
MEASURE THEORY AND INTEGRATION

1 Measuring Sets

First we define a rudimentary "measure" for a rectangle. Then, by means of cube coverings of sets, we define a rudimentary measure for arbitrary sets in \mathbb{R}^d , called the *outer measure*, and results that follow, reg. open sets etc. leads us to define a restriction on the *outer measure*, and this gives us the *Lebesgue Measure*.

1.1 Primitive Measurement of Rectangles

Definition 1.1. (Rectangle (Open) in \mathbb{R}^d)

$$R := \prod_{i=1}^d (a_i, b_i)$$

Definition 1.2. (Rectangle (Closed) in \mathbb{R}^d)

$$\overline{R} = \prod_{i=1}^d [a_i, b_i]$$

Definition 1.3. (Almost disjoint collection of rectangles $\{R_i\}$) A collection of rectangles $\{R_i\}$ is said to be *almost disjoint* if $R_i^\circ \cap R_j^\circ = \emptyset$ for all $i, j \in \mathbb{N}$.

Definition 1.4. (Volume of a rectangle (both closed and open))

$$|R| = \overline{R} := \prod_{i=1}^d |b_i - a_i|$$

Lemma 1.5. If $R \subset \mathbb{R}^d$ is a rectangle, and $\{R_i\}_{i=1}^N$ is a collection of almost disjoint rectangles such that $R = \cup_{i=1}^N R_i$, then $|R| = \sum_{i=1}^N |R_i|$

Proof. Let $R = \prod_{i=1}^d [a_i, b_i]$ and subdivide each $[a_i, b_i]$ into $t_{i0} = a_i < t_{i1} < t_{i2} < \dots < t_{iN-1} < b_i = t_{iN}$ where $[t_{ik}, t_{ik+1}]$ is the interval that forms the i -th coordinate of some rectangle R_q in our collection. Let L_{kj} represent the j -th subdivision interval of the k -th interval of R . All possible products taken of L_{kj} , call it R_{pq} yields a division of R into smaller rectangles, and by virtue of the definition of $|R|$, this is equal to $\sum_{p,q} |R_{pq}|$. In noting that each R_i can in the same sense be subdivided, and is eventually covered exactly by a subcollection of R_{pq} and that each R_i is robustly covered so, we have that $|R_i| = \sum_q |R_{pq}|$, which would yield $|R| = \sum |R_i|$.

□

A corollary of this is:

Remark 1.6. If $R = \cup_{i=1}^N R_i$ for rectangles R_i (where R itself is a rectangle), then

$$|R| \leq \sum |R_i|$$

Lemma 1.7. (*Dyadic Rationals are Dense in \mathbb{R} and \mathbb{R}^d*) The set

$$T = \left\{ \frac{1}{2^k} (m_1, m_2, \dots, m_d) : k \in \mathbb{N}, m_1, m_2, \dots, m_d \in \mathbb{Z} \right\}$$

is dense in \mathbb{R}^d

Proof. Let $\vec{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, and let $\varepsilon > 0$. Choose a cube $\pi_{i=1}^d [p_i, p_i + e]$ that is sitting inside $B_\varepsilon(x)$. Let a_j be the smallest distance between x_j and the endpoints of the j -th interval of the cube. Let $a = \inf(a_j)$. Now choose $k \in \mathbb{N}$ such that $2^k(a) > 2$, and choose m_j such that $2^k(p_j) < m_j < 2^k p_j + 2^k e$, which yields a point $\frac{1}{2^k}(m_1, m_2, \dots, m_d)$ in this cube, in this ball. Hence, dense.

□

Theorem 1.8. (*Open sets are approximately a bunch of cubes*) Let $\mathbb{O} \subseteq \mathbb{R}^d$, then there exists $\{Q_i\}_{i=1}^\infty$, a countable collection of almost disjoint closed cubes such that

$$\mathbb{O} = \cup_{i=1}^\infty Q_i$$

Proof. Subdivide \mathbb{R}^d into almost disjoint 1-cubes, making a grid of 1-cubes. Let A_0 be the set of all grid cubes that fall inside \mathbb{O} , and let T_0 be the set of all grid cubes that fall both inside and outside \mathbb{O} . In step 2, subdivide the cubes in T_0 into halves, and let A_1 be the set of all cubes obtained such that they are fully in \mathbb{O} , and let T_1 be the set of all cubes such that they fall both in and out of \mathbb{O} . Keep going as such, subdividing in each step by half. Take all A_1, A_2, \dots and union all sets in all of them. This is denoted by $\cup_{i=1}^\infty Q_i$ which is obviously contained in \mathbb{O} . Now let $x \in \mathbb{O}$. $x \in B_r(x) \subseteq \mathbb{O}$. Choose a cube inside this, do the same procedure as above proof, and get an a as the above proof (by taking $Q = \prod_{g=1}^d [p_g, p_g + e]$ and setting $a = \inf(a_j)$ where a_j is the shortest distance between x_j and the j -th interval's endpoints). Choose k large enough such that $2^k a > 2$ and choose two integers m_j and $m_j + 1$ in between $2^k p_j < m_j < m_j + 1 < 2^k p_j + 2^k e$. Now take the product $\prod_{j=1}^d [m_j/2^k, (m_j + 1)/2^k]$, which would end up being a cube in some A_i (accepted cubes), implying $\mathbb{O} \subseteq \cup_{i=1}^\infty Q_i$. Hence

$$\mathbb{O} = \cup_{i=1}^\infty Q_i$$

□

1.2 Primitive Measurement of *Sets*- The Outer Measure

Now we define, equipped with our primitive notion of a measure for a rectangle, what it means to *measure a set*.

Definition 1.9. (Outer Measure M_*) Let $E \subseteq \mathbb{R}^d$.

$$M_*(E) := \inf \left\{ \sum_{i=1}^{\infty} |Q_i| : \{Q_i\} \text{ is a cube covering of } E \right\}$$

Remark 1.10. This definition is intuitive because it takes into account all possible cube coverings, and finds the smallest such in the sense of "measurement". The name "outer measure" encompasses the fact that we are approximating the set E by coverings of E , essentially from the outside rather than the inside.

Theorem 1.11. Let $M_*^R(E) := \inf \{ \sum_{i=1}^{\infty} |R_i| : \{R_i\} \text{ a collection of rectangles covering } E \}$ and $M_*^C(E) := \inf \{ \sum_{i=1}^{\infty} |Q_i| : \text{ where } \{R_i\} \text{ is a collection of cubes covering } E \}$. Then:

$$M_*^R = M_*^C$$

Proof. It is immediately obvious that any cube covering is a rectangle covering, implying that $M_*^R \leq M_*^C$. For the other direction, we start with a rectangle covering $\{R_i\}$ of E , and then slightly extend these rectangles, by some δ such that the extended rectangles have side length ratios to be rational (This can be done, as the next throwaway result suggests), and their volumes $|S_j| \leq |R_j| + \varepsilon/2^j$. Once this is done, We can subdivide $|SS_j|$ into smaller closed, almost disjoint, finite cubes $S_j = \cup_{i=1}^{N_j} Q_{ij}$ and conclude that $\sum |Q_{ij}| = |S_j|$ and eventually say that $\{Q_{ij}\}$ is a covering of E such that $\sum |Q_{ij}| \leq \sum |R_k| + \varepsilon$.

So to conclude, given a rectangle covering $\{R_i\}$, for every $\varepsilon > 0$ we have a cube covering $\{Q_i\}$ such that

$$\sum |Q_{ij}| \leq \sum |R_j| + \varepsilon$$

which means that

$$M_*^C \leq M_*^R$$

which concludes the proof.

□

Lemma 1.12. (throwaway) Given a rectangle R , and an $\varepsilon > 0$, there exists a larger rectangle $R \subset R'$ such that $|R'| \leq |R| + \varepsilon$ and the side-side ratios of $|R'|$ are rational, or rather, we can have R' to be equal to the union of a finite number of almost disjoint closed cubes Q_i .

Proof. DRAG (WORK IN PROGRESS)

□

Theorem 1.13. 1. Outer measure of a closed cube Q is $|Q|$

2. Outer measure of an open cube Q is $|Q|$
3. Outer measure of a closed rectangle R is $|R|$
4. Outer measure of an open rectangle R is $|R|$

Proof. 1. Let Q being a closed cube covers itself, therefore $M_*(Q) \leq |Q|$. Consider any cube covering $Q \subseteq \cup_i Q_i$ of Q . Let S_j be a slightly larger open cube than Q_j such that $|S_j| \leq |Q_j|(1+\varepsilon)$. $\{S_j\}$ covers Q , and hence a finite subcovering $\{S_i\}_{i=1}^N$ covers Q , which means $|Q| \leq \sum_{i=1}^N |S_i| \leq (1+\varepsilon) \sum_{i=1}^N |Q_i| \leq (1+\varepsilon)(\sum_{i=1}^\infty |Q_i|)$ which gives us that for every ε and a covering Q_i , we have $|Q| \leq \sum_i |Q_i|(1+\varepsilon)$ which proves our result.

2. Let Q be open. \overline{Q} covers Q , implying $M_*(Q) \leq |\overline{Q}| = |Q|$. Let Q_0 be any closed set inside Q . We have $M_*(Q_0) = |Q_0| \leq M_*(Q)$ from the fact that M_* is monotonic (under the inclusion relation). Since we can make $|Q_0|$ go arbitrarily close to $|Q|$, we have that $|Q| \leq M_*(Q)$, which proves (2).

(3) and (4) for rectangles follow similarly.

□

Lemma 1.14. (Two light lemmas)

1. If $E_1 \subseteq E_2$ then $M_*(E_1) \leq M_*(E_2)$
2. Given $\varepsilon > 0$, there exists a covering $\{Q_i\}$ of E such that $\sum_{i=1}^\infty |Q_i| \leq M_*(E) + \varepsilon$

Proof. $E_1 \subseteq E_2 \implies$ any covering of E_2 is a covering for E_1 , which ultimately gives us (1).

(2) follows from the definition.

□

Theorem 1.15. (Countable sub-additivity of outer measure) Let $E = \cup_{i=1}^\infty E_i$. Then $M_*(E) \leq \sum_{i=1}^\infty M_*(E_i)$

Proof. Let $\{Q_{i,j}\}_{i=1}^{\infty}$ be a closed almost disjoint cube covering for E_j such that $\sum_{j=1}^{\infty} |Q_{ij}| \leq m_*(E_j) + \varepsilon/2^j$. $\{Q_{ij} : i, j \in \mathbb{N}\}$ is a covering for E implying $m_*(E) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |Q_{ij}| \leq \sum_{j=1}^{\infty} m_*(E_j) + \varepsilon$ which implies that $m_*(E) \leq \sum_{i=1}^{\infty} m_*(E_j)$

□

Theorem 1.16. (*Open Set criteria*)

$$m_*(E) = \inf(P) = \inf\{m_*(\mathbb{O}) : \mathbb{O} \text{ open and } E \subseteq \mathbb{O}\}$$

Proof. From monotonicity, it is clear that $M_*(E) \leq \inf(P)$. Let $\{Q_i\}$ be such that $\sum_{i=1}^{\infty} |Q_i| \leq M_*(E) + \varepsilon/2$ and for each Q_i find open cube S_i so that $|S_i| \leq |Q_i| + \varepsilon/2^{i+1}$. Then $\sum_i |S_i| \leq \sum_i |Q_i| + \varepsilon/2$. Note that $\{S_i\}$ is an open cover for E , so let $O = \cup_i S_i$. We have $M_*(O) \leq \sum_i |S_i| \leq M_*(E) + \varepsilon$. We are done.

□

Theorem 1.17. *If $E = E_1 \cup E_2$ with $d(E_1, E_2) > 0$ and $E_1 \cap E_2 = \emptyset$, then $M_*(E) = M_*(E_1) + M_*(E_2)$*

Proof. From countable additivity we have $m_*(E) \leq m_*(E_1) + m_*(E_2)$. For the other direction, take an open covering $\{Q_i\}$ for E such that $\sum_i |Q_i| \leq m_*(E) + \varepsilon$. We subdivide each Q_i into Q_{ij} (a finite number of them for each i) such that each cube is of side length less than $d(E_1, E_2)$. This means that a subcollection of Q_{ij} covers E_1 , and another, disjoint subcollection covers E_2 , which means that $m_*(E_1) + m_*(E_2) \leq \sum_{ij} |Q_{ij}| \leq m_*(E) + \varepsilon$. We are done.

□

Lemma 1.18. *If $E_1, E_2 \dots E_k$ are disjoint sets such that $d(E_i, E_j) > 0$ for all i and j , then $m_*(\cup_{i=1}^k E_i) = \sum_{i=1}^k m_*(E_i)$*

Proof. Induction

□

Theorem 1.19. *If $E = \cup_{i=1}^{\infty} Q_i$ for almost disjoint closed cubes, then $m_*(E) = \sum_i |Q_i|$*

Proof. From subadditivity we have $m_*(E) \leq \sum_i |Q_i|$. We construct $S_j \subset Q_j$ slightly smaller than Q_j , i.e, $|Q_j| \leq |S_j| + \varepsilon/2^j$. These S_j are all disjoint, and their distance from each other are all positive. For any $n \in \mathbb{N}$, we have $\cup_{i=1}^n S_i \subseteq E$, which gives $\sum_{i=1}^n |S_i| \leq m_*(E)$. This gives $\sum_{i=1}^n |Q_i| \leq m_*(E) + \sum_{i=1}^n \varepsilon/2^i$, which, in the limit, yields our desired result.

□

1.3 The Lebesgue Measure

Definition 1.20. (Lebesgue Measurable Set) A set $E \subseteq \mathbb{R}^d$ is said to be Lebesgue measurable if for every $\varepsilon > 0$, there exists an open set O such that $E \subseteq O$ and $m_*(O - E) < \varepsilon$

If E is lebesgue measurable, then its measure is defined as

$$\text{Lebesgue measure} = m(E) = m_*(E)$$

Lemma 1.21. *Every open set is lebesgue measurable.*

Proof. Trivial

□

Lemma 1.22. *Every set E such that $m_*(E) = 0$ is lebesgue measurable (and are called measure zero sets).*

Proof. If $m_*(E) = 0$, then for an $\varepsilon > 0$, there exists an open set O such that $E \subset O$ and $m_*(O) \leq \varepsilon$, but $O - E$ is a subset of O , which gives us the desired result.

□

Theorem 1.23. *Countable union of measurable sets are measurable.*

Proof. Let $E = \cup_i E_i$ where each E_i are measurable. For E_i , take an open set $E_i \subseteq O_i$ such that $m_*(O_i - E_i) \leq \varepsilon/2^i$. Consider $(\cup_i O_i) - (\cup_i E_i) \subseteq \cup_i (O_i - E_i)$ which gives $m_*((\cup_i O_i) - (\cup_i E_i)) \leq \sum_i (m_*(O_i - E_i)) \leq \varepsilon$ which yields us the result.

□

Lemma 1.24. *If F is closed and K is compact, with $F \cap K = \emptyset$ then $d(F, K) > 0$.*

Proof. If $d(F, K) > 0$, then there exists sequences $x_n \in F$ and $y_n \in K$ such that $d(x_n, y_n) \rightarrow 0$. Since y_n is in K , from heine borel we have $y_{n_k} \rightarrow y \in K$ for some subsequence y_{n_k} . This means that $x_{n_k} \rightarrow y$, but F being closed gives us $y \in F$, a contradiction.

□

Lemma 1.25. *In \mathbb{R}^d , any closed set F is a countable union of compact sets.*

Proof. Let $B_i[0]$ be an increasing sequence of compact balls. $F = \cup_i F \cap B_i[0]$. Closed subsets of compact sets are compact, so we are done.

□

Theorem 1.26. *Compact sets in \mathbb{R}^d are measurable.*

Proof. Let $\varepsilon > 0$ and O be such that $m_*(O) \leq m_*(E) + \varepsilon$. $O - E$ is actually an open set, so $O - E = \cup_i Q_i$ for a collection (countable) of almost disjoint closed cubes. Let $S_j \subset Q_j$ such that $|Q_j| \leq |S_j| + \varepsilon/2^j$. Note that, since S_j are closed, they are compact and disjoint from other S_i . For $n \in \mathbb{N}$, we have $E \cup_{i=1}^n S_i \subseteq O$, or $m_*(E \cup_{i=1}^n S_i) = m_*(E) + \sum_{i=1}^n |S_i| \leq m_*(O) \implies m_*(E) + \sum_{i=1}^n |Q_i| \leq m_*(O) + \sum_{i=1}^n \varepsilon/2^i$. Taking the n limit yields

$$m_*(E) + \sum_{i=1}^{\infty} |Q_i| = m_*(E) + m_*(O - E) \leq m_*(O) + \varepsilon$$

or $m_*(E) + m_*(O - E) \leq m_*(O)$ (or even $= m_*(O)$). This tells us that $m_*(O - E) \leq \varepsilon$. Hence E is measurable. □

Remark 1.27. Since compact sets are measurable, and closed sets are a countable union of compact sets, closed sets are also measurable. Thus far, our collection of measurable sets are those that are open, closed, and a countable union of those that are already measurable.

Theorem 1.28. *Complement of a measurable set is measurable.*

Proof. Let E be measurable. Let O_n be such that $m_*(O_n - E) \leq 1/n$. Consider $E \subseteq \cap_i O_i$, or $\cup_i O_i^c \subseteq E^c$. We have $m_*(E^c - \cup_i O_i^c) = m_*(\cap_i O_i - E) \leq 1/n$ for all n , which means $E^c - \cup_i O_i^c = J$ is a measure 0 set, hence measurable. $J \subseteq E^c$, and $J \cup_i (O_i^c) = E^c$, implying that E^c is a union of two measurable sets, hence itself is measurable. □

Lemma 1.29. *Countable intersection of measurable sets is measurable.*

Proof. $\cap_i E_i = (\cup_i E_i^c)^c$. □

Remark 1.30. Thus far, we have open sets and closed sets to be measurable, as are countable unions, intersections and complements of measurable sets.

Theorem 1.31. (Countable additivity) *Let $E = \cup_i E_i$ for measurable, disjoint sets $\{E_i\}$. Then,*

$$m(E) = \sum_{i=1}^{\infty} m(E_i)$$

Proof. One direction $m(E) \leq \sum_{i=1}^{\infty} m(E_i)$ is trivial from subadditivity.

Assume that each E_i are bounded

Consider E_i^C , which are measurable, and find O_i such that $m_*(O_i - E_i) \leq \varepsilon/2^i$. Let $F_i = O_i^C$ be closed sets such that $F_i \subset E_i$ and $m_*(E_i - F_i) = m_*(O_i - E_i^C) \leq \varepsilon/2^i$. for every n , $\cup_{i=1}^n F_i \subseteq E$, implying $m_*(\cup_{i=1}^n F_i) \leq m_*(E)$ which gives (since F_i are closed and bounded, and disjoint with each other) $\sum_{i=1}^n m_*(F_i) \leq m_*(E)$. Also, $E_i = E_i - F_i \cup F_i$ implying that $m_*(E_i) \leq m_*(E_i - F_i) + m_*(F_i) \leq \varepsilon/2^i + m_*(F_i)$, which means $\sum_{i=1}^n m_*(E_i) \leq \sum_{i=1}^n m_*(F_i) + \sum_{i=1}^n \varepsilon/2^i$. In the limit we have

$$\sum_{i=1}^{\infty} m_*(E_i) \leq \sum_{i=1}^{\infty} m_*(F_i) + \varepsilon \leq m_*(E) + \varepsilon$$

which concludes the result for the case where each E_i are bounded.

Suppose they are not bounded. Consider $Q_1 \subset Q_2 \subset \dots$ a sequence of increasing cubes converging to \mathbb{R}^d centered at origin. Let $S_1 = Q_1$, $S_2 = Q_2 - Q_1$, $S_n = Q_n - Q_{n-1}$ and so on. These S_i are disjoint, bounded, and measurable sets. Define $E_{jk} = E_j \cap S_k$. These too, are each measurable, and bounded. $E_j = E_j \cap (\cup_k S_k) = \cup_k E_{jk}$ implying from the previous proof that $m_*(E_j) = \sum_k m_*(E_{jk})$. We know also that E_{jk} are disjoint, bounded, measurable and its union equals E . So $m_*(E) = \sum_j (\sum_k m_*(E_{jk})) = \sum_j (m_*(E_j))$

□

Theorem 1.32. (*Limit Exchange*)

1. Let $E_1 \subseteq E_2 \subset \dots$ be an increasing sequence of measurable sets. Then $m(\cup_i E_i) = \lim_{n \rightarrow \infty} m(E_n)$.
2. Let $F_{n+1} \subseteq F_n \subset \dots \subset F_3 \subseteq F_2 \subseteq F_1$ be a decreasing sequence of measurable sets such that $m(F_1) < \infty$. Then $m(\cup_i F_i) = \lim_{n \rightarrow \infty} m(F_n)$

Proof. (1) Define $Q_1 = E_1$, $Q_2 = E_2 - E_1$, $Q_3 = E_3 - E_2$ and so on. These are disjoint measurable sets. $m(\cup_{i=1}^{\infty} E_i) = m(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(Q_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n m(Q_i) = \lim_{n \rightarrow \infty} m(E_n)$ (since $m(Q_i) = m(E_i - E_{i-1}) = m(E_i) - m(E_{i-1})$).

For (2), we consider $F_1 - F_1 \subset F_1 - F_2 \subset \dots$ an increasing sequence of measurable sets. We then have $m(\cup_i (F_1 - F_i)) = \lim_{n \rightarrow \infty} m(F_1 - F_i) = m(F_1) - \lim m(F_i)$. This immediately gives us the result.

□

Theorem 1.33. Let E be a measurable set, and $\varepsilon > 0$. Then,

1. \exists an open set O such that $E \subset O$ and $m(O - E) \leq \varepsilon$

2. \exists closed set F such that $F \subset E$ and $m(E - F) \leq \varepsilon$
3. if $m(E) < \infty$, then there exists a compact set $K \subset E$ such that $m(E - K) \leq \varepsilon$
4. if $m(E) < \infty$, then there exists $F = \cup_{i=1}^N Q_i$, a union of almost disjoint closed sets, such that $m(E - F) + m(F - E) < \varepsilon$

Proof. (1) Obvious from the definition of measurability.

(2) Apply definition of measurability for E^c to get an open set $E^c \subseteq F^c$ such that $m(F^c - E^c) = m(E - F) \leq \varepsilon$. The requisite closed set is F .

(3) Let F be such that $m(E - F) < \varepsilon/2$ where F is a closed set. Consider $Q_1 \subset Q_2 \subset Q_3 \dots$. $F = \cup_i F \cap Q_i$ where each $F \cap Q_i$ are compact. F is contained in E , implying $m(F) = \lim_{n \rightarrow \infty} m(F \cap Q_n) < \infty$ so the limit exists. For large enough N , we have $m(F) - m(F \cap Q_N) < \varepsilon/2$. We then have $m(E) - m(F) + m(F) - m(F \cap Q_N) = m(E - F) + m(F) - m(F \cap Q_N) = m(E - F \cap Q_N) \leq \varepsilon$.

(4) Let $m(E) < \infty$. There exists almost disjoint closed cubes Q_i such that $\sum_{i=1}^{\infty} |Q_i| \leq m(E) + \varepsilon/2$. This series converges, so we take that N such that $\sum_{i=N+1}^{\infty} |Q_i| < \varepsilon/2$, and define $F = \cup_{i=1}^N Q_i$. Note that $E - F$ is fully contained in $\cup_{i=N+1}^{\infty} Q_i$, so $m(E - F)$ is bounded by $\varepsilon/2$. Note that $F - E = \cup_{i=1}^N Q_i - E \subset \cup_{i=1}^{\infty} Q_i - E$, which means that $m(F - E) \leq m(\cup_i Q_i - E) \leq \varepsilon/2$, therefore, the sum $m(E \Delta F) = m(E - F \cup F - E) = m(E - F) + m(F - E) \leq \varepsilon$

□

Invariance properties of the Lebesgue Measure

Definition 1.34. ($E + h$ translate, $h \in \mathbb{R}^d$)

$$E + h := \{x + h : x \in E\}$$

Definition 1.35. (δE dialation, $\delta \in \mathbb{R}$)

$$\delta E := \{\delta x : x \in E\}$$

Theorem 1.36. If Q is a cube, $Q + h$ is also a cube, and $|Q| = |Q + h|$

Proof. Trivial definition chasing.

□

Theorem 1.37. If B is a ball, then $B + h$ is also a ball, of the same radius. Moreover, $B_r(x) + h = B_r(x + h)$

Proof. Also trivial

□

Theorem 1.38. *If O is an open set, so is $O + h$, and their measures are the same*

Proof. First show that if O is a union of balls B_i then $O + h$ is a union of balls $B_i + h$. Moreover, if O is a disjoint (almost) union of closed cubes $\cup_i Q_i$, then $O + h = \cup_i (Q_i + h)$. We then can easily see that their measures are the same from additivity.

□

Theorem 1.39. $m_*(E) = m_*(E + h)$

Proof. Given any open set O covering E , there exists an open set $O + h$ covering $E + h$, with the same measures. The reverse direction too is possible. So $\inf\{m_*(O) : E \subset O\} = \inf\{m_*(O') : E + h \subseteq O' + h\}$ which concludes equality.

□

Theorem 1.40. *If E is measurable, so is $E + h$ and their measures are the same.*

Proof. Given $\varepsilon > 0$, there exists $E \subset O$ open such that $m_*(O - E) \leq \varepsilon$. Consider $O + h$, which covers $E + h$. Note that $(O + h - E + h) \subseteq (O - E) + h$ so $m_*(O + h - E + h) \leq \varepsilon$. We are done with measurability, and equality comes from m_* .

□

Using similar techniques, one can show:

Theorem 1.41. $m(\delta E) = \delta^d m(E)$

1.4 Sigma Algebras

Definition 1.42. (Sigma algebra $\Sigma(\Omega)$ of a set Ω) A collection of subsets $\Sigma(\Omega)$ of Ω is a sigma algebra if:

1. $\emptyset, \Omega \in \Sigma$
2. $E_1, E_2 \dots \in \Sigma \implies \cup_i E_i \in \Sigma$
3. $E \in \Sigma \implies E^c \in \Sigma$

Definition 1.43. (Borel Sigma Algebra) Let K be the collection of all open sets (in any topology), then the Borel Sigma Algebra, denoted \mathbb{B}_Ω is the intersection of all sigma algebras containing K

Remark 1.44. The set of all lebesgue measurable subsets of \mathbb{R}^d forms a sigma algebra. Since open sets are measurable, the borel sigma algebra is sitting inside the lebesgue sigma algebra.

Definition 1.45. (Measure) Let $(\Omega, \Sigma(\Omega))$ be a set and a sigma algebra (aka a double). A function $\mu : \Sigma(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be a *measure* if:

1. $\mu(E) \geq 0$ for all $E \in \Sigma(\Omega)$
2. $\mu(\emptyset) = 0$
3. $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for a collection of disjoint subsets of $\Sigma(\Omega)$

Theorem 1.46. (Limit Exchange) Let $E_i \uparrow E$, then $\mu(E) = \lim_{i \rightarrow \infty} \mu(E_i)$. Likewise, if $E_i \downarrow E$ with $\mu(E_1) < \infty$, then $\mu(E) = \lim_{i \rightarrow \infty} \mu(E_i)$

Proof. The proof follows similarly to the case where Σ was lebesgue and Ω was \mathbb{R}^d (see above).

□

Theorem 1.47. *Limit exchange + finite additivity is equivalent to countable additivity (given other axioms for a measure stay fixed).*

Proof. \Leftarrow Countable additivity takes care of finite additivity, and previous theorem gives limit exchange.

\Rightarrow Let E_1, E_2, \dots be disjoint sets in the sigma algebra. We have $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$. Let $Q_1 = E_1, Q_2 = E_1 \cup E_2, Q_3 = E_1 \cup E_2 \cup E_3$ and so on. We have $Q_i \uparrow$. $\cup_{i=1}^n Q_i = \cup_{i=1}^n E_i$. $\mu(\cup_{i=1}^n Q_i) = \sum_{i=1}^n \mu(E_i)$. $\mu(\cup_i E_i) = \mu(\cup_{i=1}^{\infty} Q_i) = \lim_{n \rightarrow \infty} \mu(Q_i) = \sum_{i=1}^{\infty} \mu(E_i)$

□

Theorem 1.48. *A set $E \subseteq \mathbb{R}^d$ is measurable iff*

1. *E differs from a G_δ set by a set of zero measure*
2. *E differs from a F_σ set by a set of zero measure*

Proof. If measurable, then obviously we have for every $n \in \mathbb{N}$ such that $m(O_n - E) \leq 1/n$ so $\cap_n O_n$ is our desired G_δ set. Likewise, if we perform the same for E^c and get $\cap_n O'_n$ there, then $\cup_n O_n^{c'}$ is our desired F_σ set.

If we have E differ from a G_δ set by a set of measure 0, we mean that $E - G$ and $G - E$ are individually measure 0. Note that $E \cap G = G - (G - E)$ which means $E \cap G$ is measurable. Now, $E \cap G \cup (E - G) = E$, which means E is measurable. Same argument applies to an F_σ set F .

□

Theorem 1.49. (An equivalent formulation of measurability) Let E be measurable and $E_0 \subseteq E$. If $m^*(E) = m^*(E_0) + m^*(E - E_0)$ then E_0 is measurable and naturally, so is $E - E_0$. The converse statement is obviously true.

Proof. Consider $E \subset O$ such that $m(O - E) \leq \varepsilon$. Let O_1 and O_2 be so that $m^*(O_1) \leq m^*(E_0) + \varepsilon$ and $m^*(O_2) \leq m^*(E - E_0) + \varepsilon$. $E_0 \subseteq O_1$, $E - E_0 \subseteq O_2$.

$$m^*(E_0) + m^*(E - E_0) = m^*(E) \leq m^*(O_1 \cup O_2) = m^*(O_1) + m^*(O_2) - m^*(O_1 \cap O_2)$$

which yields $m^*(O_1 \cap O_2) \leq 2\varepsilon$. Note that $O_1 - E_0 \subseteq (O_1 \cap O_2) \cup ((O_1 \cup O_2) - E)$ which gives

$$m^*(O_1 - E_0) \leq m^*(O_1 \cap O_2) + m^*(O - E)$$

which are each dominated by some factor of ε . Hence, E_0 is measurable.

□

1.5 A non measurable set in $[0, 1]$

Define the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Let ϵ_α be the equivalence classes, so that $[0, 1] = \cup_\alpha \epsilon_\alpha$. Pick $x_\alpha \in \epsilon_\alpha$ (by axiom of choice). Take these x_α to make a set \mathcal{N} .

Let $\{r_k\}$ be an enumeration of rationals in $[-1, 1]$. Define $N_k = N + r_k$ (translate). Note that, if $N_l \cap N_k \neq \emptyset$, then $N + r_l \cap N + r_k \neq \emptyset$ or there exists $x = x_\alpha + r_l = x_\beta + r_k$, but that means $x_\alpha \sim x_\beta$, only possible if $\alpha = \beta$ but contradiction to the assumption that $l \neq k$, so for different rationals, N_k is disjoint to N_j . Note that $[0, 1] \subseteq \cup_{i=1}^\infty N_i \subseteq [-1, 2]$. Assuming N is measurable, it tells us N_k each are measurable, and yields $1 \leq \sum_{i=1}^\infty m(N_i) \leq 4$ but $m(N_k) = m(N)$, which gives us a contradiction.

This set is called the **Vitali Set**.

2 Measurable Functions

Definition 2.1. (Measurable Function) A function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is said to be *measurable* if $f^{-1}([-\infty, a))$ is measurable in \mathbb{R}^d .

Remark 2.2. Note that, in the above definition, it is easy to see that it is equivalent to saying any function is Lebesgue measurable if it inverse maps borel sets to lebesgue sets.

A borel measurable function on the other hand, inverse maps borel sets to borel sets.

Definition 2.3. (Characteristic function of a set $E \subseteq \mathbb{R}^d$)

$$\chi_E(x) := \begin{cases} 1 & : x \in E \\ 0 & : \text{else} \end{cases}$$

We now define the elementary functions that make up Riemann theory:

Definition 2.4. (Step function) f is a step function if there exists $N \in \mathbb{N}$, real numbers $\{a_i\}_1^N$ and N rectangles R_i such that $f(x) = \sum_{i=1}^N a_i \chi_{R_i}(x)$

The elementary functions that make up lebesgue theory on the other hand, are:

Definition 2.5. (Simple functions) f is a simple function if there exists $N \in \mathbb{N}$, N reals $\{a_i\}$ and N measurable sets $\{E_i\}$ such that $f = \sum_{i=1}^N a_i \chi_{E_i}(x)$

Theorem 2.6. *Equivalent criteria for measurability of a function:*

1. $f^{-1}([-\infty, a))$ is measurable
2. $f^{-1}([-\infty, a])$ is measurable
3. $f^{-1}(a, \infty]$ is measurable
4. $f^{-1}[a, \infty]$ is measurable

Proof. These are rather trivial, given that $[-\infty, a] = \bigcap_{r_k} [-\infty, a + r_k)$ and $f^{-1}(I^c) = (f^{-1}(I))^c$

□

Lemma 2.7. *If f is measurable, then $-f$ is measurable.*

Theorem 2.8. *If $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are measurable, then f is measurable if and only if $f^{-1}(O)$, for all open sets $O \subset \mathbb{R} \cup \{-\infty, \infty\}$ (order topology).*

Proof. \implies) Let f be measurable. Then $f^{-1}(a, b)$ is measurable, and $f^{-1}[-\infty, a)$ and $f^{-1}(a, \infty]$ is also measurable for all a . The basis sets are inverse mapped to measurable sets.

\impliedby) Let $f^{-1}(O)$ be measurable for all open sets O . This means $f^{-1}[-\infty, a)$ is measurable for all finite a , as is $f^{-1}[-\infty, a]$. For $f^{-1}[-\infty, \infty) = f^{-1}(\bigcup_i [-\infty, i])$

□

Theorem 2.9. *If f is continuous, then f is measurable. If f is finite valued and measurable, and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $\phi \circ f$ is measurable.*

Proof. (1) Is obvious

$(\phi \circ f)^{-1}(-\infty, a) = f^{-1}(\text{an open set})$ which is measurable.

□

Definition 2.10. (Almost everywhere) We say a property $P(x)$ holds almost everywhere on a measure space $x \in \Omega$ if the set $E \subseteq \Omega := \{x \in \Omega : P(x) \text{ is not true}\}$ is a set of measure 0.

For example, we say $f \equiv g$ a.e. if $\{x : f(x) \neq g(x)\}$ is a set of measure 0.

Theorem 2.11. Let $f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be a sequence of measurable functions defined on a measurable set E . Then the following functions are measurable:

1. $\sup_n f_n(x)$
2. $\inf_n f_n(x)$
3. $\limsup_n f_n(x)$
4. $\liminf_n f_n(x)$

Proof. (1) Note that $\{x : \sup_n f_n(x) > a\}$ is the same as $\cup_n \{x : f_n(x) > a\}$. To see this, note that if x is such that $\sup_n f_n(x) > a$, that means there is some n so that $f_n(x) > a$. The other direction is just as trivial.

(2) $\inf_n f_n(x) = -\sup_n -f_n(x)$.

(3) and (4) follow from $\limsup_n f_n(x) := \inf_{n \in \mathbb{N}} (\sup_{i \geq n+1} f_i(x))$ and $\liminf_n f_n(x) := \sup_{n \in \mathbb{N}} (\inf_{i \geq n+1} f_i(x))$

□

Theorem 2.12. If $\{f_n : E \rightarrow \mathbb{R} \cup \{\pm\infty\}\}$ is a sequence of measurable functions such that E is measurable and $f_n \rightarrow f$ pointwise, then f is measurable.

Proof. $f(x) = \lim_{n \rightarrow \infty} f_n(x) = \limsup_n f_n(x) = \liminf_n f_n(x)$.

□

Theorem 2.13. If f and g are measurable, then $f^k(x) := f(x)^k$ is measurable, $f + g$ is measurable, and fg is measurable.

Proof. If k is odd, $\{f^k > a\} = \{f > a^{1/k}\}$. If k is even, $\{f^k > a\}$ for negative a is just whole reals, so we take positive a , $\{f^k > a\} = \{f > a^{1/k}\} \cup \{f < -a^{1/k}\}$. Measurable.

$\{f + g > a\} = \cup_{r_k} \{f > r_k\} \cap \{g > a - r_k\}$ for rationals r_k . Proof is that, if $f + g > a$, then $f + g > r_k > a$, which yields the result one side. $f > r_k$ and $g > a - r_k$ trivially gives the equality.

$$fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$$

□

Theorem 2.14. *Let g be a measurable function, and*

$$f \equiv g \text{ a.e.}$$

Then f is measurable.

Proof. $\{f < a\} = \{f < a : f \equiv g\} \cup \{f < a : f \not\equiv g\}$. Note that $\{f < a : f \not\equiv g\}$ is a measure 0 set. So the result holds.

□

2.1 Approximating Measurable Functions

Theorem 2.15. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be measurable and non negative. There exists a sequence of simple functions φ_k such that $\varphi_k \leq \varphi_{k+1}$ and*

$$\varphi_k \rightarrow f \text{ pointwise}$$

Proof. Some commentary:

We would like to subdivide the range of f and inverse map these to get measurable sets on \mathbb{R}^d , with which we can then make characteristic functions and then simple functions, by taking a sum of the kind:

$$\sum_{i=1}^K \inf\{f(x) : x \in E_i\} \chi_{E_i}$$

where E_i is one of the measurable sets we get from inverse maps.

But the issue with this is that, the range may not be a bounded set. To mitigate this issue, we create a sequence of bounded functions that converge pointwise to f , and then approximate these functions, and take the diagonal sequence.

End of commentary.

Let N be a natural number and Q_N be the N sided cube centered at origin.

$$F_N := \begin{cases} f(x) & : x \in Q_N, f(x) < N \\ N & : x \in Q_N, f(x) \geq N \\ 0 & \text{else} \end{cases}$$

$F_N \rightarrow f$ pointwise as $N \rightarrow \infty$. The range of F_N is $[0, N]$. For $k \in \mathbb{N}$, subdivide $[0, N]$ into $2^k N$ parts, i.e, subintervals $[0, 1/2^k], [1/2^k, 2/2^k], \dots [l/2^k, (l+1)/2^k] \dots [N - 1/2^k, N]$. Now take the inverse image of each of these, to get measurable sets

$$E_{l,N,2^k} = F_N^{-1}[l/2^k, (l+1)/2^k]$$

Define

$$\phi_{N,2^k} = \sum_l \frac{l}{2^k} \chi_{E_{l,N,2^k}} = \sum_l \frac{l}{2^k} \chi_{F_N^{-1}[(l/2^k, (l+1)/2^k]}$$

Note that, $|F_N(x) - \phi_{N,2^k}(x)| \leq 1/2^k$ for all x . So $\phi_{N,2^k} \rightarrow F_N$ (for fixed N) uniformly.

The sequence $\phi_{2^k,2^k}$ ($N = 2^k$) is monotonic non decreasing by virtue of the subdivision becoming finer, and $\phi_{2^k} := \phi_{2^k,2^k} \rightarrow f$ pointwise. Hence, we are done.

□

Remark 2.16. (**Previous result strenghtens to "uniform convergence" if f were bounded**) Note that, in the previous case, if f were bounded, we would get that the convergence $\phi_k \rightarrow f$ would be uniform, from the fact that $\phi_{N,2^k} \rightarrow F_N$ was uniform. We needn't have to "pass" to a sequence of bounded functions that approximate f in the first place.

Theorem 2.17. *If f is measurable on \mathbb{R}^d , there exists a sequence of simple functions ϕ_k such that $\phi_k \rightarrow f$ pointwise and $|\phi_k| \leq |\phi_{k+1}|$.*

Proof. Let $f^+(x) := \max(f(x), 0)$ and $f^-(x) := \max(-f(x), 0)$. Note that $f \equiv f^+ - f^-$. Both f^+ and f^- are non decreasing. We can make

$$F_N^+ := \begin{cases} f^+(x) & : x \in Q_N, f^+(x) < N \\ N & : x \in Q_N, f^+(x) \geq N \\ 0 & \text{else} \end{cases}$$

and

$$F_N^- := \begin{cases} f^-(x) & : x \in Q_N, f^-(x) < N \\ N & : x \in Q_N, f^-(x) \geq N \\ 0 & \text{else} \end{cases}$$

We note the following property of f^+ and f^- , which is that, whenever f^+ is positive, f^- is 0, and likewise whenever f^- is positive, f^+ is 0. Likewise, since F_N^+ and F_N^- are just constraints on f^+ and f^- , they too obey the same property.

We then define

$$\phi_{N,2^k}^+(x) = \sum_l l/2^k \chi_{(F_N^+)^{-1}(l/2^k, l+1/2^k]}(x)$$

and

$$\phi_{N,2^k}^-(x) = \sum_l l/2^k \chi_{(F_N^-)^{-1}(l/2^k, l+1/2^k]}(x)$$

$\phi_{N,2^k}^\pm \rightarrow F_N^\pm$ respectively (uniform). Note that, if $\phi_{N,2^k}^+$ is positive, that means that for some l , $F_N^+ \in (l/2^k]$, implying that F_N^- is zero. Therefore, the same property that f follows, is followed by $\phi_{N,2^k}^+$ and $\phi_{N,2^k}^-$. For such functions, note that $|f| = f^+ + f^-$. Therefore, $|\phi_k| = |\phi_{2^k,2^k}| = |\phi_{2^k,2^k}^+ - \phi_{2^k,2^k}^-| = \phi_{2^k,2^k}^+ + \phi_{2^k,2^k}^-$. We are done.

□

Theorem 2.18. *Let f be a measurable function on \mathbb{R}^d . There exists a sequence of **step functions** ϕ_k such that $\phi_k \rightarrow f$ a.e.*

Proof. It suffices to show that χ_E for a measurable set E of finite measure can be approximated by step functions, from the previous theorem (We took the simple function's characteristics to be from finite measure sets). For a given k , there exists $F_k = \cup_{i=1}^{n_k} Q_{i,k}$ a closed set that is a union of almost disjoint closed cubes $Q_{i,k}$ such that $m(E \Delta F_k) \leq 1/2(2^k)$. Take smaller cubes $S_{i,k} \subset Q_{i,k}$ so that they are closed, of very close volume to $Q_{i,k}$ but strictly inside $Q_{i,k}$. In such a case, we then have a disjoint collection of finite closed cubes such that $m(E \Delta G_k) \leq 1/2^k$. We then consider

$$\phi_k := \chi_{G_k} = \sum_{i,k} \chi_{S_{i,k}}$$

This function is the same as χ_E everywhere except a set $E \Delta G_k$ whose measure is $m(E \Delta G_k) < 1/2^k$. Let $F_k = \cup_{k+1}^{\infty} E \Delta G_i$ and $F = \cap_k F_k$. Note that $m(F) \leq m(F_K) \leq \sum_{K+1}^{\infty} 1/2^k \leq 1/2^K$. So $m(F) = 0$. On the complement of F , i.e, $\cup_k F_k^c$ which is $\cup_k (\cap_{k+1}^{\infty} (E \Delta G_i)^c)$ on which convergence happens.

□

2.2 Littlewood's Three Principles

1. Every measurable set is "almost" a union of finite intervals.
2. Every measurable function is "almost" continuous.
3. Every sequence of measurable functions that converges pointwise, "almost" does so uniformly.

The precise formulation of these are:

Theorem 2.19. *Given $\varepsilon > 0$ and a measurable set E of finite measure, we can find $F = \cup_{i=1}^K Q_i$ a union of closed cubes such that $m(E \Delta F) \leq \varepsilon$*

Theorem 2.20. (Egorov) *Let f_k be a sequence of measurable functions defined on a measurable set E ($m(E) < \infty$), converging to f pointwise. Given $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset E$ such that $m(E - A_\varepsilon) < \varepsilon$ and $f_k \rightarrow f$ uniformly on A_ε .*

Proof. Define

$$E_k^n := \{x \in E : |f_j(x) - f(x)| < 1/n \ \forall j \geq k\}$$

These sets are measurable (since $|\cdot|$ is measurable and differences are measurable). Note that for a fixed n , $E_k^n \subseteq E_{k+1}^n$ and these sets ultimately converge to E itself since

for any x in E , and for a given $1/n$ there exists k such that $|f_j(x) - f(x)| < 1/n$ for all $j > k$. From limit exchange, $m(E) = \lim_{k \rightarrow \infty} m(E_k^n) < \infty$. Therefore, for any n , there exists n_k so that $m(E - E_{k_n}^n) = m(E) - m(E_{k_n}^n) < 1/2^n$. We then have,

$$|f_j(x) - f(x)| < 1/n \text{ whenever } j > k_n \text{ and } x \in E_{k_n}^n$$

Let N be so that $\sum_N^\infty 1/2^n \leq \varepsilon/2$ (for some given ε). Consider

$$F_\varepsilon = \cap_N^\infty E_{k_n}^n$$

$$E - F_\varepsilon \subseteq \cup_N^\infty E - E_{k_n}^n \text{ which gives}$$

$$m(E - F_\varepsilon) \leq \varepsilon/2$$

Let $\delta > 0$, choose $n \geq N$ so that $1/n \leq \delta$. Suppose $x \in F_\varepsilon$. This implies $x \in \cap_N^\infty E_{k_n}^n$ which means that

$$|f_j(x) - f(x)| < 1/n \leq \delta \text{ whenever } j \geq k_n$$

which means that we have uniformity on F_ε . Now we pass to a smaller closed set $A_\varepsilon \subseteq F_\varepsilon$ so that $m(F_\varepsilon - A_\varepsilon) \leq \varepsilon$, and on A_ε convergence is uniform.

□

Theorem 2.21. (Lusin) *Let $f : E \rightarrow \mathbb{R}$ be a finite valued, measurable function defined on a set of finite measure E . For all $\varepsilon > 0$ there exists a closed set A_ε so that $m(E - A_\varepsilon) < \varepsilon$ and $f|_{A_\varepsilon}$ is continuous.*

Proof. Note that, this does not state that f is continuous on the set A_ε but simply that when f is viewed as a function on A_ε , it is continuous.

Let f_n be a sequence of step functions such that $f_n \rightarrow f$ pointwise a.e. We shall assume that the convergence happens on E itself, since the remainder set is of measure 0 anyways. For f_n , we can remove a set E_n from the domain to make f_n continuous (This can be done since step functions, defined on \mathbb{R}^d , are discontinuous on the $d - 1$ dimensional subspaces that are the boundaries of each rectangle on which they are defined. We can remove a finite number of small rectangles containing each of these subspaces) such that $m(E_n) \leq 1/2^n$. Let N be such that $\sum_N^\infty 1/2^n \leq \varepsilon/2$ (for some given ε). We look at

$$F_N = \cup_N^\infty E_n \text{ such that } m(E - F_N) = m(E - \cup_N^\infty E_n) = m(E) - m(\cup_N^\infty E_n) \geq m(E) - \varepsilon/2$$

Let $G_N = E - F_N$, our set of consideration for a given ε . We then have $m(E - G_N) \leq \varepsilon/2$, and on G_N , which is $E - F_N$, all f_n after N are continuous. From Egorov, we have a set $A_{\varepsilon/2} \subset G_N$ so that $m(G_N - A_{\varepsilon/2}) \leq \varepsilon/2$ and f_n beyond N are uniformly

convergent to f . $m(E - A_{\varepsilon/2}) = m(E - G_N) + m(G_N - A_{\varepsilon/2}) \leq \varepsilon$. So for a given $\varepsilon > 0$, we have a closed set $A_{\varepsilon/2}$ so that f_n beyond a certain N_ε converges uniformly on $A_{\varepsilon/2}$, and f_n are all continuous on $A_{\varepsilon/2}$. Therefore, on $A_{\varepsilon/2}$ the uniform limit f of the sequence of continuous functions is continuous on A_ε . (See next Lemma)

□

Lemma 2.22. (Throwaway) *If $f_n : (X, d) \rightarrow \mathbb{R}$ is a sequence of continuous functions such that $f_n \rightarrow f$ uniformly, then f is continuous.*

Proof. Let $x \in X$ and $\varepsilon > 0$. We want a $\delta > 0$ such that $|f(x) - f(z)| < \varepsilon$ whenever $z \in B_\delta(x)$. Note that $|f_n(x) - f(x)| < \varepsilon/3$ and $|f_n(z) - f(z)| < \varepsilon/3$ whenever $n \geq N_\varepsilon$, and there is a δ_0 so that $|f_n(x) - f_n(z)| < \varepsilon/3$ whenever $z \in B_{\delta_0}(x)$. Consider $|f(x) - f(z)| = |f(x) - f_n(x) + f_n(x) - f_n(z) + f_n(z) - f(z)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(z)| + |f_n(z) - f(z)| \leq \varepsilon$ whenever $n \geq N_\varepsilon$, and $z \in B_{\delta_0}(x)$. We have found our δ . Hence, f is continuous.

□

Lemma 2.23. (Borel-Cantelli Lemma) *Let E_1, E_2, \dots be a collection of measurable sets such that*

$$\sum_{i=1}^{\infty} m(E_i) < \infty$$

then $\limsup(E_i) := \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} E_i$ is measurable and is of measure 0.

Proof. We know that $m(E_i) \rightarrow 0$ from convergence of the series. Since the series converges, we can make $\sum_{i=N+1}^{\infty} m(E_i)$ arbitrarily small for large enough N . Therefore, for a given ε , there exists N such that

$$m(\bigcup_{i=N+1}^{\infty} E_i) \leq \sum_{i=N+1}^{\infty} m(E_i) \leq \varepsilon$$

Since the limsup is smaller than the left side above for every N , we have that $m(\limsup(E_i)) = 0$.

□

2.3 Miscellaneous Theorems

Theorem 2.24. *Let $E \subseteq \mathbb{R}^d$ and O_n be the open set*

$$O_n = \{x \in \mathbb{R}^d : d(x, E) < 1/n\}$$

If E were compact, then

$$\lim_{n \rightarrow \infty} m(O_n) = m(E)$$

But there exists counter examples when:

1. E is closed and unbounded
2. E is open and bounded

Proof. Note that $O_{n+1} \subseteq O_n \cdots$ a decreasing sequence. If E were compact, all these sets have finite measure. So,

$$m(\cap_n O_n) = \lim_{n \rightarrow \infty} m(O_n)$$

But note that $\cap_n O_n = \{x : d(x, E) = 0\}$ which is $cls(E)$, which is in this case E itself. Therefore,

$$\lim_{n \rightarrow \infty} m(O_n) = m(E)$$

For the counter examples:

1. Consider $\mathbb{R} \subseteq \mathbb{R}^2$, whose closure is itself. Note that $m(\mathbb{R}) = 0$ in \mathbb{R}^2 . But $O_n := \{x : d(x, \mathbb{R}) < 1/n\}$ is an open set (its a strip) whose measure is ∞ . This is true for all O_n , so $\lim_{n \rightarrow \infty} m(O_n) = \infty$ but $m(\mathbb{R}) = 0$.
2. For the 2nd counter example: Let \hat{C} be the fat cantor set, i.e, the cantor set created by removing intervals of size $l_0, l_1 \cdots$ such that $1 - (l_0 + 2l_1 + 2^2l_2 \cdots) > 0$. Consider the union of all the deleted neighbourhoods. Note that, the fat cantor set is perfect, i.e, every point of \hat{C} is a limit point of \hat{C} , so for $x \in \hat{C}$ there exists a monotonic sequence $x_n \in \hat{C}$ that converges to $x \in \hat{C}$. We also know \hat{C} is totally disconnected, so between any two elements of \hat{C} , there is an element of \hat{C}^c , which is an element of the union of the deleted intervals. Hence, every element of \hat{C} is a limit point of the union of the deleted neighbourhoods. Let U denote this union of all deleted neighbourhoods at each step. We note that $cls(U) = [0, 1]$. $m(cls(U)) = m(\cap_n O_n) = 1$, but $m(U) < 1$ (because fat cantor set is of finite measure).

□

Example 2.25. (There exists a sequence of decreasing, continuous functions whose pointwise limit is not Riemann integrable) Let \hat{C} denote the fat cantor set. Consider F_0 to be 1 on all of $[0, 1]$. Let F_1 to be 1 on C_1 (2 intervals each of size $1 - l_0/2$), 0 at the centre of the deleted neighbourhood, and linear, connecting 0 and 1 on the deleted neighbourhood. Likewise, make F_2 so that it is 1 on C_2 , and 0 on the centres of each deleted neighbourhoods, and is linear on these deleted neighbourhoods. We then define $f_n = F_1 \cdot F_2 \cdot \cdots \cdot F_n$, which is a sequence of decreasing functions. Note that, if $x \in \hat{C}$, there exists large enough n such that $f_n = 1$ on x . Therefore, on \hat{C} , the pointwise limit (which exists since each function

is bounded) is 1. We note that f is discontinuous at every point of \hat{C} , since there exists a sequence x_n outside \hat{C} such that $f(x_n) \rightarrow 0$. Since the set of discontinuities is uncountable, the function f is not riemann integrable (See appendix).

Theorem 2.26. *Let f_n be a sequence of functions on $[0, 1]$ such that*

$$|f(x)| < \infty \text{ a.e.}$$

then there exists a sequence c_n of real numbers such that

$$f_n/c_n \rightarrow 0 \text{ a.e.}$$

Proof. Consider

$$E_n^k := \{x \in [0, 1] : |f_n(x)| > k/n\}$$

For increasing k this sequence of sets is decreasing, and $\cap_k E_n^k = \{x : |f_n(x)| = \infty\}$. Also, E_n^k are measurable and of finite measure since they're subsets of $[0, 1]$. Hence, $\lim_{k \rightarrow \infty} m(E_n^k) = m(\cap_k E_n^k) = 0$. This means that, for a given j , there exists n_0 such that

$$m(E_n^k) < 2^{-j} \quad \forall k \geq n_0$$

or simply, there exists, for every j , a k_j such that

$$m(E_j^{k_j}) = m(\{x \in [0, 1] : |f_j(x)/k_j| > 1/j\}) < 2^{-j}$$

Which means $\sum_j m(E_j^{k_j}) < \infty$, which means from Borel-Cantelli,

$$m(\limsup_j E_j^{k_j}) = m(\cap_{n \in \mathbb{N}} (\cup_{i \geq n} E_i^{k_i})) = m(\cap_{n \in \mathbb{N}} (\cup_{i \geq n} \{x \in [0, 1] : |f_i(x)/k_i| > 1/i\})) = 0$$

This means that the set $R = \cup_{n \in \mathbb{N}} (\cap_{i \geq n} \{x : |f_i(x)/k_i| \leq 1/i\})$ is a set of full measure, whose complement is null. If x is in R , $|f_i(x)/k_i| \rightarrow 0$ eventually. Thus, a.e.

$$f_i(x)/k_i \rightarrow 0 \text{ a.e.}$$

□

Theorem 2.27. *Every measurable function is an a.e. limit of a sequence of continuous functions.*

Proof. Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Let f_n be a sequence of step functions such that

$$f_n \rightarrow f \text{ pointwise a.e.}$$

Look at χ_R for a rectangle R . In $1-d$, it is easy to construct a sequence of continuous functions that converge to χ_R ; Let $R = \prod_{i=1}^d [a_i, b_i]$. For the 1-d case, say on $[a_i, b_i]$, we

can construct the function:

$$\phi_{i,n}(x) := \begin{cases} 0 & : x \notin [a_i - 1/n, b_i + 1/n] \\ 1 & : x \in [a_i, b_i] \\ n(x - a_i + 1/n) & : x \in [a_i - 1/n, a_i] \\ -n(x - b_i - 1/n) & : x \in [b_i, b_i + 1/n] \end{cases}$$

This function is continuous, and converges to $\chi_{[a_i, b_i]}$ pointwise, on the real line. Consider $f_n : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\pm\infty\}$ $f_n(x_1, x_2, \dots, x_d) = \prod_{i=1}^d \phi_{i,n}(x_i)$. Does f_n approximate χ_R well? Let $\vec{x} = (x_1, x_2, \dots, x_d) \in R$. That means $a_i \leq x_i \leq b_i$. Inside R , $\chi_R(x) = 1$, which, of course, f_n is. Outside R , χ_R is 0. Let $\vec{x} = (x_1, x_2, \dots, x_d) \in R^c$

□

3 Integration

3.1 Simple Functions

Definition 3.1. (Canonical form of a simple function) We say ϕ , a simple function, is in canonical form if

$$\phi = \sum_{i=1}^N a_i \chi_{E_i}$$

where $\{a_i\}$ are distinct constants and $\{E_i\}$ are disjoint.

Lemma 3.2. *Canonical form for a function always exists, and is unique.*

Proof. Let c_1, c_2, \dots, c_r be the finite values ϕ takes. Let $E_i := \{x \in \mathbb{R}^d : \phi(x) = c_i\}$. We then can conclude $\phi = \sum_{i=1}^N c_i \chi_{E_i}$.

As for uniqueness, if a function were to have two different canonical forms, it is easy to see why they would necessarily have to disagree on some point on \mathbb{R}^d .

□

Definition 3.3. (Lebesgue integral of a simple function) If φ is a simple function in canonical form, then

$$\int_{\mathbb{R}^d} \varphi = \int_{\mathbb{R}^d} \left(\sum_{i=1}^N a_i \chi_{E_i} \right) := \sum_{i=1}^N a_i m(E_i)$$

Theorem 3.4. *If φ is a simple function, the following holds:*

1. *If $\varphi = \sum_{i=1}^N a_i \chi_{E_i}$, then $\int \varphi = \sum_{i=1}^N a_i m(E_i)$ regardless of whether the representation is canonical or not.*

2. If φ and ψ are two simple functions, then $\int a\varphi + b\psi = a \int \varphi + b \int \psi$.
3. If $f \leq g$ where f and g are simple functions, then $\int f \leq \int g$.
4. $|\int f| \leq \int |f|$ for simple function f .
5. If E and F are disjoint measurable sets, then $\int_{E \cup F} f = \int_E f + \int_F f$.

Proof. (1) First, let us assume that E_i are all disjoint, but a_i may not be distinct in $\phi = \sum_{i=1}^N a_i \chi_{E_i}$. Let $E_0 := \{a_n : n \leq N\}$ as a set (distinct elements). Consider $a \in E_0$, and let E_a be $\cup E_i$ such that, that particular a corresponds to E_i . We then see that $\sum_{a \in E_0} a E_a$ is the canonical form of ϕ , and $\int \phi = \sum_{a \in E_0} a m(E_a) = \sum_{a \in E_0} a (\sum_{\text{times } a \text{ occurs as } a_i} m(E_i))$ which clearly is $\sum_{i=1}^N a_i m(E_i)$.

Now, let us assume that E_i need not be disjoint. We can find disjoint E'_j -s such that $\cup_{j=1}^M E'_j = \cup_{i=1}^N E_i$, and $E_i = \cup_{j: E'_j \subset E_i} E'_j$. Let $a'_j := \sum_{k: E'_j \subset E_k} a_k$. Therefore, we can see that $\phi = \sum_j a'_j \chi_{E'_j}$ (Since $\sum_{j=1}^M a'_j \chi_{E'_j} = \sum_{j=1}^M (\sum_{k: E'_j \subset E_k} a_k) \chi_{E'_j}$ where E'_j are disjoint. $\int \phi = \sum_{i=1}^M \sum_{(k: E'_j \subset E_k)} a_k m(E'_j) = \sum_{i=1}^N a_i m(E_i)$).

(2) If $f = \sum_i a_i \chi_{E_i}$ and $g = \sum_i b_i \chi_{F_i}$, then we note that $af + bg$ is also a simple function, whence this property falls immediately.

(3) If $f \leq g$, $g - f \geq 0$ and $g - f$ is a simple function which is nonnegative, which means their canonical forms have nonnegative coefficients. Hence, $\int(g - f) = \int g - \int f \geq 0$

(4) Let $f = \sum_i a_i \chi_{E_i}$ in canonical form. Hence, $|f| = \sum_i |a_i| \chi_{E_i}$. $|\int f| = |\sum_i a_i m(E_i)| \leq \sum_i |a_i| m(E_i) = \int |f|$

(5) $\int_{E \cup F} f$ for disjoint E and F is $\int f \chi_{E \cup F} = \int f \chi_E + \int f \chi_F$ which yields the result.

□

Lemma 3.5. If f and g agree a.e. and they are simple functions, then $\int f = \int g$.

Proof. To see this, note that f and g possibly disagree on a set E of measure 0. Which means $f - g$ is zero almost everywhere. i.e, $f - g = \sum_{i=1}^N a_i \chi_{E_i}$ where each E_i are disjoint and measure 0. Therefore, $\int f - g = 0$ or $\int f = \int g$.

□

3.2 Bounded functions supported by sets of finite measure

Definition 3.6. (Support of a measurable function f)

$$\text{supp}(f) := \{x \in \mathbb{R}^d : f(x) \neq 0\}$$

Since f is measurable, $f^{-1}(\mathbb{R} \cup \{\pm\infty\} - \{0\})$, or rather, the support, is also measurable.

Theorem 3.7. *If f is a bounded function (bounded by M) which is finitely supported by E , then there exists a sequence of simple functions ϕ_k , each of which is bounded by M , and is supported in E , and*

$$\phi_k \rightarrow f$$

Proof. $\phi_k := \sum_l l/2^k \chi_{f^{-1}(l/2^k, l+1/2^k]}$ from an earlier proof. Note that there, we could generate a sequence of functions such that $|\phi_k| \leq |\phi_{k+1}|$ and $\phi_k \rightarrow f$ pointwise. Therefore, all ϕ_k are uniformly bounded by M and converge pointwise to f . Note that outside E , f is 0, and $|\phi_k| \leq |\phi_{k+1}| \cdots$, and on $x \in E^c$, $|\phi_k(x)| \rightarrow 0$ which means that $|\phi_k(x)| = 0$, therefore, ϕ_k are supported inside E .

□

Lemma 3.8. *Let f be a function that is finitely supported on E and bounded by M . Let ϕ_k be a sequence of simple functions, finitely supported on E and uniformly bounded by N , and that*

$$\phi_k(x) \rightarrow f(x) \text{ a.e.}$$

then,

1. $\lim_{n \rightarrow \infty} \int \phi_n$ exists
2. If $f \equiv 0$ a.e. then $\lim_{n \rightarrow \infty} \int \phi_k = 0$

Proof. From Egorov, for a given ε , there exists $A_\varepsilon \subseteq E$ such that $m(E - A_\varepsilon) < \varepsilon$ and on A_ε , $\phi \rightarrow f$ is uniform. $|\int \phi_n - \phi_m| \leq \int_{A_\varepsilon} |\phi_n - \phi_m| + \int_{E-A_\varepsilon} |\phi_n - \phi_m| \leq \int_{A_\varepsilon} |\phi_n - \phi_m| + 2M\varepsilon$ (since $|\phi_m - \phi_n|$ is bounded by $2M$). This is in turn less than

$$m(E)\varepsilon + 2M\varepsilon$$

for large enough m, n since convergence is uniform in A_ε . Hence, we are done.

For the 2nd part, note that $|\int \phi_k| \leq \int |\phi_k| = \int_{A_\varepsilon} |\phi_k| + \int_{E-A_\varepsilon} |\phi_k|$ which is less than or equal to $\varepsilon m(E) + 2M\varepsilon$, but since $m(E) = 0$, we have the requisite result.

□

Definition 3.9. (Lebesgue integral for bonded funtions on finite measure support)

Let f be bounded by M and $m(\text{supp}(f) = E) < \infty$. If ϕ_k is a sequence of simple functions, bounded by M , supported on E , and $\phi_k \rightarrow f$ a.e. Then

$$\int f := \lim_{n \rightarrow \infty} \int \phi_n$$

Lemma 3.10. *This definition makes sense, i.e. if ϕ_i and ψ_i are two sequences of simple functions converging to f , supported on E and bounded by M , then*

$$\lim_{n \rightarrow \infty} \int \phi_n = \lim_{n \rightarrow \infty} \int \psi_n$$

Proof. We know $\phi_n \rightarrow f$ and $\psi_n \rightarrow f$, which means $\phi_n - \psi_n \rightarrow 0$, and $\phi_n - \psi_n$ is a sequence of simple functions bounded by $2M$, and converges to 0 almost everywhere. $\varepsilon > 0$ gives $\int_{A_\varepsilon} |\phi_n - \psi_n| + \int_{E-A_\varepsilon} |\phi_n - \psi_n| \leq \varepsilon m(E) + 2M\varepsilon$ (from Egorov and uniform convergence) hence $\lim_{n \rightarrow \infty} \int |\phi_n - \psi_n| = 0$ or

$$\lim_{n \rightarrow \infty} \int \phi_n = \lim_{n \rightarrow \infty} \int \psi_n$$

□

Theorem 3.11. *Suppose f and g are bounded functions supported by a set of finite measure, then the following hold:*

1. (**Linearity**) $\int af + bg = a \int f + b \int g$
2. (**Additivity**) If E and F are disjoint, then $\int_{E \cup F} f = \int_E f + \int_F f$
3. (**Monotonicity**) $\int f \leq \int g$ whenever $f \leq g$
4. (**Triangle**) $|\int f| \leq \int |f|$

Proof. (1) $f = \lim_{n \rightarrow \infty} \phi_n$ and $g = \lim_{n \rightarrow \infty} \psi_n$, and $af + bg = \lim_{n \rightarrow \infty} (a\phi_n + b\psi_n)$, where $a\phi_n + b\psi_n$ is bounded, and is supported on E . Hence, $\int af + bg = \lim_{n \rightarrow \infty} \int a\phi_n + b\psi_n = a \int f + b \int g$

(2) E and F are disjoint, so $\chi_{E \cup F} = \chi_E + \chi_F$, which would yield the result.

(3) If $f \leq g$, then $g - f \geq 0$, so the sequence of simple functions converging to $g - f$, would also be each non-negative, which means $\lim_{n \rightarrow \infty} \phi_n \geq 0$ which yields the result.

(4) Let $\phi_n \rightarrow f$ be a sequence of simple functions uniformly bounded and supported on the support of f . $|\lim_{n \rightarrow \infty} \int \phi_n| = \lim_{n \rightarrow \infty} |\int \phi_n| \leq \lim_{n \rightarrow \infty} \int |\phi_n|$ (notice that in the canonical form of $\phi_n \rightarrow f$, we get $|\phi_n| \rightarrow |f|$) which yields $|\int f| \leq \int |f|$.

□

Theorem 3.12. (Bounded Convergence) Let f_n be a sequence of functions uniformly bounded by M , and each supported on a set E of finite measure. Suppose $f_n \rightarrow f$ a.e (pointwise), f is itself bounded by M , is supported on E and

$$\int |f_n - f| \xrightarrow{n \rightarrow \infty} 0$$

Proof. Let $\varepsilon > 0$. Consider A_ε from Egorov such that $f_n \rightarrow f$ and $m(E - A_\varepsilon) < \varepsilon$. Look at $\int |f_n - f| = \int_{A_\varepsilon} |f_n - f| + \int_{E - A_\varepsilon} |f_n - f| \leq \int_{A_\varepsilon} |f_n - f| + 2M\varepsilon$, and for large enough n (since inside A_ε convergence is uniform) we have $|f_n - f| < \varepsilon$ for large enough n for all x (uniformly) which gives us $\int |f_n - f| < \varepsilon$ for large enough N , which means $\int |f_n - f| \rightarrow 0$. In other words: For a convergent sequence of uniformly bounded functions on the same support of finite measure,

$$\int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n$$

□

Lemma 3.13. If $f \geq 0$, is bounded with its support E being of finite measure and $\int f = 0$, then $f = 0$ almost everywhere

Proof. Consider $E_k := \{x : f(x) \geq 1/k\}$. Note that $1/k \chi_{E_k} \leq f \implies 1/k m(E_k) \leq \int f = 0$. For every k , $m(E_k) = 0$, and this sequence is increasing. Therefore, $\cup_n E_n = \{x : f(x) > 0\}$ is of measure 0. Hence, $f = 0$ almost everywhere.

□

3.3 S Appendix: Riemann Integrable functions

Definition 3.14. (Partition) A partition P of an interval $[a, b]$ is a collection of points $t_0, t_1, t_2, \dots, t_{P_k}$ such that $a = t_0 < t_1 < t_2 < \dots < t_{P_k-1} < t_{P_k} = b$

Given a partition, we define I_j , the j th interval of the partition as $[t_{j-1}, t_j]$

Definition 3.15. (Upper and Lower Riemann Sum, given a Partition) Let f be a **Bounded** function defined on a closed and bounded interval $[a, b]$, and P be a given partition of $[a, b]$. We define

$$\mathcal{L}(P, f) := \sum_{j=1}^{P_k} \inf_{x \in I_j} \{f(x)\} |I_j|$$

and

$$\mathcal{U}(P, f) := \sum_{j=1}^{P_k} \sup_{x \in I_j} \{f(x)\} |I_j|$$

Definition 3.16. (P' , the refinement of P) Let P be a partition, $a = t_0 < t_1 < t_2 \cdots t_{P_k-1} < t_{P_k} = b$. We say P' is a refinement if, $P' \subseteq P$ in the sense that $P' = a = u_0 < u_1 < \cdots u_{P'_k-1} < u_{P'_k} = b$ where a subsequence of u_j is t_j

Definition 3.17. (Riemann Integrability) We say a bounded function f defined on $[a, b]$ is riemann integrable if for every $\varepsilon > 0$, there exists a partition such that

$$\mathcal{L}(P, f) - \mathcal{U}(P, f) < \varepsilon$$

Definition 3.18. ((Alternate) Riemann Integrability) We say a bounded function f defined on $[a, b]$ is Riemann integrable if

$$\sup_P \mathcal{U}(P, f) = \inf_P \mathcal{L}(P, f)$$

Theorem 3.19. *These two definitions are equivalent*

Proof. \implies) Let f be Riemann Integrable in the sense that, given $\varepsilon > 0$, there exists partition such that $\mathcal{L}(P, f) - \mathcal{U}(P, f) < \varepsilon$. Note that, for any refinement P' of P , $\mathcal{U}(P, f) \leq \mathcal{U}(P', f)$ and $\mathcal{L}(P', f) \leq \mathcal{L}(P, f)$. Also note that, given any two partitions P_1 and P_2 , we have

$$\mathcal{L}(P_2, f) \geq \mathcal{U}(P_1, f)$$

To see this, take a common refinement of P_1 and P_2 , namely P' such that $\mathcal{U}(P_1, f) \leq \mathcal{U}(P', f)$ and $\mathcal{L}(P', f) \leq \mathcal{L}(P_2, f)$. Together with $\mathcal{U}(P', f) \leq \mathcal{L}(P', f)$, we get the desired result.

Therefore, we have that

$$\sup_P \mathcal{U}(P, f) \leq \inf_P \mathcal{L}(P, f)$$

Note that, the 1st definition intuitively speaks to how we can make the upper and lower sum arbitrarily close to each other, which speaks to the 2nd definition, which is that, the supremum and infimum resp. can be brought close together arbitrarily. Hence, the equality falls.

\impliedby) Let $\varepsilon > 0$, consider $\sup_P \mathcal{U}(P, f) = \inf_P \mathcal{L}(P, f) < \sup_P \mathcal{U}(P, f) + \varepsilon$. This means there exists some P_0 such that $\mathcal{L}(P_0, f) \leq \sup_P \mathcal{U}(P, f) + \varepsilon$ which then again gives P_1 such that $\mathcal{L}(P_0, f) \leq \mathcal{U}(P_1, f) + \varepsilon$. Since \mathcal{L} is a decreasing function for increasing partitions (more finer partitions) and \mathcal{U} is decreasing for finer partitions, we combine P_0 and P_1 to yield a finer partition P''' such that $\mathcal{L}(P''', f) - \mathcal{U}(P''', f) < \varepsilon$. We are done.

□