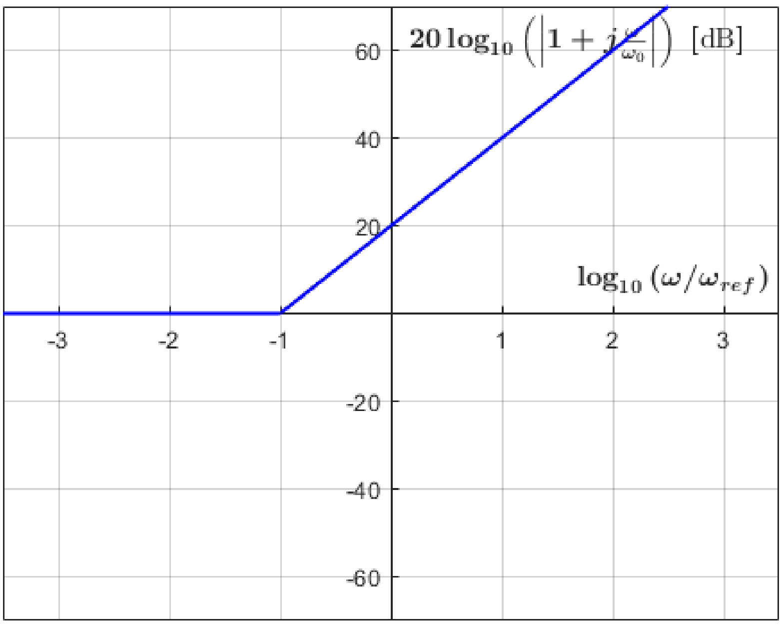
Solving each one separately:

- $$20log_{10}\left|1 + j\frac{\omega}{\omega_0}\right|$$

  In order to simplify it we consider:

  1. $\frac{\omega}{\omega_0} \ll 1 \quad \Rightarrow \quad 1 + j\frac{\omega}{\omega_0} \approx 1 \quad \Rightarrow \quad 20log_{10}|1| \quad = 0$

  2. $$\frac{\omega}{\omega_0} \gg 1 \quad \Rightarrow \quad 1 + j\frac{\omega}{\omega_0} \approx j\frac{\omega}{\omega_0} \quad \Rightarrow \quad 20log_{10}\left|j\frac{\omega}{\omega_0}\right| = 20log_{10}\left(\frac{\omega}{\omega_0}\right)$$
  $$= 20log_{10}\left(\frac{\omega}{\omega_0}\frac{\omega_r}{\omega_r}\right)$$
  $$= 20log_{10}\left(\frac{\omega}{\omega_r}\right) + 20log_{10}\left(\frac{\omega_r}{\omega_0}\right)$$
  $$= 20log_{10}\left(\frac{\omega}{\omega_r}\right) + 20$$

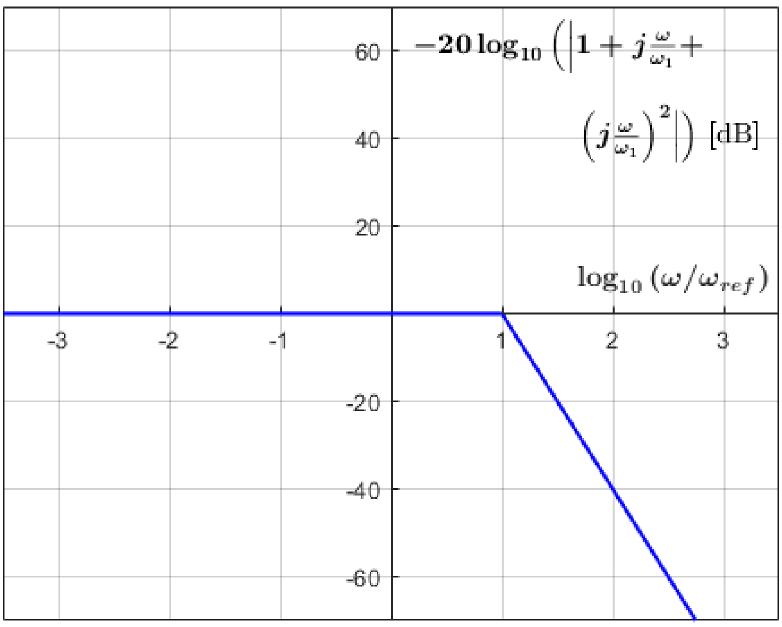


- $$-20log_{10}\left|1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\right|$$

In order to simplify it we consider:

$$\textbf{1.}\quad \frac{\omega}{\omega_1}\ll 1 \quad\Rightarrow\quad 1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\approx 1 \quad\Rightarrow\quad -20log_{10}|1| = 0$$

$$\begin{aligned}\textbf{2.}\quad \frac{\omega}{\omega_1}\gg 1 \quad\Rightarrow\quad 1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\approx\left(j\frac{\omega}{\omega_1}\right)^2 \quad\Rightarrow\quad -20log_{10}\left(\frac{\omega}{\omega_1}\right)^2 &= -40log_{10}\left(\frac{\omega}{\omega_1}\right)\\ &= -40log_{10}\left(\frac{\omega}{\omega_1}\frac{\omega_r}{\omega_r}\right)\\ &= -40log_{10}\left(\frac{\omega}{\omega_r}\right)-40log_{10}\left(\frac{\omega_r}{\omega_1}\right)\\ &= -40log_{10}\left(\frac{\omega}{\omega_r}\right)+40\end{aligned}$$



Phase plot:

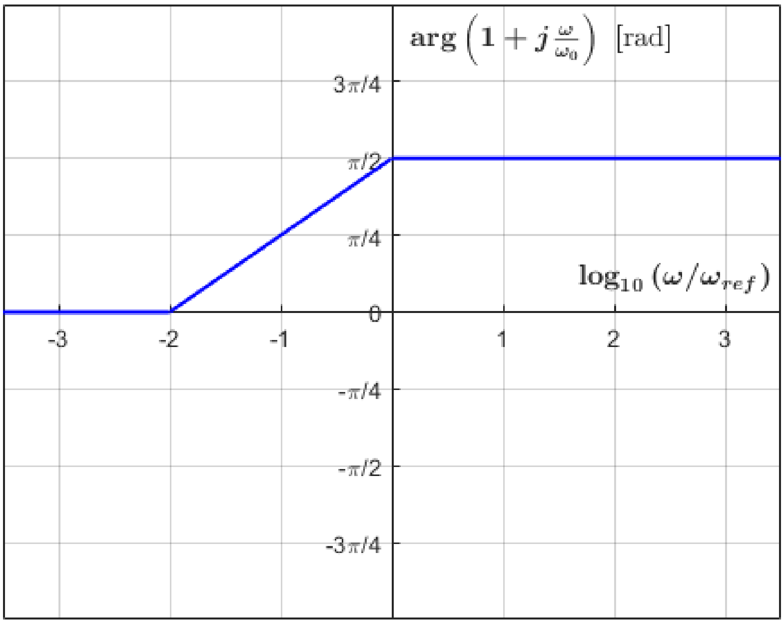
$$\arg\left(H(j\omega)\right) = \arg\left(1 + j\frac{\omega}{\omega_0}\right) - \arg\left(1 + j\frac{\omega}{\omega_1} + \left(j\frac{\omega}{\omega_1}^2\right)\right)$$

Solving each one separately:

- $$\arg\left(1 + j\frac{\omega}{\omega_0}\right) = \arctan\left(\frac{\omega}{\omega_0}\right)$$

  Here we consider three cases:

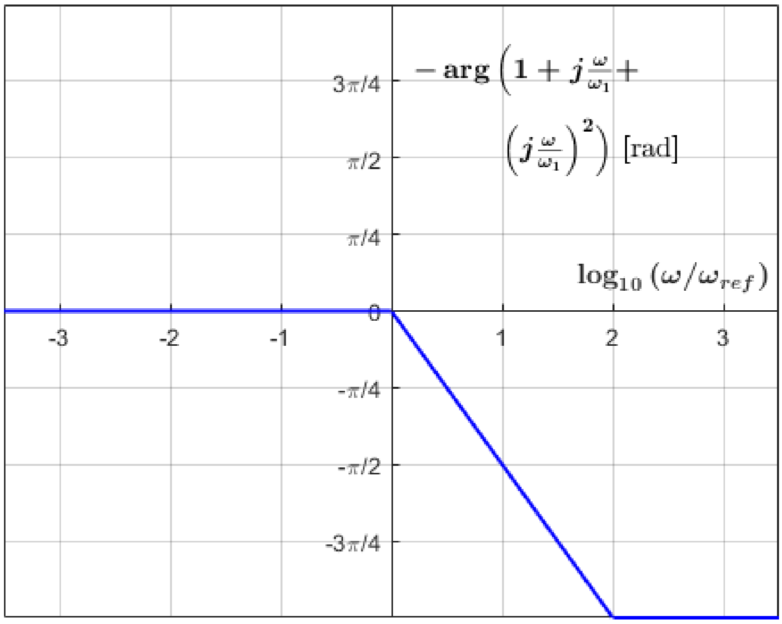
  $$\begin{array}{llllll}
  \textbf{1.} & \omega \ll \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} \approx 0 & \Rightarrow & \arctan(0) = 0 \\
  \textbf{2.} & \omega \gg \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} \approx \infty & \Rightarrow & \arctan(\infty) = \dfrac{\pi}{2} \\
  \textbf{3.} & \omega = \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} = 1 & \Rightarrow & \arctan(1) = \dfrac{\pi}{4}
  \end{array}$$



- $$-\arg\left(1 + j\frac{\omega}{\omega_1} + \left(j\frac{\omega}{\omega_1}^2\right)\right) = -\arctan\left(\frac{\frac{\omega}{\omega_1}}{1 - \left(\frac{\omega}{\omega_1}\right)^2}\right)$$

  Here we consider three cases:

  $$\begin{array}{llllll}
  \textbf{1.} & \omega \ll \omega_1 & \Rightarrow & \dfrac{\omega}{\omega_1} \approx 0 & \Rightarrow & -\arctan(0) & = 0 \\
  \textbf{2.} & \omega \gg \omega_1 & \Rightarrow & \dfrac{\omega}{\omega_1} \approx \infty & \Rightarrow & -\left(\arctan\left(\dfrac{1}{-\frac{\omega}{\omega_1}}\right) + \pi\right) & = -\left(\arctan(0) + \pi\right) \\
  & & & & & & = -\pi \\
  \textbf{3.} & \omega = \omega_1 & \Rightarrow & \dfrac{\omega}{\omega_1} = 1 & \Rightarrow & -\arctan\left(\dfrac{1}{1-1}\right) & = -\arctan\left(\dfrac{1}{0}\right) \\
  & & & & & & = -\arctan(\infty) \\
  & & & & & & = -\dfrac{\pi}{2}
  \end{array}$$



x-axis	$1+j\frac{\omega}{\omega_0}$		$1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2$		SUM	
	A_{dB}	Φ	A_{dB}	Φ	A_{dB}	Φ
-3	0	0	0	0	0	0
-2	0	0	0	0	0	0
-1	0	$\pi/4$	0	0	0	$\pi/4$
0	20	$\pi/2$	0	0	20	$\pi/2$
1	40	$\pi/2$	0	$-\pi/2$	40	0
2	60	$\pi/2$	-40	$-\pi$	20	-$\pi/2$
3	80	$\pi/2$	-80	$-\pi$	0	-$\pi/2$

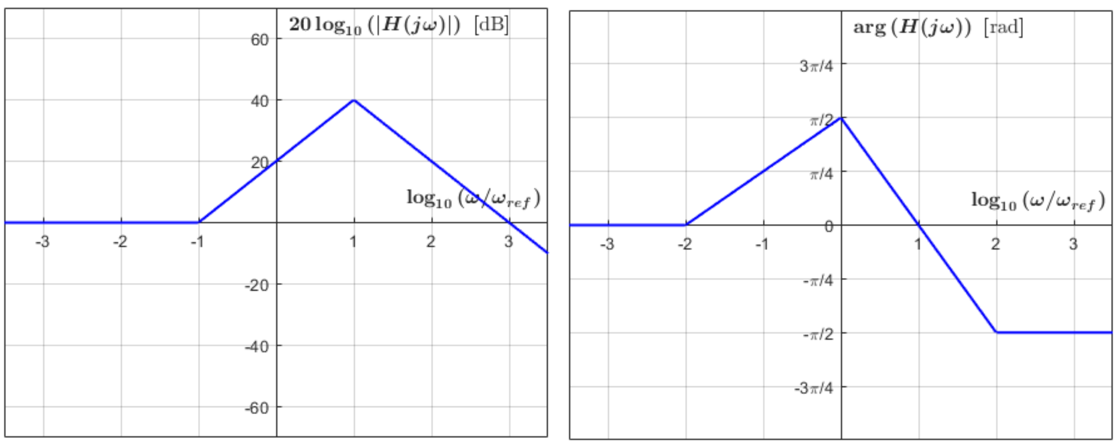


Figure 10.2: Modulus and phase plots of $H(j\omega)$

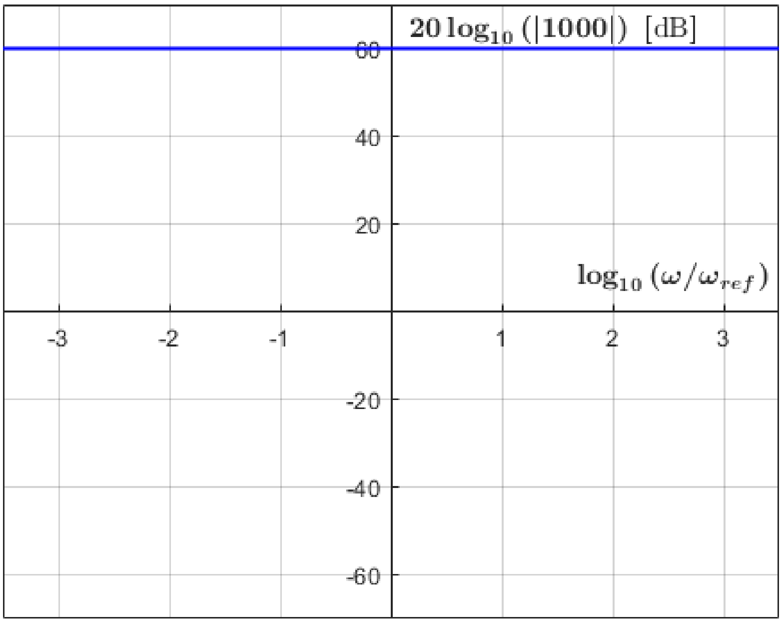
Exercise 10.5

Modulus plot:

$$20log_{10}|H(j\omega)| = 20log_{10}(1000) + 20log_{10}\left|1+j\frac{\omega}{\omega_0}\right| - 20log_{10}\left|1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\right| - 20log_{10}\left|j\frac{\omega}{\omega_2}\right|$$

Solving each one separately:

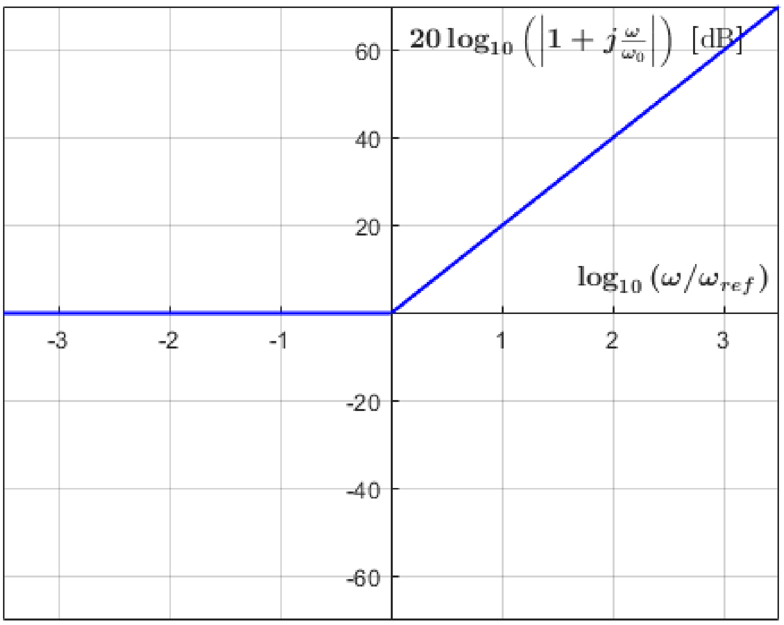
- $$20log_{10}(1000) = 60[dB]$$



- $$20log_{10}\left|1 + j\frac{\omega}{\omega_0}\right|$$

  In order to simplify it we consider:

  $$\begin{aligned}
  &\textbf{1.}\quad \frac{\omega}{\omega_0} \ll 1 \quad \Rightarrow \quad 1 + j\frac{\omega}{\omega_0} \approx 1 \quad \Rightarrow \quad 20log_{10}|1| &&= 0\\
  &\textbf{2.}\quad \frac{\omega}{\omega_0} \gg 1 \quad \Rightarrow \quad 1 + j\frac{\omega}{\omega_0} \approx j\frac{\omega}{\omega_0} \quad \Rightarrow \quad 20log_{10}\left|j\frac{\omega}{\omega_0}\right| &&= 20log_{10}\left(\frac{\omega}{\omega_0}\right)\\
  & &&= 20log_{10}\left(\frac{\omega}{\omega_0}\frac{\omega_r}{\omega_r}\right)\\
  & &&= 20log_{10}\left(\frac{\omega}{\omega_r}\right) + 20log_{10}\left(\frac{\omega_r}{\omega_0}\right)\\
  & &&= 20log_{10}\left(\frac{\omega}{\omega_r}\right)
  \end{aligned}$$



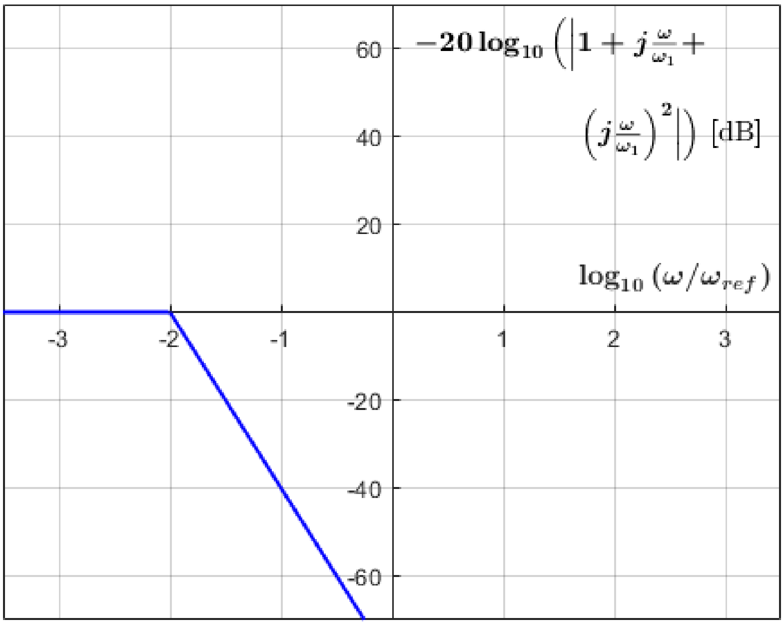
- $$-20log_{10}\left|1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\right|$$

  In order to simplify it we consider:

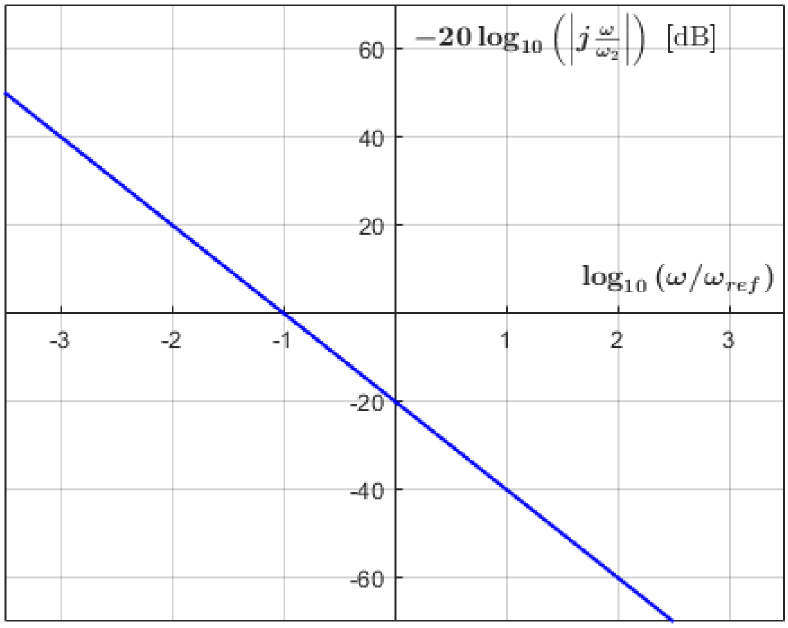
1. $\frac{\omega}{\omega_1} \ll 1 \quad \Rightarrow \quad 1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2 \approx 1 \quad \Rightarrow \quad -20log_{10}|1| = 0$

2. $\frac{\omega}{\omega_1} \gg 1 \quad \Rightarrow \quad 1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2 \approx \left(j\frac{\omega}{\omega_1}\right)^2 \quad \Rightarrow \quad -20log_{10}\left(\frac{\omega}{\omega_1}\right)^2$

$$\begin{aligned}
&= -40log_{10}\left(\frac{\omega}{\omega_1}\right)\\
&= -40log_{10}\left(\frac{\omega}{\omega_1}\frac{\omega_r}{\omega_r}\right)\\
&= -40log_{10}\left(\frac{\omega}{\omega_r}\right) - 40log_{10}\left(\frac{\omega_r}{\omega_1}\right)\\
&= -40log_{10}\left(\frac{\omega}{\omega_r}\right) - 80
\end{aligned}$$



- $$\begin{aligned} -20log_{10}\left|j\frac{\omega}{\omega_2}\right| &= -20log_{10}\left(\frac{\omega}{\omega_2}\right) \\ &= -20log_{10}\left(\frac{\omega}{\omega_2}\frac{\omega_r}{\omega_r}\right) \\ &= -20log_{10}\left(\frac{\omega}{\omega_r}\right) - 20log_{10}\left(\frac{\omega_r}{\omega_2}\right) \\ &= -20log_{10}\left(\frac{\omega}{\omega_r}\right) - 20 \end{aligned}$$

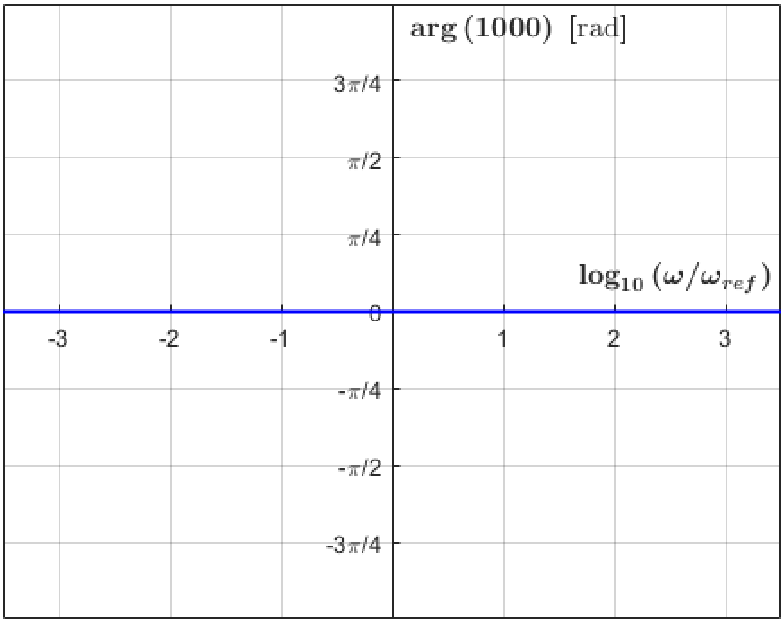


Phase plot:

$$\arg\left(H(j\omega)\right) = \arg(1000) + \arg\left(1+j\frac{\omega}{\omega_0}\right) - \arg\left(1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}^2\right)\right) - \arg\left(j\frac{\omega}{\omega_2}\right)$$

Solving each one separately:

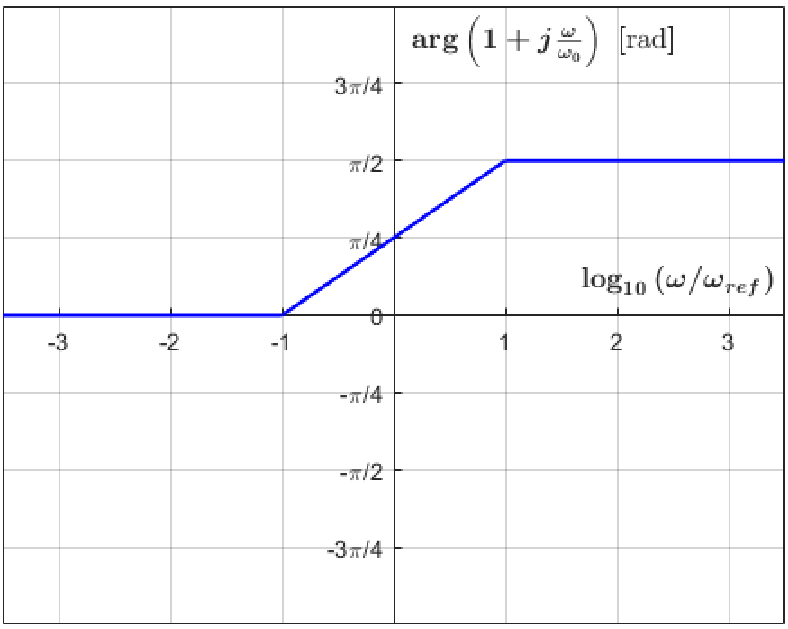
- $$\begin{aligned}\arg(1000) &= \arctan\left(\frac{0}{1000}\right)\\ &= 0\end{aligned}$$



- $$\arg\left(1 + j\frac{\omega}{\omega_0}\right) = \arctan\left(\frac{\omega}{\omega_0}\right)$$

Here we consider three cases:

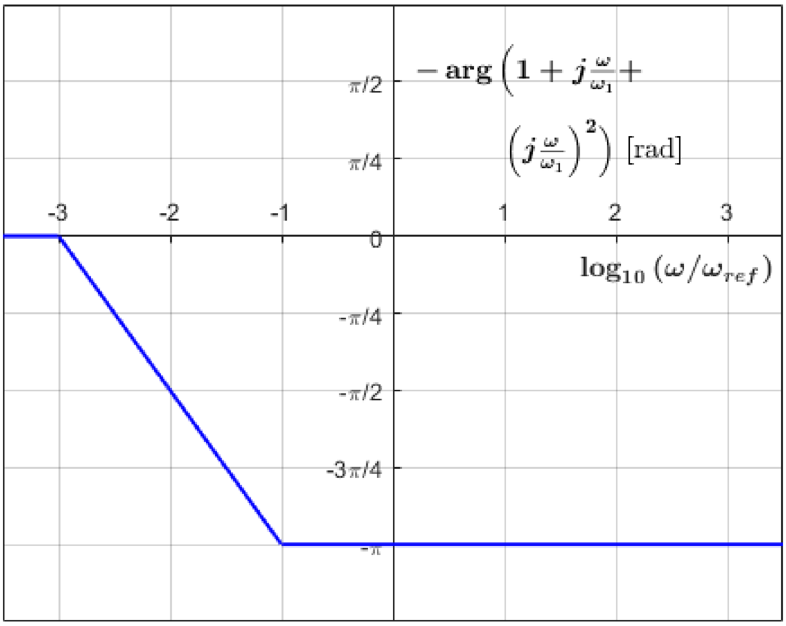
$$\begin{array}{llllll}
\mathbf{1.} & \omega \ll \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} \approx 0 & \Rightarrow & \arctan(0) = 0 \\
\mathbf{2.} & \omega \gg \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} \approx \infty & \Rightarrow & \arctan(\infty) = \dfrac{\pi}{2} \\
\mathbf{3.} & \omega = \omega_0 & \Rightarrow & \dfrac{\omega}{\omega_0} = 1 & \Rightarrow & \arctan(1) = \dfrac{\pi}{4}
\end{array}$$



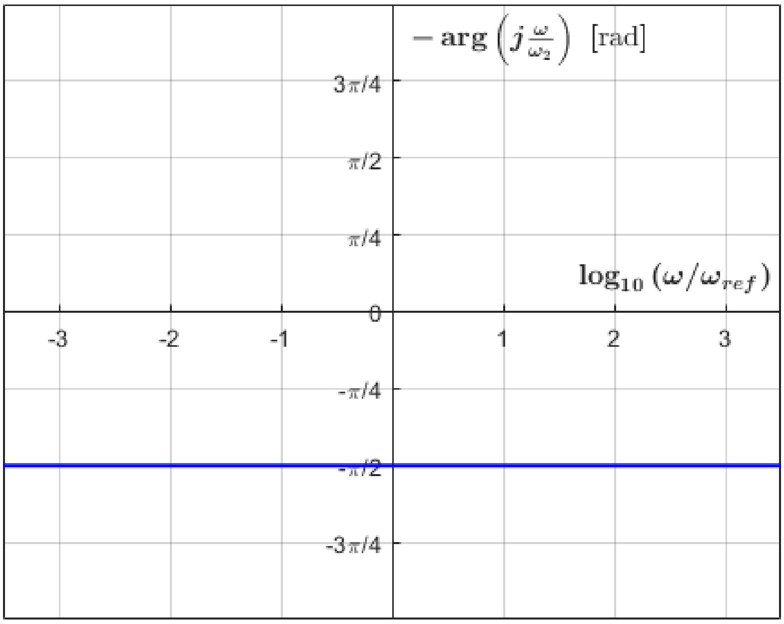
- $$-\arg\left(1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2\right) = -\arctan\left(\frac{\frac{\omega}{\omega_1}}{1-\left(\frac{\omega}{\omega_1}\right)^2}\right)$$

  Here we consider three cases:

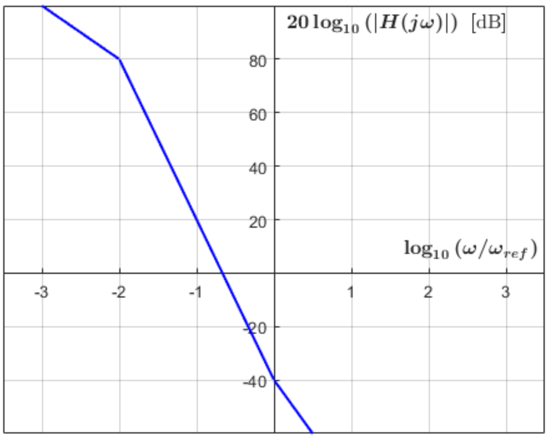
  $$\begin{aligned}
  &\textbf{1.}\quad \omega \ll \omega_1 \quad \Rightarrow \quad \frac{\omega}{\omega_1}\approx 0 \quad \Rightarrow \quad -\arctan(0) && = 0\\
  &\textbf{2.}\quad \omega \gg \omega_1 \quad \Rightarrow \quad \frac{\omega}{\omega_1}\approx \infty \quad \Rightarrow \quad -\left(\arctan\left(\frac{1}{-\frac{\omega}{\omega_1}}\right)+\pi\right) && = -\left(\arctan(0)+\pi\right)\\
  & && = -\pi\\
  &\textbf{3.}\quad \omega = \omega_1 \quad \Rightarrow \quad \frac{\omega}{\omega_1} = 1 \quad \Rightarrow \quad -\arctan\left(\frac{1}{1-1}\right) && = -\arctan\left(\frac{1}{0}\right)\\
  & && = -\arctan(\infty)\\
  & && = -\frac{\pi}{2}
  \end{aligned}$$



- $$\begin{aligned}
  -\arg\left(j\frac{\omega}{\omega_2}\right) &= -\arctan\left(\frac{\frac{\omega}{\omega_2}}{0}\right)\\
  &= -\arctan(\infty)\\
  &= -\frac{\pi}{2}
  \end{aligned}$$



x-axis	1000		$1+j\frac{\omega}{\omega_0}$		$1+j\frac{\omega}{\omega_1}+\left(j\frac{\omega}{\omega_1}\right)^2$		$j\frac{\omega}{\omega_2}$		SUM	
	A_{dB}	Φ	A_{dB}	Φ	A_{dB}	Φ	A_{dB}	Φ	A_{dB}	Φ
-3	60	0	0	0	0	0	40	$-\pi/2$	100	$-\pi/2$
-2	60	0	0	0	0	$-\pi/2$	20	$-\pi/2$	80	$-\pi$
-1	60	0	0	0	-40	$-\pi$	0	$-\pi/2$	20	$-3\pi/2$
0	60	0	0	$\pi/4$	-80	$-\pi$	-20	$-\pi/2$	-40	$-5\pi/4$
1	60	0	20	$\pi/2$	-120	$-\pi$	-40	$-\pi/2$	-80	$-\pi$
2	60	0	40	$\pi/2$	-160	$-\pi$	-60	$-\pi/2$	-120	$-\pi$
3	60	0	60	$\pi/2$	-200	$-\pi$	-80	$-\pi/2$	-160	$-\pi$



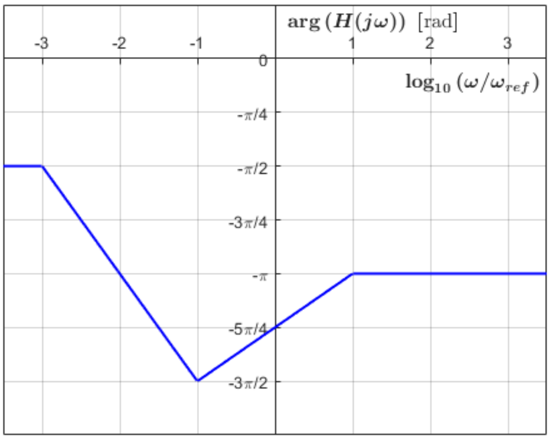


Figure 10.3: Modulus and phase plots of $H(j\omega)$

Exercise 10.6

a)

$$V_{out}(t) = |H(j\omega_i)|\, V_m \cos\left(\omega_i t + \phi_i + \arg(H(j\omega_i))\right)$$

step 1: Based on ω_i and ω_{ref}, find the correct coordinate point on the frequency axis.

step 2: Find $|H(j\omega_i)|$ from the modulus bode plot for the found coordinate point in step 1.

step 3: Find $\arg(H(j\omega_i))$ from the phase plot for the found coordinate point in step 1. **b)**

$$V_{in}(t) = 30\cos\left(100t + \frac{\pi}{6}\right)$$

From $V_{in}(t)$ we can deduce that $\omega_i = 100[rad/s])$ and $\phi_i = \frac{\pi}{6}$. It is also given that $\omega_{ref} = 1000[rad/s]$

$$\omega_i = 0.1\omega_{ref}$$

So, the coordinate point is at -1 on the frequency axes. At that point $20\log_{10}|H(j\omega_i)| \approx 12[dB]$. Converting that to linear scale gives $|H(j\omega_i)| \approx 3.98$. The phase at the -1 point is $\arg(H(j\omega_i)) = \frac{\pi}{4}$

$$\begin{aligned} V_{out}(t) &= 3.98 \cdot 30\cos\left(100t + \frac{\pi}{6} + \frac{\pi}{4}\right) \\ &= 119.4\cos\left(100t + \frac{5}{12}\pi\right) \end{aligned}$$

Chapter 11

Fourier series I

Exercise 11.1

Step 1 & Step 2: the period is equal to T. The signal $x(t)$ is given by

$$x(t) = \begin{cases} 1 & , 0 \le t < \frac{T}{2} \\ 0 & , \frac{T}{2} \le t < T \end{cases}$$

Step 3 & Step 4: the function does not have any symmetry properties. This means that all three Fourier coefficients have to be evaluated.

Step 5 & Step 6:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T x(t)\; dt \\ &= \frac{1}{T} \int_0^{\frac{T}{2}} 1\; dt \\ &= \frac{1}{T} \frac{T}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\; dt \\ &= \frac{2}{T} \int_0^{\frac{T}{2}} \cos(n\omega_0 t)\; dt \\ &= \frac{2}{T} \left[\frac{1}{n\omega_0} \sin(n\omega_0 t) \right]_0^{\frac{T}{2}} \\ &= \frac{2}{T} \frac{1}{n\omega_0} \sin(n\pi) = 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\ dt \\
&= \frac{2}{T}\int_0^{\frac{T}{2}} \sin(n\omega_0 t)\ dt \\
&= \frac{2}{T}\Big[\frac{-1}{n\omega_0}\cos(n\omega_0 t)\Big]_0^{\frac{T}{2}} \\
&= \frac{2}{T}\frac{1}{n\omega_0}\Big[1-\cos(n\pi)\Big] \\
&= \frac{1}{n\pi}\Big[1-\cos(n\pi)\Big] \\
&= \begin{cases} \frac{2}{n\pi} & \text{, n = odd} \\ 0 & \text{, n = even} \end{cases}
\end{aligned}
$$

Step 7:

$$
x(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi}\sin(n\omega_0 t)
$$

Exercise 11.2

Step 1 & Step 2: the period is still equal to T. The signal $x(t)$ is now given by

$$
x(t) = \begin{cases} 1 & , 0 \le t < \frac{T}{4} \\ 0 & , \frac{T}{4} \le t < \frac{3T}{4} \\ 1 & , \frac{3T}{4} \le t < T \end{cases}
$$

Step 3 & Step 4: the function only has even symmetry. This means that only a_0 and a_n have to be evaluated, while $b_n = 0$

Step 5 & Step 6:

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_0^T x(t)\ dt \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} 1\ dt + \frac{1}{T}\int_{\frac{3T}{4}}^{T} 1\ dt \\
&= \frac{1}{T}\frac{T}{4} + \frac{1}{T}\frac{T}{4} = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\ dt \\
&= \frac{4}{T}\int_0^{\frac{T}{4}} \cos(n\omega_0 t)\ dt \\
&= \frac{4}{T}\Big[\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big]_0^{\frac{T}{4}} \\
&= \frac{4}{T}\frac{1}{n\omega_0}\sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} \frac{2}{n\pi}\sin(\frac{n\pi}{2}) & \text{, n = odd} \\ 0 & \text{, n = even} \end{cases}
\end{aligned}
$$

Step 7:

$$x(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)\ cos(n\omega_0 t)$$

Exercise 11.3

The Fourier series of exercise 11.1 were given by:

$$x(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin(n\omega_0 t)$$

substituting $t = t^{'} + \frac{T}{4}$ gives

$$\begin{aligned}
x(t) &= \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(n\omega_0\left(t^{'} + \frac{T}{4}\right)\right) \\
x(t) &= \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(n\omega_0 t^{'} + n\omega_0 \frac{T}{4}\right) \\
x(t) &= \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(n\omega_0 t^{'} + n\frac{2\pi}{T}\frac{T}{4}\right) \\
x(t) &= \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(n\omega_0 t^{'} + n\frac{\pi}{2}\right) \\
x(t) &= \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\omega_0 t^{'})
\end{aligned}$$

where the last step is achieved by writing out $\sin(n\omega_0 t^{'} + n\frac{\pi}{2})$ using the sum formula for the sine function and taking into account that n is odd.

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

This is a general property. A shift in the time-domain corresponds to a phase-shift in the frequency domain. This phase-shift is seen in the fourth line in the above derivation.

Exercise 11.4

The signal is given by:

$$\begin{aligned}
x(t) &= \sin(2\pi t) + 2\cos(6\pi t) \\
&= \sin(\omega_1 t) + 2\cos(\omega_2 t)
\end{aligned}$$

The signal is periodic because we can find numbers k and l such that such $k\omega_1 = l\omega_2$. Since $\omega_2 = 3\omega_1$ we choose (note that many possible numbers exist) $k = 3$ and $l = 1$. Therefore, the period of the signal is given by:

$$T = \frac{2\pi k}{\omega_2} = \frac{2\pi l}{\omega_1} = 1 \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = 2\pi$$

In order to determine the Fourier coefficients, we write the Fourier series of $x(t)$ as:

$$\begin{aligned}
x_F(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \\
&= a_0 + a_1\cos(\omega_0 t) + b_1 \sin(\omega_0 t) + a_2\cos(2\omega_0 t) + b_2\sin(2\omega_0 t) + a_3\cos(3\omega_0 t) + b_3\sin(3\omega_0 t) + \dots \\
&= a_0 + a_1\cos(2\pi t) + b_1\sin(2\pi t) + a_2\cos(4\pi t) + b_2\sin(4\pi t) + a_3\cos(6\pi t) + b_3\sin(6\pi t) + \dots
\end{aligned}$$

For $x(t) = x_F(t)$ to hold, it is required that the Fourier coefficients of the sinusoidal-functions that are not present in $x(t)$ to be zero. In other words, we have:

$$\begin{aligned}
a_n &= 0, \quad n = 1, \dots, \infty,\ n \neq 3 \\
a_3 &= 2 \\
b_n &= 0, \quad n = 1, \dots, \infty,\ n \neq 1 \\
b_1 &= 1
\end{aligned}$$

Exercise 11.5

Step 1 & Step 2: the period is equal to 4. The signal $x(t)$ is given by

$$x(t) = \begin{cases} -\frac{1}{2}t + 2 & , 0 \leq t < 2 \\ 1 & , 2 \leq t < 4 \end{cases}$$

Step 3 & Step 4: the function does not have any symmetry properties. This means that all three Fourier coefficients have to be evaluated.

Step 5 & Step 6:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_0^T x(t)\ dt \\
&= \frac{1}{4} \int_0^2 -\frac{1}{2}t + 2\ dt + \frac{1}{4} \int_2^4 1\ dt \\
&= \frac{1}{4} \Big[-\frac{1}{4}t^2 + 2t \Big]_0^2 + \frac{1}{4} 2 \\
&= \frac{3}{4} + \frac{1}{2} = \frac{5}{4}
\end{aligned}$$

Evaluating a_n:

$$\begin{aligned}
a_n &= \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\ dt \\
&= \frac{2}{T} \int_0^2 \left(-\frac{1}{2}t + 2 \right) \cos(n\omega_0 t)\ dt + \frac{2}{T} \int_2^4 \cos(n\omega_0 t)\ dt \\
&= \frac{2}{T} \int_0^2 -\frac{1}{2}t \left(\frac{1}{n\omega_0} \sin(n\omega_0 t) \right)'\ dt + \frac{4}{T} \int_0^2 \cos(n\omega_0 t)\ dt + \frac{2}{T} \int_2^4 \cos(n\omega_0 t)\ dt
\end{aligned}$$

First integral:

$$\begin{aligned}
\frac{2}{T} \int_0^2 -\frac{1}{2}t \left(\frac{1}{n\omega_0} \sin(n\omega_0 t) \right)'\ dt &= -\frac{1}{T} \Big[\frac{t}{n\omega_0} \sin(n\omega_0 t) \Big]_0^2 + \frac{1}{T} \int_0^2 \frac{1}{n\omega_0} \sin(n\omega_0 t)\ dt \\
&= -\frac{1}{T} \Big[\frac{2}{n\omega_0} \sin(n\pi) \Big] + \frac{1}{T} \Big[-\frac{1}{(n\omega_0)^2} \cos(n\omega_0 t) \Big]_0^2 \\
&= \frac{1}{T} \Big[-\frac{1}{(n\omega_0)^2} \cos(n\pi) + \frac{1}{(n\omega_0)^2} \Big] \\
&= \frac{1}{(n\pi)^2} \Big[1 - \cos(n\pi) \Big]
\end{aligned}$$

Second & third integral:

$$\begin{aligned}
\frac{4}{T}\int_0^2 \cos(n\omega_0 t)\;dt + \frac{2}{T}\int_2^4 \cos(n\omega_0 t)\;dt &= \frac{4}{T}\Big[\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big]_0^2 + \frac{2}{T}\Big[\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big]_2^4 \\
&= \frac{4}{T}\Big[\frac{1}{n\omega_0}\sin(n\pi)\Big] + \frac{2}{T}\Big[\frac{1}{n\omega_0}\sin(n2\pi) - \frac{1}{n\omega_0}\sin(n\pi)\Big] \\
&= 0
\end{aligned}$$

Therefore, we have that:

$$\begin{aligned}
a_n &= \frac{1}{(n\pi)^2}\Big[1-\cos(n\pi)\Big] \\
&= \begin{cases} \frac{2}{(n\pi)^2} & \text{, n = odd} \\ 0 & \text{, n = even} \end{cases}
\end{aligned}$$

Evaluating b_n:

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\;dt \\
&= \frac{2}{T}\int_0^2 -\frac{1}{2}t\left(-\frac{1}{n\omega_0}\cos(n\omega_0 t)\right)'\;dt + \frac{4}{T}\int_0^2 \sin(n\omega_0 t)\;dt + \frac{2}{T}\int_2^4 \sin(n\omega_0 t)\;dt
\end{aligned}$$

First integral:

$$\begin{aligned}
\frac{2}{T}\int_0^2 -\frac{1}{2}t\left(-\frac{1}{n\omega_0}\cos(n\omega_0 t)\right)'\;dt &= -\frac{1}{T}\Big[-\frac{t}{n\omega_0}\cos(n\omega_0 t)\Big]_0^2 - \frac{1}{T}\int_0^2 \frac{1}{n\omega_0}\cos(n\omega_0 t)\;dt \\
&= -\frac{1}{T}\Big[-\frac{2}{n\omega_0}\cos(n\pi)\Big] - \frac{1}{T}\Big[\frac{1}{(n\omega_0)^2}\sin(n\omega_0 t)\Big]_0^2 \\
&= \frac{1}{n\pi}\cos(n\pi) - \frac{1}{T}\Big[\frac{1}{(n\omega_0)^2}\sin(n\pi)\Big] \\
&= \frac{1}{n\pi}\cos(n\pi)
\end{aligned}$$

Second & third integral:

$$\begin{aligned}
\frac{4}{T}\int_0^2 \sin(n\omega_0 t)\;dt + \frac{2}{T}\int_2^4 \sin(n\omega_0 t)\;dt &= \frac{4}{T}\Big[-\frac{1}{n\omega_0}\cos(n\omega_0 t)\Big]_0^2 + \frac{2}{T}\Big[-\frac{1}{n\omega_0}\cos(n\omega_0 t)\Big]_2^4 \\
&= \frac{4}{T}\Big[-\frac{1}{n\omega_0}\cos(n\pi) + \frac{1}{n\omega_0}\Big] + \frac{2}{T}\Big[-\frac{1}{n\omega_0}\cos(n2\pi) + \frac{1}{n\omega_0}\cos(n\pi)\Big] \\
&= \frac{2}{n\pi}\Big[1-\cos(n\pi)\Big] + \frac{1}{n\pi}\Big[\cos(n\pi) - \cos(n2\pi)\Big] \\
&= \frac{1}{n\pi}(1-\cos(n\pi))
\end{aligned}$$

Thus, b_n is equal to:

$$b_n = \frac{1}{n\pi}(\cos(n\pi) - \cos(n\pi) + 1) = \frac{1}{n\pi} \quad \text{for all n} \geq 1 \tag{11.1}$$

Step 7:

$$x(t) = \frac{5}{4} + \sum_{n\ odd} \frac{2}{(n\pi)^2}\cos(n\omega_0 t) + \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin(n\omega_0 t)$$

Exercise 11.6

Step 1 & Step 2: the period is equal to T. The signal $x(t)$ is given by

$$x(t) = \begin{cases} 0 & , -\frac{T}{2} \leq t < -\frac{T}{4} \\ 1 & , -\frac{T}{4} \leq t < -\frac{T}{8} \\ 0 & , -\frac{T}{8} \leq t < \frac{T}{8} \\ 1 & , \frac{T}{8} \leq t < \frac{T}{4} \\ 0 & , \frac{T}{4} \leq t < \frac{T}{2} \end{cases}$$

Step 3 & Step 4: the function has only even symmetry. This means that only a_0 and a_n have to be evaluated, while $b_n = 0$.

Step 5 & Step 6:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt \\ &= \frac{1}{T} \int_{-\frac{T}{4}}^{-\frac{T}{8}} 2\, dt + \frac{1}{T} \int_{\frac{T}{8}}^{\frac{T}{4}} 2\, dt \\ &= \frac{2}{T}\frac{T}{8} + \frac{2}{T}\frac{T}{8} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt \\ &= \frac{4}{T} \int_{\frac{T}{8}}^{\frac{T}{4}} 2\ \cos(n\omega_0 t)\, dt \\ &= \frac{8}{T} \Big[\frac{1}{n\omega_0} \sin(n\omega_0 t) \Big]_{\frac{T}{8}}^{\frac{T}{4}} \\ &= \frac{8}{T} \Big[\frac{1}{n\omega_0} \sin\left(n\frac{\pi}{2}\right) - \frac{1}{n\omega_0} \sin\left(n\frac{\pi}{4}\right) \Big] \\ &= \frac{4}{n\pi} \Big[\sin\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{4}\right) \Big] \quad \text{for all n} \geq 1 \end{aligned}$$

Step 7:

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n\pi}\Big[\sin\Big(n\frac{\pi}{2}\Big) - \sin\Big(n\frac{\pi}{4}\Big)\Big]\cos(n\omega_0 t)$$

Exercise 11.7

Step 1 & Step 2: the period is equal to T. The signal $x(t)$ is given by

$$x(t) = \begin{cases} 0 & , -\frac{T}{2} \le t < -\frac{T}{4} \\ -2 & , -\frac{T}{4} \le t < -\frac{T}{8} \\ -1 & , -\frac{T}{8} \le t < 0 \\ 1 & , 0 \le t < \frac{T}{8} \\ 2 & , \frac{T}{8} \le t < \frac{T}{4} \\ 0 & , \frac{T}{4} \le t < \frac{T}{2} \end{cases}$$

Step 3 & Step 4: the function has only odd symmetry. This means that only b_n has to be evaluated, while $a_n = 0$ for all n ≥ 0.

Step 5 & Step 6:

$$\begin{aligned} b_n &= \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\ dt \\ &= \frac{4}{T}\int_0^{\frac{T}{8}} \sin(n\omega_0 t)\ dt + \frac{4}{T}\int_{\frac{T}{8}}^{\frac{T}{4}} 2\ \sin(n\omega_0 t)\ dt \\ &= \frac{4}{T}\Big[-\frac{1}{n\omega_0}\cos(n\omega_0 t)\Big]_0^{\frac{T}{8}} + \frac{8}{T}\Big[-\frac{1}{n\omega_0}\cos(n\omega_0 t)\Big]_{\frac{T}{8}}^{\frac{T}{4}} \\ &= \frac{4}{T}\Big[-\frac{1}{n\omega_0}\cos\Big(n\frac{\pi}{4}\Big) + \frac{1}{n\omega_0}\Big] + \frac{8}{T}\Big[-\frac{1}{n\omega_0}\cos\Big(n\frac{\pi}{2}\Big) + \frac{1}{n\omega_0}\cos\Big(n\frac{\pi}{4}\Big)\Big] \\ &= \frac{2}{n\pi}\Big[1 + \cos\Big(n\frac{\pi}{4}\Big)\Big] - \frac{4}{n\pi}\cos\Big(n\frac{\pi}{2}\Big) \quad \text{for all n} \ge 1 \end{aligned}$$

Step 7:

$$x(t) = \sum_{n=1}^{\infty}\Big[\frac{2}{n\pi}\Big(1 + \cos\Big(n\frac{\pi}{4}\Big)\Big) - \frac{4}{n\pi}\cos\Big(n\frac{\pi}{2}\Big)\Big]\sin(n\omega_0 t)$$

Exercise 11.8

Step 1 & Step 2: the period is equal to 4. The signal $x(t)$ is given by

$$x(t) = \begin{cases} -t & , 0 \leq t < 1 \\ t-2 & , 1 \leq t < 3 \\ -t+4 & , 3 \leq t < 4 \end{cases}$$

Step 3 & Step 4: the function has odd and half-wave symmetry. This means that only b_n has to be evaluated for odd n, while $a_n = 0$ for all n ≥ 0.

Step 5 & Step 6:

$$\begin{aligned} b_n &= \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \sin(n\omega_0 t)\, dt \\ &= -\frac{4}{T} \int_0^1 t\ \sin(n\omega_0 t)\, dt + \frac{4}{T} \int_1^2 t\ \sin(n\omega_0 t)\, dt - \frac{4}{T} \int_1^2 2\ \sin(n\omega_0 t)\, dt \end{aligned}$$

First integral:

$$\begin{aligned} -\frac{4}{T} \int_0^1 t\ \sin(n\omega_0 t)\, dt &= -\int_0^1 t\, \Big(-\frac{1}{n\omega_0} \cos(n\omega_0 t) \Big)'\, dt \\ &= -\Big[-\frac{t}{n\omega_0} \cos(n\omega_0 t) \Big]_0^1 - \int_0^1 \frac{1}{n\omega_0} \cos(n\omega_0 t)\, dt \\ &= -\Big[-\frac{1}{n\omega_0} \cos\Big(n\frac{\pi}{2}\Big) \Big] - \Big[\frac{1}{(n\omega_0)^2} \sin(n\omega_0 t) \Big]_0^1 \\ &= \frac{1}{n\omega_0} \cos\Big(n\frac{\pi}{2}\Big) - \frac{1}{(n\omega_0)^2} \sin\Big(n\frac{\pi}{2}\Big) \\ &= \frac{2}{n\pi} \Big(\cos\Big(n\frac{\pi}{2}\Big) - \frac{2}{n\pi} \sin\Big(n\frac{\pi}{2}\Big) \Big) \end{aligned}$$

Second integral: Using the intermediate integration result of the first integral, we have:

$$\begin{aligned} \frac{4}{T} \int_1^2 t\ \sin(n\omega_0 t)\, dt &= \int_1^2 t\, \Big(-\frac{1}{n\omega_0} \cos(n\omega_0 t) \Big)'\, dt \\ &= \Big[-\frac{t}{n\omega_0} \cos(n\omega_0 t) \Big]_1^2 + \int_1^2 \frac{1}{n\omega_0} \cos(n\omega_0 t)\, dt \\ &= \Big[-\frac{2}{n\omega_0} \cos(n\pi) + \frac{1}{n\omega_0} \cos\Big(n\frac{\pi}{2}\Big) \Big] + \Big[\frac{1}{(n\omega_0)^2} \sin(n\omega_0 t) \Big]_1^2 \\ &= \Big[-\frac{2}{n\omega_0} \cos(n\pi) + \frac{1}{n\omega_0} \cos\Big(n\frac{\pi}{2}\Big) \Big] + \Big[\frac{1}{(n\omega_0)^2} \sin(n\pi) - \frac{1}{(n\omega_0)^2} \sin\Big(n\frac{\pi}{2}\Big) \Big] \\ &= \frac{2}{n\pi} \Big(\cos\Big(n\frac{\pi}{2}\Big) - 2\cos(n\pi) - \frac{2}{n\pi} \sin\Big(n\frac{\pi}{2}\Big) \Big) \end{aligned}$$

Third integral:

$$\begin{aligned} -\frac{4}{T} \int_1^2 2\ \sin(n\omega_0 t)\, dt &= -2\Big[-\frac{1}{n\omega_0} \cos(n\omega_0 t) \Big]_1^2 \\ &= -2\Big[-\frac{1}{n\omega_0} \cos(n\pi) + \frac{1}{n\omega_0} \cos\Big(n\frac{\pi}{2}\Big) \Big] \\ &= \frac{4}{n\pi} \Big(\cos(n\pi) - \cos\Big(n\frac{\pi}{2}\Big) \Big) \end{aligned}$$

Adding the above three results, gives the following expression for b_n:

$$\begin{aligned} b_n &= \frac{2}{n\pi}\Big(\cos\left(n\frac{\pi}{2}\right) - \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right)\Big) + \frac{2}{n\pi}\Big(\cos\left(n\frac{\pi}{2}\right) - 2\cos(n\pi) - \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right)\Big) + \frac{4}{n\pi}\Big(\cos(n\pi) - \cos\left(n\frac{\pi}{2}\right)\Big) \\ &= -\frac{8}{(n\pi)^2}\sin\left(n\frac{\pi}{2}\right) \end{aligned}$$

Step 7:

$$x(t) = \sum_{n\ odd} -\frac{8}{(n\pi)^2}\sin\left(n\frac{\pi}{2}\right)\sin(n\omega_0 t)$$

Chapter 12

Fourier series II

Exercise 12.1

From exercise 11.1 we have:

$$a_0 = \frac{1}{2}$$
$$a_n = 0$$
$$b_n = \begin{cases} \frac{2}{n\pi} & \text{, n = odd} \\ 0 & \text{, n = even} \end{cases}$$

This means that we can now calculate the Fourier cosine form coefficients from these coefficients:

$$c_0 = a_0 = \frac{1}{2}$$
$$c_n = \sqrt{a_n^2 + b_n^2} = \begin{cases} \frac{2}{n\pi} & \text{, n = odd} \\ 0 & \text{, n = even} \end{cases}$$
$$\phi_n = \arctan\left(\frac{b_n}{a_n}\right) = \arctan(\infty) = \frac{\pi}{2} \quad \text{for odd n}$$

Filling in these coefficients in the Fourier cosine form formula gives

$$x(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \cos\left(n\omega_0 t - \frac{\pi}{2}\right)$$

Exercise 12.2

The function of exercise 11.2 was given by

$$x(t) = \begin{cases} 1 & , 0 \le t < \frac{T}{4} \\ 0 & , \frac{T}{2} \le t < \frac{3T}{4} \\ 1 & , \frac{3T}{4} \le t < T \end{cases}$$

The Fourier series exponential form coefficient F_n can be calculated directly by:

$$\begin{aligned} F_0 &= \frac{1}{T}\int_0^T x(t)\, e^{-j0\omega_0 t}\, dt \\ &= \frac{1}{T}\int_0^{\frac{T}{4}} 1\, dt + \frac{1}{T}\int_{\frac{3T}{4}}^T 1\, dt \\ &= \frac{1}{T}\frac{T}{4} + \frac{1}{T}\frac{T}{4} = \frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
F_n &= \frac{1}{T}\int_0^T x(t)\; e^{-jn\omega_0 t}\; dt \\
&= \frac{1}{T}\int_0^{\frac{T}{4}} e^{-jn\omega_0 t}\; dt + \frac{1}{T}\int_{\frac{3T}{4}}^{T} e^{-jn\omega_0 t}\; dt \\
&= \frac{1}{T}\Big[-\frac{1}{jn\omega_0}e^{-jn\omega_0 t}\Big]_0^{\frac{T}{4}} + \frac{1}{T}\Big[-\frac{1}{jn\omega_0}e^{-jn\omega_0 t}\Big]_{\frac{3T}{4}}^{T} \\
&= \frac{1}{T}\Big[-\frac{1}{jn\omega_0}\cos\Big(n\frac{\pi}{2}\Big) + \frac{1}{n\omega_0}\sin\Big(n\frac{\pi}{2}\Big) + \frac{1}{jn\omega_0}\Big] + \frac{1}{T}\Big[-\frac{1}{jn\omega_0}\cos(n2\pi) + \frac{1}{n\omega_0}\sin(n\pi) + \\
&\qquad \frac{1}{jn\omega_0}\cos\left(n\frac{3\pi}{2}\right) - \frac{1}{n\omega_0}\sin\left(n\frac{3\pi}{2}\right)\Big] \\
&= \frac{1}{2n\pi}\left(j\;\cos\Big(n\frac{\pi}{2}\Big) + \sin\Big(n\frac{\pi}{2}\Big) - j + j - j\;\cos\left(n\frac{3\pi}{2}\right) - \sin\left(n\frac{3\pi}{2}\right)\right) \\
&= \begin{cases} \frac{1}{n\pi}\sin\left(n\frac{\pi}{2}\right) & ,\text{n} = \text{odd} \\ 0 & ,\text{n} = \text{even} \end{cases}
\end{aligned}
$$

Thus, the Fourier series exponential form is given by:

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} F_n\; e^{jn\omega_0 t} \\
&= \frac{1}{2} + \sum_{n=-\infty,\text{n odd},n\neq 0}^{\infty} \frac{1}{n\pi}\sin\Big(n\frac{\pi}{2}\Big)\; e^{jn\omega_0 t}
\end{aligned}
$$

As a check, you can now calculate a_0, a_n and b_n from F_n and compare the results with those found in exercise 11.2:

$$
\begin{aligned}
a_0 &= F_0 = \frac{1}{2}^* \\
a_n &= 2Re\Big(F_n\Big) = \frac{2}{n\pi}\sin(n\frac{\pi}{2}) \\
b_n &= 2Im\Big(F_n\Big) = 0
\end{aligned}
$$

* The expression for F_n is a special expression called the "sinc" function (except for a scaling value of 1/2), whose value at $n = 0$ is equal to 1 and with the scaling factor of 1/2 the result is equal to 1/2.

Exercise 12.3

From exercise 11.8 the Fourier series coefficients were found to be:

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= 0 \\
b_n &= \begin{cases} -\frac{8}{(n\pi)^2}\sin(n\frac{\pi}{2}) & ,\text{n} = \text{odd} \\ 0 & ,\text{n} = \text{even} \end{cases}
\end{aligned}
$$

From these coefficients, you can calculate the Fourier series exponential form coefficient F_n as:

$$
\begin{aligned}
F_0 &= a_0 = 0 \\
F_n &= \frac{a_n - jb_n}{2} = j\frac{4}{(n\pi)^2}\sin\Big(n\frac{\pi}{2}\Big),\quad \text{n} = \text{odd} \\
F_{-n} &= F_n^* = -j\frac{4}{(n\pi)^2}\sin\Big(n\frac{\pi}{2}\Big),\quad \text{n} = \text{odd}
\end{aligned}
$$

This results in the input voltage $v_{in}(t)$ to be given by:

$$v_{in}(t) = \sum_{n=-\infty,\text{n odd},n\neq 0}^{\infty} j\frac{4}{(n\pi)^2}\sin\left(n\frac{\pi}{2}\right)\, e^{jn\omega_0 t}$$

From exercise 11.8, we have that the period $T = 4$ and, therefore, $\omega_0 = 2\pi/4 = \pi/2$. With the transfer function of the circuit given by:

$$H(j\omega_0) = \frac{j\omega_0}{j\omega_0+1} \Rightarrow H(jn\omega_0) = \frac{jn\omega_0}{jn\omega_0+1}$$

The output voltage $v_{out}(t)$ is then given by:

$$v_{out}(t) = \sum_{n=-\infty,\text{n odd},n\neq 0}^{\infty} \frac{jn\omega_0}{jn\omega_0+1}\, j\frac{4}{(n\pi)^2}\sin\left(n\frac{\pi}{2}\right)\, e^{jn\omega_0 t}$$

Approximating the output voltage by the first two non-zero harmonics is then given by:

$$v_{out}(t) \approx F_1H(j\omega_0)e^{j\omega_0 t} + F_{-1}H(-j\omega_0)e^{-j\omega_0 t} + F_3H(3j\omega_0)e^{j3\omega_0 t} + F_{-3}H(-3j\omega_0)e^{-j3\omega_0 t}$$

First term:

$$\begin{aligned}
F_1H(j\omega_0)e^{j\omega_0 t} &= j\frac{4}{\pi^2}\frac{j\frac{\pi}{2}}{1+j\frac{\pi}{2}}\Big(\cos(\omega_0 t) + j\sin(\omega_0 t)\Big)\\
&= \left(\frac{-2}{\pi(1+(\frac{\pi}{2})^2)} + j\frac{1}{(1+(\frac{\pi}{2})^2)}\right)\Big(\cos(\omega_0 t) + j\sin(\omega_0 t)\Big)\\
&= (A+jB)\Big(\cos(\omega_0 t) + j\sin(\omega_0 t)\Big)
\end{aligned}$$

Second term:

$$\begin{aligned}
F_{-1}H(-j\omega_0)e^{-j\omega_0 t} &= -j\frac{4}{\pi^2}\frac{-j\frac{\pi}{2}}{1-j\frac{\pi}{2}}\Big(\cos(\omega_0 t) - j\sin(\omega_0 t)\Big)\\
&= \left(\frac{-2}{\pi(1+(\frac{\pi}{2})^2)} - j\frac{1}{(1+(\frac{\pi}{2})^2)}\right)\Big(\cos(\omega_0 t) - j\sin(\omega_0 t)\Big)\\
&= (A-jB)\Big(\cos(\omega_0 t) - j\sin(\omega_0 t)\Big)
\end{aligned}$$

Third term:

$$\begin{aligned}
F_3H(3j\omega_0)e^{j3\omega_0 t} &= -j\frac{4}{9\pi^2}\frac{j\frac{3\pi}{2}}{1+j\frac{3\pi}{2}}\Big(\cos(3\omega_0 t) + j\sin(3\omega_0 t)\Big)\\
&= \left(\frac{2}{3\pi(1+(\frac{3\pi}{2})^2)} - j\frac{1}{(1+(\frac{3\pi}{2})^2)}\right)\Big(\cos(3\omega_0 t) + j\sin(3\omega_0 t)\Big)\\
&= (C-jD)\Big(\cos(3\omega_0 t) + j\sin(3\omega_0 t)\Big)
\end{aligned}$$

Fourth term:

$$\begin{aligned}
F_{-3}H(-3j\omega_0)e^{-j3\omega_0 t} &= j\frac{4}{9\pi^2}\frac{-j\frac{3\pi}{2}}{1-j\frac{3\pi}{2}}\Big(\cos(3\omega_0 t) - j\sin(3\omega_0 t)\Big)\\
&= \left(\frac{2}{3\pi(1+(\frac{3\pi}{2})^2)} + j\frac{1}{(1+(\frac{3\pi}{2})^2)}\right)\Big(\cos(3\omega_0 t) - j\sin(3\omega_0 t)\Big)\\
&= (C+jD)\Big(\cos(3\omega_0 t) - j\sin(3\omega_0 t)\Big)
\end{aligned}$$

Adding all four terms results in:

$$\begin{aligned}
v_{out}(t) &\approx 2A\cos(\omega_0 t) - 2B\sin(\omega_0 t) + 2C\cos(3\omega_0 t) + 2D\sin(3\omega_0 t)\\
&= -0.3672\cos(\omega_0 t) - 0.5768\sin(\omega_0 t) + 0.0183\cos(3\omega_0 t) + 0.0862\sin(3\omega_0 t)\\
&= -0.3672\cos\left(\frac{\pi}{2}t\right) - 0.5768\sin\left(\frac{\pi}{2}t\right) + 0.0183\cos\left(3\frac{\pi}{2}t\right) + 0.0862\sin\left(3\frac{\pi}{2}t\right)
\end{aligned}$$

Exercise 12.5

a) The function of exercise 11.1 was given by (where $T = 2\pi$ in this case):

$$v(t) = \begin{cases} 1 & , 0 \leq t < \pi \\ 0 & , \pi \leq t < 2\pi \end{cases}$$

In order to find the expression for the current, we have to use $v(t)$ in its Fourier cosine form. From exercise 12.1, we then have:

$$v(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \cos\left(n\omega_0 - \frac{\pi}{2}\right)$$

The current in the phasor domain is then given by:

$$\begin{aligned} I &= \frac{1}{Z\left(jn\omega_0\right)} V \\ &= \frac{V}{\frac{1}{jn\omega_0} + jn\omega_0 + 1} \\ &= \frac{jn\omega_0}{1 - \left(n\omega_0\right)^2 + jn\omega_0} V, \qquad \omega_0 = \frac{2\pi}{T} = 1\ rad/s \\ &= \frac{jn}{1 - n^2 + jn} V \end{aligned}$$

This means that the current is given by:

$$i(t) = \sum_{n\ odd} \frac{2}{n\pi} \frac{1}{\left|Z(jn\omega_0)\right|} \cos\left(n\omega_0 - \frac{\pi}{2} + \arg\left(\frac{1}{Z(jn\omega_0)}\right)\right)$$

Note that the factor 1/2 in $v(t)$ gives a zero current, as the capacitor behaves as an open circuit for DC. This can also be seen by considering the factor 1/2 as a signal with a frequency of zero, thereby giving a current of $I = V/\infty = 0$A.

The magnitude is given by:

$$\left|Z(jn\omega_0)\right| = \frac{n}{\sqrt{(1-n^2)^2 + n^2}}$$

The argument is given by:

$$\arg\left(\frac{1}{Z(jn\omega_0)}\right) = \arg\left(jn\right) - \arg\left(1 - n^2 + jn\right)$$

For the case of $n = 1$, the argument is equal to:

$$\arg\left(\frac{1}{Z(j\omega_0)}\right) = \frac{\pi}{2} - \arctan\left(\frac{1}{0}\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0\ rad$$

For the case $n \geq 3$, the argument is equal to:

$$\begin{aligned} \arg\left(\frac{1}{Z(j\omega_0)}\right)_{n\geq 3} &= \frac{\pi}{2} - \left(\arctan\left(\frac{n}{1-n^2}\right) + \pi\right), \qquad \frac{n}{1-n^2} < 0 \\ &= -\frac{\pi}{2} - \arctan\left(\frac{n}{1-n^2}\right), \qquad \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2} - \arctan\left(x\right)\ \text{for } x < 0 \\ &= \arctan\left(\frac{1-n^2}{n}\right) \end{aligned}$$

Note that substituting $n = 1$ now gives the same argument as calculated above. Therefore, the current is now given by:

$$
\begin{aligned}
i(t) &= \sum_{n\ odd} \frac{2}{n\pi} \frac{n}{\sqrt{(1-n^2)^2+n^2}} \cos\left(n\omega_0 - \frac{\pi}{2} + \arctan\left(\frac{1-n^2}{n}\right)\right) \\
&= \sum_{n\ odd} \frac{2}{\pi} \frac{1}{\sqrt{(1-n^2)^2+n^2}} \cos\left(n\omega_0 - \frac{\pi}{2} + \arctan\left(\frac{1-n^2}{n}\right)\right)
\end{aligned}
$$

b) With $v(t)$ and $i(t)$ known in the cosine form, the average power can be calculated using:

$$
\begin{aligned}
P &= V_{dc}I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos\left(\theta_{in} - \theta_{vn}\right) \\
&= \frac{1}{2}\cdot 0 + \sum_{n\ odd} \frac{2}{n\pi}\,\frac{1}{\pi}\,\frac{1}{\sqrt{(1-n^2)^2+n^2}} \cos\left(-\frac{\pi}{2} + \arctan\left(\frac{1-n^2}{n}\right) + \frac{\pi}{2}\right) \\
&= \sum_{n\ odd} \frac{2}{n\pi^2} \frac{1}{\sqrt{(1-n^2)^2+n^2}} \cos\left(\arctan\left(\frac{1-n^2}{n}\right)\right), \quad \cos\left(\arctan\left(x\right)\right) = \frac{1}{\sqrt{1+x^2}} \\
&= \sum_{n\ odd} \frac{2}{n\pi^2} \frac{1}{\sqrt{(1-n^2)^2+n^2}} \frac{1}{\sqrt{1+\left(\frac{1-n^2}{n}\right)^2}}
\end{aligned}
$$

Exercise 12.6

From exercise 12.1, we have:

$$
x(t) = \frac{1}{2} + \sum_{n\ odd} \frac{2}{n\pi} \cos\left(n\omega_0 - \frac{\pi}{2}\right)
$$

The RMS value can be calculated using:

$$
\begin{aligned}
F_{rms} &= \sqrt{c_0^2 + \sum_{n=1}^{\infty} \left(\frac{c_n}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{1}{4} + \sum_{\text{n odd}} \frac{2}{(n\pi)^2}} \\
&\approx \sqrt{\frac{1}{4} + \frac{2}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}\right)} \\
&\approx 0.6999
\end{aligned}
$$

Chapter 13

Convolution I

Exercise 13.1

a)

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t)(t^2+2)\ dt &= \int_{-\infty}^{\infty} \delta(t)(0^2+2)\ dt \\ &= 2\int_{-\infty}^{\infty} \delta(t)\ dt = 2 \cdot 1 \\ &= 2\end{aligned}$$

b)

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-1)cos(\pi t)\ dt &= \int_{-\infty}^{\infty} \delta(t-1)cos(\pi)\ dt \\ &= -\int_{-\infty}^{\infty} \delta(t-1)\ dt \\ &= -1\end{aligned}$$

c)

$$\begin{aligned}\int_{0}^{\infty} \frac{\delta(t+2)}{(t^4+1)}\ dt &= \int_{0}^{\infty} \frac{\delta(t+2)}{((-2)^4+1)}\ dt \\ &= \frac{1}{17}\int_{0}^{\infty} \delta(t+2)\ dt \\ &= 0\end{aligned}$$

d)

$$\begin{aligned}\int_{\tau=-\infty}^{\tau=\infty} \delta(\tau)h(t-\tau)\ d\tau &= \int_{\tau=-\infty}^{\tau=\infty} \delta(\tau)h(t)\ d\tau \\ &= h(t)\int_{\tau=-\infty}^{\tau=\infty} \delta(\tau)\ d\tau \\ &= h(t)\end{aligned}$$

Exercise 13.2

a) $u(t-1)$
b) $u(t+1) \cdot u(-t+3)$
c) $2u(t) \cdot u(-t+2) - u(t-2) \cdot u(-t+3)$
d) $-u(t) \cdot u(-t+1) - 2u(t-1) \cdot u(-t+2) + 3u(t-2) \cdot u(-t+5) + u(-t) + u(t-5)$
e) $2u(-t+1) \cdot \int u(t)dt + (\int -u(t)dt + 3) \cdot (u(-t+2) \cdot u(t-1)) + u(t-2) \cdot u(-t+5)$

Exercise 13.3

a)

$$\begin{aligned}\int_{\tau=-\infty}^{\tau=\infty} 3\delta(\tau)h(2-\tau)\ d\tau &= 3\ h(2)\int_{\tau=-\infty}^{\tau=\infty} \delta(\tau)\ d\tau \\ &= 3\ h(2) \\ &\approx 0.4\end{aligned}$$

b) $t = 2$ and $v_{in}(t) = 3\delta(t)$

c)

$$v_{in}(t) = 2u(t)u(-t+1) - u(t-1)u(-t+3)$$

d)

$$\begin{aligned}v_{out}(t) &= \int_{\tau=-\infty}^{\tau=\infty} v_{in}(\tau)h(t-\tau)\ d\tau \\ &= \int_{\tau=-\infty}^{\tau=\infty} \Big(2u(\tau)u(-\tau+1) - u(\tau-1)u(-\tau+3)\Big)h(t-\tau)\ d\tau \\ &= \int_{\tau=-\infty}^{\tau=\infty} 2u(\tau)u(-\tau+1)h(t-\tau)\ d\tau - \int_{\tau=-\infty}^{\tau=\infty} u(\tau-1)u(-\tau+3)h(t-\tau)\ d\tau\end{aligned}$$

By taking only the regions into account where $u(\tau)u(-\tau+1)$ in the integrand of the first integral and $u(\tau-1)u(-\tau+3)$ in the integrand of the second integral are not equal to zero, the integrals simplify to

$$\begin{aligned}v_{out}(t) &= \int_{\tau=0}^{\tau=1} 2h(t-\tau)\ d\tau - \int_{\tau=1}^{\tau=3} h(t-\tau)\ d\tau \\ &= \int_{\tau=0}^{\tau=1} 2e^{-(t-\tau)}u(t-\tau)\ d\tau - \int_{\tau=1}^{\tau=3} e^{-(t-\tau)}u(t-\tau)\ d\tau\end{aligned}$$

- For t < 0, $v_{out}(t)$ clearly is equal to 0.
- For $t \in [0,1]$:

$$\begin{aligned}v_{out}(t) &= \int_{\tau=0}^{\tau=t} 2e^{-(t-\tau)}u(t-\tau)\ d\tau \\ &= \int_{\tau=0}^{\tau=t} 2e^{-(t-\tau)}\ d\tau \\ &= 2\Big(1 - e^{-t}\Big)\end{aligned}$$

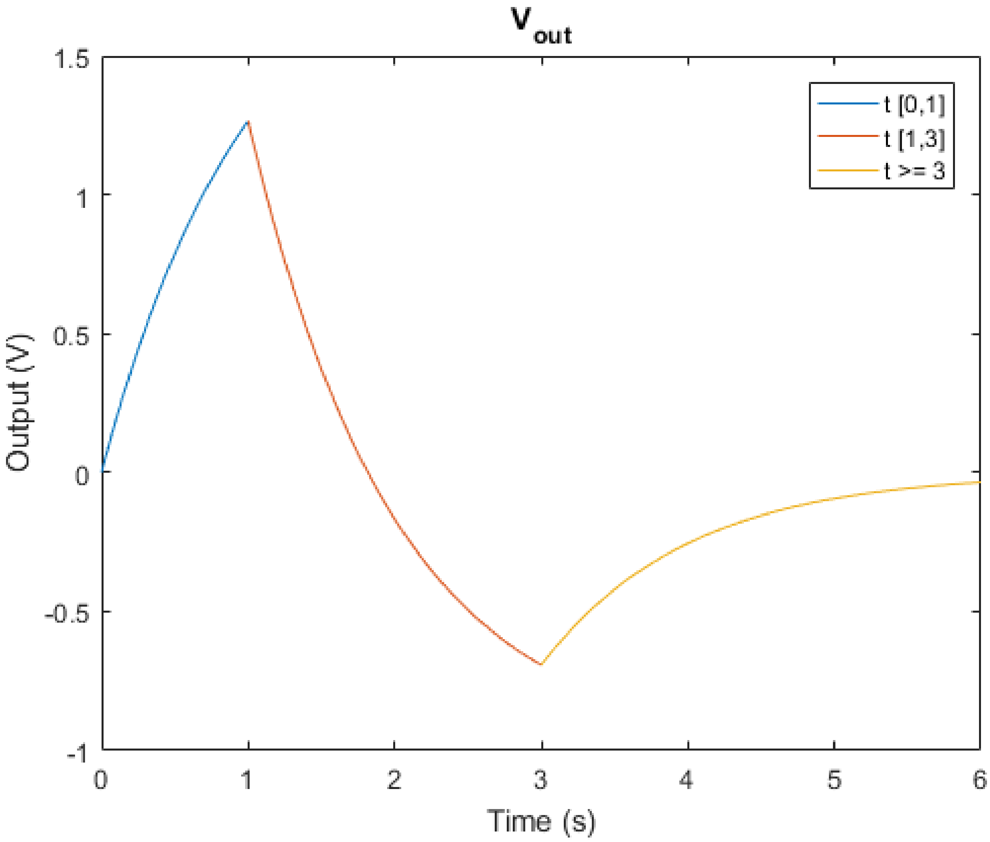
- For $t \in [1,3]$:

$$\begin{aligned}
v_{out}(t) &= \int_{\tau=0}^{\tau=1} 2e^{-(t-\tau)}u(t-\tau)\,d\tau - \int_{\tau=1}^{\tau=t} e^{-(t-\tau)}u(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=1} 2e^{-(t-\tau)}\,d\tau - \int_{\tau=1}^{\tau=t} e^{-(t-\tau)}\,d\tau \\
&= -1 - 2e^{-t} + 3e^{-(t-1)}
\end{aligned}$$

- For $t \geq 3$:

$$\begin{aligned}
v_{out}(t) &= \int_{\tau=0}^{\tau=1} 2e^{-(t-\tau)}u(t-\tau)\,d\tau - \int_{\tau=1}^{\tau=3} e^{-(t-\tau)}u(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=1} 2e^{-(t-\tau)}\,d\tau - \int_{\tau=1}^{\tau=3} e^{-(t-\tau)}\,d\tau \\
&= -2e^{-t} + 3e^{-(t-1)} - e^{-(t-3)}
\end{aligned}$$

e)



Exercise 13.4

a)

$$
\begin{aligned}
v_{out}(t) &= \int_{\tau=-\infty}^{\tau=\infty} v_{in}(\tau)h(t-\tau)\,d\tau \\
&= \int_{\tau=-\infty}^{\tau=\infty} u(\tau-a)\cdot Ae^{-B(t-\tau)}u(t-\tau)\,d\tau \\
&= u(t-a)\cdot\int_a^t Ae^{-B(t-\tau)}\,d\tau \\
&= u(t-a)\cdot Ae^{-Bt}\cdot\left[\frac{1}{B}e^{B\tau}\right]_a^t \\
&= \frac{A}{B}\left(1-e^{-B(t-a)}\right)u(t-a)
\end{aligned}
$$

b) When the input signal becomes a summation of pulses, from the linear property we find that the output is the sum of the individual responses, i.e.:

$$
\begin{aligned}
v_{out}(t) &= \int_{\tau=-\infty}^{\tau=\infty} v_{in}(\tau)h(t-\tau)\,d\tau \\
&= \int_{\tau=-\infty}^{\tau=\infty} \left(\sum_{i=1}^{n} c_i u(\tau-a_i)\right)\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=-\infty}^{\tau=\infty} c_1 u(\tau-a_1)h(t-\tau)\,d\tau+\cdots+c_n u(\tau-a_n)h(t-\tau)\,d\tau \\
&= u(t-a_1)\cdot\int_{a_1}^{\infty} h(t-\tau)\,d\tau+\cdots+u(t-a_n)\cdot\int_{a_n}^{\infty} h(t-\tau)\,d\tau \\
&= c_1\frac{A}{B}\left(1-e^{-B(t-a_1)}\right)u(t-a_1)+\cdots+c_n\frac{A}{B}\left(1-e^{-B(t-a_n)}\right)u(t-a_n) \\
&= \sum_{i=1}^{n} c_i\frac{A}{B}\left(1-e^{-B(t-a_i)}\right)u(t-a_i)
\end{aligned}
$$

Exercise 13.5

a)

$$
\begin{aligned}
v_{out}(t) &= \int_{\tau=-\infty}^{\tau=\infty} v_{in}(\tau)h(t-\tau)\,d\tau \\
&= \int_{\tau=-\infty}^{\tau=0} 0\cdot h(t-\tau)\,d\tau \;+\; \int_{\tau=0}^{\tau=1} -1\cdot h(t-\tau)\,d\tau \;+\; \int_{\tau=1}^{\tau=2} 1\cdot h(t-\tau)\,d\tau \;+\; \int_{\tau=2}^{\tau=3} (-\tau+3)\cdot h(t-\tau)\,d\tau \\
&\quad + \int_{\tau=3}^{\tau=\infty} 0\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=1} -h(t-\tau)\,d\tau \;+\; \int_{\tau=1}^{\tau=2} h(t-\tau)\,d\tau \;+\; \int_{\tau=2}^{\tau=3} (-\tau+3)\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=1} -2\,e^{-3(t-\tau)}\,u(t-\tau)\,d\tau+\int_{\tau=1}^{\tau=2} 2\,e^{-3(t-\tau)}\,u(t-\tau)\,d\tau+\int_{\tau=2}^{\tau=3} (-\tau+3)\cdot 2\,e^{-3(t-\tau)}\,u(t-\tau)\,d\tau
\end{aligned}
$$

- For t $<$ 0, $v_{out}(t)$ clearly is equal to 0.
- For $t \in [0, 1]$:

$$\begin{aligned} v_{out}(t) &= \int_0^t -2\ e^{-3(t-\tau)} d\tau \\ &= -2\ e^{-3t}\ \int_0^t e^{3\tau} d\tau \\ &= -2\ e^{-3t}\ \left[\frac{1}{3} \cdot e^{3\tau}\right]_0^t \\ &= -2\ e^{-3t}\ \left[\frac{1}{3} \cdot e^{3t} - \frac{1}{3}\right] \\ &= -\frac{2}{3} + \frac{2}{3}\ e^{-3t} \end{aligned}$$

- For $t \in [1, 2]$:

$$\begin{aligned} v_{out}(t) &= \int_0^1 -2\ e^{-3(t-\tau)} d\tau + \int_1^t 2\ e^{-3(t-\tau)} d\tau \\ &= -2\ e^{-3t}\ \left[\frac{1}{3}\ e^{3\tau}\right]_0^1 + 2\ e^{-3t}\ \left[\frac{1}{3}\ e^{3\tau}\right]_1^t \\ &= -2\ e^{-3t}\ \left[\frac{1}{3}\ e^{3} - \frac{1}{3}\right] + 2\ e^{-3t}\ \left[\frac{1}{3}\ e^{3t} - \frac{1}{3}\ e^{3}\right] \\ &= -\frac{2}{3}\ e^{-3(t-1)} + \frac{2}{3}\ e^{-3t} + \frac{2}{3} - \frac{2}{3}\ e^{-3(t-1)} \\ &= \frac{2}{3} + \frac{2}{3}\ e^{-3t} - \frac{4}{3}\ e^{-3(t-1)} \end{aligned}$$

- For $t \in [2, 3]$:

$$v_{out}(t) = \int_0^1 -2\ e^{-3(t-\tau)} d\tau + \int_1^2 2\ e^{-3(t-\tau)} d\tau + \int_2^t (-\tau + 3) \cdot 2\ e^{-3(t-\tau)} d\tau$$

The first integral is solved in the previous case and it is found to be:

$$\int_0^1 -2 \cdot e^{-3(t-\tau)} d\tau = -\frac{2}{3} e^{-3(t-1)} + \frac{2}{3} e^{-3t}$$

Evaluating the second integral:

$$\begin{aligned} \int_1^2 2 \cdot e^{-3(t-\tau)} d\tau &= 2e^{-3t} \cdot \left[\frac{1}{3} e^{3\tau}\right]_1^2 \\ &= \frac{2}{3} e^{-3(t-2)} - \frac{2}{3} e^{-3(t-1)} \end{aligned}$$

Evaluating the third integral:

$$
\begin{aligned}
\int_2^t (-\tau+3)\cdot 2\, e^{-3(t-\tau)} d\tau &= \int_2^t -2\tau \cdot e^{-3(t-\tau)} d\tau + \int_2^t 6\, e^{-3(t-\tau)}\, d\tau \\
&= -2\, e^{-3t} \cdot \int_2^t \tau \cdot e^{3\tau} d\tau + 6e^{-3t} \int_2^t e^{3\tau}\, d\tau \\
&= -2\, e^{-3t} \left(\left[\frac{\tau}{3} \cdot e^{3\tau} \right]_2^t - \int_2^t \frac{1}{3} \cdot e^{3\tau} d\tau \right) + 6e^{-3t} \cdot \left[\frac{1}{3} \cdot e^{3\tau} \right]_2^t \\
&= -2\, e^{-3t} \left(\left[\frac{\tau}{3} \cdot e^{3\tau} \right]_2^t - \left[\frac{1}{9} \cdot e^{3\tau} \right]_2^t \right) + 6e^{-3t} \cdot \left[\frac{1}{3} \cdot e^{3\tau} \right]_2^t \\
&= -2\, e^{-3t} \left(\frac{t}{3} \cdot e^{3t} - \frac{2}{3} \cdot e^{6} - \frac{1}{9} \cdot e^{3t} + \frac{1}{9} \cdot e^{6} \right) + 2 - 2\, e^{-3(t-2)} \\
&= -\frac{2}{3}\, t + \frac{4}{3}\, e^{-3(t-2)} + \frac{2}{9} - \frac{2}{9}\, e^{-3(t-2)} + 2 - 2\, e^{-3(t-2)} \\
&= \frac{20}{9} - \frac{2}{3}\, t - \frac{8}{9}\, e^{-3(t-2)}
\end{aligned}
$$

Now adding all three integrals gives:

$$
\begin{aligned}
v_{out}(t) &= -\frac{2}{3}\, e^{-3(t-1)} + \frac{2}{3}\, e^{-3t} + \frac{2}{3}\, e^{-3(t-2)} - \frac{2}{3}\, e^{-3(t-1)} + \frac{20}{9} - \frac{2}{3}\, t - \frac{8}{9}\, e^{-3(t-2)} \\
&= \frac{20}{9} - \frac{2}{3}\, t + \frac{2}{3}\, e^{-3t} - \frac{4}{3}\, e^{-3(t-1)} - \frac{2}{9}\, e^{-3(t-2)}
\end{aligned}
$$

- For $t \geq 3$:

$$
v_{out}(t) = \int_0^1 -2\, e^{-3(t-\tau)} d\tau + \int_1^2 2\, e^{-3(t-\tau)} d\tau + \int_2^3 (-\tau+3) \cdot 2\, e^{-3(t-\tau)} d\tau
$$

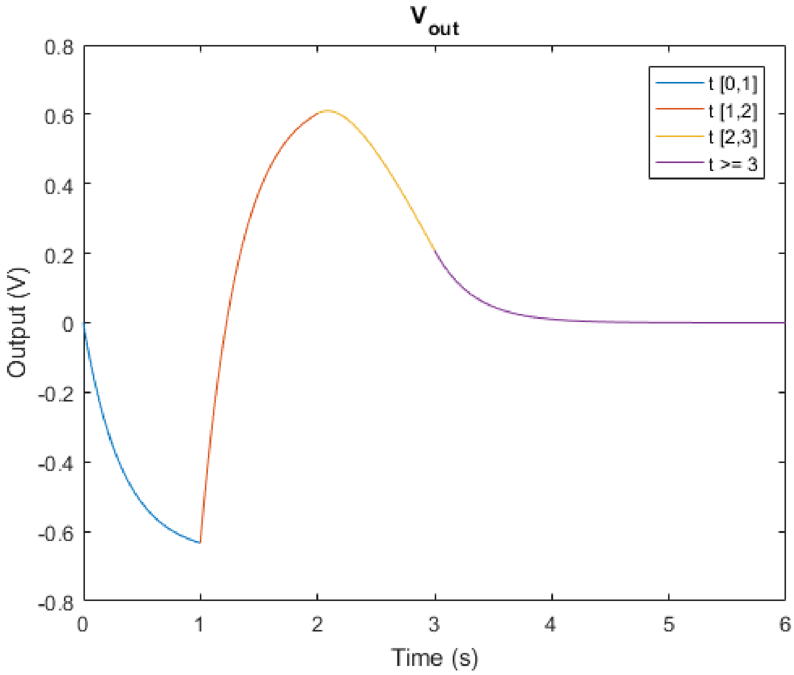
The first two integrals have been calculated. For the third one we have:

$$
\int_2^3 (-\tau+3) \cdot 2 \cdot e^{-3(t-\tau)} d\tau = -\frac{8}{9} e^{-3(t-2)} + \frac{2}{9} e^{-3(t-3)}
$$

Now adding all three integrals gives:

$$
v_{out}(t) = \frac{2}{3} \cdot e^{-3t} - \frac{4}{3} \cdot e^{-3(t-1)} - \frac{2}{9} \cdot e^{-3(t-2)} + \frac{2}{9} \cdot e^{-3(t-3)}
$$

b)



V out
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Output (V)
0
1
2
3
4
5
6
Time (s)
t [0,1]
t [1,2]
t [2,3]
t >= 3

Chapter 14

Convolution II

Exercise 14.3

a)

$$\begin{aligned}
v_{out}(t) &= \int_{\tau=-\infty}^{\tau=\infty} v_{in}(\tau)h(t-\tau)\,d\tau \\
&= \int_{\tau=-\infty}^{\tau=0} 0\cdot h(t-\tau)\,d\tau \;+\; \int_{\tau=0}^{\tau=2} -\tau(\tau-2)\cdot h(t-\tau)\,d\tau + \int_{\tau=2}^{\tau=\infty} 0\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=2} -\tau(\tau-2)\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=2} -\tau^2\cdot h(t-\tau)\,d\tau + \int_{\tau=0}^{\tau=2} 2\tau\cdot h(t-\tau)\,d\tau \\
&= \int_{\tau=0}^{\tau=2} -\tau^2\cdot 3\,e^{-5(t-\tau)}\,u(t-\tau)\,d\tau + \int_{\tau=0}^{\tau=2} 2\tau\cdot 3\,e^{-5(t-\tau)}\,u(t-\tau)\,d\tau
\end{aligned}$$

- For t $<$ 0, $v_{out}(t)$ clearly is equal to 0.
- For $t \in [0,2]$:

$$v_{out}(t) = \int_0^t -\tau^2\cdot 3\,e^{-5(t-\tau)}\,d\tau + \int_0^t 2\tau\cdot 3\,e^{-5(t-\tau)}\,d\tau$$

Starting with evaluating the first integral:

$$
\begin{aligned}
\int_0^t -\tau^2 \cdot 3\, e^{-5(t-\tau)}\, d\tau &= -3\, e^{-5t} \int_0^t \tau^2 \cdot e^{5\tau}\, d\tau \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^t - \int_0^t \frac{2\tau}{5} e^{5\tau}\, d\tau \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^t - \frac{2}{5} \int_0^t \tau \cdot e^{5\tau}\, d\tau \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^t - \frac{2}{5} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^t - \int_0^t \frac{1}{5} e^{5\tau}\, d\tau \right) \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^t - \frac{2}{5} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^t - \left[\frac{1}{25} e^{5\tau} \right]_0^t \right) \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^t - \frac{2}{5} \left[\frac{\tau}{5} e^{5\tau} \right]_0^t + \frac{2}{5} \left[\frac{1}{25} e^{5\tau} \right]_0^t \right) \\
&= -3\, e^{-5t} \left(\frac{t^2}{5} e^{5t} - \frac{2}{25} t \cdot e^{5t} + \frac{2}{125} e^{5t} - \frac{2}{125} \right) \\
&= -\frac{6}{125} + \frac{6}{25}\, t - \frac{3}{5}\, t^2 + \frac{6}{125}\, e^{-5t}
\end{aligned}
$$

Now for the second integral:

$$
\begin{aligned}
\int_0^t 2\tau \cdot 3\, e^{-5(t-\tau)}\, d\tau &= 6e^{-5t} \int_0^t \tau \cdot e^{5\tau}\, d\tau \\
&= 6\, e^{-5t} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^t - \int_0^t \frac{1}{5} e^{5\tau}\, d\tau \right) \\
&= 6\, e^{-5t} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^t - \frac{1}{5} \left[\frac{1}{5} e^{5\tau} \right]_0^t \right) \\
&= 6\, e^{-5t} \left(\frac{t}{5} e^{5t} - \frac{1}{25} e^{5t} + \frac{1}{25} \right) \\
&= -\frac{6}{25} + \frac{6}{5}\, t + \frac{6}{25} e^{-5t}
\end{aligned}
$$

Now adding both of them gives:

$$
\begin{aligned}
v_{out}(t) &= -\frac{6}{125} - \frac{6}{25} + \frac{6}{25}\, t + \frac{6}{5}\, t - \frac{3}{5}\, t^2 + \frac{6}{125}\, e^{-5t} + \frac{6}{25} e^{-5t} \\
&= -\frac{36}{125} + \frac{36}{25}\, t - \frac{3}{5}\, t^2 + \frac{36}{125}\, e^{-5t}
\end{aligned}
$$

- For $t \geq 2$:

$$
v_{out}(t) = \int_0^2 -\tau^2 \cdot 3\, e^{-5(t-\tau)}\, d\tau + \int_0^2 2\tau \cdot 3\, e^{-5(t-\tau)}\, d\tau
$$

Starting with evaluating the first integral:

$$\begin{aligned}
\int_0^2 -\tau^2 \cdot 3\, e^{-5(t-\tau)}\, d\tau &= -3\, e^{-5t} \int_0^2 \tau^2 \cdot e^{5\tau}\, d\tau \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^2 - \int_0^2 \frac{2\tau}{5} e^{5\tau}\, d\tau \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^2 - \frac{2}{5} \int_0^2 \tau \cdot e^{5\tau}\, d\tau \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^2 - \frac{2}{5} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^2 - \int_0^2 \frac{1}{5} e^{5\tau}\, d\tau \right) \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^2 - \frac{2}{5} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^2 - \left[\frac{1}{25} e^{5\tau} \right]_0^2 \right) \right) \\
&= -3\, e^{-5t} \left(\left[\frac{\tau^2}{5} e^{5\tau} \right]_0^2 - \frac{2}{5} \left[\frac{\tau}{5} e^{5\tau} \right]_0^2 + \frac{2}{5} \left[\frac{1}{25} e^{5\tau} \right]_0^2 \right) \\
&= -3\, e^{-5t} \left(\frac{4}{5} e^{10} - \frac{4}{25}\, e^{10} + \frac{2}{125} e^{10} - \frac{2}{125} \right) \\
&= \frac{6}{125}\, e^{-5t} - \frac{246}{125}\, e^{-5(t-2)}
\end{aligned}$$

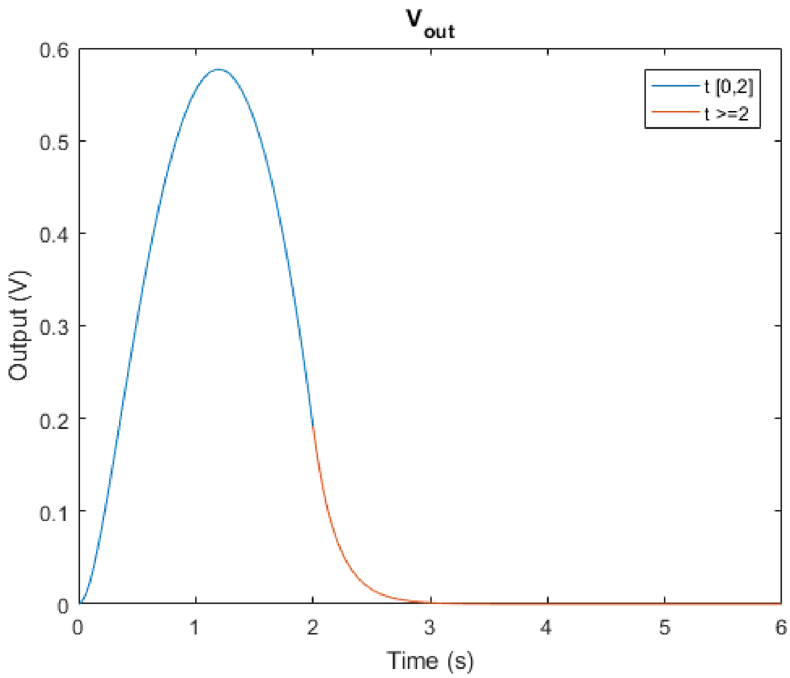
Now for the second integral:

$$\begin{aligned}
\int_0^2 2\tau \cdot 3\, e^{-5(t-\tau)}\, d\tau &= 6e^{-5t} \int_0^2 \tau \cdot e^{5\tau}\, d\tau \\
&= 6\, e^{-5t} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^2 - \int_0^2 \frac{1}{5} e^{5\tau}\, d\tau \right) \\
&= 6\, e^{-5t} \left(\left[\frac{\tau}{5} e^{5\tau} \right]_0^2 - \frac{1}{5} \left[\frac{1}{5} e^{5\tau} \right]_0^2 \right) \\
&= 6\, e^{-5t} \left(\frac{2}{5} e^{10} - \frac{1}{25} e^{10} + \frac{1}{25} \right) \\
&= \frac{6}{25}\, e^{-5t} + \frac{54}{25}\, e^{-5(t-2)}
\end{aligned}$$

Now adding both of them gives:

$$\begin{aligned}
v_{out}(t) &= \frac{6}{125}\, e^{-5t} + \frac{6}{25}\, e^{-5t} - \frac{246}{125}\, e^{-5(t-2)} + \frac{54}{25}\, e^{-5(t-2)} \\
&= \frac{36}{125}\, e^{-5t} + \frac{24}{125}\, e^{-5(t-2)}
\end{aligned}$$

b)


V_{out}
Output (V)
Time (s)
t [0,2]
t >=2
0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
5
6

Chapter 15

Two-port circuits

Exercise 15.1

The z-parameters are given by:

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

- By setting $I_2 = 0$ we can find z_{11} and z_{21}. Setting $I_2 = 0$ means that R_1 and R_3 are now in series with each other. So, $V_1 = (R_1 + R_3) \cdot I_1$ and $V_2 = R_3 \cdot I_1$.

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = R_1 + R_3 \quad [\Omega]$$
$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = R_3 \quad [\Omega]$$

- By setting $I_1 = 0$ we can find z_{12} and z_{22}. Setting $I_1 = 0$ means that R_2 and R_3 are now in series with each other. So, $V_1 = R_3 \cdot I_2$ and $V_2 = (R_2 + R_3) \cdot I_2$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = R_3 \quad [\Omega]$$
$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = R_2 + R_3 \quad [\Omega]$$

Exercise 15.2

The y-parameters are given by:

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

- By setting $V_2 = 0$ we can find y_{11} and y_{21}. Setting $V_2 = 0$ means that R_1 and $R_2//R_3$ are now in series ($R_1 + R_2//R_3 = \frac{R_2R_3+R_2R_1+R_3R_1}{R_2+R_3}$). So, $V_1 = \frac{R_2R_3+R_2R_1+R_3R_1}{R_2+R_3} \cdot I_1$ and $I_2 = \frac{V_2}{R_2} = -\frac{\frac{R_2R_3}{R_2R_3+R_1R_2+R_1R_3} \cdot V_1}{R_2}$.

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{I_1}{\frac{R_2R_3+R_2R_1+R_3R_1}{R_2+R_3} \cdot I_1} = \frac{R_2 + R_3}{R_2R_3 + R_1R_2 + R_1R_3} \quad [S]$$
$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{\frac{R_2R_3}{R_2R_3+R_1R_2+R_1R_3} \cdot V_1}{R_2 \cdot V_1} = -\frac{R_3}{R_2R_3 + R_1R_2 + R_1R_3} \quad [S]$$

- By setting $V_1 = 0$ we can find y_{12} and y_{22}. Setting $V_1 = 0$ means that R_2 and $R_1//R_3$ are now in series ($R_2 + R_1//R_3 = \frac{R_2R_3+R_2R_1+R_3R_1}{R_1+R_3}$). So, $I_1 = -\frac{V_1}{R_1} = -\frac{\frac{R_1R_3}{R_2R_3+R_1R_2+R_1R_3} \cdot V_2}{R_1}$ and $V_2 = \frac{R_2R_3+R_2R_1+R_3R_1}{R_1+R_3} \cdot I_2$.

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{\frac{R_1R_3}{R_2R_3+R_1R_2+R_1R_3} \cdot V_2}{R_1 \cdot V_2} = -\frac{R_3}{R_2R_3 + R_1R_2 + R_1R_3} \quad [S]$$
$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{I_2}{\frac{R_2R_3+R_2R_1+R_3R_1}{R_1+R_3} \cdot I_2} = \frac{R_1 + R_3}{R_2R_3 + R_1R_2 + R_1R_3} \quad [S]$$

The relations between the z and y parameters are:

$$y_{11} = \frac{z_{22}}{\Delta z} = \frac{R_2 + R_3}{R_2R_3 + R_1R_2 + R_1R_3 + R_3^2 - R_3^2} = \frac{R_2 + R_3}{R_2R_3 + R_1R_2 + R_1R_3}$$
$$y_{12} = \frac{-z_{12}}{\Delta z} = \frac{-R_3}{R_2R_3 + R_1R_2 + R_1R_3 + R_3^2 - R_3^2} = \frac{-R_3}{R_2R_3 + R_1R_2 + R_1R_3}$$
$$y_{21} = \frac{-z_{21}}{\Delta z} = \frac{-R_3}{R_2R_3 + R_1R_2 + R_1R_3 + R_3^2 - R_3^2} = \frac{-R_3}{R_2R_3 + R_1R_2 + R_1R_3}$$
$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{R_1 + R_3}{R_2R_3 + R_1R_2 + R_1R_3 + R_3^2 - R_3^2} = \frac{R_1 + R_3}{R_2R_3 + R_1R_2 + R_1R_3}$$

Exercise 15.3

The h-parameters are given by:

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

- For $V_2 = 0$ we can find h_{11} and h_{21}.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{20mV}{4uA} = 5k\Omega$$
$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{0.5uA}{4uA} = 0.125$$

- For $I_1 = 0$ we can find h_{12} and h_{22}.

$$
\begin{aligned}
h_{12} &= \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{300mV}{75mV} = 4 \\
h_{22} &= \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{7.5uA}{75mV} = 0.1mS
\end{aligned}
$$

Exercise 15.4

In order to find the resistor values of R_1, R_2 and R_3 such that the given conditions for the g-parameters are satisfied, the expressions of the g-parameters in terms of the resistors have to be evaluated.

The g-parameters are given by-

$$
\begin{aligned}
I_1 &= g_{11}V_1 + g_{12}I_2 \\
V_2 &= g_{21}V_1 + g_{22}I_2
\end{aligned}
$$

- By setting $I_2 = 0$ we can find g_{11} and g_{21}. Setting $I_2 = 0$ means that R_1 and R_3 are now in series with each other and together in parallel with R_2

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{I_1}{I_1\Big((R_1 + R_3)//R_2\Big)} = \frac{R_1 + R_2 + R_3}{R_1R_2 + R_2R_3} \tag{15.1}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{\frac{R_3}{R_1+R_3}V_1}{V_1} = \frac{R_3}{R_1 + R_3} \tag{15.2}$$

- By setting $V_1 = 0$ we can find g_{12} and g_{22}. Setting $V_1 = 0$ means that R_2 is "short-circuited" and can be ignored, as it does not influence the rest of the circuit. I_1 is then the current through R_1

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} = \frac{-\frac{R_3}{R_1+R_3}I_2}{I_2} = -\frac{R_3}{R_1 + R_3} \tag{15.3}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \frac{I_2\Big(R_1//R_3\Big)}{I_2} = \frac{R_1R_3}{R_1 + R_3} \tag{15.4}$$

Using the given value of $g_{21} = 0.5$ with Equation 15.2 gives-

$$0.5 = \frac{R_3}{R_1 + R_3} \quad \Rightarrow \quad R_1 = R_3 \tag{15.5}$$

Equation 15.3 with the given value of $g_{12} = -0.5$ gives the same result. Using Equation 15.4 with the given value of $g_{22} = 6$ and the above result in Equation 15.5 gives-

$$\begin{aligned} 6 &= \frac{R_1 R_3}{R_1 + R_3} \\ 6 &= \frac{R_1^2}{2R_1} \\ 6 &= \frac{R_1}{2} \\ &\Rightarrow R_1 = R_3 = 12\ \Omega \end{aligned} \tag{15.6}$$

Filling in the found value for R_1 and R_3 in Equation 15.1 and using the given value of $g_{11} = 0.125$ gives-

$$\begin{aligned} 0.125 &= \frac{R_1 + R_2 + R_3}{R_1 R_2 + R_2 R_3} \\ 0.125 &= \frac{24 + R_2}{24 R_2} \\ R_2 &= \frac{24}{2} = 12\ \Omega \end{aligned}$$

Exercise 15.5

The h-parameters are given by-

$$\begin{aligned} V_1 &= h_{11} I_1 + h_{12} V_2 \\ I_2 &= h_{21} I_1 + h_{22} V_2 \end{aligned}$$

- Set $V_2 = 0$. This results in the $-100j\Omega$-capacitor being "short-circuited". The 200Ω-resistor is then in parallel with the dependent voltage source, and together they are in series with the 10Ω-resistor and the $j20\Omega$-inductor. I_2 is the current through the 200Ω-resistor. In this situation h_{11} and h_{21} can be determined.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \quad \text{and} \quad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

Finding h_{11}

I_1 can be found by first finding the voltage over the series equivalent of the 10Ω-resistor and the $j20\Omega$-inductor using KVL for the loop at the left-hand side of the circuit. Dividing this voltage by the equivalent impedance of the resistor and the inductor results in I_1

$$I_1 = \frac{V_1 - 50I_2}{10 + j20} \tag{15.7}$$

I_2 can be found by considering the loop at the right-hand side of the circuit and using KVL.

$$50I_2 = -200I_2 \Rightarrow I_2 = 0\ A \tag{15.8}$$

Filling in the found value for I_2 in Equation 15.7 results in-

$$I_1 = \frac{V_1}{10+j20} \quad \Rightarrow \quad h_{11} = \frac{V_1}{\frac{V_1}{10+j20}} = 10 + j20\ \Omega$$

Finding h_{21}

Using the the result of Equation 15.8 results in-

$$h_{21} = \frac{0}{I_1} = 0$$

- Set $I_1 = 0$. This results in a circuit where the 200Ω-resistor and the dependent voltage source are in series, and together in parallel with the $-100j\Omega$-capacitor. Also, this means that $V_1 = 50I_2$. h_{12} and h_{22} can now be determined.

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \quad \text{and} \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

As V_1 is already given in terms of I_2, both parameters can easily be found if V_2 can also be expressed in terms of I_2. This can be done by finding the current through the 200Ω-resistor and the current through the $-100j\Omega$-capacitor, whose sum is equal to I_2.

$$\begin{aligned}
I_2 &= I_{200\Omega} + I_{-100j\Omega} \\
I_2 &= \frac{V_2 - 50I_2}{200} + \frac{V_2}{-100j} \\
I_2 &= \frac{1}{200}V_2 - \frac{1}{4}I_2 + \frac{1}{100}j\ V_2 \\
\frac{5}{4}I_2 &= \left(\frac{1}{200} + \frac{1}{100}j\right)V_2 \\
V_2 &= \frac{I_2}{0.004 + 0.008j}
\end{aligned} \tag{15.9}$$

Using this expression for V_2 gives the following results for h_{12} and h_{22}-

$$\begin{aligned}
h_{12} &= \frac{50I_2}{\frac{I_2}{0.004+0.008j}} = 0.2 + 0.4j \\
h_{22} &= \frac{I_2}{\frac{I_2}{0.004+0.008j}} = 4 + 8j\ mS
\end{aligned}$$

Exercise 15.6

The a-parameters are given by-

$$\begin{aligned}
V_1 &= a_{11}V_2 - a_{12}I_2 \\
I_1 &= a_{21}V_2 - a_{22}I_2
\end{aligned}$$

a) Using the second equation of the a-parameters set of equations and the fact that $V_2 = -I_2Z_L$ gives-

$$
\begin{aligned}
I_1 &= a_{21}V_2 - a_{22}I_2 \\
I_1 &= -a_{21}Z_L I_2 - a_{22}I_2 \\
I_1 &= -\Big(a_{21}Z_L + a_{22}\Big)I_2 \\
\frac{I_2}{I_1} &= \frac{-1}{a_{21}Z_L + a_{22}}
\end{aligned}
$$

b) Using the first equation of the a-parameters set of equations and the fact that $V_2 = -I_2 Z_L$ gives-

$$
\begin{aligned}
V_1 &= a_{11}V_2 - a_{12}I_2 \\
V_1 &= a_{11}V_2 - a_{12}\frac{-V_2}{Z_L} \\
V_1 &= \Big(\frac{a_{11}Z_L + a_{12}}{Z_L}\Big)V_2 \\
\frac{V_2}{V_1} &= \frac{Z_L}{a_{11}Z_L + a_{12}}
\end{aligned}
$$